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Exact analysis of the Liouvillian gap and dynamics in the dissipative SU(*N*) Fermi-Hubbard model

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HY and Hosho Katsura, arXiv:2209.03743 (2022)

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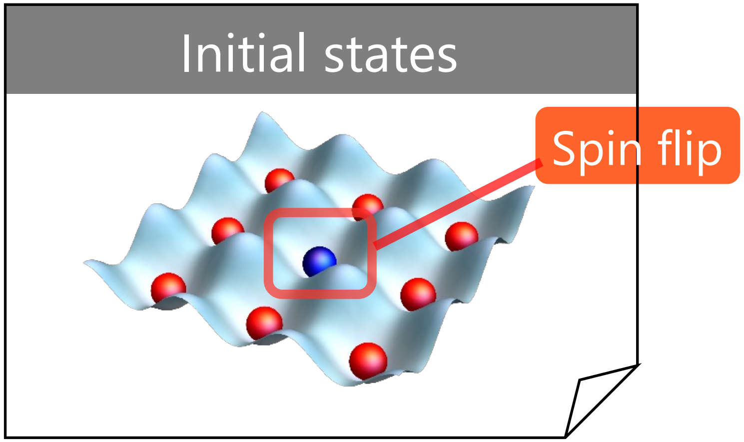
Take-home message

- We consider **the Fermi-Hubbard model** with **two-particle loss.**
- In the unit filling, **the Liouvillian gap** and **a single spin-flip dynamics** can be calculated **analytically**.
- A single spin-flip dynamics serves as an exactly solvable example of **the continuous quantum Zeno effect**.



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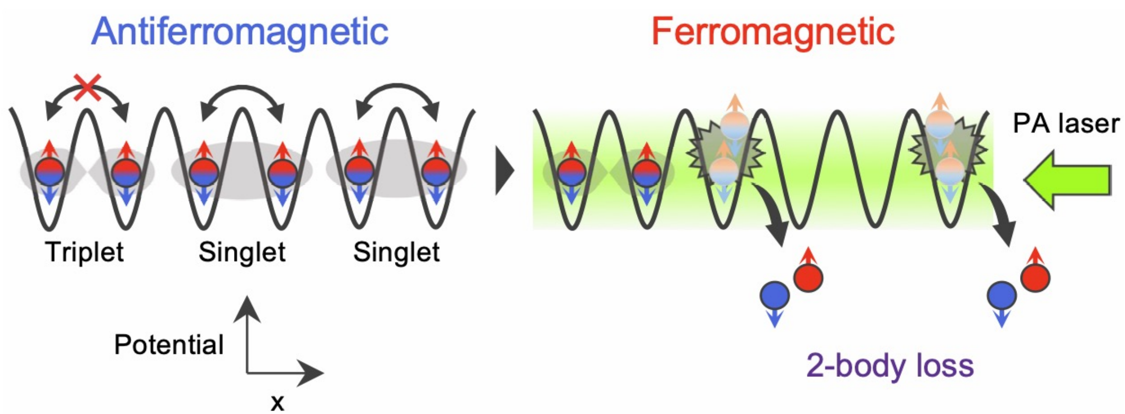
Open quantum systems

- **Quantum many body systems + dissipation**
  - Interaction with environments → **dissipation**
  - **Non-equilibrium steady states ?**
  - **Relaxation dynamics ?**
- **Ex) Fermi-Hubbard model + on-site two-body loss**
  - Realized with **cold atoms** in an optical lattice [1,2,3]
  - Controlled with **photoassociation(PA) laser**

[3]



[1] B. Yan *et al.*, Nature **501**, 521 (2013), [2] K. Sponselee *et al.*, Quantum Sci. Technol. **4**, 014002 (2019), [3] K. Honda *et al.*, arXiv:2205.13162 (2022).

Dynamics of open quantum systems

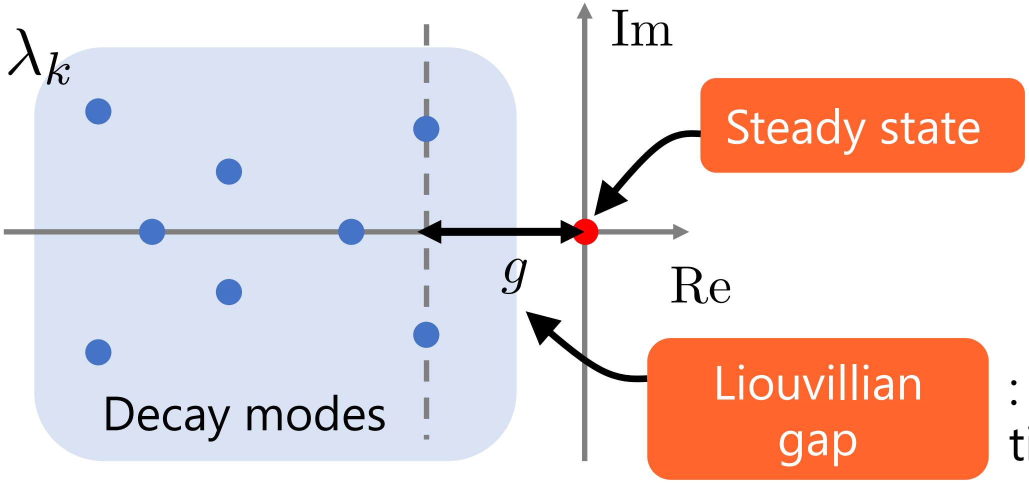
GKSL quantum master equation

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \sum_j \left(\hat{L}_j \hat{\rho} \hat{L}_j^\dagger - \frac{1}{2} \left\{ \hat{L}_j^\dagger \hat{L}_j, \hat{\rho} \right\} \right) =: \mathcal{L}\hat{\rho}$$

$\hat{\rho}$: Density matrix $\hat{H}$: Hamiltonian $\hat{L}_j$: Lindblad operator

$\mathcal{L}$: **Liouvillian**

$\mathcal{L}\hat{\rho}_k = \lambda_k \hat{\rho}_k$ λ_k : eigenvalue $\hat{\rho}_k$: eigenmode



: inverse of the longest relaxation time of decay modes.

SU(*N*) Fermi-Hubbard model + particle loss

Hamiltonian

$$\hat{H} = \hat{H}_{\text{hop}} + \hat{H}_{\text{int}}$$

$$\hat{H}_{\text{hop}} = -J \sum_{\langle x,y \rangle} \sum_{\sigma=1}^{N} \hat{c}^{\dagger}_{x,\sigma} \hat{c}_{y,\sigma}$$

$$\hat{H}_{\text{int}} = U \sum_{x \in \Lambda} \sum_{1 \le \sigma < \tau \le N} \hat{n}_{x,\sigma} \hat{n}_{x,\tau}$$

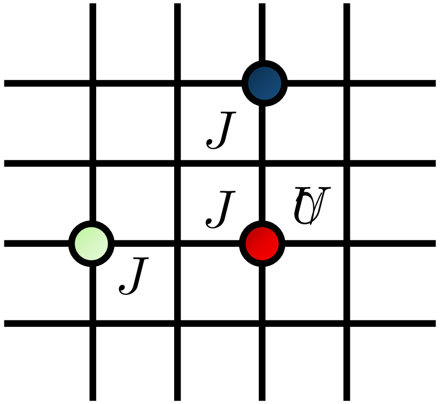
N : the number of internal degrees of freedoms

Solids:
N=2 (spins of electrons)
Cold atoms:
N=6 (nuclear spins of ^{173}Yb)
N=10 (nuclear spins of ^{87}Sr)

Lindblad operators

$$\hat{L}_x = \sqrt{2\gamma} \sum_{1 \le \sigma < \tau \le N} \hat{c}_{x,\sigma} \hat{c}_{x,\tau}$$

$\gamma > 0$: loss rate



Problems and our approach

In one dimension,
The Liouvillian gap and dynamics are analyzed with

- Exact solutions [1]
- Loss-rate equations[2,3]

[Problems] In higher dimensions, it is much harder to analyze them because

- some methods are not applicable
- numerically costly

[Our approach] By focusing on **low-energy excitations** from the steady states, the Liouvillian gap and the a single-spin flip dynamics can be calculated **analytically** or **numerically with low cost.**

[1] M. Nakagawa *et al.*, Phys. Rev. Lett. **126**, 110404 (2021),
[2] K. Sponselee *et al.*, Quantum Sci. Technol. **4**, 014002 (2019)
[3] L. Rosso *et al.*, arXiv:2206.06837 (2022).

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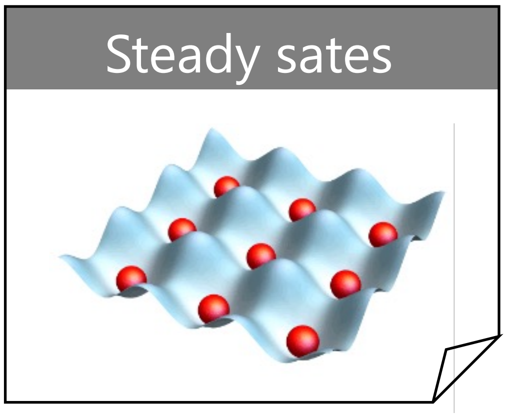
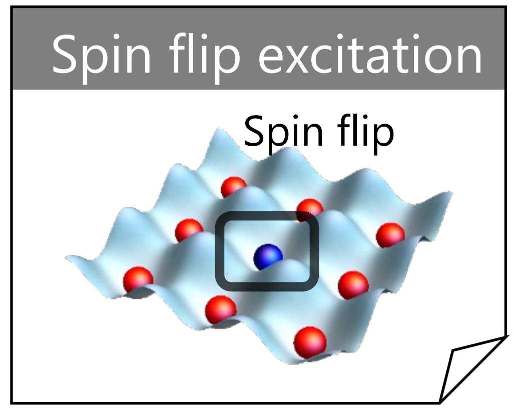
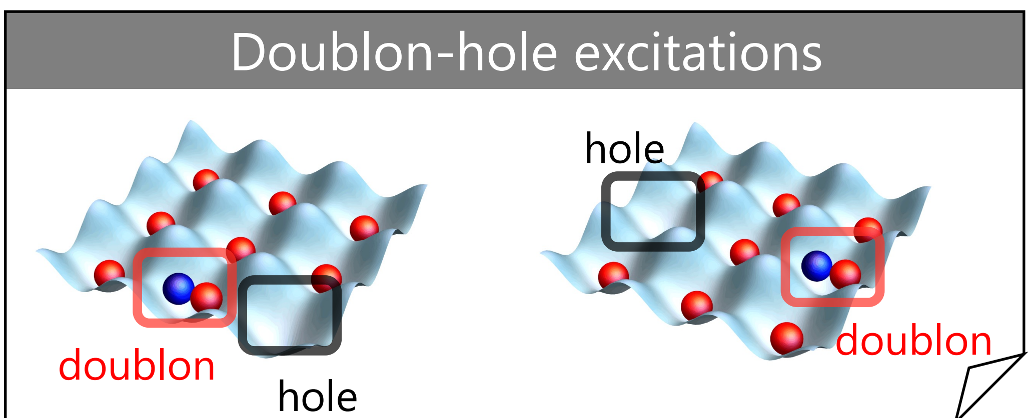
Slowly decaying states

- We focus on **the unit filling**
- **Steady state**: ferromagnetic states
- **Slowly decaying states**:
  States with a spin flip excitation or a doublon-hole excitation







Results 1: Liouvillian gap

We consider the SU(N) Fermi-Hubbard model on a ***d*-dimensional** hypercubic lattice with **linear size *L***. In the unit filling sector, the **Liouvillian gap *g*** satisfies

$$g = \frac{8\pi^2 J^2 \gamma}{(U^2 + \gamma^2)\, L^2} + o(L^{-2})$$

J : hopping amplitude

U : strength of interaction

γ : loss rate

Points

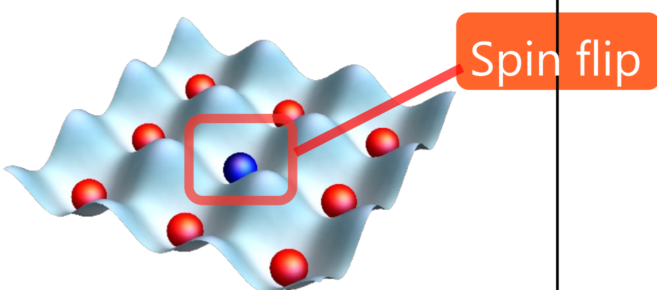
- g does not depend on d or N.
- g is proportional to $1/L^2$ "gapless"

Results 2: single spin-flip dynamics

- **Initial state**: states with a spin flip
- The dynamics can be exactly captured by a spin flip and a doublon-hole excitation.

$\mathcal{N}$: **survival probability** of a spin flip $\quad t$: time

Initial states



$$\mathcal{N}_0 = 1$$

Dynamics?

Steady states

$$\mathcal{N}_\infty = \frac{1}{L^d}$$

Results 2: single spin-flip dynamics

- Analytical results in two limits $L \gg 1$

Strong interaction, dissipation:

$$\mathcal{N}(t) = \left[e^{-\Gamma t} I_0(\Gamma t)\right]^d \sim (2\pi\Gamma t)^{-d/2}$$

$$\Gamma = \frac{8J^2\gamma}{U^2+\gamma^2}$$

Power-law decay !

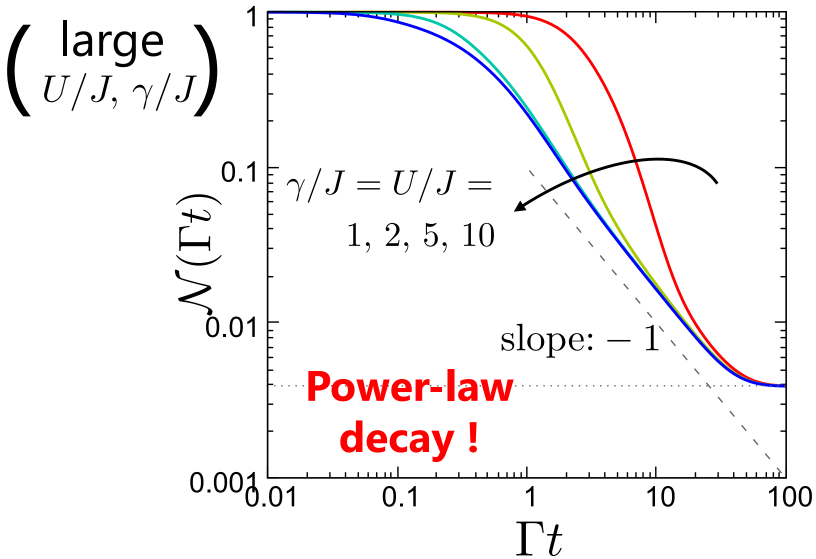
$I_0(x)$: modified Bessel function of the first kind

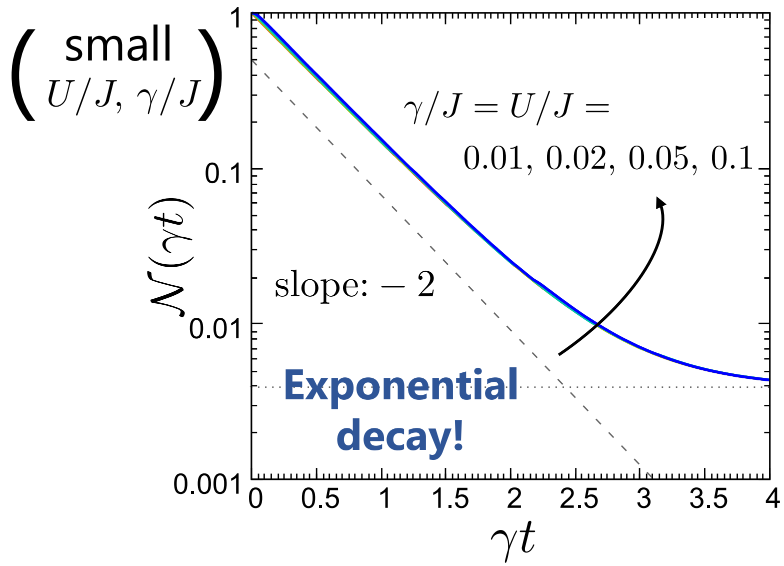
Weak interaction, dissipation:

$$\mathcal{N}(t) = e^{-2\gamma t}$$

Exponential decay!

- Numerical results (16×16)





Strong dissipations make the dynamics slower

Weak interaction, dissipation : $J \gg U, \gamma$ $\mathcal{N}(t) = e^{-2\gamma t}$

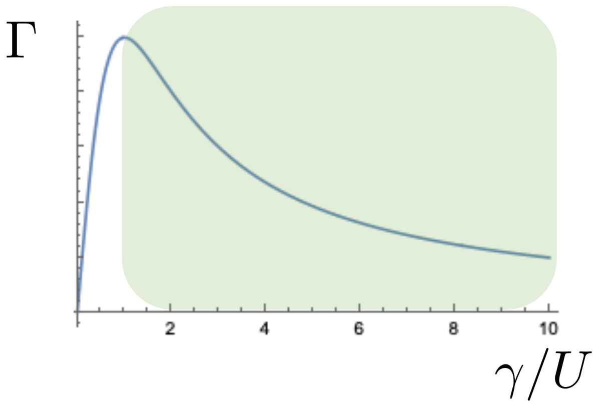
Increasing γ : dissipation becomes **faster**

Strong interaction, dissipation : $J \ll U, \gamma$

$$\mathcal{N}(t) = \left[e^{-\Gamma t} I_0(\Gamma t)\right]^d \sim (2\pi\Gamma t)^{-d/2}$$

$$\Gamma = \frac{8J^2\gamma}{U^2 + \gamma^2}$$

Fixing J, U and changing γ



When $\gamma/U \geq 1$

γ increase → Γ decrease

→ The dissipation becomes **slower**

Related to continuous quantum Zeno effect

Continuous quantum Zeno effect

- **Quantum Zeno effect** : Measurements suppress the coherent dynamics
- **Two body loss** can be interpreted as **continuous (local) measurements of the double occupations**
- Hopping that changes the number of **double occupations** is suppressed
  → Two body loss is suppressed

Experiments : N. Syassen et. al., Science **320**, 1329 (2008)
B. Zhu et al., Phys. Rev. Lett. **112**, 070404 (2014)
T. Tomita, et. al., Sci. Adv. **3**, e1701513 (2017)

Review : A. J. Daley, *Advances in Physics* 63.2 (2014): 77-149.

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Calculation of the Liouvillian gap

(1) Using the **block-triangular structure** of the Liouvillian

(2) Assume a state with **a doublon+a hole: two-body problem!**

(3) Separation of the center-of-mass and relative coordinates：
the relative coordinate satisfies **tight binding model with complex impurity**

(4) From the **energy of the bound states**, we can calculate the Liouvillian gap

(1) Diagonalization of the Liouvillian

- **Hamiltonian** (**loss**) **conserves** (**decreases**) the particle number
- The Liouvillian can be decomposed as [1,2,3]

$$\mathcal{L} = \mathcal{K} + \mathcal{J}$$

$$\mathcal{K}\hat{\rho} := -i\left(\hat{H}_{\mathrm{eff}}\hat{\rho} - \hat{\rho}\hat{H}_{\mathrm{eff}}^{\dagger}\right)$$

$$\mathcal{J}\hat{\rho} := \sum_{x\in\Lambda} \hat{L}_x\hat{\rho}\hat{L}_x^{\dagger}$$

$$\hat{H}_{\mathrm{eff}} := \hat{H} - \frac{i}{2}\sum_{x\in\Lambda}\hat{L}_x^{\dagger}\hat{L}_x$$: effective Hamiltonian

Matrix representation (in a proper basis)

Conserves the particle number
→block-diagonal

Decreases the particle number
→strictly upper block-triangular

- Eigenvalues of a **block-triangular matrix (Liouvillian)** can be calculated from **block-diagonal part (Effective Hamiltonian)**

[1] J. M. Torres, PRA **89**, 052133 (2014), [2] T. Yoshida, et. al., PRR **2**, 033428 (2020), [3] M. Nakagawa, et. al., PRL **126**, 110404 (2021).

(2) Effective Hamiltonian : Non-Hermitian Hubbard model

$$\hat{H}_{\mathrm{eff}} = \hat{H}_{\mathrm{hop}} + \hat{H}_{\mathrm{int}}$$

$$\hat{H}_{\mathrm{hop}} = -J \sum_{\langle x,y \rangle} \sum_{\sigma=1}^{N} \hat{c}^{\dagger}_{x,\sigma} \hat{c}_{y,\sigma}$$

$$\hat{H}_{\mathrm{int}} = (U - i\gamma) \sum_{x \in \Lambda} \sum_{1 \le \sigma < \tau \le N} \hat{n}_{x,\sigma} \hat{n}_{x,\tau}$$

- Exactly solved with Bethe ansatz in one dimension [1]
- Cannot be exactly solved in two dimension or higher.
- But exactly solved by assuming the following wave function

$$|\psi\rangle = \sum_{\boldsymbol{x},\boldsymbol{y} \in \Lambda} g(\boldsymbol{x}, \boldsymbol{y}) \hat{c}^{\dagger}_{\boldsymbol{x},\sigma} \hat{c}_{\boldsymbol{y},\tau} |\mathrm{FM}_{\tau}\rangle. \qquad |\mathrm{FM}_{\tau}\rangle = \prod_{\boldsymbol{x} \in \Lambda} \hat{c}^{\dagger}_{\boldsymbol{x},\tau} |0\rangle$$: Steady states

[1] M. Nakagawa *et al.*, PRL **126**, 110404 (2021)

(3) Reducing to a tight-binding model

- Separation of the **center-of-mass** and **relative** coordinates

$$\boldsymbol{m} = (\boldsymbol{x}+\boldsymbol{y})/2,\ \boldsymbol{n} = \boldsymbol{x}-\boldsymbol{y}$$

$$g(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{m}} f_{\boldsymbol{k}}(\boldsymbol{n})$$

$\boldsymbol{k}$: **center-of-mass** momentum

$k^{\mu} = 2\pi l_{\mu}/L \quad (l_{\mu} = 1,\dots,L)$

- $f_{\boldsymbol{k}}(\boldsymbol{n})$ satisfies **tight-binding model** with **complex-valued impurity**

$$\sum_{\mu=1}^{d} r_{\boldsymbol{k}}^{\mu}[f_{\boldsymbol{k}}(\boldsymbol{n}+\boldsymbol{e}_{\mu}) + f_{\boldsymbol{k}}(\boldsymbol{n}-\boldsymbol{e}_{\mu})] = [E_{\boldsymbol{k}} - u\,(1-\delta_{\boldsymbol{n},\boldsymbol{0}})]\, f_{\boldsymbol{k}}(\boldsymbol{n}).$$

where $r_{\boldsymbol{k}}^{\mu} = 2J\sin(k^{\mu}/2) \quad u = U - i\gamma$

(4) Calculate the energy of the bound state

- Energy $E_{\boldsymbol{k}}$ of the **bound state** is a solution of

$$\frac{1}{\pi^d}\int_0^{\pi} d^d q \frac{u}{E_{\boldsymbol{k}} - u - \sum_{\mu=1}^{d} 2 r_{\boldsymbol{k}}^{\mu} \cos q_{\mu}} = -1$$

Energy without the impurity

$|\boldsymbol{k}| \ll 1$ ⇒

$$E_{\boldsymbol{k}} = -\frac{8J^2 \sum_{\mu=1}^{d} \sin^2\left(k^{\mu}/2\right)}{u}$$

- By substituting $k^1 = 2\pi/L \quad k^2 = \ldots = k^d = 0$

Result 1 : $$g = \frac{8J^2\gamma \sin^2(\pi/L)}{U^2+\gamma^2} = \frac{8\pi^2 J^2 \gamma}{\left(U^2+\gamma^2\right) L^2} + O\left(L^{-3}\right)$$

A single spin-flip dynamics

- **Point!** In the initial state with **a single-spin flip** ,
  - The quantum jump (particle loss) only occurs **once**.
  - The number of the spin flip after the jump is **zero.**
- **The survival probability** can be calculated as [Appendix]

$$\mathcal{N}(t) = \mathrm{Tr}[|\psi(t)\rangle\langle\psi(t)|] \qquad |\psi(t)\rangle = e^{-i\hat{H}_{\mathrm{eff}}t}\,|\psi_0\rangle$$

- From the separation of the center-of mass and relative coordinates,

$$\mathcal{N}(t) = \frac{1}{L^d}\sum_{\boldsymbol{k},j} |d_{\boldsymbol{k},j}|^2\, e^{2\,\mathrm{Im}[E_{\boldsymbol{k},j}]t} \qquad \delta_{\boldsymbol{n},\boldsymbol{0}} = \sum_j d_{\boldsymbol{k},j} f_{\boldsymbol{k},j}(\boldsymbol{n})$$

$E_{\boldsymbol{k},j},\ f_{\boldsymbol{k},j}(\boldsymbol{n})$: j-th solutions of the **tight-binding model** with impurity

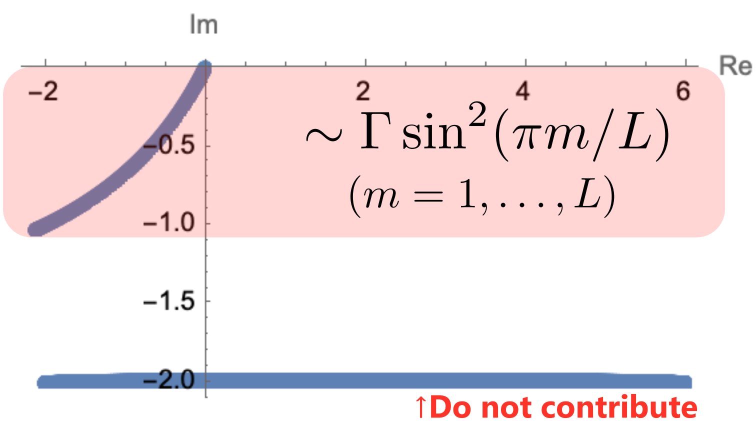
$$\sum_{\mu=1}^{d} r_{\boldsymbol{k}}^{\mu}[f_{\boldsymbol{k}}(\boldsymbol{n}+\boldsymbol{e}_\mu) + f_{\boldsymbol{k}}(\boldsymbol{n}-\boldsymbol{e}_\mu)] = [E_{\boldsymbol{k}} - u\,(1-\delta_{\boldsymbol{n},\boldsymbol{0}})]\, f_{\boldsymbol{k}}(\boldsymbol{n}). \quad r_{\boldsymbol{k}}^{\mu} = 2J\sin(k^\mu/2)$$

Dynamics in the two limits

- Spectrum of the tight binding model (1D)

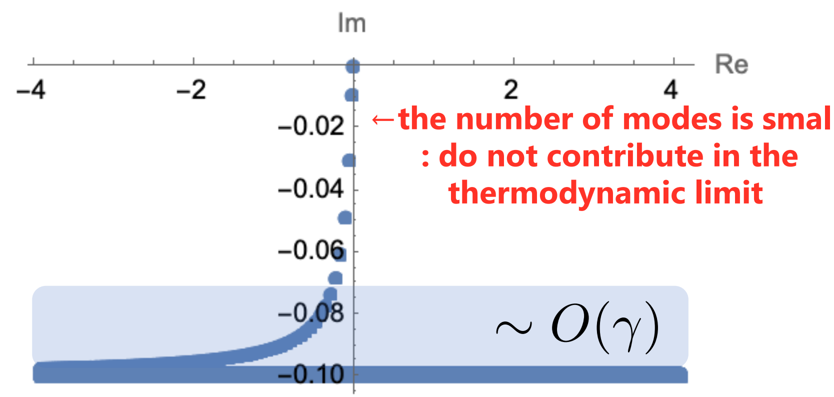
Strong interaction and dissipation

$$t=1,\ U=2,\ \gamma=2$$



Weak interaction and dissipation

$$t=1,\ U=0.1,\ \gamma=0.1$$



$$\mathcal{N}(t)=\frac{1}{L^d}\sum_{\boldsymbol{k}} e^{-2\Gamma t\sum_{\mu=1}^{d}\sin^2(k^\mu/2)}$$

$$\to \left(\int_0^1 e^{-2\Gamma t\sin^2(\pi x)}dx\right)^d$$

$$=\left[e^{-\Gamma t}I_0(\Gamma t)\right]^d$$

$$\mathcal{N}(t)=e^{-2\gamma t}$$

Result 2 !

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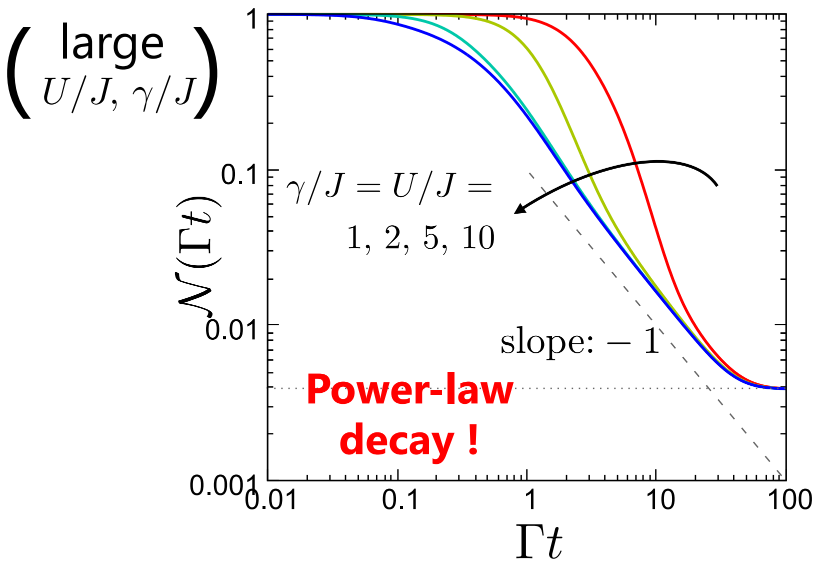
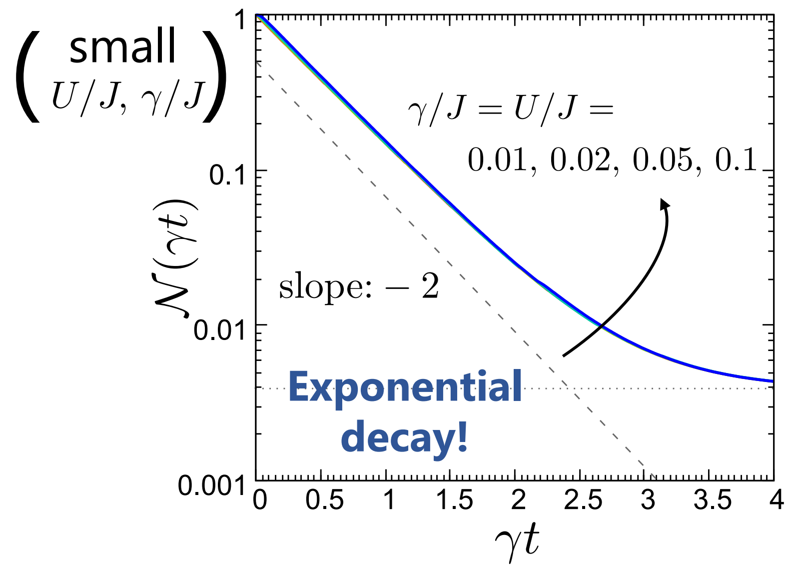
- **Introduction**
  - Open quantum system and GKSL equation
  - SU(N) Fermi-Hubbard model with two-body loss
- **Results**
  - Liouvillian gap
  - A single spin flip dynamics
- **Derivations**
  - Liouvillian gap
  - A single spin flip dynamics
- **Summary and outlook**

Summary

arXiv: 2209.03743

- For the dissipative SU(*N*) Fermi-Hubbard model on a *d*-dimensional lattice, we obtained
  - **Closed form of the Liouvillian gap in the unit filling**
    - Does not depend on *d* or *N* and proportional to $1/L^2$
  - **A single spin-flip dynamics**
    - Crossover from the power-law to exponential decay with the strength of interaction and dissipations





Summary

arXiv: 2209.03743

The range of applications

- Applicable to **a general lattice** (in principle).
- Even without translational symmetry, a spin-flip dynamics can be analyzed (with increased numerical costs)
  →Applicable to a system with **a confinement potential**

Outlook

- Experimental realizations?
- **Beyond the two-body dynamics**
  - Below the unit filling?
  - Multibody excitation→truly SU(N) phenomena?

Appendix

I. DERIVATION OF EQ. (23)

Here, we derive Eq. (23) in the main text. By decomposing $\mathcal{L}$ as $\mathcal{L} = \mathcal{K} + \mathcal{J}$, $\mathcal{N}_\sigma(t)$ can be expanded as

$$\mathcal{N}_\sigma(t) = \mathrm{Tr}\left[\hat{N}_\sigma e^{\mathcal{L}t}\hat{\rho}_0\right] = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{Tr}\left[\hat{N}_\sigma(\mathcal{K}+\mathcal{J})^n\hat{\rho}_0\right]. \tag{S1}$$

Note that $\mathcal{K}$ ($\mathcal{J}$) always preserves (decreases) the particle number. Since $\mathrm{Tr}\left[\hat{N}_\sigma\hat{\rho}_0\right] = \mathrm{Tr}\left[\hat{\rho}_0\right] = 1$, we have $\mathrm{Tr}\left[\hat{N}_\sigma\mathcal{K}^n\hat{\rho}_0\right] = \mathrm{Tr}\left[\mathcal{K}^n\hat{\rho}_0\right]$ for all $n = 0, 1, \ldots$ and the other terms appearing in Eq. (S1) such as $\mathrm{Tr}\left[\hat{N}_\sigma\mathcal{J}\mathcal{K}\hat{\rho}_0\right]$ equal to 0. Thus

$$\mathcal{N}_\sigma(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{Tr}\left[\mathcal{K}^n\hat{\rho}_0\right] = \mathrm{Tr}\left[e^{\mathcal{K}t}\hat{\rho}_0\right] = \mathrm{Tr}\left[|\psi(t)\rangle\langle\psi(t)|\right], \tag{S2}$$

where $|\psi(t)\rangle = e^{-i\hat{H}_{\mathrm{eff}}t}|\psi_0\rangle$.