GRADUATE SCHOOL OF / FACULTY OF SCIENCE KYOTO UNIVERSITY

Extreme Universe
A New Paradigm for Spacetime and Matter from Quantum Information
Grant-in-Aid for Transformative Research Areas (A)

Many-body localization and quantum error correction in Sachdev-Ye-Kitaev type models

Novel Quantum States in Condensed Matter 2022

1 December 2022

Masaki TEZUKA (Kyoto Univ.)

SYK-related publications and collaborators

- Sachdev-Ye-Kitaev model
  - Proposal for experiment: PTEP 2017, 083I01 and arXiv:1709.07189
    - with Ippei Danshita and Masanori Hanada
  - Black Holes and Random Matrices: JHEP 1705(2017)118
    - with J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, and A. Streicher
  - Scrambling time: JHEP 1807(2018)124 with Hrant Gharibyan, M. Hanada, and S. H. Shenker
- SYK4+2
  - **Chaotic-integrable transition**: PRL **120**, 241603 (2018)
    - with Antonio M. García-García, Bruno Loureiro, and Aurelio Romero-Bermúdez
  - Characterization of quantum chaos: JHEP 1904(2019)082 and Phys. Rev. E **102**, 022213 (2020)
    - with Hrant Gharibyan, M. Hanada, and Brian Swingle
  - Related setups:
    - [short-range interactions] Phys. Rev. B **99**, 054202 (2019) with A. M. García-García
    - Uniform couplings: Phys. Lett. B **795**, 230 (2019) and J. Phys. A **54**, 095401 (2021) with Pak Hang Chris Lau, Chen-Te Ma, and Jeff Murugan

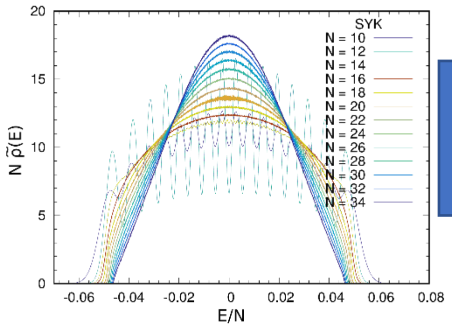
Also see our reply [PRL **126**, 109102 (2021)] to the comment by J. Kim and X. Cao [PRL **126**, 109101 (2021)]

- Quantitative analysis of **Fock-space localization in SYK4+2**
  - Many-body transition point and inverse participation ratio
    - Phys. Rev. Research **3**, 013023 (2021) with Felipe Monteiro, Tobias Micklitz, and Alexander Altland
  - Entanglement entropy
    - Phys. Rev. Lett. **127**, 030601 (2021) with F. Monteiro, A. Altland, David A. Huse, and T. Micklitz
- **Sparse SYK**: arXiv:2208.12098 with Onur Oktay, Enrico Rinaldi, M. Hanada and Enrico Nori
- **Quantum error correction**: in preparation with Yoshifumi Nakata

Introduction

Chaos and scrambling in quantum many-body systems

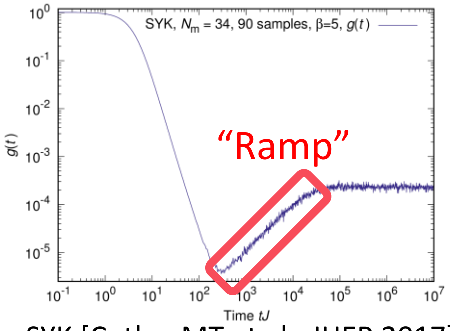
Density of states: not universal
(cf. Wigner semi-circle law for
Gaussian random matrices)



Fourier transformation of 2pt corr.

<ins>Energy spectrum in chaotic systems</ins>
≈ random matrix level statistics Wigner, … BGS conjecture

Spectral form factor



SYK [Cotler, MT et al., JHEP 2017]

Neighboring gap ratio

$$r = \frac{\min(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}$$

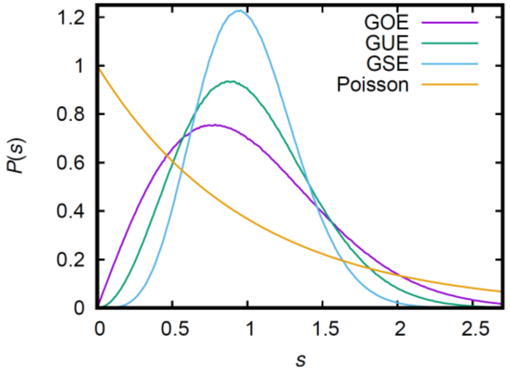
	Uncorrelated	GOE ($\mathbb{R}$)	GUE ($\mathbb{C}$)	GSE ($\mathbb{H}$)
$\langle r \rangle$	0.3863	0.5307	0.5996	0.6744

[Atas et al., PRL 2013]

Normalized gap distribution

$s_j = \frac{e_{j+1} - e_j}{\Delta(\bar{e})}$: universal

G*E: Gaussian ensembles



$e^{-it\hat{H}}$
$\approx 1 - it\hat{H} - \cdots$

<ins>Scrambling dynamics:</ins>
Delocalization of quantum information
cf. OTOC, Lyapunov spectrum
🡐 eigenstate wavefunctions?

$e^{-it\hat{H}}, |\epsilon_j| t \gg 2\pi$

$(\hbar = 1)$

Circular unitary ensemble
(level repulsion on unit circle)
not realized by t-independent
Hamiltonian time evolution

[D. A. Roberts and B. Yoshida, 1610.04903]

Early time Late time

ΔE large ΔE small

Contents

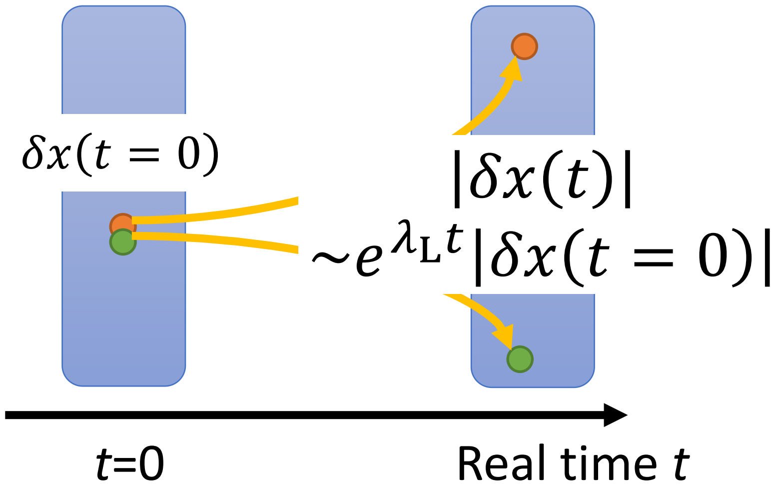
- Out-of-time-ordered correlators and the SYK model
- Chaotic-integrable transition in SYK_{4+2}
- Fock space localization: eigenstate localization and entanglement
- Quantum error correction with SYK-type models
- Case of binary-coupling sparse SYK

Lyapunov exponent and out-of-time-order correlators (OTOC)

$$F(t) = \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle \quad W(t) = e^{iHt}We^{-iHt}$$

Classical chaos:

Infinitesimally different initial coords



λ_{L}: Lyapunov exponent

$$\left(\frac{\partial x(t)}{\partial x(0)}\right)^2 = \{x(t), p(0)\}_{\mathrm{PB}}{}^2 \to e^{2\lambda_{\mathrm{L}} t}$$

Quantum dynamics:

$$C_T(t) = \langle [\hat{x}(t), \hat{p}(0)]^2\rangle$$

For operators V and W, consider

$$C(t) = \langle |[W(t), V(t=0)]|^2\rangle = \langle W^\dagger(t)V^\dagger(0)W(t)V(0)\rangle + \cdots$$

[Wiener 1938][Larkin & Ovchinnikov 1969]

OTOC $\sim e^{2\lambda_{\mathrm{L}} t}$ at long times, $\lambda_{\mathrm{L}} > 0$: chaotic

“Black holes are fastest quantum scramblers”
[P. Hayden and J. Preskill 2007] [Y. Sekino and L. Susskind 2008]
[Shenker and Stanford 2014]

$\lambda_{\mathrm{L}} \le 2\pi k_{\mathrm{B}} T/\hbar$ (chaos bound)
[J. Maldacena, S. H. Shenker, and D. Stanford, JHEP08(2016)106]

Sachdev-Ye-Kitaev model

N Majorana- or Dirac- fermions with all-to-all Gaussian random couplings

[Majorana version]

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

[A. Kitaev: talks at KITP (2015)]

cf. SY model [S. Sachdev and J. Ye, PRL 1993]

arXiv:cond-mat/**9212**030 (>1300 citations after 2015)

[Dirac version]

$$\hat{H} = \frac{1}{(2N)^{3/2}} \sum_{ij;kl} J_{ij;kl} \hat{c}_i^{\dagger} \hat{c}_j^{\dagger} \hat{c}_k \hat{c}_l$$

[A. Kitaev's talk]

[S. Sachdev: PRX **5**, 041025 (2015)]

Studied for long time in the nuclear theory context

[French and Wong (1970)][Bohigas and Flores (1971)]

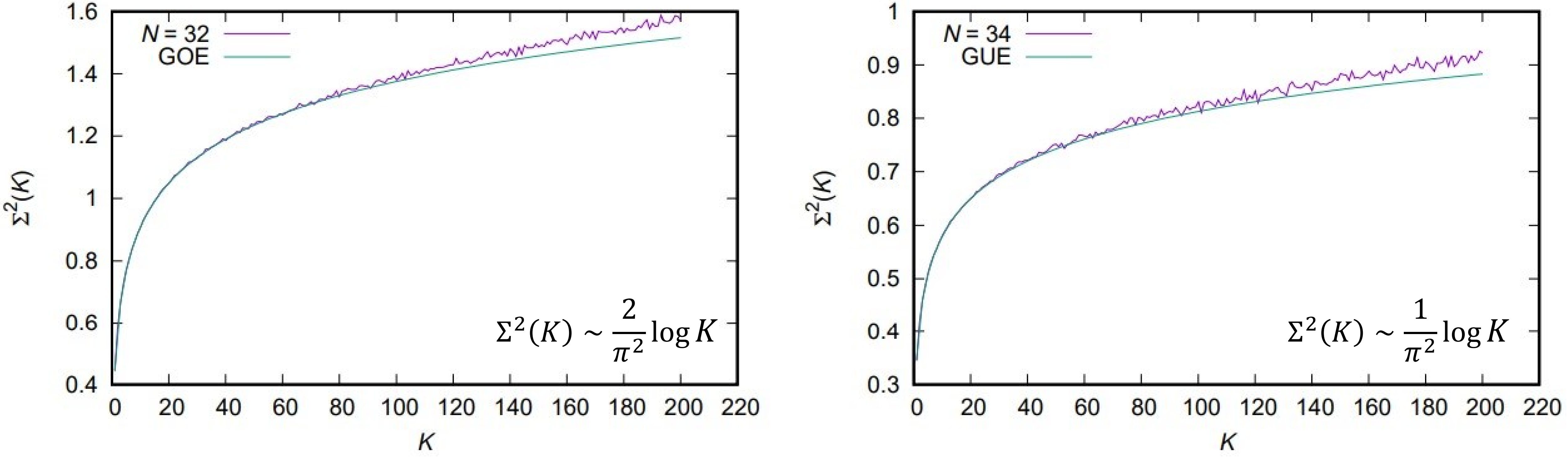
"Two-body Random Ensemble"

Solvable in the large-*N* limit, maximally chaotic, holographic correspondence to 1+1d gravity

➔ Many variants of the model: bosonic, multiflavor, supersymmetric, nonhermitian, …

Appendix of [Gharibyan, Hanada, Shenker, Tezuka: JHEP07(2018)124]
cf. [S. Torquato, Phys. Rep. **745**, 1 (2018)]; S. Sakai's talk earlier today

Hyperuniform distribution of eigenvalues



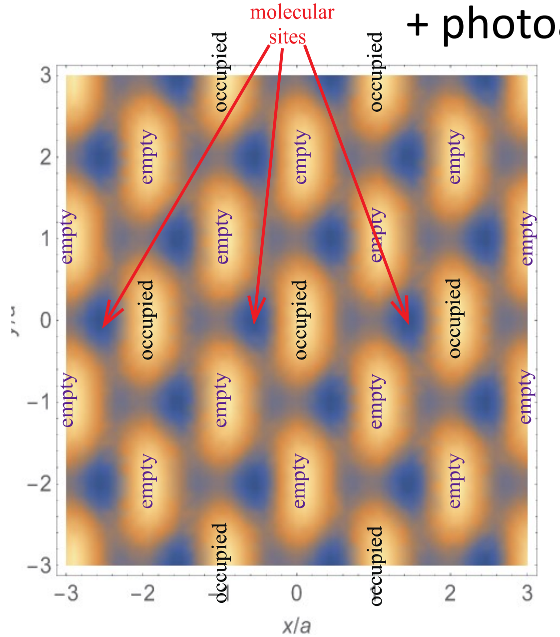
- Shift the origin so that $\langle \epsilon_j \rangle = 0$, rescale so that $\mathrm{Tr} H^2 = \sum_j \epsilon_j^2 = \text{const}.$
- Unfold each sample using the density profile $\langle \rho(E) \rangle$.
- Compute the number variance $\Sigma^2(K) = \langle n^2(E,K) \rangle - \langle n(E,K) \rangle^2 = \langle n^2(E,K) \rangle - K^2$.

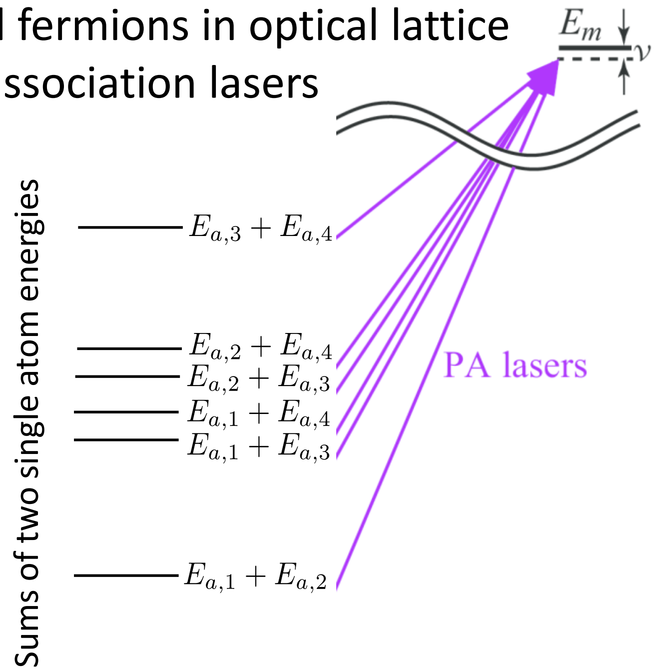
$n(E,K)$: number of levels in $[E, E + K\bar{\Delta}]$

Proposals for experimental realization

[I. Danshita, M. Hanada, MT: PTEP **2017**, 083I01 (2017)]

Ultracold fermions in optical lattice
+ photoassociation lasers





s: molecular levels

$$\hat{H}_{\rm m}=\sum_{s=1}^{n_s}\left\{\nu_s\hat{m}_s^\dagger\hat{m}_s+\sum_{i,j}g_{s,ij}\left(\hat{m}_s^\dagger\hat{c}_i\hat{c}_j-\hat{m}_s\hat{c}_i^\dagger\hat{c}_j^\dagger\right)\right\}.$$

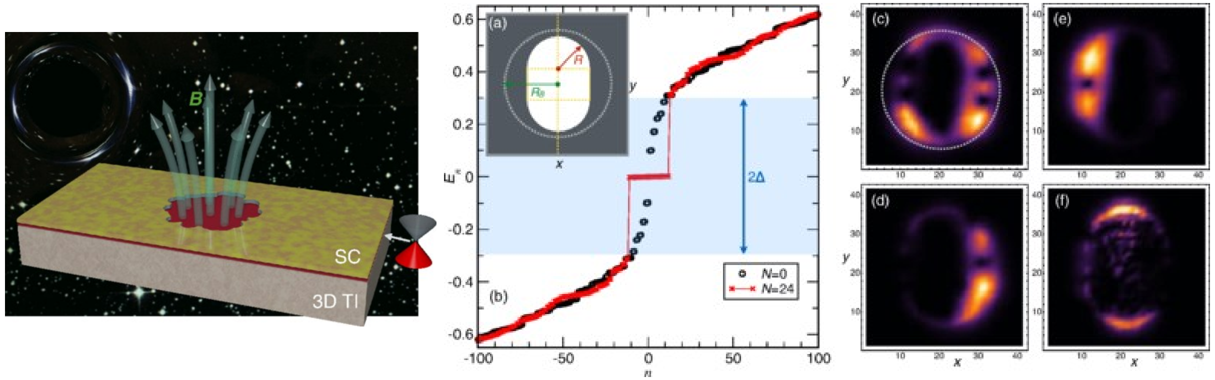
$|\nu_s|\gg|g_{s,ij}|$

$$\hat{H}_{\rm eff}=\sum_{s,i,j,k,l}\frac{g_{s,ij}g_{s,kl}}{\nu_s}\hat{c}_i^\dagger\hat{c}_j^\dagger\hat{c}_k\hat{c}_l.$$

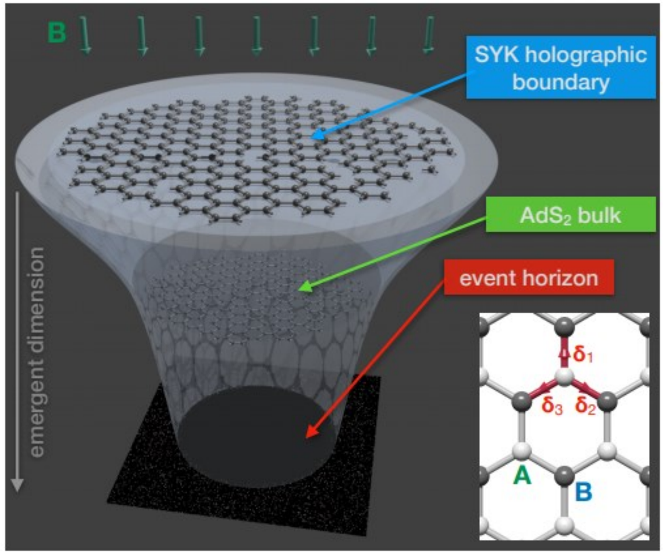
Approximately Gaussian if many levels are used

[D. I. Pikulin and M. Franz, PRX **7**, 031006 (2017)]

N quanta of magnetic flux through a nanoscale hole



[A. Chen, R. Ilan, F. de Juan, D.I. Pikulin, M. Franz, PRL **121**, 036403 (2018)]



Graphene flake with an irregular boundary in magnetic field

NMR experiment for the SYK model

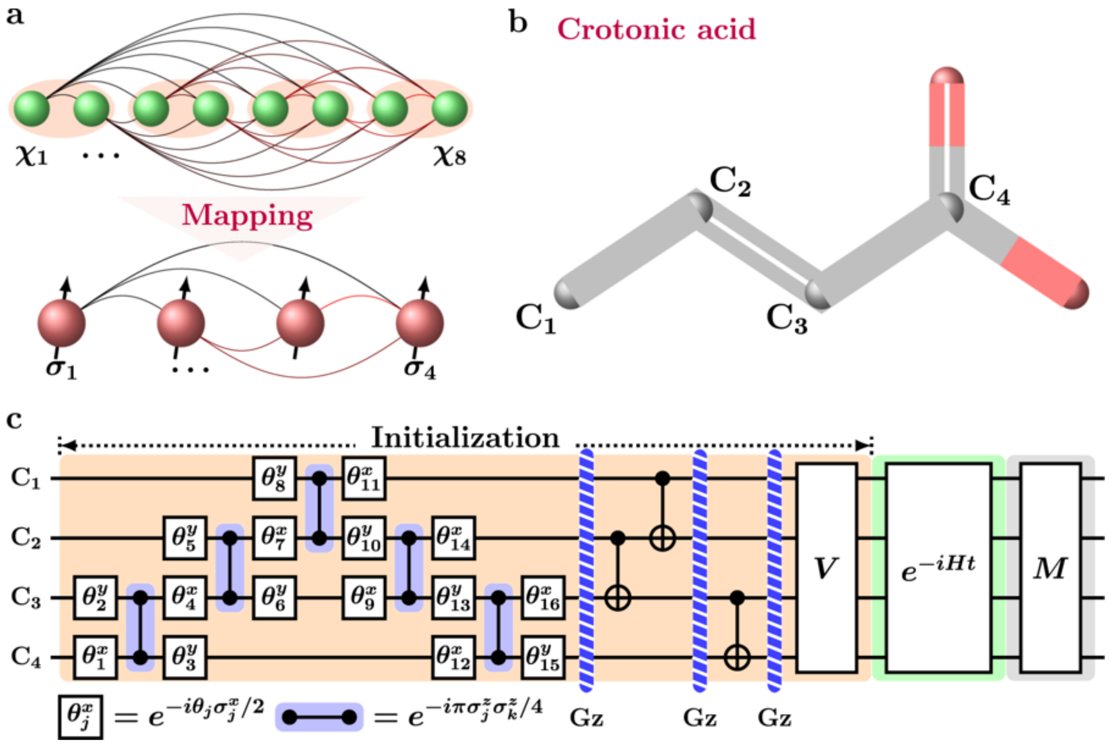
"Quantum simulation of the non-fermi-liquid state of Sachdev-Ye-Kitaev model" Zhihuang Luo, Yi-Zhuang You, Jun Li, Chao-Ming Jian, Dawei Lu, Cenke Xu, Bei Zeng and Raymond Laflamme, npj Quantum Information **5**, 53 (2019)

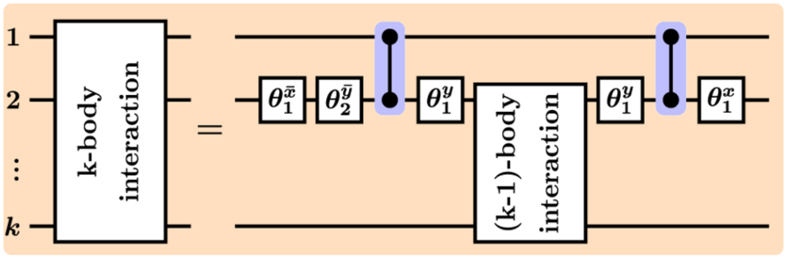
$$H = \frac{J_{ijkl}}{4!}\chi_i\chi_j\chi_k\chi_l + \frac{\mu}{4}C_{ij}C_{kl}\chi_i\chi_j\chi_k\chi_l$$

$$\chi_{2i-1} = \frac{1}{\sqrt{2}}\sigma_x^1\sigma_x^2\cdots\sigma_x^{i-1}\sigma_z^i,\ \chi_{2i} = \frac{1}{\sqrt{2}}\sigma_x^1\sigma_x^2\cdots\sigma_x^{i-1}\sigma_y^i.$$

$$H = \sum_{s=1}^{70} H_s = \sum_{s=1}^{70} a_{ijkl}^s\sigma_{\alpha_i}^1\sigma_{\alpha_j}^2\sigma_{\alpha_k}^3\sigma_{\alpha_l}^4$$

$$e^{-iH\tau} = \left(\prod_{s=1}^{70} e^{-iH_s\tau/n}\right)^n + \sum_{s<s'}\frac{[H_s, H_{s'}]\tau^2}{2n} + O(|a|^3\tau^3/n^2),$$





A. M. García-García, A. Romero-Bermúdez, B. Loureiro, and MT, Phys. Rev. Lett. **120**, 241603 (2018)

SYK_{4+2}

Also see our reply [PRL **126**, 109102 (2021)] to the comment by J. Kim and X. Cao [PRL **126**, 109101 (2021)]

Q.: Minimum requirements for chaotic behavior? (→ gravity interpretation?)
Study a simple model with analytical + numerical methods

$$\hat{H} = \underbrace{\sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}_{\mathrm{SYK}_4} + i\underbrace{\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b}_{\mathrm{SYK}_2}$$

Gaussian random couplings

J_{abcd}: average 0, standard deviation $\frac{\sqrt{6}J}{N^{3/2}}$

K_{ab}: average 0, standard deviation $\frac{K}{\sqrt{N}}$

$J = 1$: unit of energy

Normalization here: $\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$

SYK_4 as unperturbed Hamiltonian,
K controls the strength of SYK_2 (one-body random term, solvable)

Here we take (GUE) $N \equiv 2,6 \pmod 8$

Both terms respect charge parity in complex fermion description
➔ Full numerical exact diagonalization (ED) of $2^{N/2-1}$-dimensional matrix, $N \lesssim 34$ possible

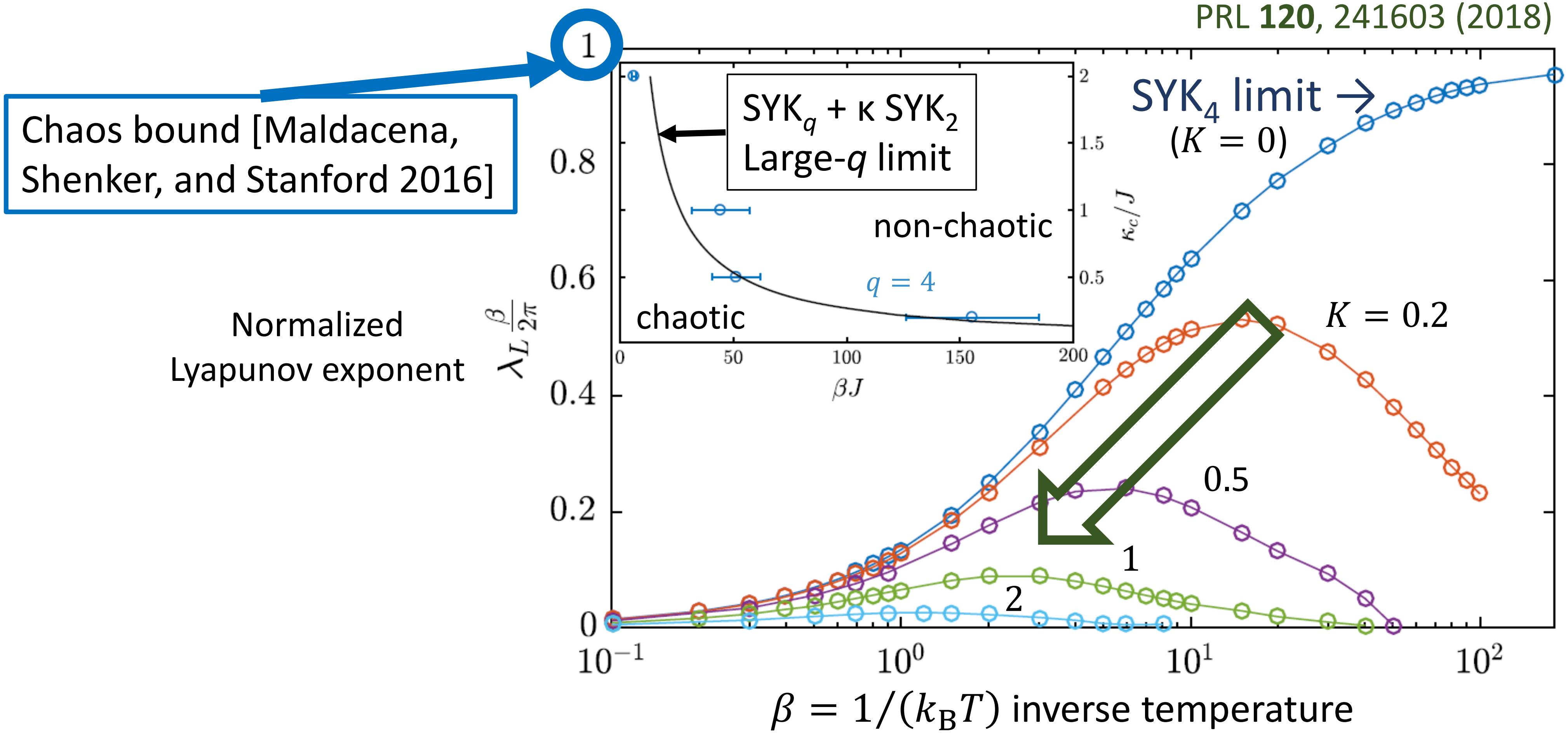
Large-N calculation for Out-of-Time Order Correlator (OTOC)

$$\hat{H} = \sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

$SYK_{q=4}$ SYK_2

K_{ab}: standard deviation $\frac{K}{\sqrt{N}}$

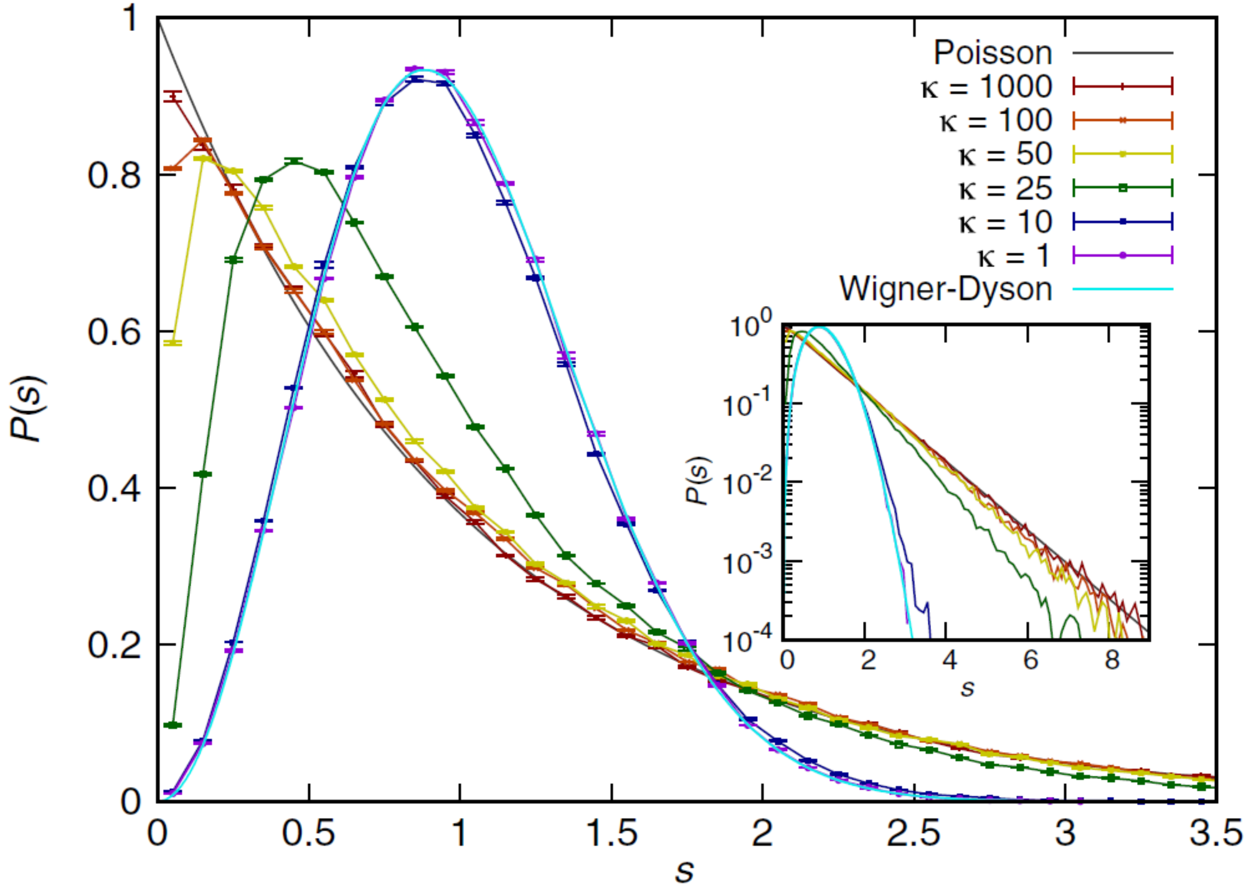
PRL **120**, 241603 (2018)



Deviation from the chaos bound as SYK_2 component is introduced

PRL **120**, 241603 (2018)

RMT-like behavior lost as SYK_2 term is introduced



N=30, Central 10 % of eigenvalues

$P(s)$: level spacing distribution

Ratio of consecutive level spacing $E_{i+1} - E_i$
to the local mean level spacing Δ
(requires unfolding of the spectrum)

SYK_4 limit (small *K*):
Obeys random matrix theory (RMT)
(GUE (Gaussian Unitary Ensemble) if $N \equiv 2{,}6 \pmod 8$)

SYK_2 (large *K*): Poisson (e^{-s})

Also see: T. Nosaka, D. Rosa, and J. Yoon, JHEP **1809**, 041 (2018) for other symmetry cases
cf. A. V. Lunkin, K. S. Tikhonov, and M. V. Feigel'man, PRL 121, 236601 (2018); Y. Yu-Xiang, F. Sun, J. Ye, and W. M. Liu, PRB **104**, 235133 (2021), …

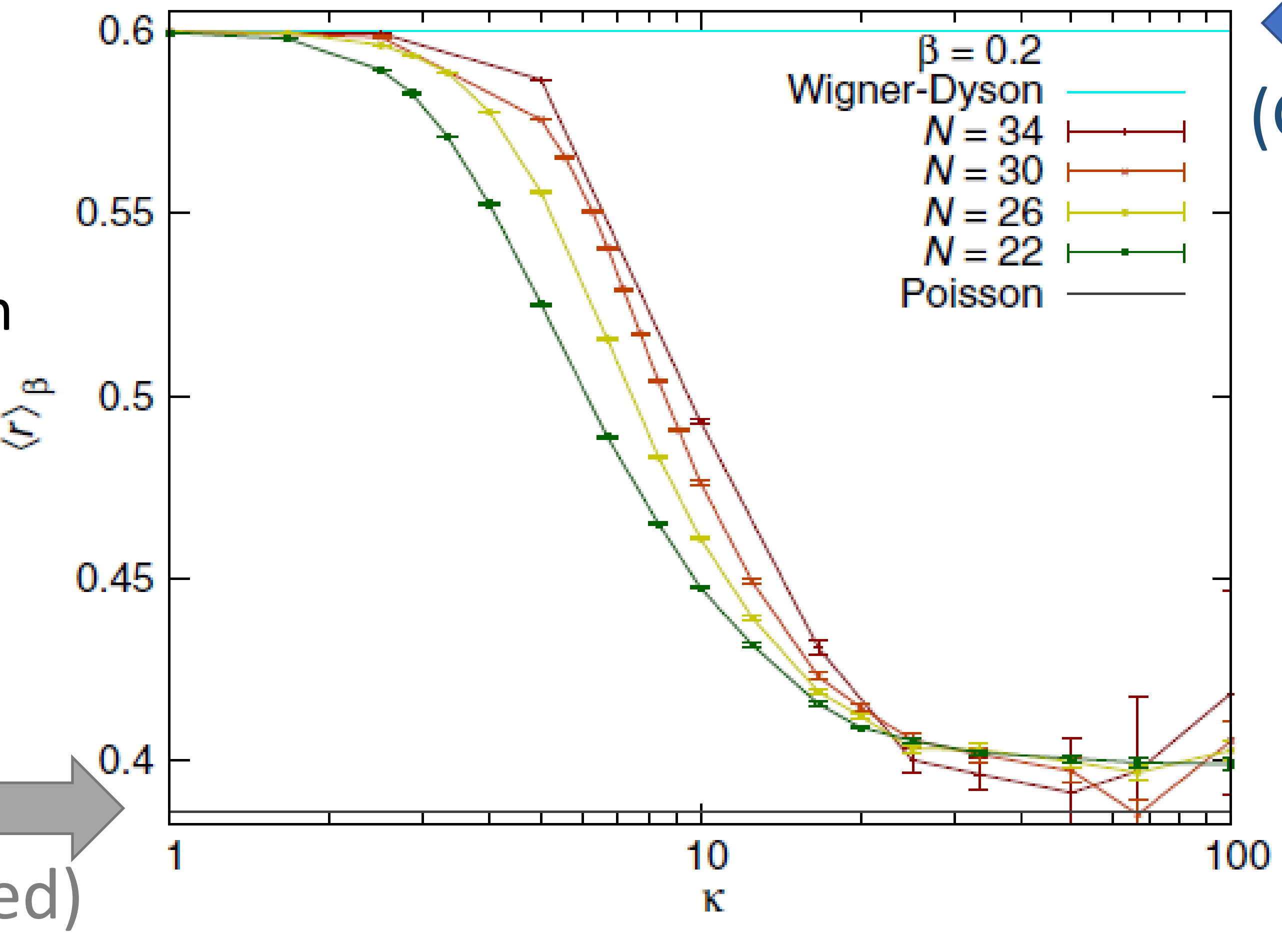
PRL **120**, 241603 (2018)

$\mathrm{SYK}_{q\geq 4} + \mathrm{SYK}_2$: breakdown of chaos

$$\hat{H} = \sum_{1\leq a<b<c<d}^{N} \underbrace{J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}_{\mathrm{SYK}_4} + i \sum_{1\leq a<b}^{N} \underbrace{K_{ab}\hat{\chi}_a\hat{\chi}_b}_{\mathrm{SYK}_2}$$

K_{ab}: standard deviation = $\kappa/\sqrt{N}$

Averaged ratio between neighboring energy level separations



GUE (Gaussian Unitary Ensemble)

Poisson (uncorrelated)

➔ Understood as localization of the many-body wave function in Fock space

Many-body localization

ETH: "(almost) all eigenstates are thermal
(expectation values of operators = microcanonical average)"

- Anderson localization: concept in non-interacting systems
  - Localization of wavefunctions due to scatterings at impurities
  - Many experiments in cold atom gases, optical fibers, etc.
- MBL: does localization occur in interacting systems?

  [Gornyi, Mirlin, Polyakov 2005, Basko, Aleiner, Altshuler 2006, Oganesyan and Huse 2007, … many others]
  - Memory of initial conditions remains accessible at long times
  - Reduced density matrix on a subsystem does not approach a thermal one
  - Energy eigenstates do not obey Eigenstate Thermalization Hypothesis (ETH)
  - Area law, rather than volume law, of entanglement entropy
  - "Standard model": spin-1/2 Heisenberg model + random field in *z* direction
    - Much debate on the location of the localization transition

$$\hat{H} = \sum_i^N \hat{S_i} \cdot \widehat{S_{i+1}} + \sum_i^N h_i \widehat{S_i^z}$$

$h_i \in [-h, h]$ uniform distribution

F. Monteiro, T. Micklitz, MT, and A. Altland, Phys. Rev. Research **3**, 013023 (2021)

Our model and choice of basis

$$\mathrm{SYK}_4 + \delta\ \mathrm{SYK}_2$$

$$\hat{H} = -\sum_{1\le a<b<c<d}^{N=2N_\mathrm{D}} J'_{abcd}\hat{\psi}_a\hat{\psi}_b\hat{\psi}_c\hat{\psi}_d + i\sum_{1\le a<b}^{N} K_{ab}\hat{\psi}_a\hat{\psi}_b$$

Block-diagonalize the SYK_2 part
(the skew-symmetric matrix (K_{ab}) has eigenvalues $\pm v_j$)

$$\hat{H} = -\sum_{1\le a<b<c<d}^{2N_\mathrm{D}} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le j\le N}^{2N_\mathrm{D}} v_j\hat{\chi}_{2j-1}\hat{\chi}_{2j}$$

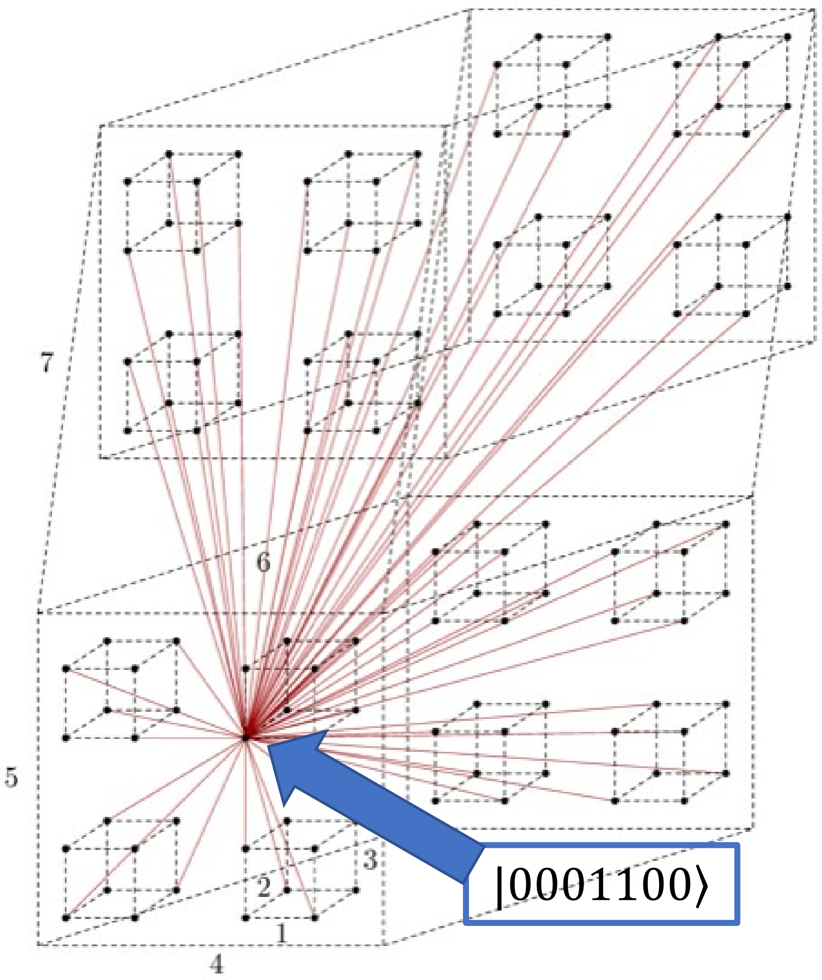
Normalization of J_{abcd}, v_j :
SYK_4 bandwidth = 1,
Width of v_j distribution = δ

We choose $\{\hat{\psi}_a,\hat{\psi}_b\} = \{\hat{\chi}_a,\hat{\chi}_b\} = 2\delta_{ab}$ as the normalization for the $N = 2N_\mathrm{D}$ Majorana fermions. For $\hat{c}_j = \frac{1}{2}\left(\hat{\chi}_{2j-1} + \mathrm{i}\hat{\chi}_{2\mathrm{j}}\right)$ we have $\{\hat{c}_i,\hat{c}_j^\dagger\} = \delta_{ij}$.

F. Monteiro, T. Micklitz, MT, and A. Altland, Phys. Rev. Research **3**, 013023 (2021)

Our model and choice of basis

$N = 2N_{\mathrm{D}} = 14$: $2^7 = 128$ states



Basis diagonalizing the complex fermion number operators $\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j$ → Sites: the $2^{N_{\mathrm{D}}}$ vertices of an N_{D}-dim. hypercube.

$$\hat{c}_j = \frac{1}{2}(\hat{\chi}_{2j-1} + \mathrm{i}\hat{\chi}_{2\mathrm{j}})$$

$$\hat{H} = -\sum_{1 \le a<b<c<d}^{2N_{\mathrm{D}}} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le j\le N}^{N_{\mathrm{D}}} v_j \hat{\chi}_{2j-1}\hat{\chi}_{2j}$$

$$= -\sum_{1 \le a<b<c<d}^{2N_{\mathrm{D}}} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + \sum_{1\le j\le N}^{N_{\mathrm{D}}} v_j(2\hat{n}_j - 1)$$

Each term of SYK_4 connects vertices with distance = 0, 2, 4.

For $N = 14$, each vertex is directly connected with
1 (distance=0, itself) + 21 (distance=2) + 35 (distance=4)
vertices out of the possible $2^N = 128$ (64 per parity).

Analytical results

Method: Exact matrix integral representation; mapping to a supersymmetric sigma model; saddle point equations; effective medium approximation

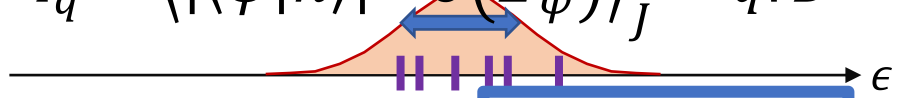
PRR 3, 013023 (2021)

$(N_{\mathrm{D}} = \frac{N}{2}, c = O(1), D = 2^{N_{\mathrm{D}}-1})$

I

- I: Average density of states (ADoS) at band center $\nu = cD$
- $I_q \equiv \langle |\langle \psi | n \rangle|^{2q} \delta(E_\psi) \rangle_J = q!\, D^{1-q}$

Fully delocalized



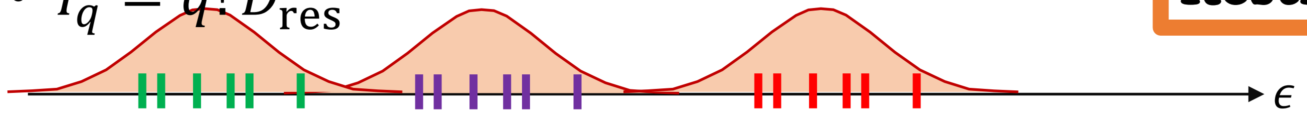
SYK4 bandwidth=1

$N_{\mathrm{D}}^{-1/2}$

II

- II: ADoS $\nu = \frac{cD}{\sqrt{N_{\mathrm{D}}}\delta}$, spread of wave functions $D_{\mathrm{res}} \simeq \frac{D}{\sqrt{N_{\mathrm{D}}}\delta}$
- $I_q = q!\, D_{\mathrm{res}}^{1-q}$

Restricted

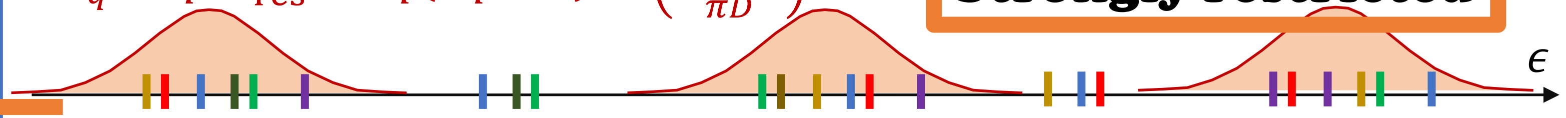


1

III

- III: ADoS $\nu = \frac{cD}{\sqrt{N_{\mathrm{D}}}\delta}$, spread of wave functions $D_{\mathrm{res}} \simeq \frac{D}{\sqrt{N_{\mathrm{D}}}\delta^2}$
- $I_q = q!\, D_{\mathrm{res}}^{1-q} = q(2q-3)!!\, \left(\frac{4\sqrt{N}\delta^2}{\pi D}\right)^{q-1}$

Strongly restricted



δ_{c}

IV

$$\delta_{\mathrm{c}} = \frac{N_{\mathrm{D}}^2}{4\sqrt{3}} \log N_{\mathrm{D}} \text{ for large } N$$

- IV: All eigenstates localized to $\mathcal{O}(1)$ sites

δ: typical fermion mass $\propto$ SYK_2 coefficient

<u>Eigenenergy spectral statistics (for odd N case for simplicity)</u>

$$\widetilde{K}(s) = 1 - \frac{\sin^2 s}{s^2} + \delta\left(\frac{s}{\pi}\right),$$

$s = \pi\omega\nu$ in I, II, III :
agrees with Gaussian Unitary Ensemble (GUE)

IV: **Poisson statistics**

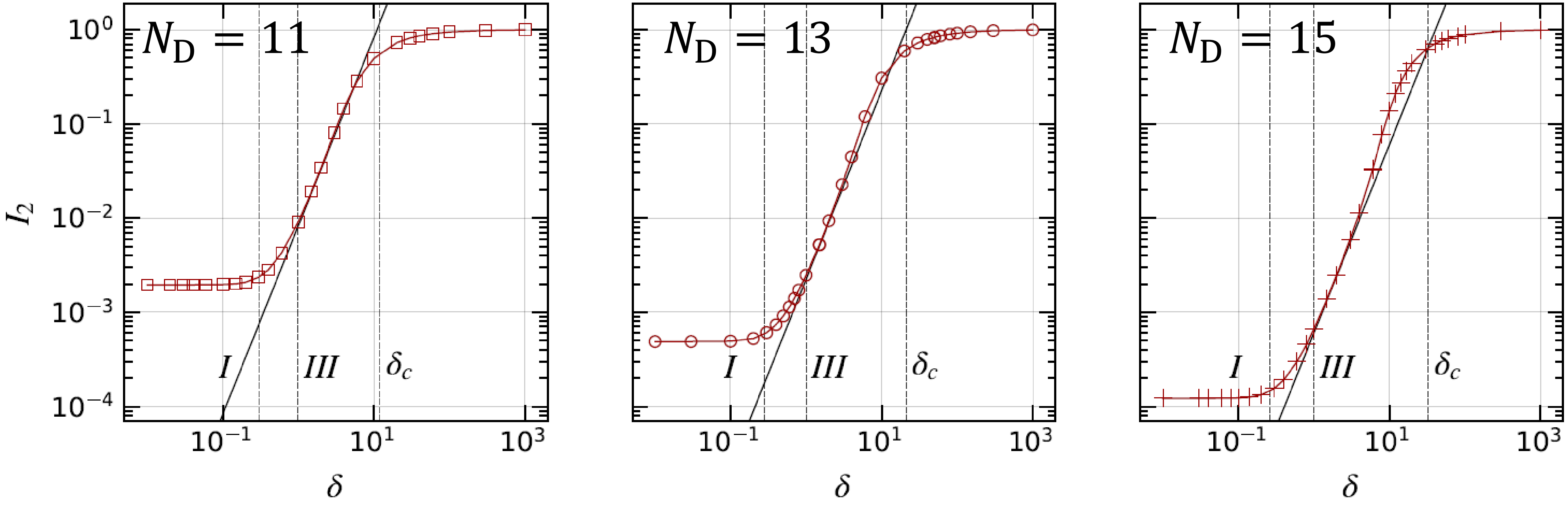
PRR 3, 013023 (2021)

Inverse participation ratio for Regime III

IPR $I_2 =$ average of $\sum_n |\langle\psi|n\rangle|^4$ for normalized ψ, $\frac{1}{D} \leq I_2 \leq 1$

Equal weights

Single non-zero element



$$I_q = \frac{q(2q-3)\,!!}{\delta^{2(1-q)}}\left(\frac{\pi D}{4\sqrt{N_D}}\right)^{1-q} = q(2q-3)\,!!\,\left(\frac{4\sqrt{N_D}\delta^2}{2^{N-1}\pi}\right)^{q-1} \text{ in III}$$

Central 1/7 of the energy spectrum

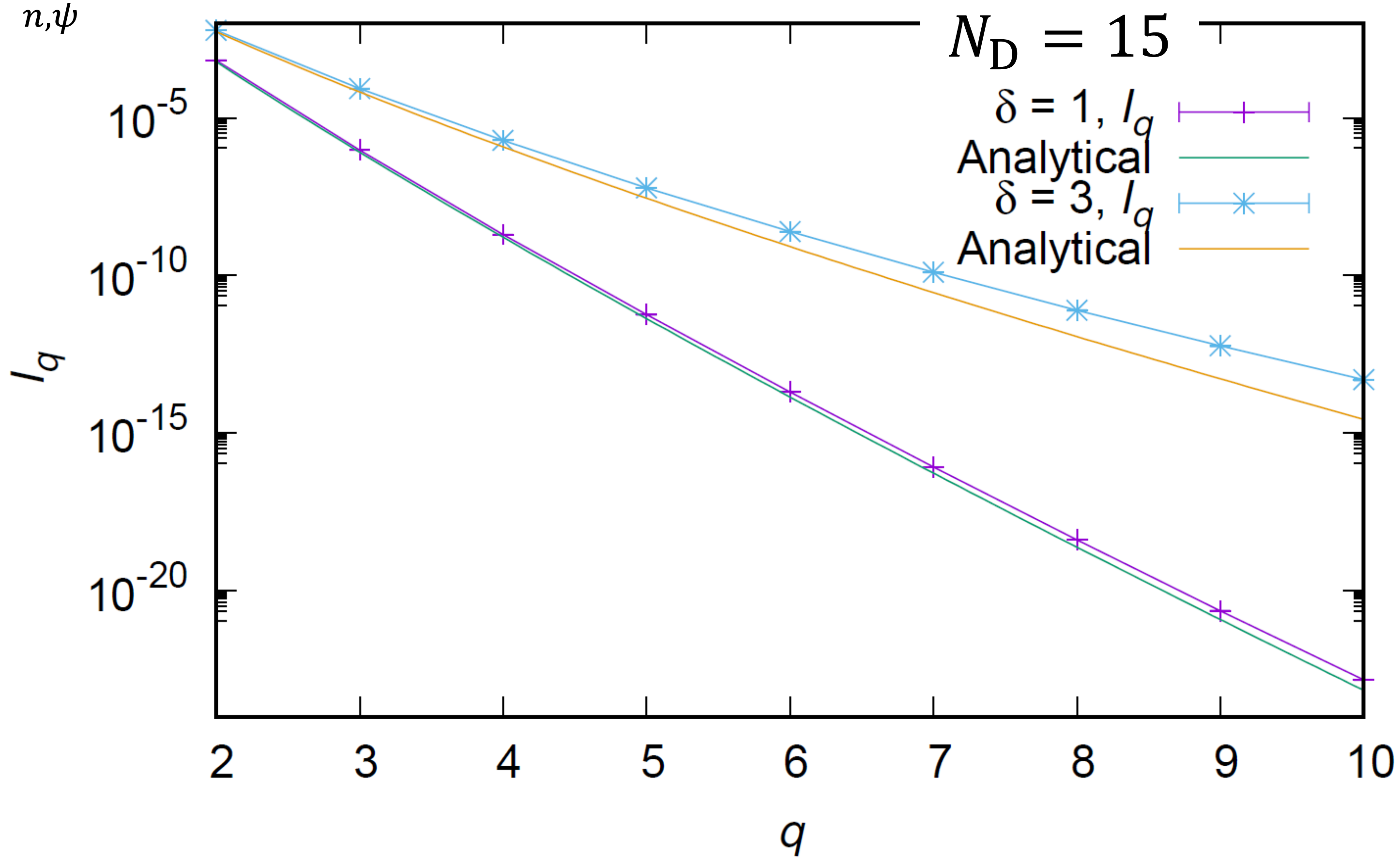
Higher moments of eigenvectors

PRR 3, 013023 (2021)

Analytical prediction:

$$I_q = \frac{q(2q-3)\,!!}{\delta^{2(1-q)}}\left(\frac{\pi D}{4\sqrt{N_\mathrm{D}}}\right)^{1-q} = q(2q-3)\,!!\left(\frac{4\sqrt{N_\mathrm{D}}\delta^2}{2^{N-1}\pi}\right)^{q-1} \text{ in III}$$

$$I_q = \nu^{-1}\sum_{n,\psi}\left\langle |\langle\psi|n\rangle|^{2q}\delta(E_\psi)\right\rangle_J$$

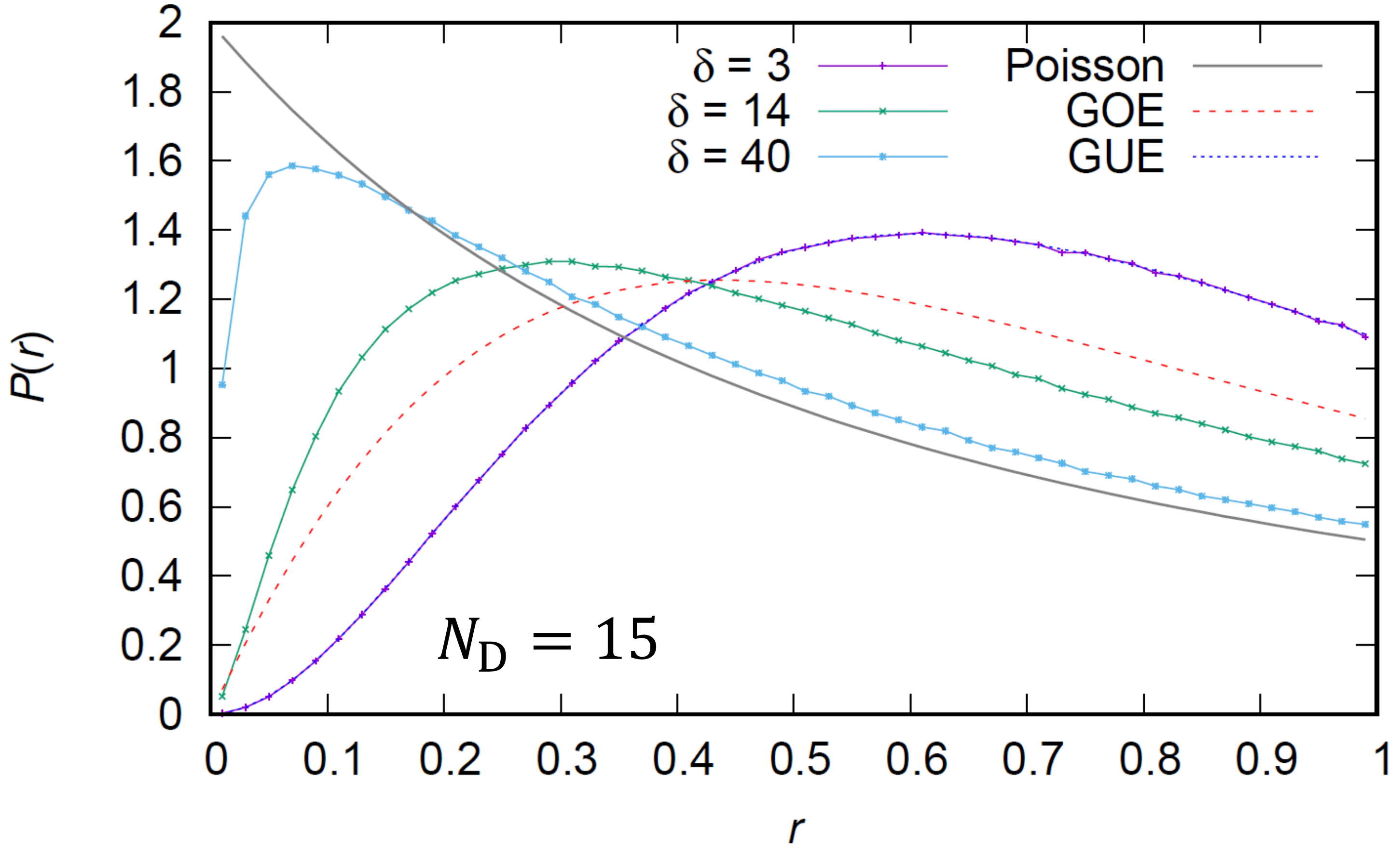


Good agreement up to large q for $\delta \sim 1$

Central 1/7 of the energy spectrum

PRR 3, 013023 (2021)

Spectral statistics: gap ratio distribution



(Analytical prediction: $\delta_{\mathrm{c}} = \frac{Z}{\sqrt{2\rho}} W(2Z\sqrt{\pi}) = 38.47$)

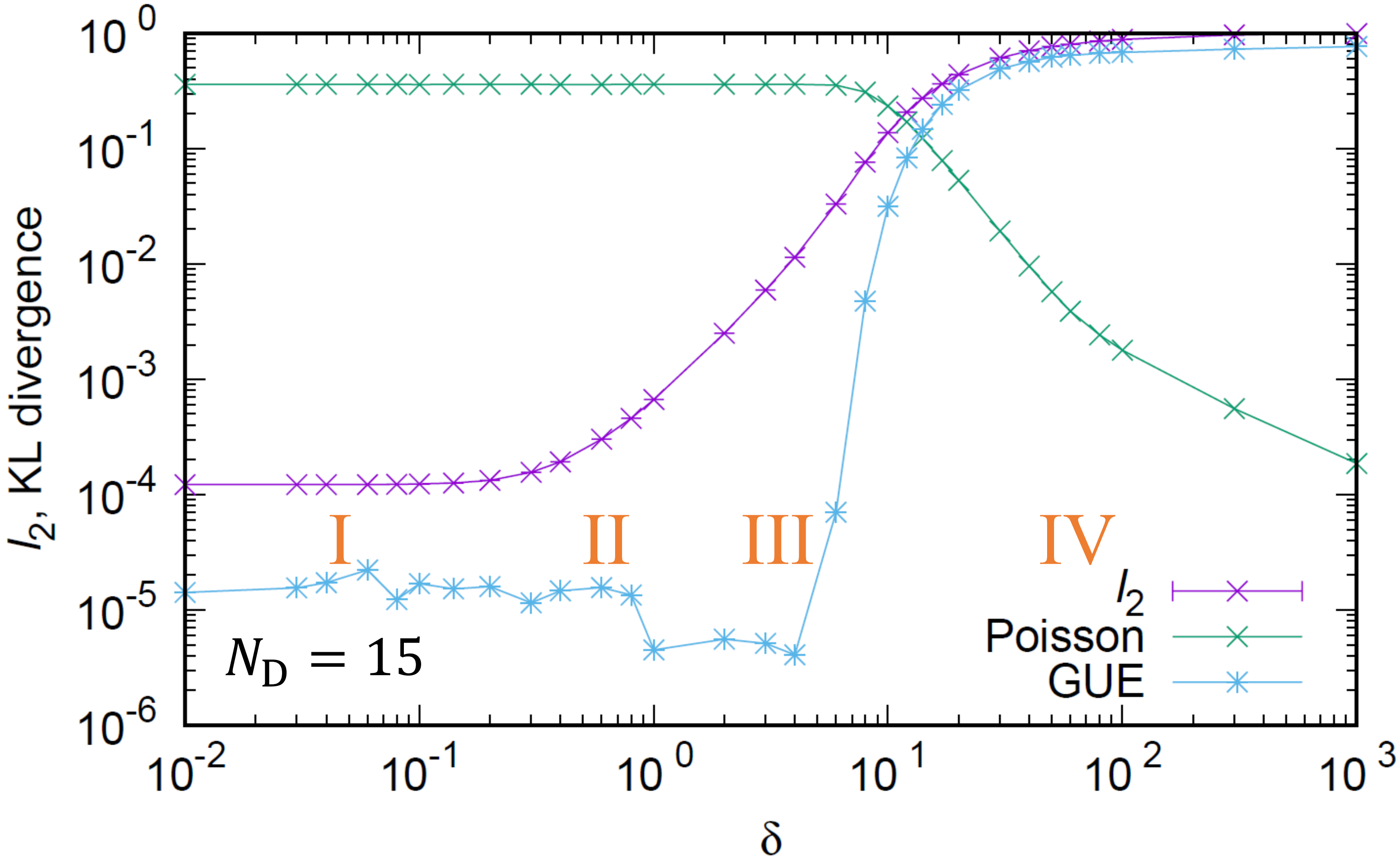
Measure difference by Kullback-Leibler (KL) divergence:

$$D_{\mathrm{KL}}(P||Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)}.$$

$$r = \frac{\min(E_{i+1} - E_i\,,\;\; E_{i+2} - E_{i+1})}{\max(E_{i+1} - E_i\,,\;\; E_{i+2} - E_{i+1})}$$

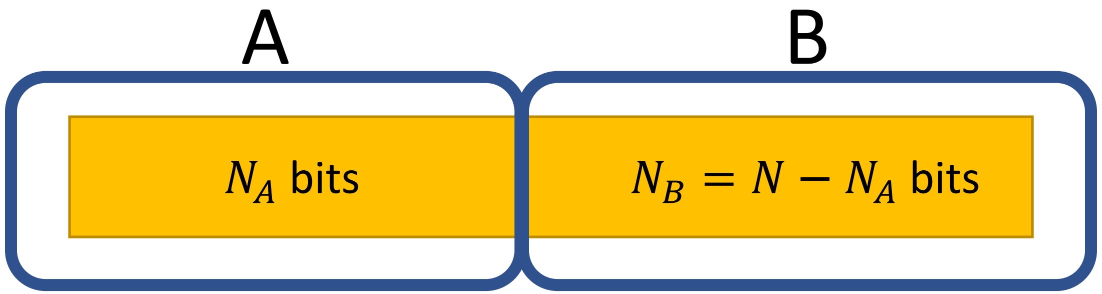
PRR 3, 013023 (2021)

Departure from random matrix $P(r)$ occurs **after** IPR (I_2) has grown significantly



F. Monteiro, MT, A. Altland, D. A. Huse, and T. Micklitz, PRL **127**, 030601 (2021)

Entanglement entropy for eigenstates



Zero-energy eigenstate $|\psi\rangle$, density matrix $\rho = |\psi\rangle\langle\psi|$

Reduced density matrix $\rho_A = \mathrm{tr}_B \rho$

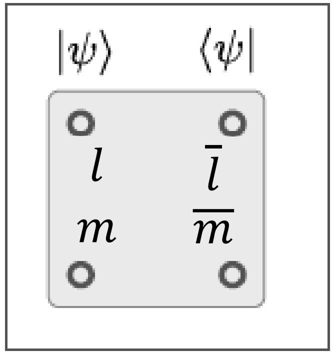
Entanglement entropy $S_A = -\mathrm{tr}_A(\rho_A \ln \rho_A)$

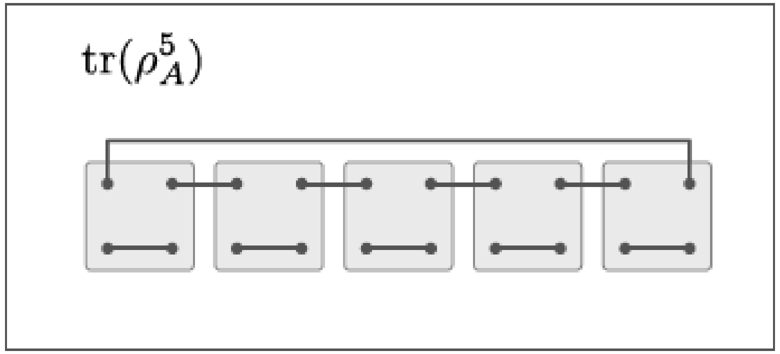
Replica method: Evaluate disorder averaged moments $M_r = \langle \mathrm{tr}_A(\rho_A^r) \rangle$, $S_A = -\partial_r M_r|_{r=1}$.

Fock space $\mathcal{F} = \mathcal{F}_A \otimes \mathcal{F}_B$

$n = (l, m)$

$\mathcal{N} = (n^1, n^2, \dots, n^r), \mathcal{N}_A = (l^1, l^2, \dots, l^r), \mathcal{N}_B = (m^1, m^2, \dots, m^r)$





$$\rho_A^r = \sum_{\substack{l^1,\dots,l^r \\ m^1,\dots,m^r}} \psi^{(l^1,m^1)}\, \overline{\psi}^{(l^2,m^1)} \psi^{(l^2,m^2)}\, \overline{\psi}^{(l^3,m^2)} \cdots \psi^{(l^r,m^r)}\, \overline{\psi}^{(l^1,m^r)}$$

PRL 127, 030601 (2021)

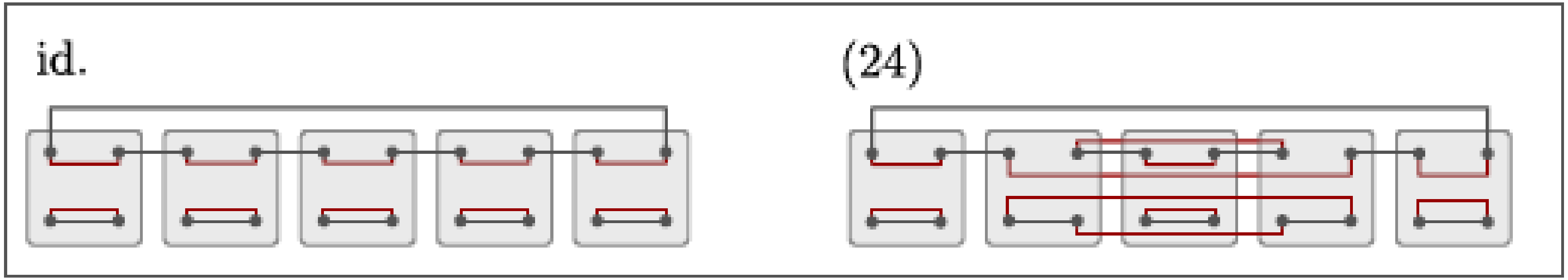
Evaluation of power of reduced density matrix

$$\rho_A^r = \sum_{\substack{l^1,\dots,l^r \\ m^1,\dots,m^r}} \overset{n^1}{\psi^{(l^1,m^1)}}\, \overset{\overline{n}^1}{\overline{\psi}^{(l^2,m^1)}}\, \overset{n^2}{\psi^{(l^2,m^2)}}\, \overset{\overline{n}^2}{\overline{\psi}^{(l^3,m^2)}} \cdots \psi^{(l^r,m^r)}\, \overline{\psi}^{(l^1,m^r)}$$

For this sum to survive disorder averaging,

$\mathcal{N} = (n^1, n^2, \dots, n^r)$ and $\overline{\mathcal{N}} = (\overline{n}^1, \overline{n}^2, \dots, \overline{n}^r)$ should be equal as sets,

$\mathcal{N}^i = \overline{\mathcal{N}}^{\sigma(i)}$



$n^1 = \overline{n}^1, n^2 = \overline{n}^2, n^3 = \overline{n}^3, n^4 = \overline{n}^4, n^5 = \overline{n}^5$

$n^1 = \overline{n}^1, \boldsymbol{n^2 = \overline{n}^4}, n^3 = \overline{n}^3, \boldsymbol{n^4 = \overline{n}^2}, n^5 = \overline{n}^5$

$$M_r = \langle \mathrm{tr}_A(\rho_A^r) \rangle = \sum_{\sigma} \sum_{\mathcal{N}} \prod_{i=1}^{r} \left\langle \left|\psi_{n^i}\right|^2 \right\rangle \delta_{\mathcal{N}_A, (\sigma\circ\tau)\mathcal{N}_A}\, \delta_{\mathcal{N}_B, \sigma\mathcal{N}_B}$$

PRL 127, 030601 (2021)

Regime I: maximally random case

$D_{A(B)} = 2^{N_{A(B)}-1}$

$M_r = \langle \mathrm{tr}_A(\rho_A^r) \rangle,\ S_A = -\partial_r M_r|_{r=1}$

Uniform distribution of wave functions, $\nu_n = \nu$

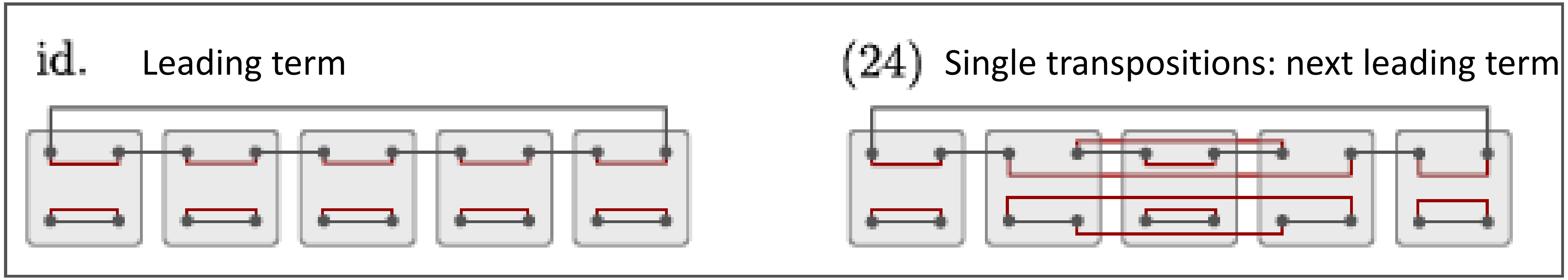
$$M_r \approx D_A^{1-r} + \binom{r}{2} D_A^{2-r} D_B^{-1}$$

Up to single transpositions

Difference from the thermal value $S_{\mathrm{th}} = \ln D_A$

$$S_A - S_{\mathrm{th}} = -\frac{D_A}{2D_B}$$

Exponentially small if $N_A \ll N_B$;
S_A very close to the thermal value



uniform

$$M_r = \langle \mathrm{tr}_A(\rho_A^r) \rangle = \sum_{\sigma} \sum_{\mathcal{N}} \prod_{i=1}^{r} \left\langle \left| \psi_{n^i} \right|^2 \right\rangle \delta_{\mathcal{N}_A,(\sigma\circ\tau)\mathcal{N}_A} \delta_{\mathcal{N}_B,\sigma\mathcal{N}_B}$$

PRL 127, 030601 (2021)

Regimes II and III: reduced effective dimension

- Assume ergodicity and calculate S_A
- Energy shell: extended cluster of resonant sites (width κ) embedded in the Fock space
- Neighboring sites of n: energy $v_m = v_n \pm \mathcal{O}(\delta)$, much more likely to be in the same shell because $\delta \ll \Delta_2 = \sqrt{N_\mathrm{D}}\delta$

Additional assumptions

- Exponentially large number of sites → self averaging
  (sum over site energies = average over approx. Gaussian distributed contributions of subsystem energies to the total energy)
- Total energy $E \sim E_A + E_B$

➔ Up to single transpositions (justified in $1 \ll N_A \ll N_\mathrm{D}$ & replica limit):

$$S_A - S_\mathrm{th} = -\frac{1}{2}\ln\left(\frac{N_\mathrm{D}}{N_B}\right) + \frac{N_A}{2N_D} - \sqrt{\frac{N_\mathrm{D}}{2N_A}\frac{D_A}{2D_B}}$$

in Regimes II, III

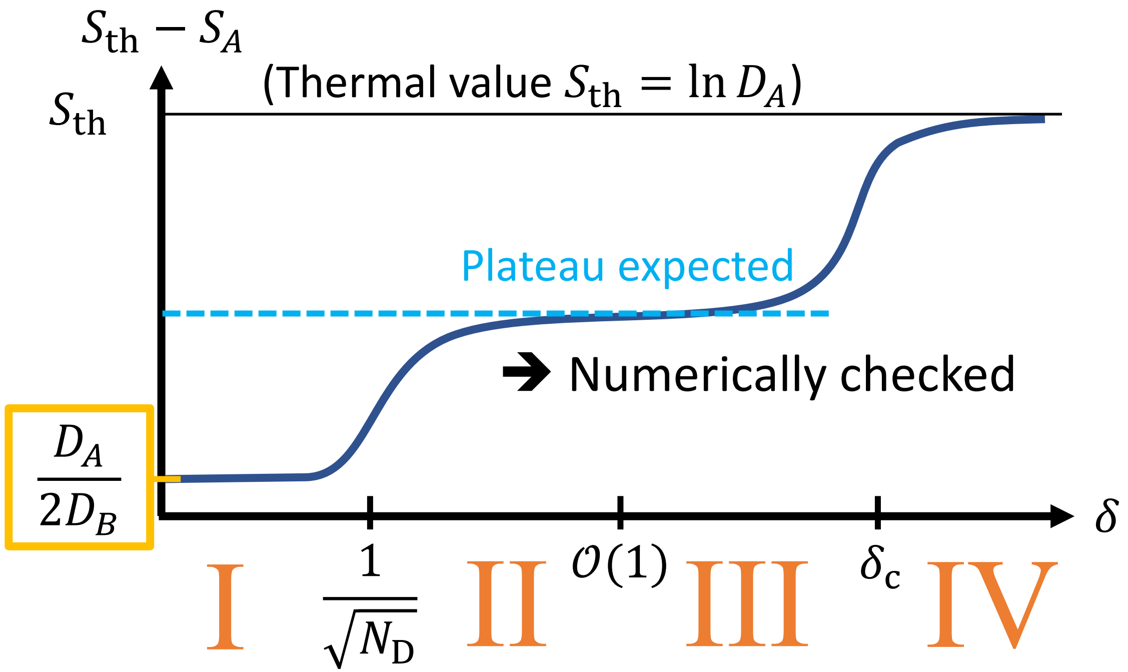
$\left(\frac{1}{\sqrt{N_\mathrm{D}}} \ll \delta < \delta_\mathrm{c} \sim N_\mathrm{D}^2 \ln N_\mathrm{D}\right)$

$$S_A - S_\mathrm{th} = -\frac{D_A}{2D_B}$$

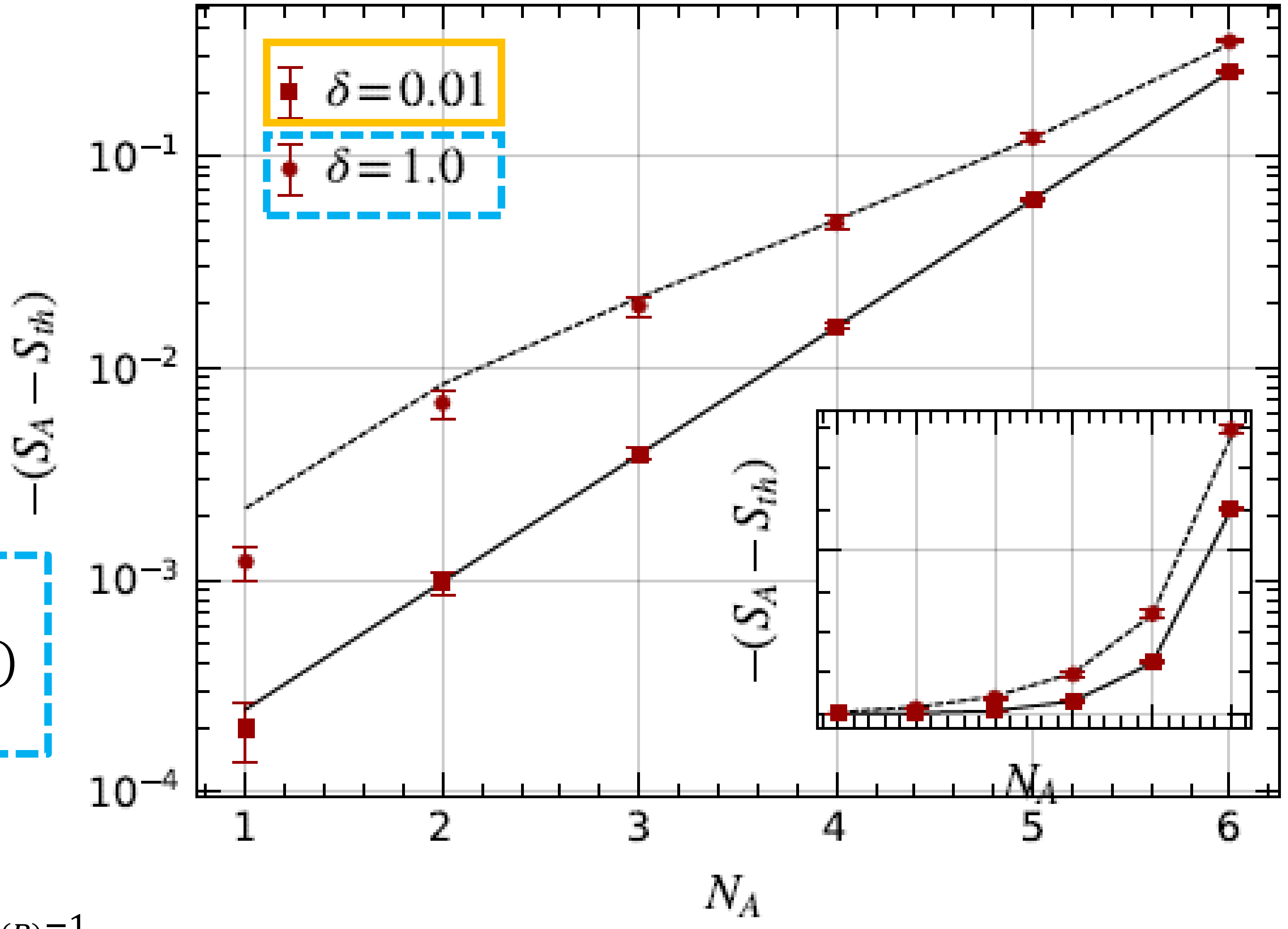
in Regime I

PRL 127, 030601 (2021)

Offset from the thermal value



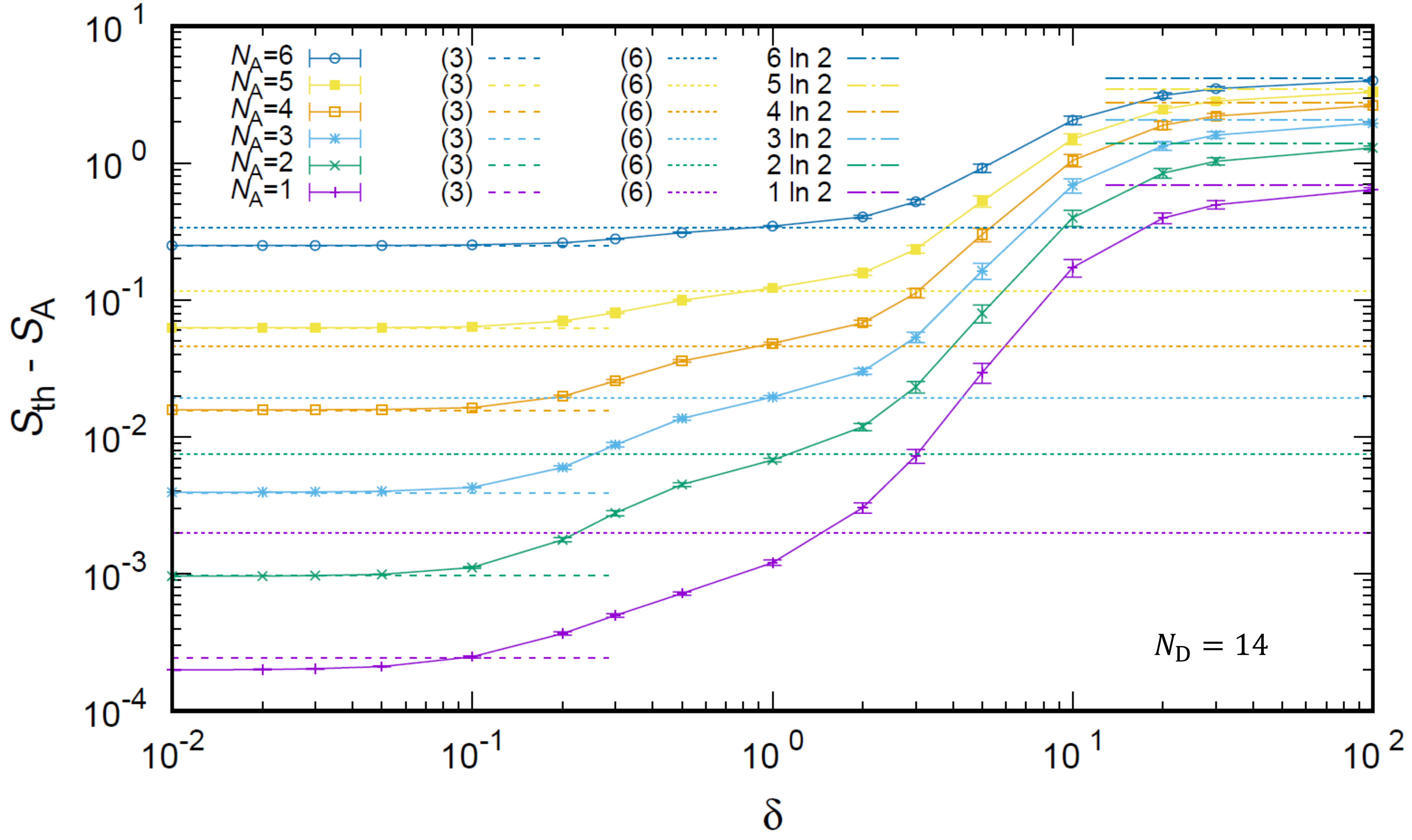
$N_{\rm D} = 14$ ($N = 28$ Majorana fermions)



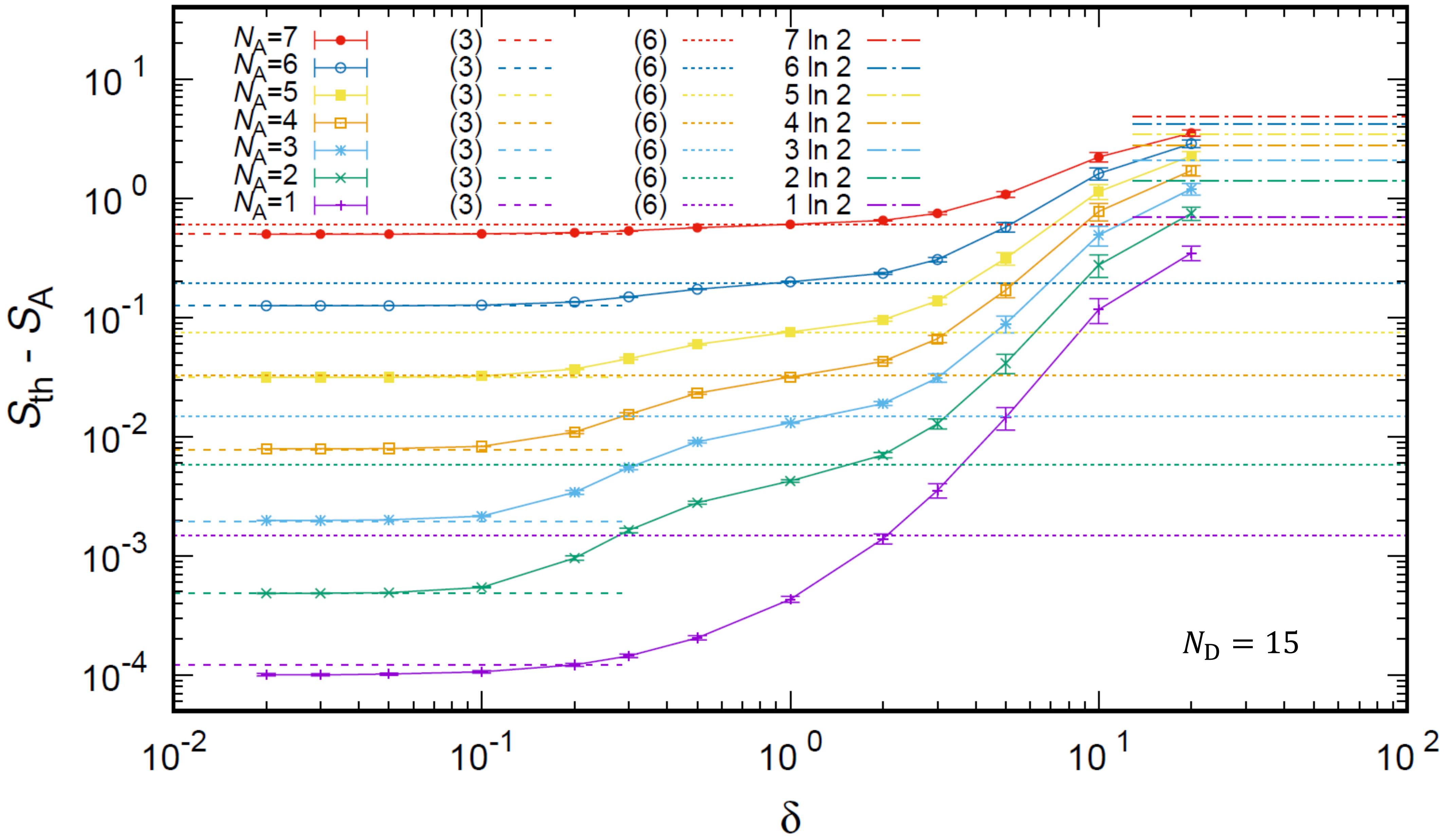
$$S_A - S_{\rm th} = -\frac{1}{2}\ln\left(\frac{N_{\rm D}}{N_B}\right) + \frac{N_A}{2N_D} - \sqrt{\frac{N_{\rm D}}{2N_A}\frac{D_A}{2D_B}} \ (< 0)$$

in Regimes II, III ($\frac{1}{\sqrt{N_{\rm D}}} \ll \delta < \delta_{\rm c} \sim N_{\rm D}^2 \ln N_{\rm D}$)

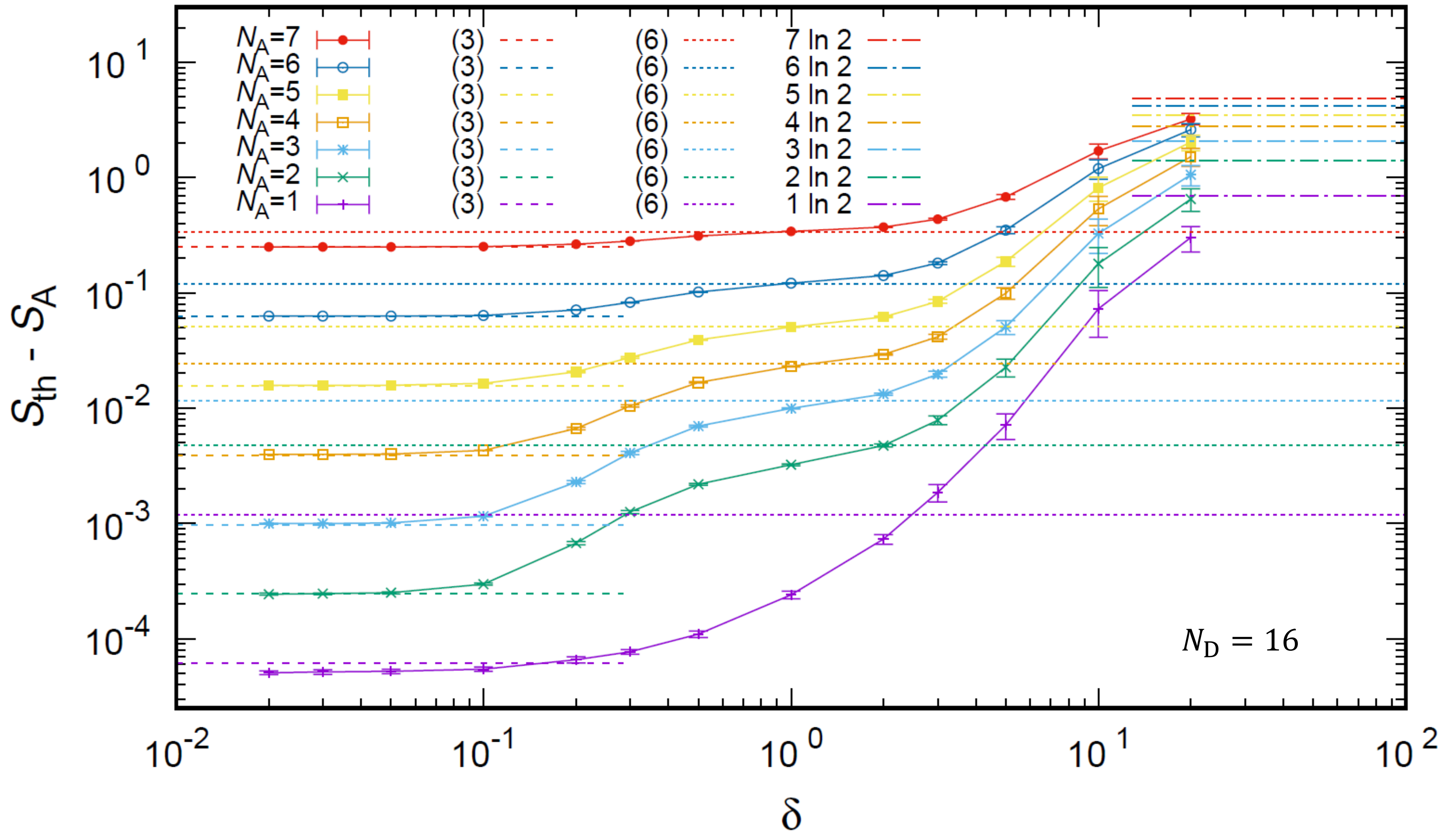
$D_{A(B)} = 2^{N_{A(B)}-1}$



N_A=6
N_A=5
N_A=4
N_A=3
N_A=2
N_A=1
(3)
(6)
6 ln 2
5 ln 2
4 ln 2
3 ln 2
2 ln 2
1 ln 2
S_{th} - S_A
δ
$N_D = 14$



N_A=7
N_A=6
N_A=5
N_A=4
N_A=3
N_A=2
N_A=1
(3)
(6)
7 ln 2
6 ln 2
5 ln 2
4 ln 2
3 ln 2
2 ln 2
1 ln 2
$N_D = 15$
S_{th} - S_A
δ



N_A=7
N_A=6
N_A=5
N_A=4
N_A=3
N_A=2
N_A=1
(3)
(6)
7 ln 2
6 ln 2
5 ln 2
4 ln 2
3 ln 2
2 ln 2
1 ln 2
S_{th} - S_A
δ
$N_D = 16$

Phys. Rev. Lett. 120, 241603 (2018) arXiv:1707.02197 (poster at NQS2017)
with A. M. García-García, A. Romero-Bermúdez, and B. Loureiro
Phys. Rev. Research 3, 013023 (2021) arXiv:2005.12809
with Felipe Monteiro, Tobias Micklitz, and Alexander Altland
Phys. Rev. Lett. 127, 030601 (2021) arXiv:2012.07884
with F. Monteiro, A. Altland, David A. Huse, and T. Micklitz

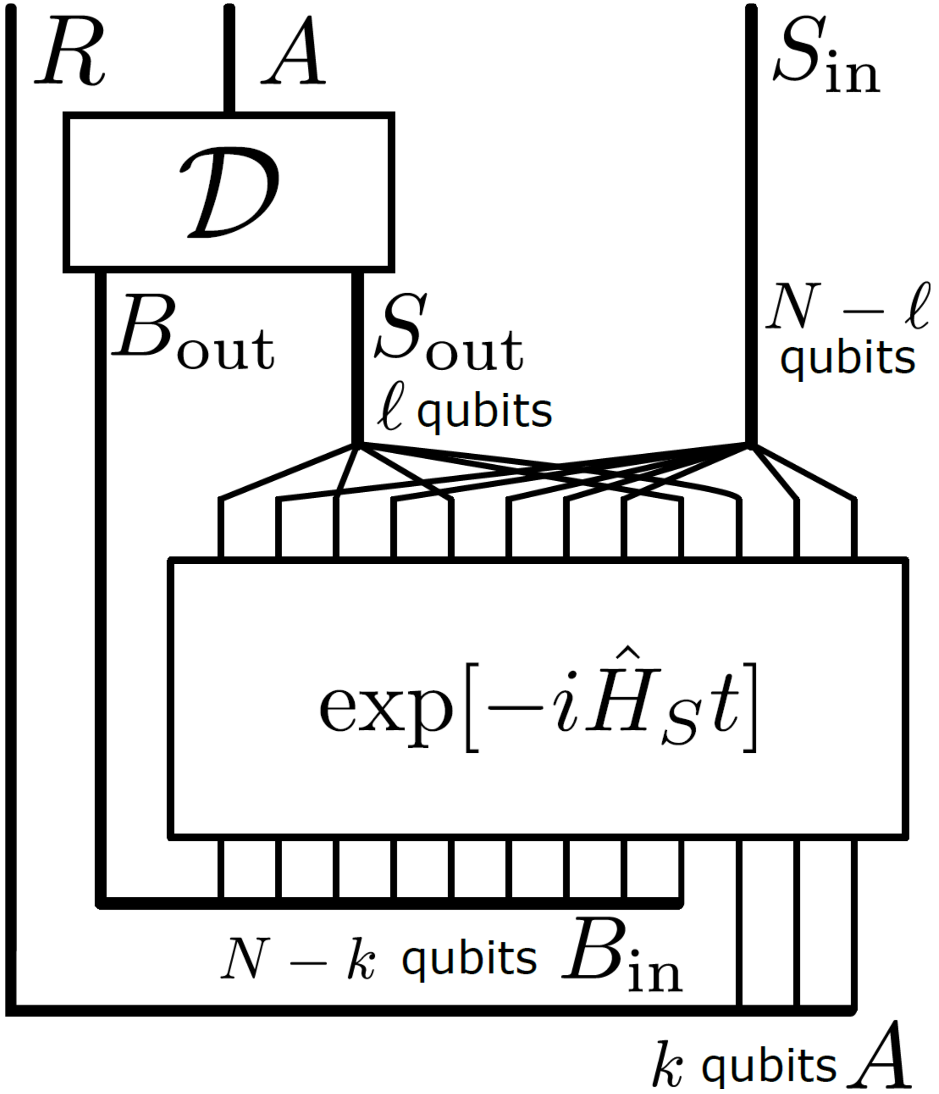
Summary so far…

- SYK_{4+2}: maximally chaotic to integrable transition
- Analytically tractable model for many-body localization (MBL)
  - Fock space: (N/2)-dimensional hypercube
- Analytical results on eigenfunction moments and MBL point

  ➔ Agreement with numerical results without free paramters
- Evaluation of entanglement entropy S_A assuming ergodicity in energy shells

  ➔ Agreement between our numerical and analytical results

Quantum error correction: The Hayden-Preskill protocol



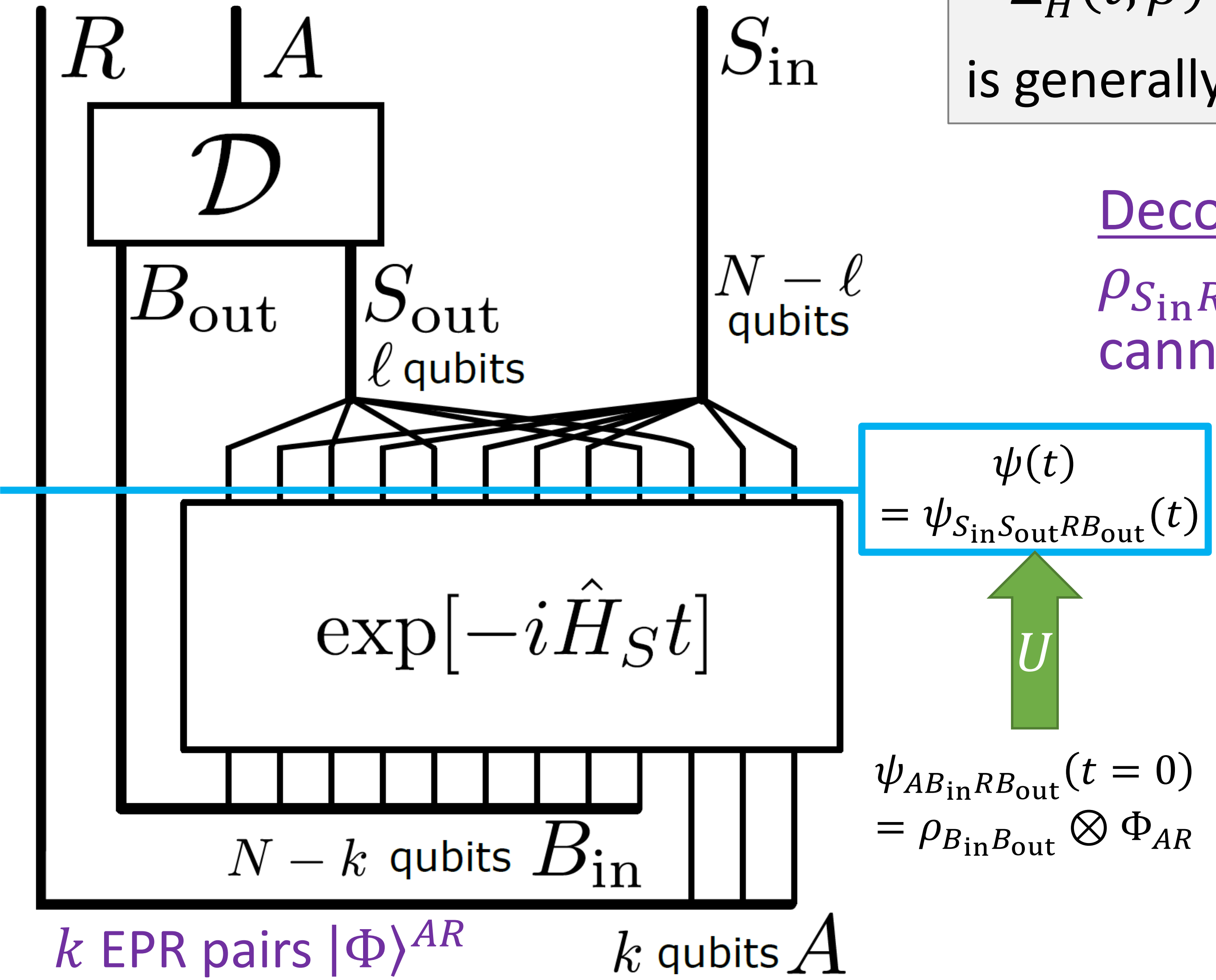
- Alice: throws k-qubit quantum information A into a box B_{in}
- Bob: knows the original state of B_{in} and the Hamiltonian $\widehat{H}_S$ of $S = A + B_{\text{in}}$
- Bob obtains ℓ qubits S_{out} after time t. Can Bob decode ($\mathcal{D}$) Alice's secret?

Black holes: fast scrambling;
information recovery for $\ell \sim k$

P. Hayden and J. Preskill, "Black holes as mirrors: quantum information in random subsystems" JHEP 0709 (2007) 120

Circular unitary (Haar) ensemble was assumed

Quantum error correction: The Hayden-Preskill protocol



Recovery error

$$\Delta_{\hat{H}}(t,\beta) = \frac{1}{2}\min_{\mathcal{D}:\mathrm{CPTP}} \left| \Phi^{AR} - \mathcal{D}\left(\Psi_{\rm fin}^{S_{\rm out}B_{\rm out}R}(t,\beta)\right)\right|_1$$

is generally hard to compute… $(|M|_1 \equiv \mathrm{Tr}\sqrt{M^\dagger M})$

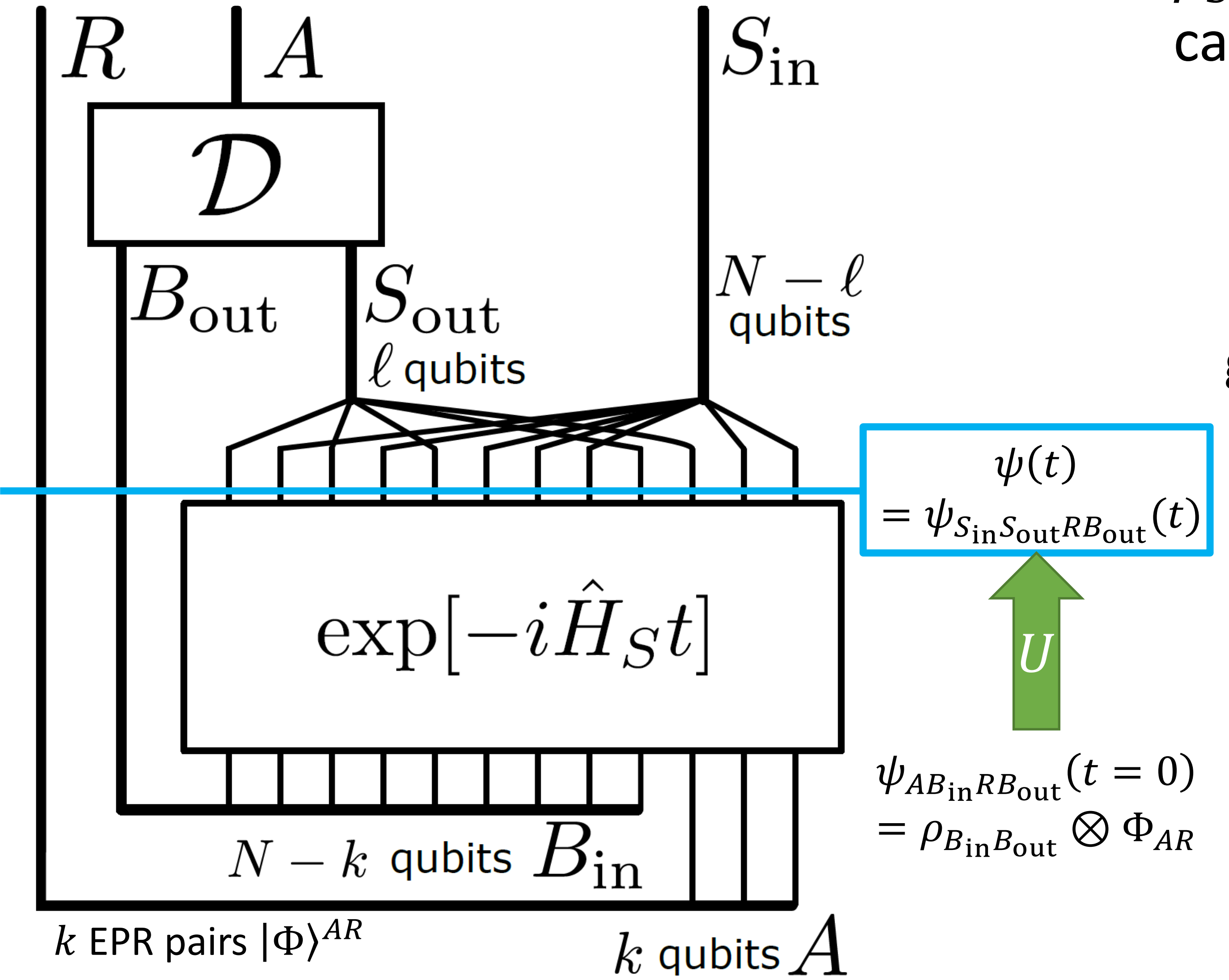
Decoupling approach: for $\mathcal{D}$ to succeed,
$\rho_{S_{\rm in}R} = \mathrm{Tr}_{B_{\rm out}}\mathrm{Tr}_{S_{\rm out}}|\psi(t)\rangle\langle\psi(t)|$
cannot have correlation between $S_{\rm in}$ and R

$$\bar{\Delta}_{\hat{H}}(t,\beta) \equiv \min\left\{1, \sqrt{\left|\rho_{S_{\rm in}R} - \rho_{S_{\rm in}} \otimes \frac{I_R}{d_R}\right|_1}\right\}$$

$(\rho_{S_{\rm in}} = \mathrm{Tr}_R \rho_{S_{\rm in}R})$

satisfies $\Delta_{\hat{H}}(t,\beta) \le \bar{\Delta}_{\hat{H}}(t,\beta)$;
$\bar{\Delta}_{\hat{H}}(t,\beta)$ gives a decoding error estimate

Quantum error correction: The Hayden-Preskill protocol



Decoupling approach: for $\mathcal{D}$ to succeed,
$\rho_{S_{\text{in}}R} = \text{Tr}_{B_{\text{out}}}\text{Tr}_{S_{\text{out}}}|\psi(t)\rangle\langle\psi(t)|$
cannot have correlation between S_{in} and R

$$\bar{\Delta}_{\hat{H}}(t,\beta) \equiv \min\left\{1, \sqrt{\left|\rho_{S_{\text{in}}R} - \rho_{S_{\text{in}}} \otimes \frac{I_R}{d_R}\right|_1}\right\}$$

gives a decoding error estimate

Haar random unitary case:

$$\bar{\Delta}_{\text{Haar}}(\beta) = \min\left\{1, 2^{\frac{1}{2}(\ell_{\text{Haar,th}}(\beta)-\ell)}\right\}$$

$$\ell_{\text{Haar,th}}(\beta) = \frac{N+k-H(\beta)}{2} \xrightarrow{\beta\to 0} k$$

$\bar{\Delta}_{\text{Haar}}$ exponentially decreases as function of ℓ after $\ell \approx k$ [HP recovery]

P. Hayden and J. Preskill, JHEP 2007

Models for $\hat{H}_S$ and quantum error correction (QEC)

1. SYK-like long-range couplings

Gaussian dense SYK$_4$

$$\hat{H} = \sum_{a<b<c<d} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

sparsify →

Binary coupling sparse SYK

$$\hat{H} \propto \sum_{\substack{(a,b,c,d)\in P \\ \#P\sim N}} (\pm 1)\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

[2208.12098]

Error decays to $\sim$ Haar value in $t \sim \sqrt{N}$

Add SYK$_2$ term ↓

SYK$_{4+2}$

$$\hat{H} = \cos\theta \sum_{a<b<c<d} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + \sin\theta \sum_{a<b} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

[PRL **120**, 241603; PRR **3**, 013023; PRL **127**, 030601]

$\delta \propto \tan\theta \ll 1$: SYK$_4$, $\delta = \mathcal{O}(1)$: chaotic spectrum but eigenstates restricted in Fock space, $\delta \gg 1$: many-body localization

Error increases before many-body localization

2. One-dimensional spin chains

Ising chain + uniform field

$$\hat{H}_{\text{Ising}} = -J\sum_{\langle j,k\rangle} S_j^z S_k^z - g\sum_j S_j^x - h\sum_k S_k^z$$

$g = 0$ or $h = 0$: integrable, far from integrable lines: chaotic

Heisenberg chain + random field

$$\hat{H}_{\text{XXZ}} = \sum_{\langle j,k\rangle} S_j \cdot S_k + \sum_j h_j S_j^z, h_j \in [-W, W]$$

$W \ll 1$: integrable, $W \sim 1$: chaotic, $W \gg 4$: MBL(?)

Efficient QEC not observed even for chaotic cases

The Sachdev-Ye-Kitaev (SYK) model

$$\hat{H} = \sum_{1\le a<b<c<d\le 2N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

[Kitaev 2015][Sachdev & Ye 1993]

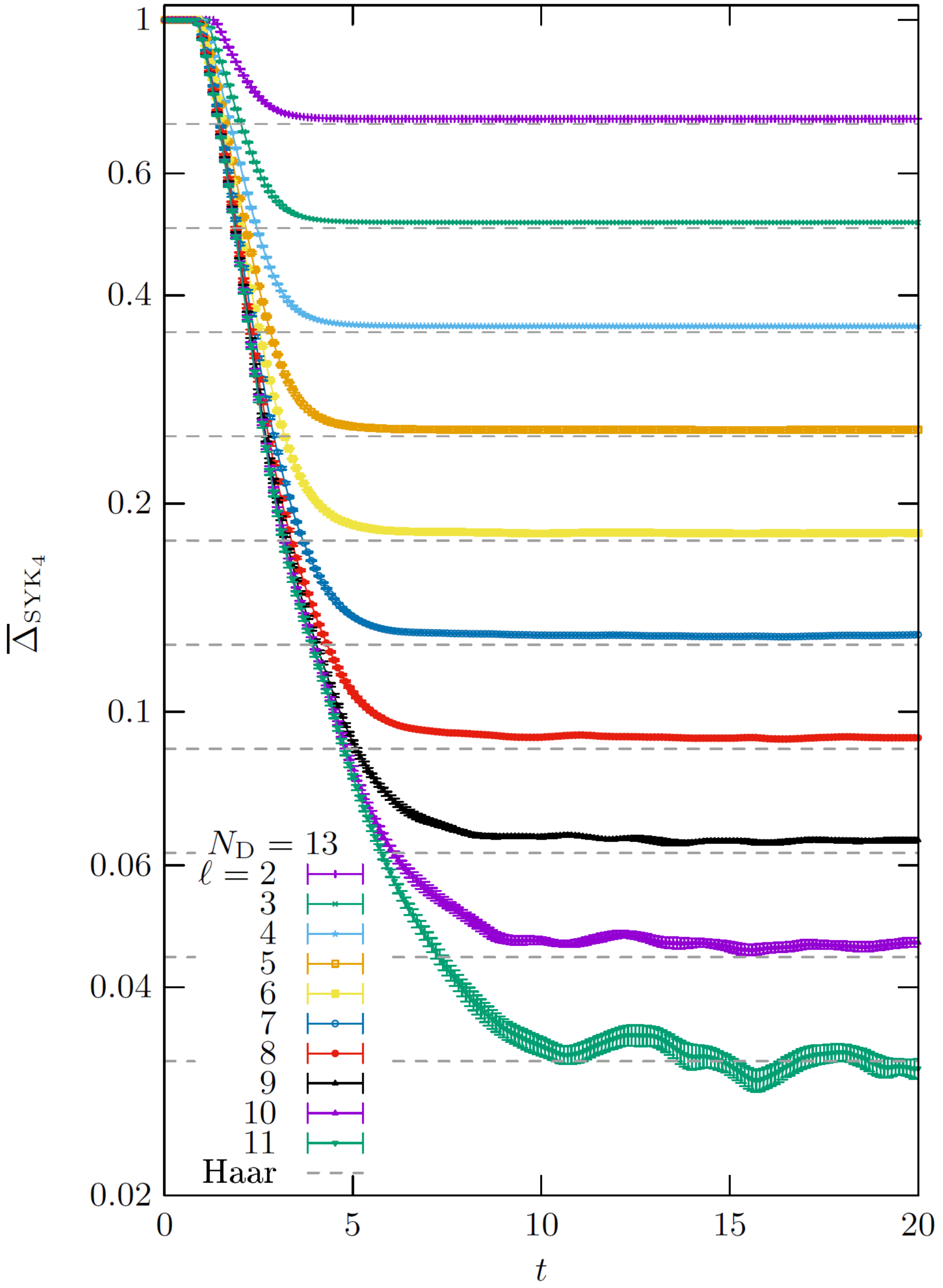
$\hat{\chi}_{a=1,2,\ldots,2N}$: $2N$ Majorana fermions ($\{\hat{\chi}_a, \hat{\chi}_b\} = 2\delta_{ab}$)

J_{abcd} : independent Gaussian random couplings ($\overline{J_{abcd}{}^2} = J^2$, $\overline{J_{abcd}} = 0$);

Normalization: SYK half-bandwidth $\sqrt{\frac{\langle \mathrm{Tr}\,\hat{H}^2\rangle}{2^N}} = 1$

- Solvable in the large-N limit [Maldacena, Shanker, and Stanford, JHEP08(2016)106]
- Maximally chaotic ($\lambda_{\mathrm{Lyapunov}} \to 2\pi k_{\mathrm{B}}T/\hbar$: chaos bound)
- Correspondence to 1+1d gravity, random matrix

➔ $\overline{\Delta}$ reaches the Haar value quickly ($t \sim \sqrt{N}$)

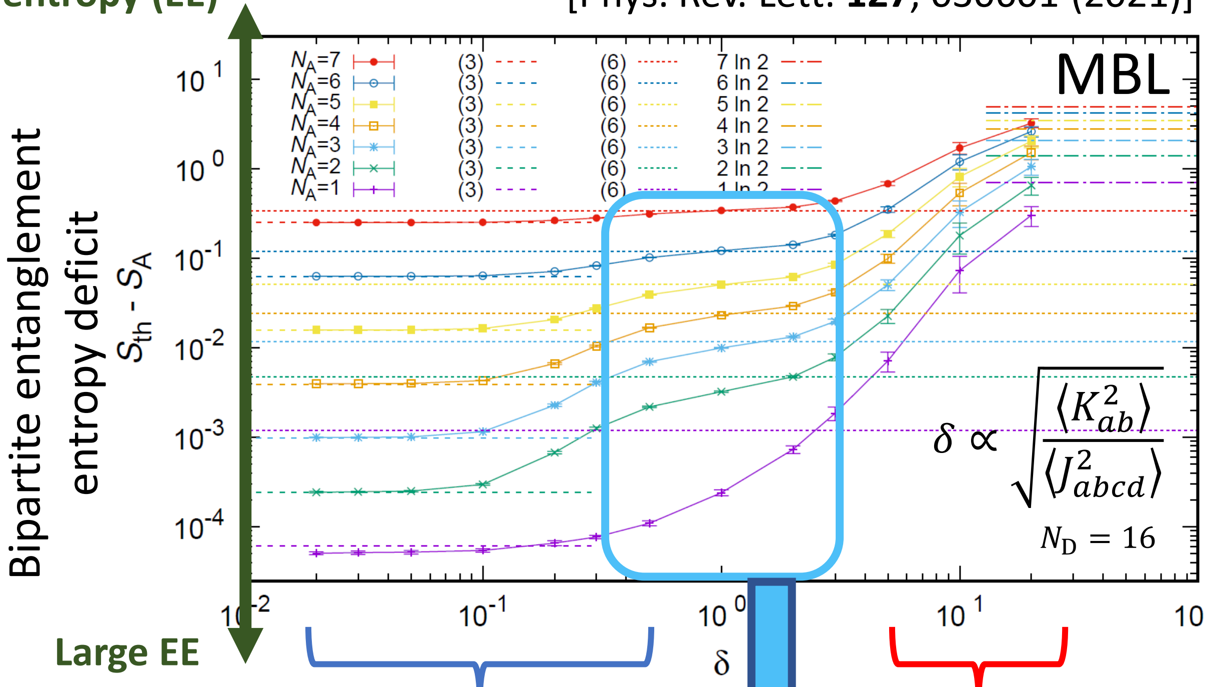


Result for SYK4+2

$$\hat{H} = \sum_{1\le a<b<c<d}^{N_{\mathrm{Maj}}=2N} J_{abcd}\,\hat{\chi}'_a \hat{\chi}'_b \hat{\chi}'_c \hat{\chi}'_d + i \sum_{1\le a<b}^{N_{\mathrm{Maj}}} K_{ab}\,\hat{\chi}'_a \hat{\chi}'_b = \cos\theta\, \hat{H}_{\mathrm{SYK}_4} + \sin\theta\, \hat{H}_{\mathrm{SYK}_2}$$

Small entanglement entropy (EE)

[Phys. Rev. Lett. **127**, 030601 (2021)]



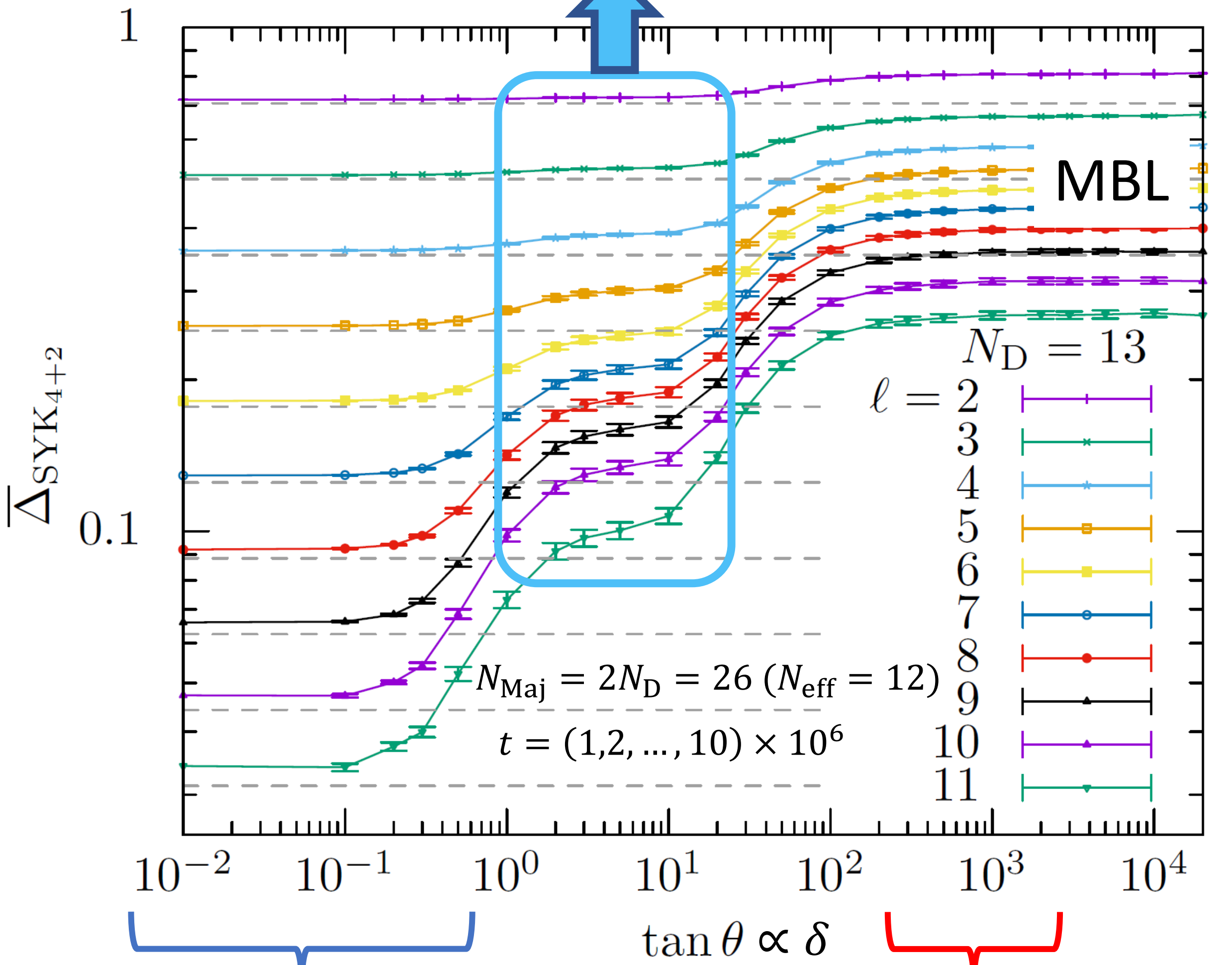
Large EE

Small δ: almost equal distribution in the entire Fock space

Large δ: each eigenstate in tiny part of the Fock space

Intermediate δ: eigenstates in restricted part of Fock space, within which they are thermally distributed

Intermediate δ: plateau!



Small error ~ Haar

Large error near MBL

Sparse (or pruned) SYK

$$\hat{H} = \sum_{a<b<c<d} x_{abcd} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d \,,\; x_{abcd} = \begin{cases} 1 & \text{(probability } p\text{)} \\ 0 & \text{(probability } 1-p\text{)} \end{cases},\; P(J_{abcd}) = \frac{\exp\left(-\frac{J_{abcd}^2}{2J^2}\right)}{\sqrt{2\pi J^2}}$$

$K_{\mathrm{cpl}} = \binom{N}{4} p$: Number of non-zero x_{abcd}

$K_{\mathrm{cpl}} \sim \mathcal{O}(1) N$ enough for
- Random matrix-like behavior
- Large entropy per fermion at low T !

$$p \sim \frac{4!}{N^3} = \mathcal{O}(N^{-3})$$

- Talk by Brian Swingle at Simons Center (18 September 2019)
- "Sparse Sachdev-Ye-Kitaev model, quantum chaos and gravity duals" A. M. García-García, Y. Jia, D. Rosa, J. J. M. Verbaarschot, Phys. Rev. D **103**, 106002 (2021)
- "A Sparse Model of Quantum Holography" S. Xu, L. Susskind, Y. Su, and B. Swingle, arXiv:2008.02303
- "Spectral Form Factor in Sparse SYK models" E. Cáceres, A. Misobuchi, and A. Raz, JHEP **2208**, 236 (2022)

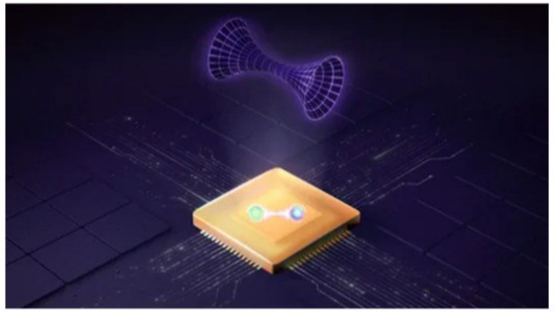
Article | Published: 30 November 2022

Traversable wormhole dynamics on a quantum processor

Daniel Jafferis, Alexander Zlokapa, Joseph D. Lykken, David K. Kolchmeyer, Samantha I. Davis, Nikolai Lauk, Hartmut Neven & Maria Spiropulu

Nature **612**, 51–55 (2022) | Cite this article

Quanta Magazine (30 November 2022)



QUANTUM GRAVITY

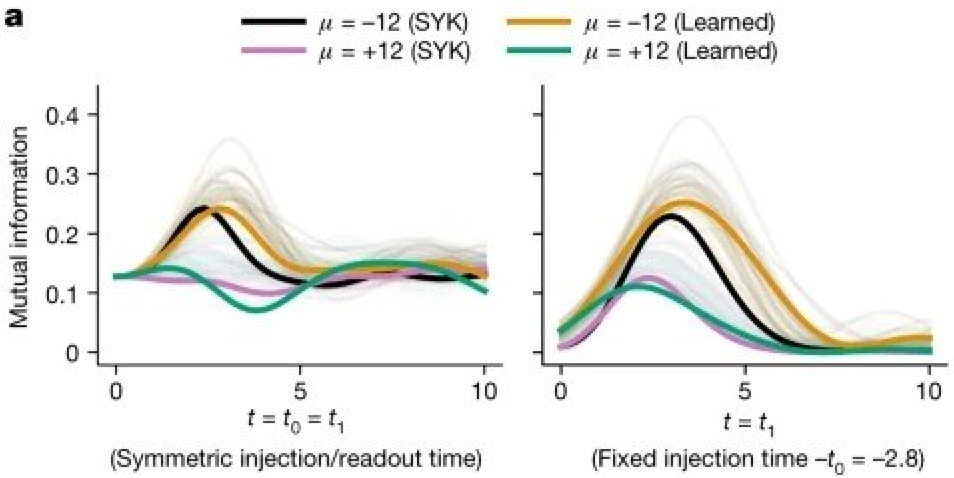
Physicists Create a Wormhole Using a Quantum Computer

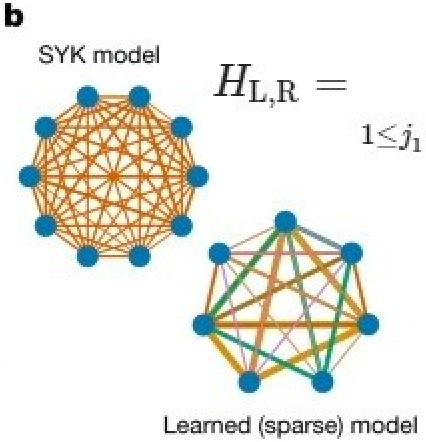
By NATALIE WOLCHOVER | NOVEMBER 30, 2022 |

3 |

The unprecedented experiment explores the possibility that space-time somehow emerges from quantum information.

Fig. 2: Learning a traversable wormhole Hamiltonian from the SYK model.





$$H_{\mathrm{L,R}} = \sum_{1 \le j_1 < \cdots < j_q \le N} J_{j_1 \ldots j_q} \psi_{\mathrm{L,R}}^{j_1} \cdots \psi_{\mathrm{L,R}}^{j_q},$$

Machine learning

$$H_{\mathrm{L,R}} = -0.36\psi^1\psi^2\psi^4\psi^5 + 0.19\psi^1\psi^3\psi^4\psi^7 - 0.71\psi^1\psi^3\psi^5\psi^6$$
$$+0.22\psi^2\psi^3\psi^4\psi^6 + 0.49\psi^2\psi^3\psi^5\psi^7,$$

➔ Realized on the Google Sycamore processor (nine-qubit circuit of 164 two-qubit, 295 single-qubit gates)

M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, arXiv:2208.12098

Sparse (or pruned) SYK with interaction $= \pm 1$

$$\hat{H} = C_{N,p} \sum_{1 \leq a<b<c<d \leq N} x_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d \, , x_{abcd} = \begin{cases} 1 & (\text{probability } p/2) \\ -1 & (\text{probability } p/2) \\ 0 & (\text{probability } 1-p) \end{cases}$$

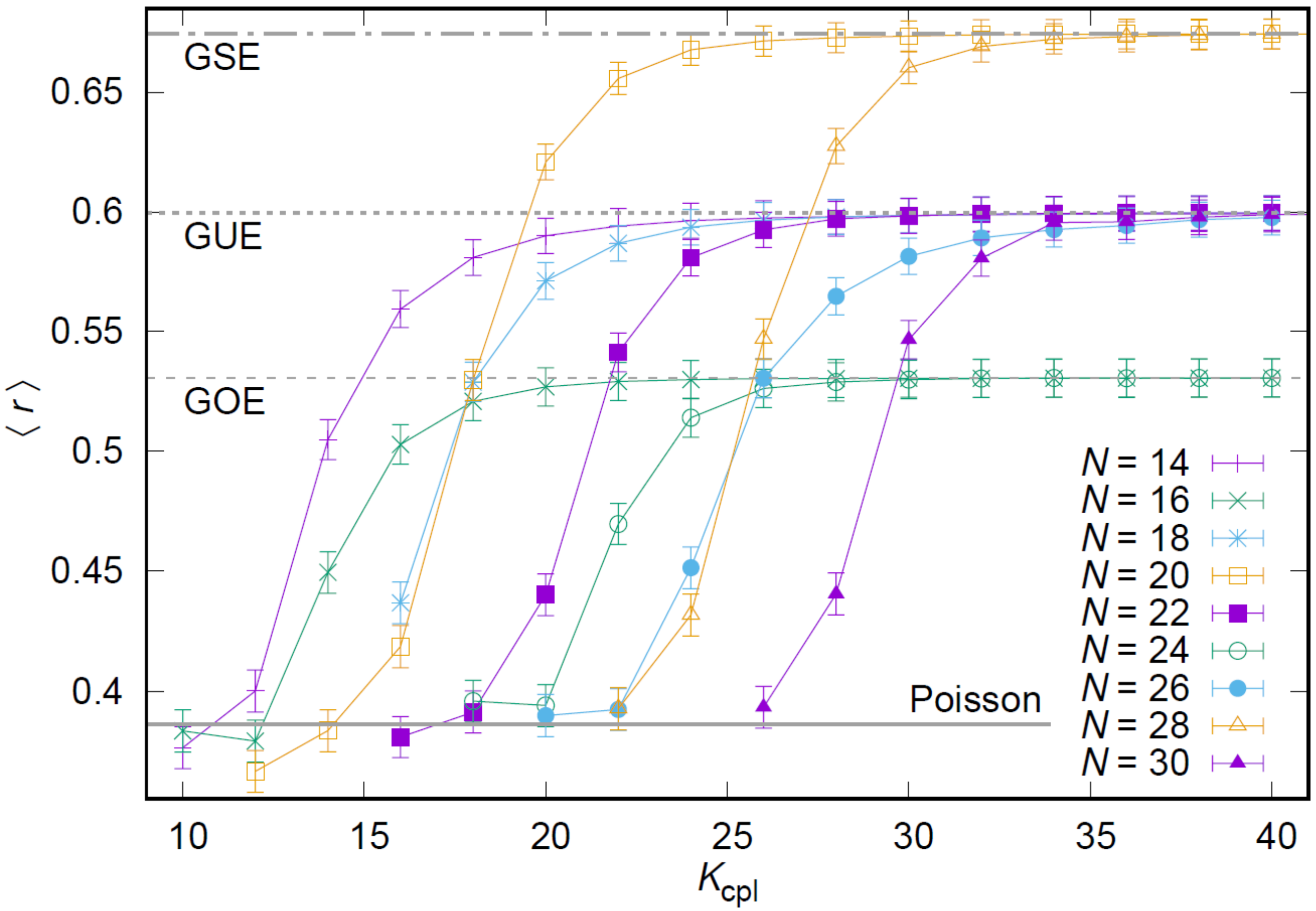
Random-matrix statistics for $K_{\mathrm{cpl}} = \binom{N}{4} p \gtrsim N$.

cf. Non-Gaussian disorder average [T. Krajewski, M. Laudonio, R. Pascalie, and A. Tanasa, PRD **99**, 126014 (2019)]; Kitaev's talk (2015)

x_{abcd} can be taken to be +1 at finite $p \ll 1$ (unary sparse SYK, see appendix of 2208.12098), however at $p = 1$, the model is not chaotic [P. H. C. Lau, C.-T. Ma, J. Murugan, and MT, J. Phys. A. **54**, 095401 (2021)]

M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, arXiv:2208.12098

$\langle r \rangle$ as a function of K_{cpl}: approach RMT value



Neighboring gap ratio

$$r = \frac{\min(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}$$

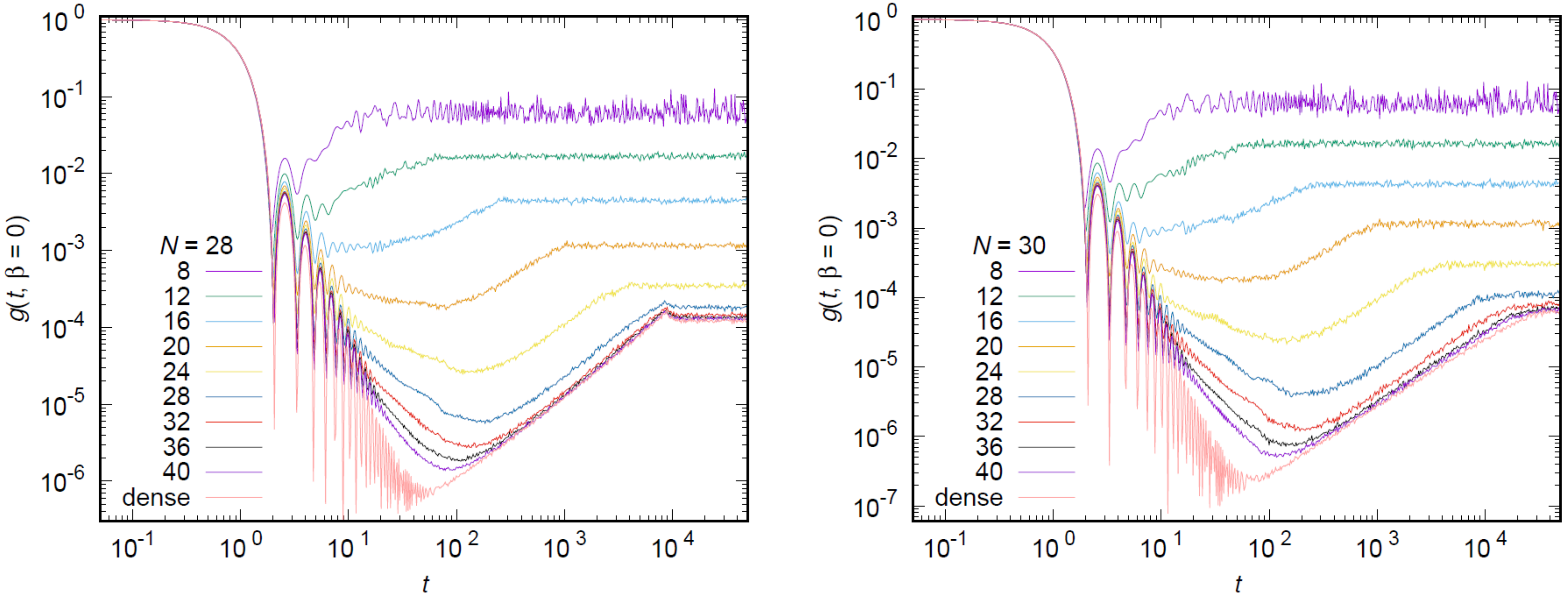
	Uncorrelated	GOE	GUE	GSE
$\langle r \rangle$	2log 2 – 1 = 0.38629…	0.5307(1)	0.5996(1)	0.6744(1)

[Y. Y. Atas *et al.* PRL 2013]

M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, arXiv:2208.12098

Spectral form factor

Clear ramp for $K_{\mathrm{cpl}} \gtrsim N$, coincides with the dense SYK as $N \to$ large

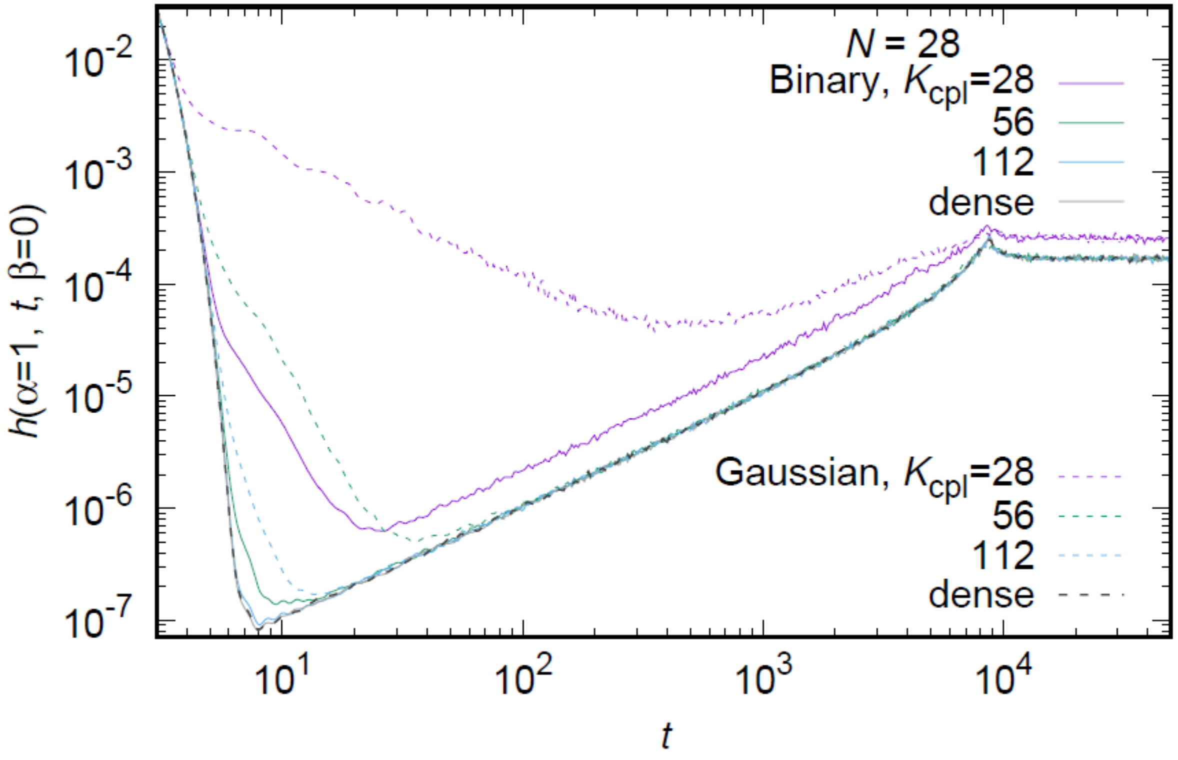


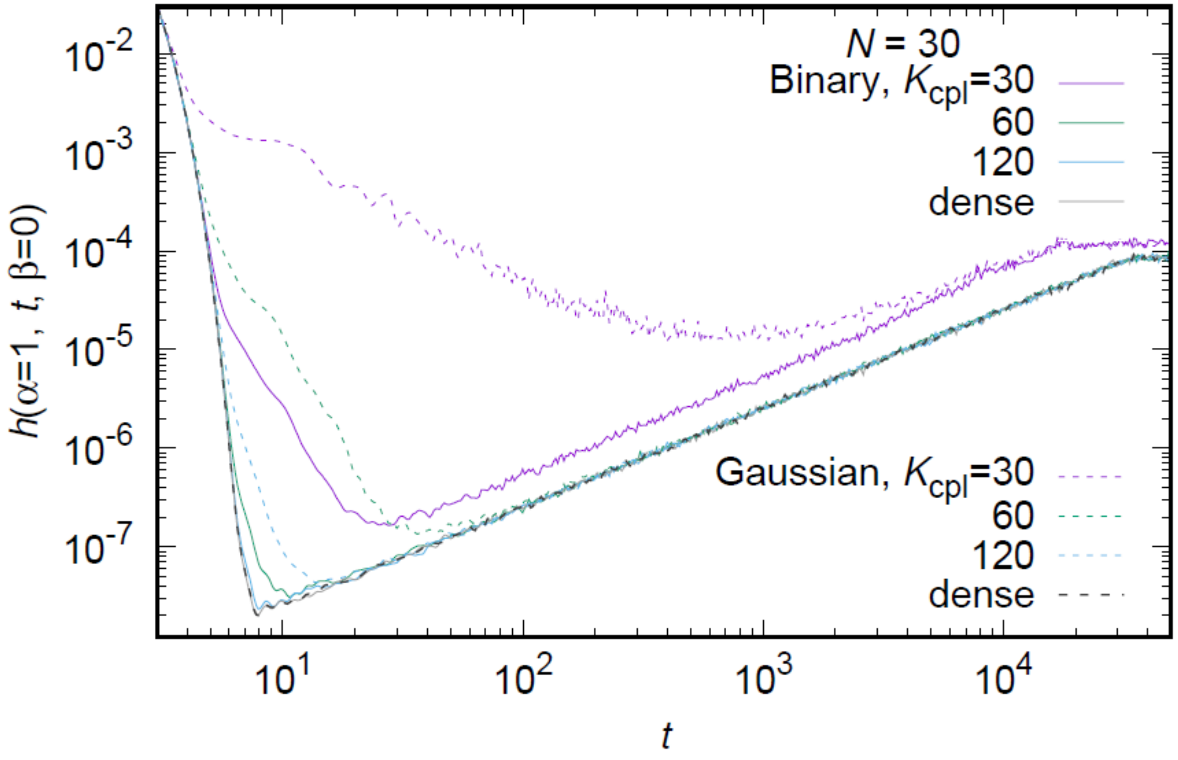
M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, arXiv:2208.12098

Modified SFF (focus on band center)

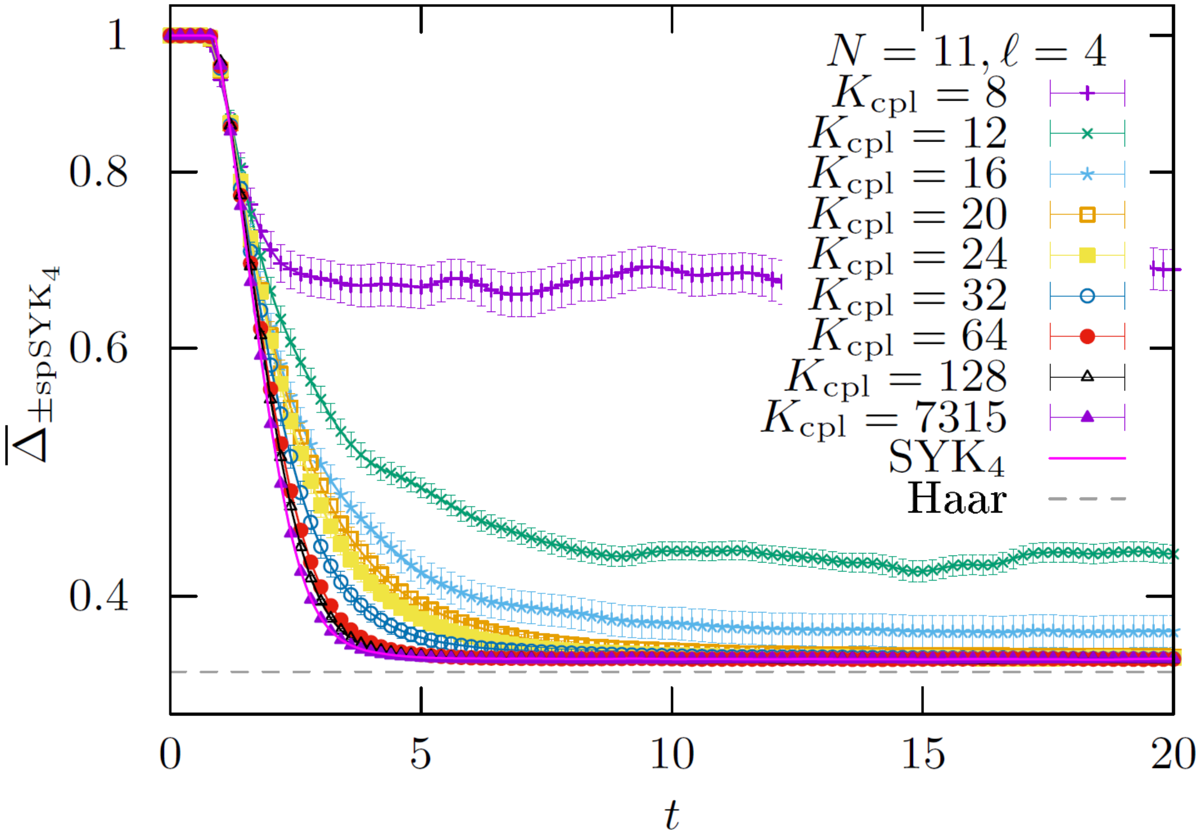
$$h(\alpha, t, \beta) = \frac{|Y(\alpha, t, \beta)|^2}{Y(\alpha, 0, \beta)^2}, Y(\alpha, 0, \beta) = \sum_j e^{-\alpha\epsilon_j^2 - (\beta + it)\epsilon_j}$$

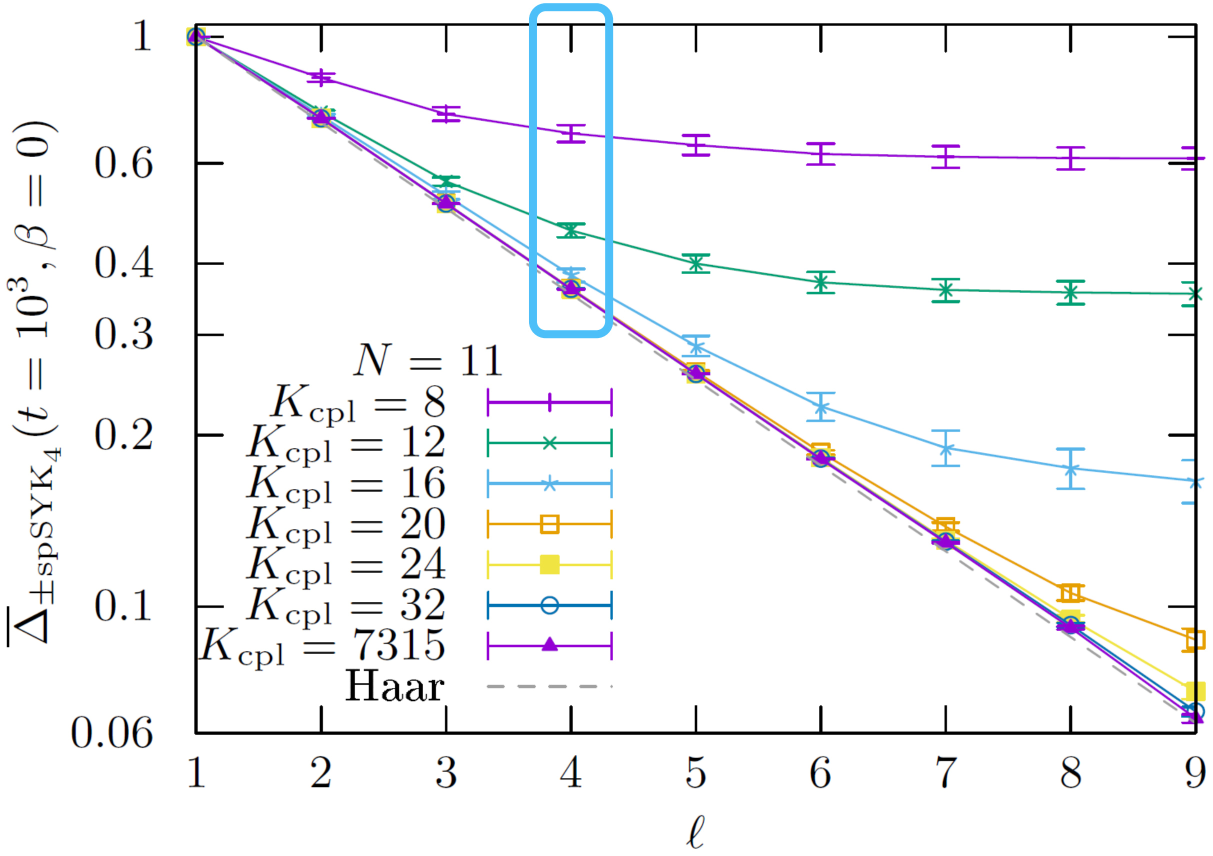
- Binary-coupling sparse model: rigidity of the eigenenergy spectrum $\sim$ Gaussian-coupling model with twice as large K_{cpl}





$\overline{\Delta}_{\widehat{H}}(t,\beta)$ for binary-coupling sparse SYK



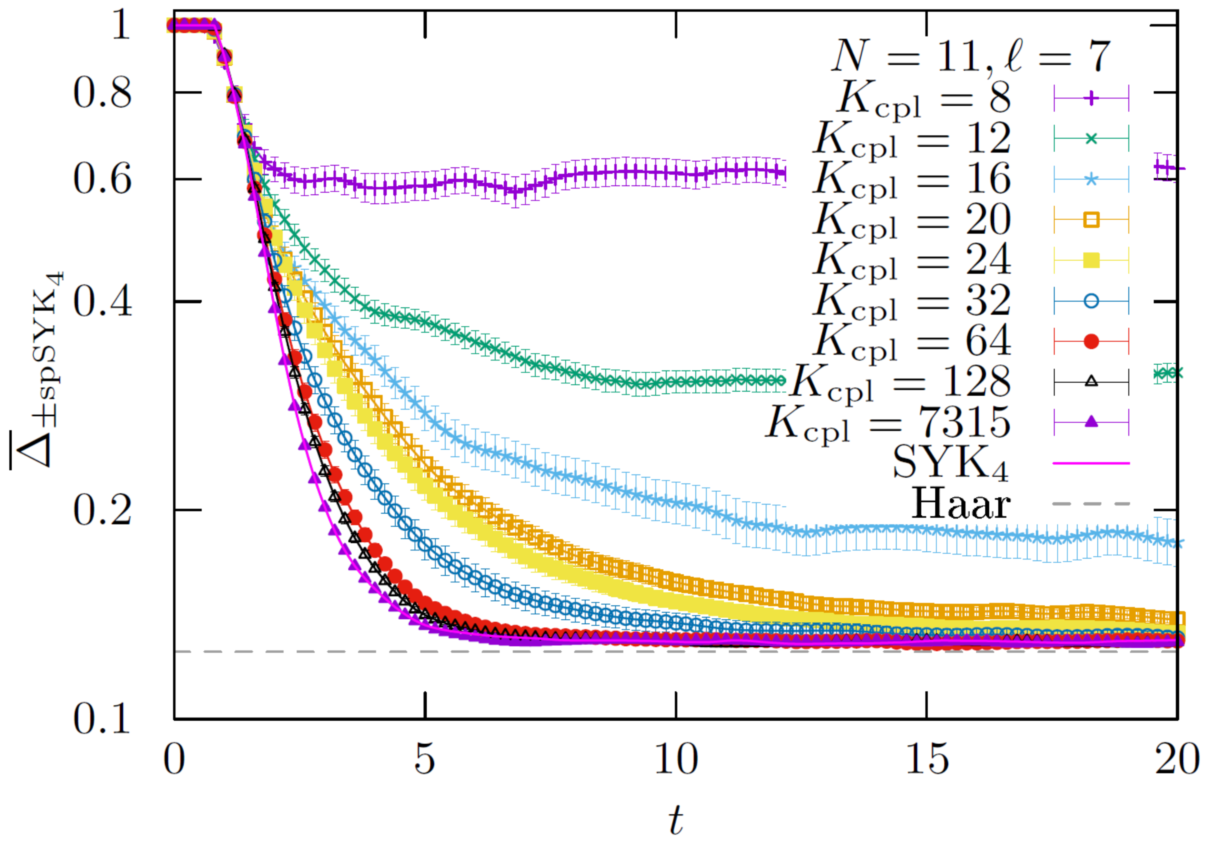


Time dependence:
approach (binary-coupling & Gaussian)
dense model as $K_{\rm cpl}$ is increased

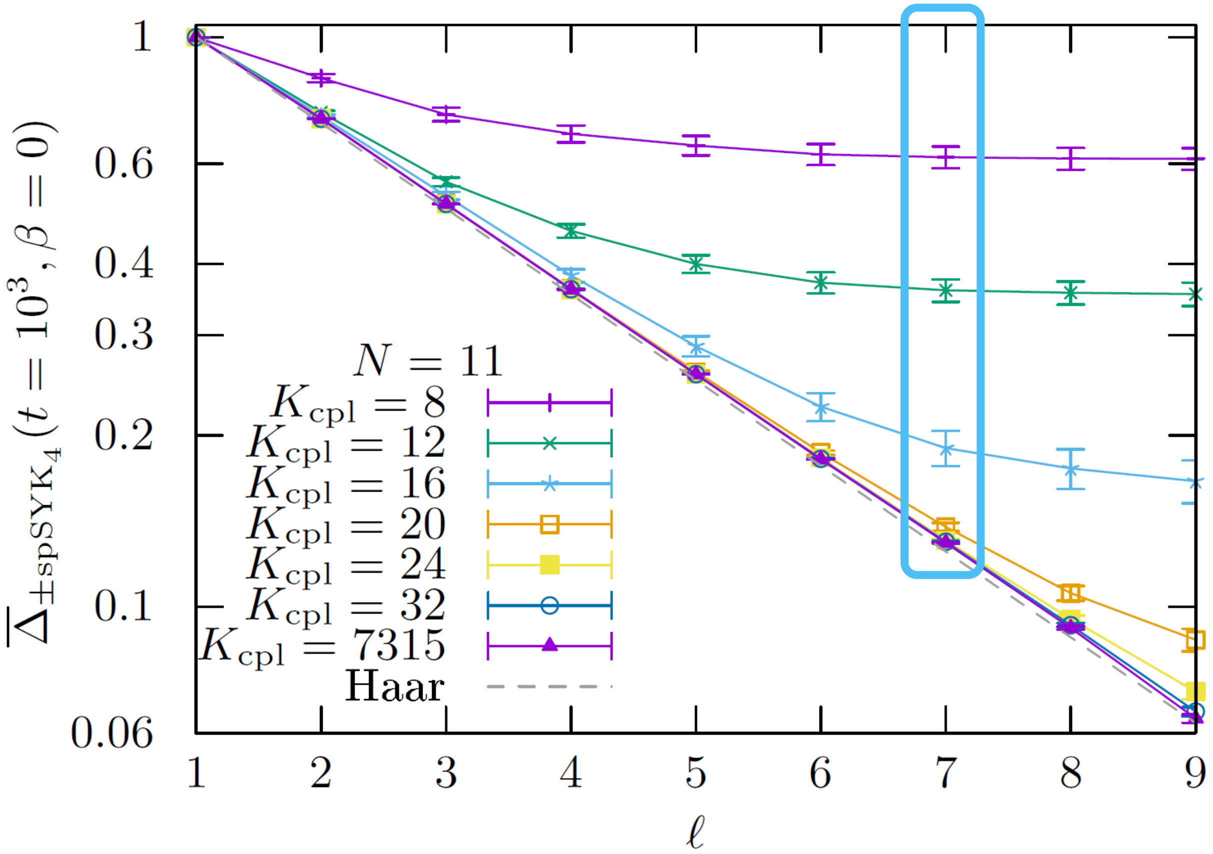
Late-time value:

very close to the Haar value $2^{\frac{1-\ell}{2}}$,
indistinguishable for $K_{\rm cpl} \gtrsim 3N$

$\overline{\Delta}_{\hat{H}}(t,\beta)$ for binary-coupling sparse SYK

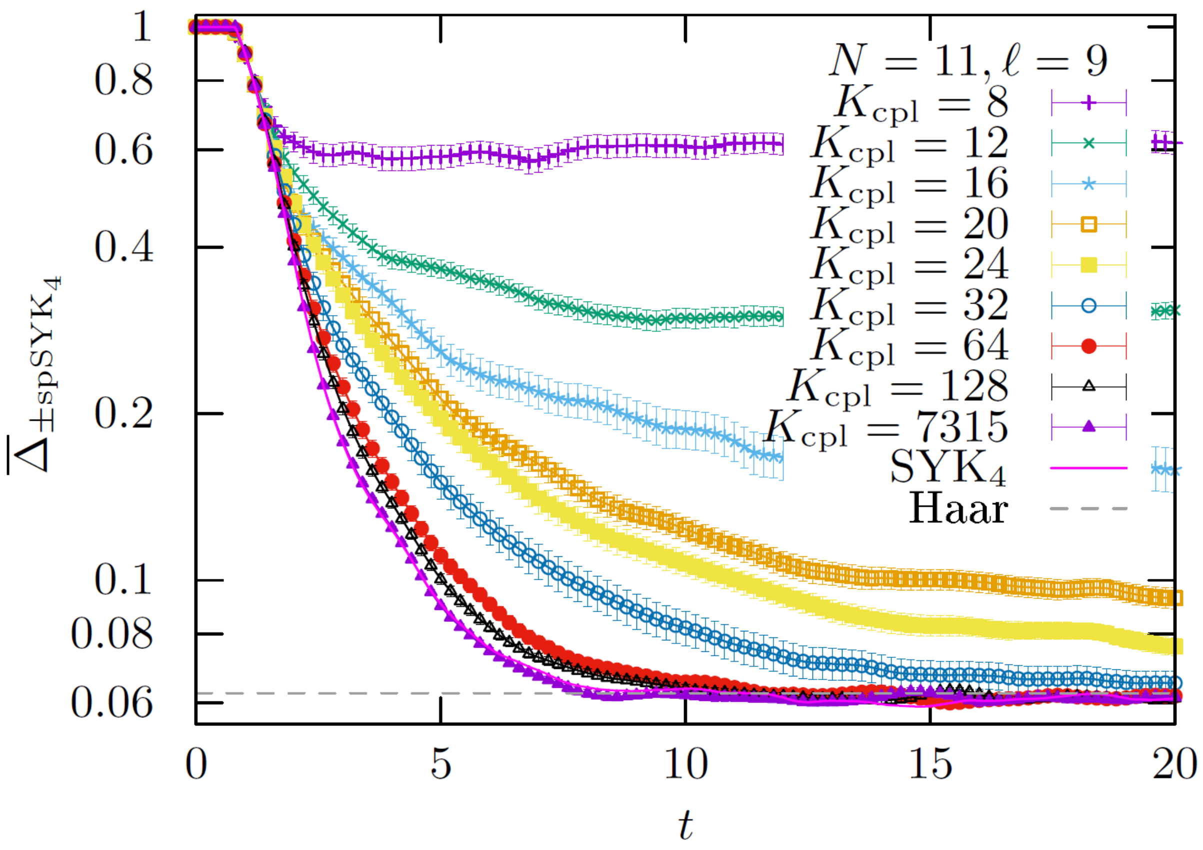


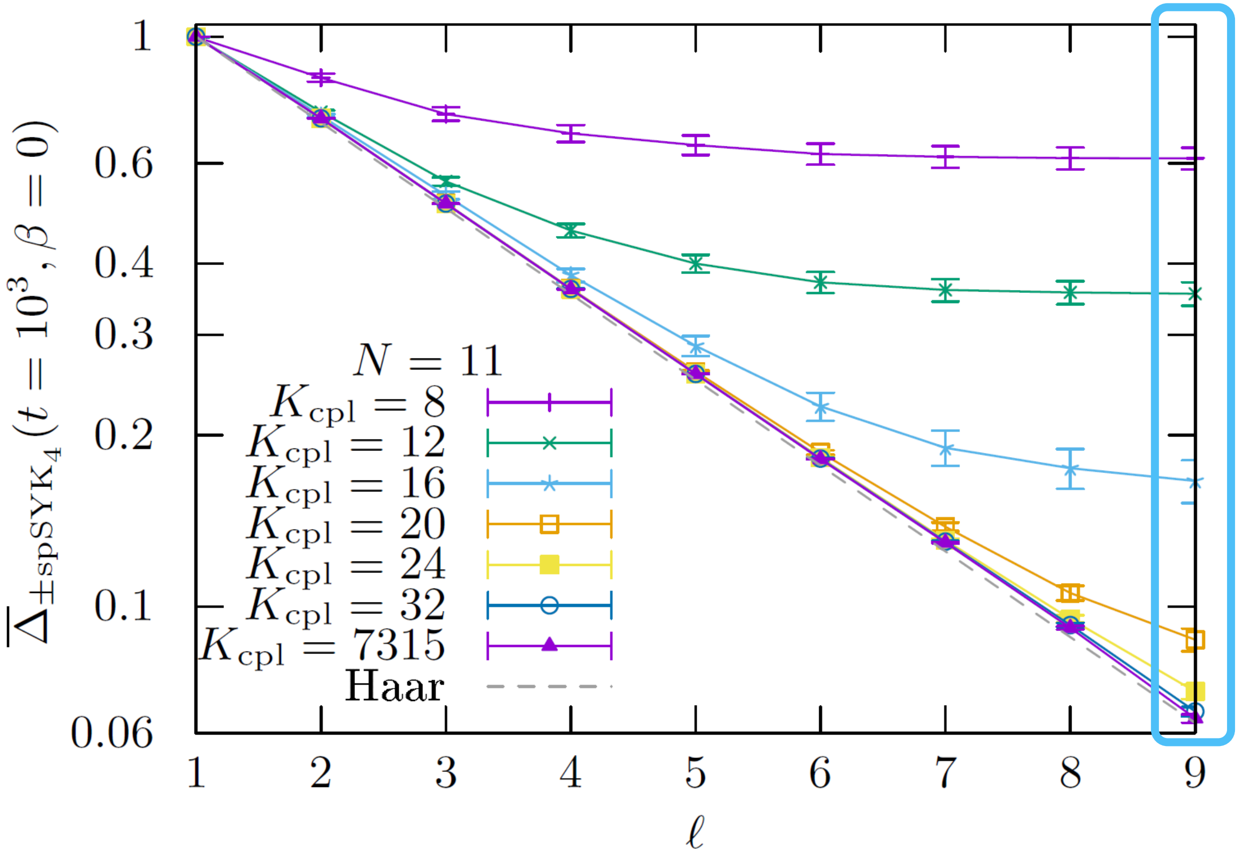
Time dependence:
approach (binary-coupling & Gaussian) dense model as $K_{\rm cpl}$ is increased



Late-time value:
very close to the Haar value $2^{\frac{1-\ell}{2}}$,
indistinguishable for $K_{\rm cpl} \gtrsim 3N$

$\overline{\Delta}_{\widehat{H}}(t,\beta)$ for binary-coupling sparse SYK

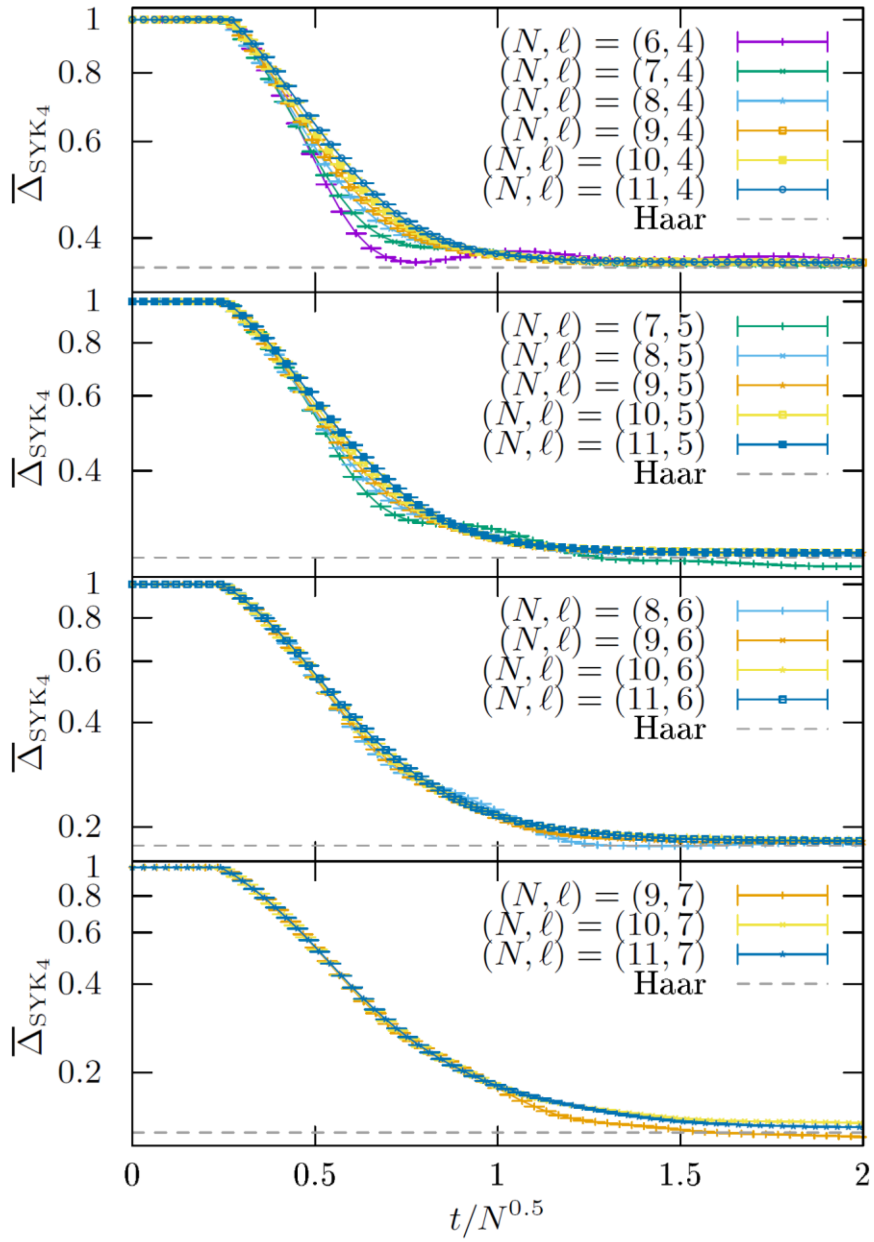


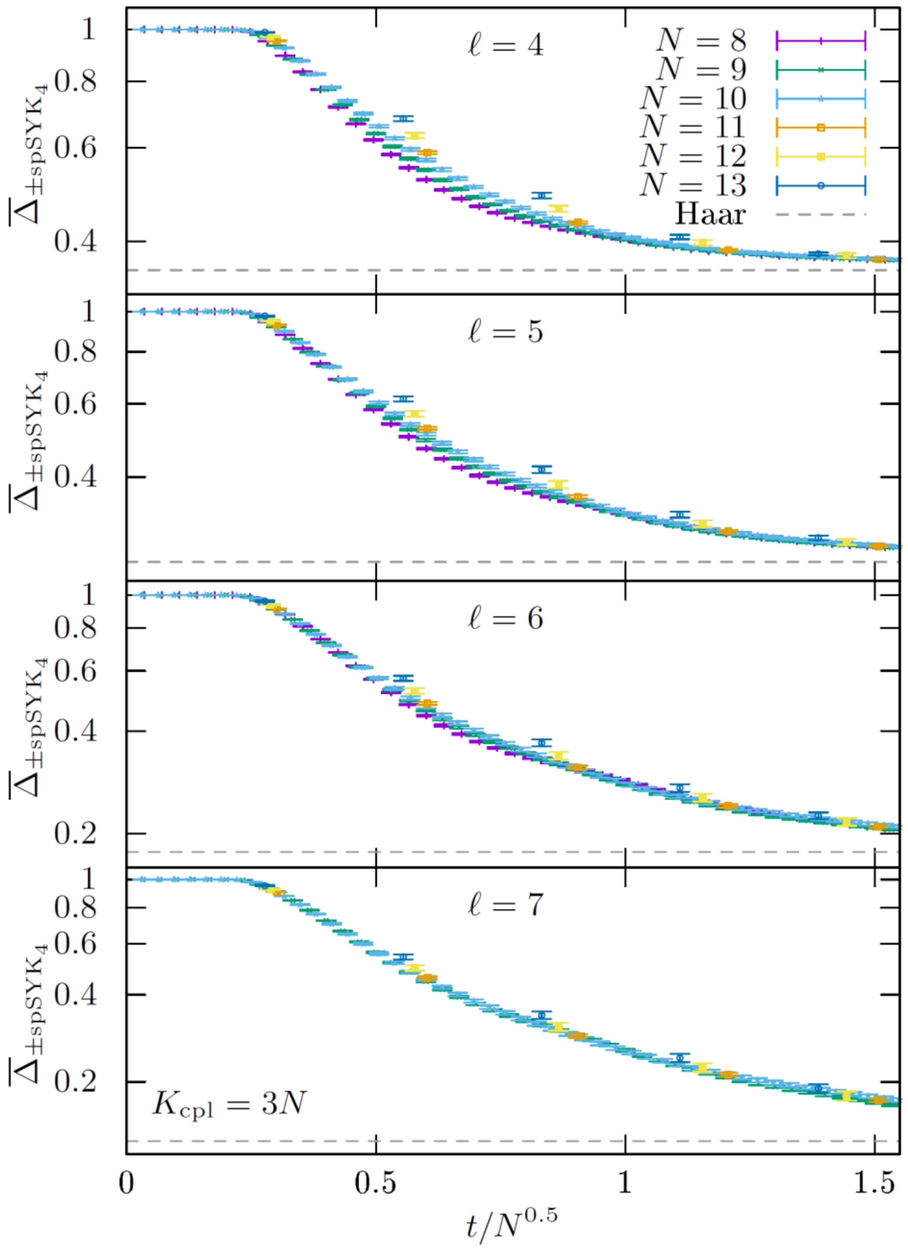


Time dependence:
approach (binary-coupling & Gaussian) dense model as $K_{\rm cpl}$ is increased

Late-time value:
very close to the Haar value $2^{\frac{k-\ell}{2}}$, indistinguishable for $K_{\rm cpl} \gtrsim 3N$

Scaling





Normalization: SYK half-bandwidth

$$\sqrt{\frac{\langle \mathrm{Tr}\,\widehat{H}^2\rangle}{2^N}} = 1, \hbar = 1$$

- The Haar value $\overline{\Delta} = 2^{\frac{k-\ell}{2}}$ is reached after $t \sim \mathcal{O}(\sqrt{N})$

Summary

SYK-like models with long-range couplings

Gaussian dense SYK$_4$

$$\hat{H} = \sum_{a<b<c<d} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

sparsify

Binary coupling sparse SYK

$$\hat{H} \propto \sum_{\substack{(a,b,c,d)\in P \\ K_{\rm cpl}=|P|\sim\mathcal{O}(N)}} (\pm 1)\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

[2208.12098]

- $K_{\rm cpl} \ll N$: additional degeneracy
- $K_{\rm cpl} \gtrsim N$: realize chaotic spectrum more efficiently than Gaussian

Error decays to $\sim$ Haar value in $t \sim \sqrt{N}$

Add SYK$_2$ term

Quantum error correction by scrambling Hamiltonian dynamics [Yoshifumi Nakata and MT, in preparation]

SYK$_{4+2}$

$$\hat{H} = \cos\theta \sum_{a<b<c<d} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + \sin\theta \sum_{a<b} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

[PRL **120**, 241603; PRR **3**, 013023; PRL **127**, 030601]

- $\delta \propto \tan\theta \ll 1$: SYK$_4$
- $\delta = \mathcal{O}(1)$: chaotic spectrum but eigenstates restricted in Fock space; entanglement entropy has plateau
- $\delta \gg 1$: many-body localization

Error increases before many-body localization