Aspects of Krylov Complexity

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Overview

Part I: Operator Growth

Part II: Unstable saddle

Part III: SYK model

Part IV: Double Scaled SYK

PART I

Operator Growth

For a given hamiltonian **H** and operator $\mathcal{O}$. Under the time evolution the operator becomes complicated, $\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt}$

$$\begin{aligned}\mathcal{O}(t) &= \mathcal{O} - it\,[H,\mathcal{O}] - \frac{t^2}{2!}\,[H,[H,\mathcal{O}]] + \frac{it^3}{3!}\,[H,[H,[H,\mathcal{O}]]] + \cdots \\ &= \mathcal{O} - it\mathcal{L}\mathcal{O} - \frac{t^2}{2!}\mathcal{L}^2\mathcal{O} + \frac{it^3}{3!}\mathcal{L}^3\mathcal{O}\cdots = e^{i\mathcal{L}t}\mathcal{O}.\end{aligned}$$

Liouvillian $\mathcal{L}(\bullet) = [H,\bullet]$

Example

$$H = -\sum_{i=1}^{N} Z_i Z_{i+1} - g \sum_{i=1}^{N} X_i - h \sum_{i=1}^{N} Z_i$$

$$\mathcal{L} Z_1 = [H, Z_1] \sim Y_1$$
$$\mathcal{L}^2 Z_1 = [H, [H, Z_1]] \sim Y_1 + X_1 Z_2$$
$$\mathcal{L}^3 Z_1 = [H, [H, [H, Z_1]]] \sim Y_1 + X_1 Y_2 + Y_1 Z_2$$
$$\mathcal{L}^4 Z_1 = [H, [H, [H, [H, Z_1]]]] \sim X_1 + Y_1 + Z_1 X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 + X_1 Z_2 + \cdots$$

Lanczos Algorithm

- Form the basis: $\{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \cdots\}$

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$$\langle A \mid B \rangle = \frac{1}{D} \operatorname{Tr}\left(A^\dagger B\right) \tag{1}$$

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- $1 \leq$Krylov Space with dimensions $\leq D^2 - D + 1$.

Lanczos coefficients

- Start with normalized vector $|\mathcal{O}_0)$[1]
- Define $|\mathcal{O}_1) := b_1^{-1}\mathcal{L}|\mathcal{O}_0)$ where $b_1^2 = (\mathcal{O}_0\mathcal{L}|\mathcal{L}\mathcal{O}_0)$.
- Repeat the procedure

$$
\begin{aligned}
|\mathcal{A}_n) &:= \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2}) \\
b_n &:= \sqrt{(\mathcal{A}_n|\mathcal{A}_n)} \\
|\mathcal{O}_n &:= b_n^{-1}|\mathcal{A}_n)
\end{aligned}
$$

Sequence $\{b_n\}$ is called **Lanczos coefficients**.
Set of orthonormal basis states $|\mathcal{O}_n)$, is called **Krylov basis**.

[1] **arXiv:1812.08657**.

[2] Time evolution: $|\mathcal{O}(t)) = \sum_{n=0}^{\mathcal{K}-1} i^n \phi_n(t) \,|\, \mathcal{O}_n\Big)$

The ϕ_n's satisfy the following equation

$$\dot{\phi}_n = b_n \phi_{n-1}(t) - b_{n+1}\phi_{n+1}(t)$$

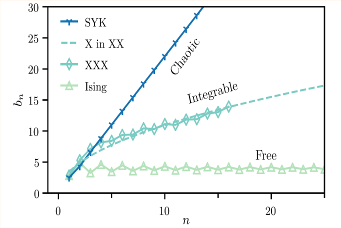
Unitarity: $\sum_{n=0}^{K-1} |\phi_n(t)|^2 = 1$

Krylov complexity: $\sum_{n=0}^{K-1} n|\phi_n(t)|^2$

[2] **arXiv:1812.08657**.

Lanczos coefficients

[3]



Behavior of lanczos coefficients

Chaotic system:
$b_n \sim \alpha n, \quad C_K(t) \sim e^{2\alpha t}$

Integrable system:
$b_n \sim \alpha n^{\delta}, \; C_K(t) \sim (\alpha t)^{1/(1-\delta)}$

[3] **arXiv:1812.08657**.

Universal Operator growth hypothesis

"For a chaotic system Lanczos coefficients grow linearly and this is the fastest growth possible"[4]

$$b_n = \alpha n + \gamma + O(1)$$

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Does linear $b_n \implies$ chaos?

No!!!

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PART II

Unstable saddle

Classical LMG model[5]:

$$H = x + 2Jz^2$$

where $X = (x, y, z)$ are classical SU(2) spin variables satisfying $\{X_i, X_j\} = \epsilon_{ijk} X_k$, $J > 1/2$ and $x^2 + y^2 + z^2 = 1$.

[5] **Does scrambling equal chaos? arXiv:1912.11063** .

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The growth exponent of the trajectories near the saddle is $\omega_{saddle} = \sqrt{2J - 1}$

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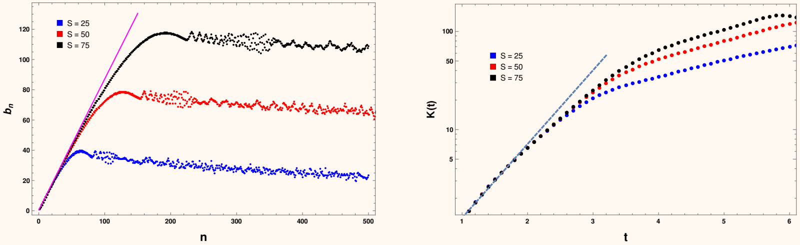


Figure: a) The growth[6] of b_n with n. b) The growth of $K(t)$ with t.

[6] **arxiv:2203.03534**.

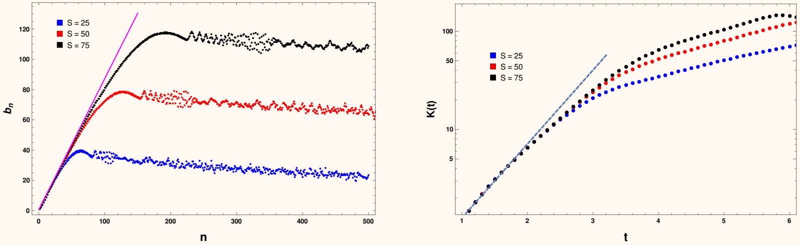


Figure: a) The growth[6] of b_n with n. b) The growth of $K(t)$ with t.

Krylov complexity detects scrambling in integrable system, due to unstable saddle point!!!

[6]**arxiv:2203.03534**.

PART III

Method of moments

The time evolution is in Krylov basis: $|O(t)\rangle = \sum_{n=0}^{K-1} i^n \varphi_n(t)\,|O_n\rangle$

Autocorrelation function

$$C(t) \equiv \varphi_0 = \langle O(t) \mid O_0\rangle$$

It contains the full information about the growth.

$$C(-it) := \sum_{k=0}^{\infty} \mu_{2k} \frac{t^{2k}}{(2k)!} \quad \Rightarrow \quad \mu_{2k} = (-1)^k \frac{d^{2k}}{dt^{2k}} C(t)\Big|_{t=0}.$$

$$b_1^2 \cdots b_n^2 = det(\mu_{i+j})_{0\leq i,j\leq n} \qquad \{\mu_{2k}\} \Rightarrow \{b_k\}$$

If we know the analytic results for the autocorrelation function then we can get some analytic results for moments, lanczos coefficients and Krylov complexity.

SYK model

The hamiltonian is given by

$$H = \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} j_{i_1 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q} \tag{2}$$

Mean:

$$\left\langle j_{i_1 \cdots i_q} \right\rangle = 0$$

Variance:

$$\left\langle j^2_{i_1 \cdots i_q} \right\rangle = 2^{q-1} \frac{(q-1)! J^2}{q N^{q-1}}$$

Consider initial operator[7] $\mathcal{O}_0 = \sqrt{2}\psi_1$ The auto-correlation function

$$C(t) = 1 + \frac{2\log((\mathcal{J}t))}{q}$$

Moments: $m_{2n} = \frac{1}{q}\mathcal{J}^{2n}T_{n-1} + O(1/q^2), \quad n \geq 1$

Tangent numbers: $\{T_{n-1}\}_{n=1}^{\infty} = 1, 2, 16, \cdots$

Lanczos coefficients

$$b_n = \begin{cases} \mathcal{J}\sqrt{2/q} + O(1/q), & n = 1 \\ \mathcal{J}\sqrt{n(n-1)} + O(1/q), & n > 1 \end{cases} \quad \xrightarrow{\text{large-}n} \quad b_n \sim \alpha n$$

K-complexity: $\frac{2}{q}\sinh^2(\mathcal{J}t) + O(1/q^2)$

[7] **arXiv:1812.08657** .

Large q-correction

We next consider the $1/q^2$ correction[8] to the auto-correlation function as follows

$$C(t) = 1 + \frac{g(t)}{q} + \frac{h(t)}{q^2}$$

[8] **Tarnopolsky arXiv:1801.06871**.

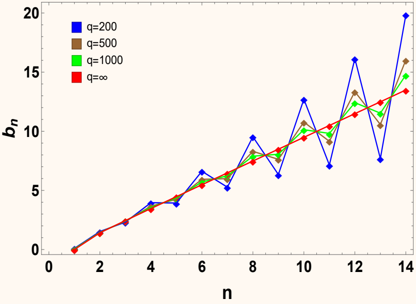
Lanczos coefficients

First few lanczos analytically[9]

$$b_1 = \mathcal{J}\sqrt{\frac{2}{q}}\,, \quad b_2 = \sqrt{2}\mathcal{J} + \frac{31}{\sqrt{2}}\frac{\mathcal{J}}{q} + O(1/q^2)\,, \quad b_3 = \sqrt{6}\mathcal{J} - \frac{65}{\sqrt{6}}\frac{\mathcal{J}}{q} + O(1/q^2)\,,$$
$$b_4 = \sqrt{12}\mathcal{J} + \frac{343}{\sqrt{12}}\frac{\mathcal{J}}{q} + O(1/q^2)\,, \quad b_5 = \sqrt{20}\mathcal{J} - \frac{8677}{18\sqrt{20}}\frac{\mathcal{J}}{q} + O(1/q^2)\,,$$
$$b_6 = \sqrt{30}\mathcal{J} + \frac{74987}{60\sqrt{30}}\frac{\mathcal{J}}{q} + O(1/q^2)\,, \quad b_7 = \sqrt{42}\mathcal{J} - \frac{18811}{12\sqrt{42}}\frac{\mathcal{J}}{q} + O(1/q^2)\,,$$
$$b_8 = \sqrt{56}\mathcal{J} + \frac{4830986}{1575\sqrt{56}}\frac{\mathcal{J}}{q} + O(1/q^2)\,, \quad b_9 = \sqrt{72}\mathcal{J} - \frac{17822817}{4900\sqrt{72}}\frac{\mathcal{J}}{q} + O(1/q^2)\,,$$
$$b_{10} = \sqrt{90}\mathcal{J} + \frac{71870293}{11760\sqrt{90}}\frac{\mathcal{J}}{q} + O(1/q^2). \tag{3}$$

[9] **arxiv:2210.02474**.

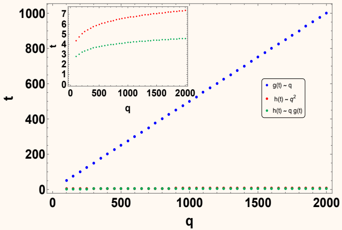
a)Behaviour of lanczos coefficients[10]



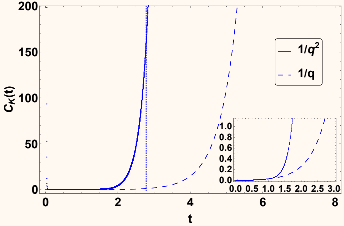
[10] **arxiv:2210.02474**.

Results[11]

a) Different time scales



b) K-complexity



[11] **arxiv:2210.02474**.

PART IV

Double Scaled SYK

- Two stage limit[12]:

  First take $N \to \infty$, with fixed q.

  Then take $q \to \infty$

[12] **arXiv:1604.07818**.

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- Double scaled limit[13]:

  Take $N, q \to \infty$ simultaneously keeping $\lambda = \frac{q^2}{N}$ fixed.

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[13] **arXiv:1811.02584**.

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- $\tilde{t}_s = q\tilde{t}_c$ [14][15]

[12] **arXiv:1604.07818**.
[13] **arXiv:1811.02584**.
[14] **arXiv:2209.09999**..
[15] **arXiv:2205.00315**..

Hyperscrambling

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Scrambling time :

$C_K(\tilde{t}_c^*) \sim O(1) \implies \tilde{t}_c^* \sim \frac{1}{2q\mathcal{J}}\ln q$

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Violation of bound on chaos?

We can have following situations[16] for $q \sim N^{\alpha}$:

$\alpha < 1/2$ K-local, $\alpha = 1/2$ transition, $\alpha > 1/2$- K-non-local

[16] **L. Erdos and D. Schroeder- [arXiv:1407.1552]**.

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Hamburger moment problem: Suppose $w(x)$ is a distribution on $\mathbb{R}$ whose moments $\{m_0, m_2, \cdots\}$ are all known. What is $w(x)$?

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Carleman's condition(sufficient)

$$\lim_{N\to\infty} \sum_{n=1}^{N} m_{2n}^{-1/2n} = \infty \quad \longrightarrow \quad \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{b_n} = \infty$$

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$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{b_n} = 0$ Hamburger problem is indeterminate

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- Proving or deriving the operator growth hypothesis.

Thank You !