QCD instanton effects in light and heavy mesons

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Abstract

Quantum chromodynamics (QCD) describes the physics of the strong interaction between quarks and gluons. QCD has a characteristic feature called "asymptotic freedom", and this feature leads to non-perturbative phenomena. How the non-perturbative effects in QCD are evaluated is one of the important issue in the elementary particle physics.

It is blindly believed that a non-trivial vacuum structure in QCD is the quantum mechanical superposition of an infinite number of vacua. The QCD instanton solution is a classical solution to Yang-Mills theory in Euclidean space-time, and is believed to describe the transition between the vacua. Although this object is very interesting and contributes to developments of mathematical and theoretical physics, its signature has not been discovered in any experiment yet. The verification of the non-trivial vacuum structure or the QCD instanton effects is important to comprehend the phenomena in the non-perturbative region in QCD.

We discuss the constraint on the size of the QCD instanton effects in a low-energy effective theory. Among various instanton effects in meson mass spectrum and dynamics, we concentrate on the instanton-induced masses of light quarks, namely up, down and strange quark. The famous instanton-induced six-quark interaction, the so-called 't Hooft vertex, could give non-perturbative quantum corrections to light quark masses. Many works have already been done to constrain the mass corrections in the light meson system, namely in the system of π, K, η and η', and we know the fact that the instanton-induced mass of up-quark is too small to realize the solution of the strong CP problem because of vanishing current mass of up-quark.

In this thesis we give a constraint on the instanton-induced mass correction to light quarks from the mass spectrum of heavy mesons, B^+, B^0, B_s and their anti-particles. To accomplish this, the complete second order chiral symmetry breaking terms are identified in the heavy meson effective theory. We find that the strength of the constraint from heavy meson masses is at

the same level as that from light mesons, and it would be made even stronger by more precise data from future B factories and lattice calculations.

Contents

Chapter 1

Introduction

The aim of elementary particle physics is to find a fundamental principle which governs all the phenomena in the real world. The principle is believed to be simple and the fundamental theory is expected to be universal. It is important that experiments and theories should be complementarily developed in physics. To discover how the real world is, a lot of collider experiments and observations as well as theoretical investigations have been done in the past. For now, the standard model of elementary particles is the most reliable theory.

The standard model of elementary particles is composed by $\mathrm{SU}(3)_{\mathrm{C}}$ gauge symmetry of the strong interaction and $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{U}(1)_{\mathrm{Y}}$ gauge symmetry of the electroweak interaction [1, 2, 3]. The standard model provides the most successful description of the physics in the energy scale which we can currently reach with particle accelerators. Especially, the discovery of Higgs particle [4, 5], which is associated with the spontaneously electroweak symmetry breaking, is one of the glorious achievements in the Large Hadron Collider [6].

Despite its successes, many open questions remain in the standard model. One of the theoretical problems is that the gravitational interaction is not contained in the standard model. There are a lot of unpredicted parameters associated with the flavor sector of the standard model. This mystery, which would be related with the origin of electroweak symmetry breaking, implies an existence of fundamental physics behind the standard model. Furthermore, in view of cosmological observations, dark matter and dark energy are also left unexplained.

There are a great deal of challenges to explain above issues with many different types of the new physics beyond the standard model (for example, supersymmetric models, models with extra dimensions, composite Higgs models and so on). It is highly important to investigate which new physics is the most suitable scenario to describe nature. However, no signature of the new physics has been discovered in the experiments and observations in the past yet.

On the basis of these circumstances, deep understanding of the standard model is more important than verifying the physics beyond the standard model. Quantum chromodynamics (QCD) is known as the system with the asymptotic freedom which the coupling constant of QCD, namely the strong coupling constant, becomes small in process with large momentum transfer [7, 8]. On the other hand, the coupling constant becomes large in process with small momentum transfer, corresponding to interactions at large distance scales. This characteristic feature leads to non-perturbative phenomena such as the confinement of quarks and gluons. How the non-perturbative effects are evaluated is one of the most challenging subject in understanding the standard model deeper than now.

The non-trivial vacuum structure in QCD, which is the quantum mechanical superposition of an infinite number of vacua, has not been discovered. The transition from one vacuum to another vacuum in the vacuum structure would be described by an instanton solution (or an instanton configuration) which are classical solutions to the non-Abelian gauge field equation defined in Euclidean space-time [9]. This tunneling effect is referred to as an "instanton effect". The verification of existences of the non-trivial vacuum structure and the QCD instanton effect is an important topic which is related to understanding of non-perturbative effects in the QCD sector. In this thesis, we give a possibility to verify the instanton effect in the non-trivial QCD vacuum in the low energy effective theory.

In the limit in which up, down and strange quark masses vanish, the QCD Lagrangian has a chiral symmetry which is an invariance under the independent phase transformation of the left-handed and right-handed fermions. The chiral symmetry is spontaneously broken due to the quark condensate and the Nambu-Goldstone bosons are expected. The light mesons, π, K and η are identified as the Nambu-Goldstone bosons. However, it is known that the physical η' mass is much larger than the theoretical prediction [10]. A possibility to solve this problem, so-called $\mathrm{U(1)_A}$ problem, by using the instanton effect has been pointed out in [11]. The statement is that as a result of the axial anomaly [12, 13] and the non-trivial vacuum structure, there is no Nambu-Goldstone boson coupled to the physical $\mathrm{U(1)_A}$ current. Therefore, η' would be heavy.

On the other hand, there are some indications which suggest that the instanton effect may not necessarily give a solution of this problem. The problem could be understood within the $1/N_c$ expansion [14, 15, 16]. There is also an indication of inconsistency between the Ward-Takahashi identity for the $\mathrm{U}(1)_A$ current and the quark condensate in the instanton configuration [17]. In addition, the instanton effect has not been directly confirmed by experiments yet.

If we believe the existence of the non-trivial vacuum structure, the instanton effect provides the so-called Θ-term which gives CP violation in QCD. Then the Θ-term should be strongly suppressed by some reasons because such a CP violating process is not observed in QCD. In fact, from the CPT theorem, CP violation leads T violation and the observed scale of T violation in physics demands $\Theta < 10^{-5}$ [18]. This is called strong CP problem. The Peccei-Quinn mechanism [19] is a possibility to solve this problem. The predicted new particle, called "axion", which is a Nambu-Goldstone boson with the spontaneous breaking of the Peccei-Quinn symmetry, has not been discovered yet. The verification of the instanton effect in the real world remains to be achieved.

It is highly important to directly observe instanton-induced effects in experiments. The instanton effect gives a six-quark interaction, which violates the $\mathrm{U}(1)_A$ symmetry in QCD, known as 't Hooft vertex [20]. The contribution of instanton-induced effects in deep inelastic scattering is investigated with instanton perturbation theory [21] and the direct searches have been made at the electron-proton collider HERA [22, 23, 24]. No signal is observed, and it gives a constraint on the cross section by the instanton-induced processes.

This is one of the quantitative result of the direct search for instanton effects.

The six-quark interaction also induces a quantum correction to light quark masses. This quantum correction is proportional to the product of different quark flavor masses. An "effective up-quark mass" of the form has been first considered in connection with the instanton effect in [25, 26]. Since the instanton effect could generate a non-zero effective up-quark mass even when $m_u = 0$, the strong CP phase could be unphysical, and there could be no strong CP problem. Here, m_u is a bare or current quark mass, and $m_u = 0$ means the existence of chiral symmetry for up quark. A hidden symmetry under the so-called instanton transformation, which is related to the instanton-induced quark mass correction, is discovered in the low-energy light meson effective theory with next-to-leading order terms in chiral Lagrangian [27]. The instanton effect on the second order coupling constant has been discussed in [28]. This attempt is one of the other quantitative result of the indirect search for the instanton effect in the light meson system.

The precise data on B meson masses, namely b-flavored pseudoscalar mesons B^+, B^0, B_s and their anti-particles, are obtained by various experiments. The purpose of this thesis is to give another quantitative result from heavy meson system. We consider a heavy meson system with the heavy meson effective theory. In order to discuss the constraint on the size of the instanton effect, the chiral symmetry breaking terms in next-to-leading order effective Lagrangian are systematically investigated. We find a hidden symmetry under the instanton transformation in the heavy meson effective theory. We estimate the upper bound of the correction to light quark masses from the instanton-induced effect under some assumptions and also discuss whether or not the instanton-induced effective mass is large enough to resolve strong CP problem by $m_u = 0$.

In the future the B-factories, such as LHCb and Super-KEKB, will give more precise data on the mass spectrum and the decay constants in the B meson system. In addition, the development of lattice calculation on heavy quarks gives useful information. These knowledge would provide more strict constraints on the instanton-induced effects.

This thesis is developed as follows. Chapters 1 to 4 represent a review of essential theory whereas chaper 4 and 5 contain the original results. In the next chapter, we review the fundamental properties of symmetries and the spontaneous symmetry breaking. We also introduce the general formulation of a low energy effective theory with spontaneous symmetry breaking. In chapter 3, the tunneling effect in quantum mechanics is briefly reviewed. The QCD instanton effects are introduced as tunneling effects between non-trivial vacua in QCD which are classified with integers n. We see that the instanton effects induce an quark effective interaction. In chapter 4, the effective theory which describes the interactions of pseudo Nambu-Goldstone bosons at low energies is introduced. The dynamical meaning of instanton transformation, which is related to instanton-induced mass correction, is discussed. The light meson mass formulae of next-to-leading order in chiral expansion are derived. We extract the value of couplings which are sensitive to the instanton effect using the formulae, and the constraint on the quark mass correction given by the instanton effect is discussed. In chapter 5, we discuss the instanton effect in the heavy meson effective theory. The effective Lagrangian which includes the next-to-leading order of chiral symmetry breaking terms is constructed. We show the invariance under the instanton transformation even in the heavy meson effective theory. The mass formulae of pseudoscalar B mesons and the formulae of their mass differences are given. The constraint on the instanton-induced effect are obtained in the B system. Chapter 6 is devoted to the conclusion.

Chapter 2

Symmetry

In the elementary particle physics, the concept of symmetry plays an important role in classifying the particles spectra and in relating the interactions between them. When the Lagrangian of a system is invariant under transformations, the symmetries of the transformations realize in the system.

In certain cases, though the Lagrangian of a system is invariant under the transformation of a symmetry group G, the ground state is not necessarily invariant under the transformation of symmetry G but invariant under the transformation of symmetry subgroup H. This phenomenon is called "spontaneously symmetry breaking".

The spontaneously symmetry breaking occurs in cases, for example, the acquiring of vacuum expectation values by one scalar field in the theory as in the breaking of local $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{U}(1)_{\mathrm{Y}}$ gauge invariance by Higgs field in the electroweak interactions. Even in the absence of scalar fields, quantum effects can lead to the dynamical breaking of a symmetry as in the case of chiral symmetry breaking by quark condensate in the strong interaction.

We give the formulation for realization of symmetries and its spontaneously breaking in this chapter. The treatment in this chapter will be used for construction of the chiral effective theory as we will see later.

2.1 Noether current and charge

Let us assume that the Lagrangian (density) is a functional of fields φ and its derivative as

$$\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi). \tag{2.1}$$

The symmetry is represented as the invariance of Lagrangian $\mathcal{L}$. The dynamical variables are the fields, and symmetries describe invariance under transformations of the fields. We consider a continuous infinitesimal transformation of the field as

$$\varphi(x) \longrightarrow \varphi'(x) = \varphi(x) + \theta^a (\delta\varphi)^a, \tag{2.2}$$

where θ^a is a transformation parameter and a is the index of the transformation. Under the transformation of eq.(2.2), the Lagrangian is transformed as

$$\mathcal{L}(\varphi, \partial_\mu \varphi) \longrightarrow \mathcal{L}(\varphi', \partial_\mu \varphi') = \mathcal{L}(\varphi, \partial_\mu \varphi) + \theta^a (\delta\mathcal{L})^a. \tag{2.3}$$

When the Lagrangian has the symmetry of the transformation, the deviation of Lagrangian vanishes so that

$$\begin{aligned} (\delta\mathcal{L})^a &= \frac{\partial \mathcal{L}}{\partial \varphi} (\delta\varphi)^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} (\delta(\partial_\mu \varphi))^a \\ &= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} (\delta\varphi)^a = 0, \end{aligned} \tag{2.4}$$

with Euler-Lagrange equation, and the Lagrangian is invariant under the transformation of eq.(2.2).

We define the Noether current as

$$J_\mu^a \equiv -\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} (\delta\varphi)^a \tag{2.5}$$

and the corresponding charge as

$$Q^a \equiv -\int d^3x J_0^a(x). \tag{2.6}$$

When the Lagrangian has the symmetry, the divergence of current vanishes,

$$\partial^\mu J_\mu^a = 0, \tag{2.7}$$

and we find that the charge is time-independent

$$\frac{d}{dt}Q^a = -\int d^3x \partial^0 J_0^a = -\int d^3x \partial^i J_i^a = 0 \tag{2.8}$$

under an assumption that the field and its derivative converge at the boundary. In the canonical quantization, the charge becomes the operator which generates the transformation of fields:

$$[i\hat{Q}^a, \varphi] = (\delta\varphi)^a, \tag{2.9}$$

where $\hat{Q}^a$ is the quantized operator.

2.2 Spontaneous symmetry breaking

In case that the Lagrangian of a system is invariant under the transformation of a symmetry group, there are two situations called by the Wigner phase or the Nambu-Goldstone phase. The situations are symbolically described by

$$\begin{aligned} Q|0\rangle &= 0 \quad \text{Wigner phase,} \\ Q|0\rangle &\neq 0 \quad \text{Nambu-Goldstone phase,} \end{aligned} \tag{2.10}$$

where Q is the generator of the symmetry and the vacuum state is defined by annihilation operators in the asymptotic fields of the theory.

In the Wigner phase the charge Q is well-defined from eq.(2.6). Since Q and the vacuum state are invariant under space and time translations, the quantity

$$\begin{aligned} \langle 0|Q|0\rangle &= -\int d^3x \langle 0|j_0(x)|0\rangle \\ &= -\langle 0|j_0(0)|0\rangle \int d^3x \end{aligned} \tag{2.11}$$

converges only in the case of $\langle 0|j_0(0)|0\rangle = 0$, that is to say, $Q|0\rangle = 0$.

The same argument does not apply in the Nambu-Goldstone phase. The charge Q is not well-defined since the volume integral in eq.(2.11) diverges. Therefore, the phenomenon of spontaneous symmetry breaking is defined in terms of the condition that there exists at least one operator Φ satisfying the commutation relation

$$[iQ, \Phi(y)] \equiv -i \int d^3x [j_0(x), \Phi(y)] = \delta\Phi(y) \tag{2.12}$$

with the finite vacuum expectation value

$$\langle 0|\delta\Phi|0\rangle \neq 0. \tag{2.13}$$

The Nambu-Goldstone theorem states that massless bosons, the Nambu-Goldstone bosons, appear and are coupled to the currents in the system with spontaneously broken symmetry. We show the consequences of the theorem in the following. Define a correlation function as

$$\int d^4x e^{iq\cdot x} \partial^\mu \langle 0|T j_\mu(x) \Phi(0)|0\rangle, \tag{2.14}$$

where T is a time ordered product. This correlation function is related to the vacuum expectation value of eq.(2.12) in the soft limit $q_\mu \to 0$,

$$\int d^4x e^{iq\cdot x} \partial^\mu \langle 0|T j_\mu(x) \Phi(0)|0\rangle = -\langle 0|[Q, \Phi]|0\rangle, \tag{2.15}$$

from the current conservation. When the theory is in the Nambu-Goldstone phase, the correlation function becomes finite. The correlation function can be also expressed as

$$\begin{aligned}\int d^4x e^{iq\cdot x} \partial^\mu \langle 0|T j_\mu(x) \Phi(0)|0\rangle &= -iq^\mu \int d^4x e^{iq\cdot x} \langle 0|T j_\mu(x) \Phi(0)|0\rangle \\ &= i \sum_n F_n G_n \frac{q^2}{q^2 - m_n^2 + i\epsilon},\end{aligned} \tag{2.16}$$

where we use the completeness of the theory

$$1 = \sum_n \int \frac{d^3p_n}{2p_n^0 (2\pi)^3} |n(p_n)\rangle\langle n(p_n)|. \tag{2.17}$$

Here F_n and G_n are defined as

$$\langle 0|j_\mu(x)|n(p_n)\rangle = ip_\mu F_n e^{-ip\cdot x}, \tag{2.18}$$

$$\langle 0|\Phi(x)|n(p_n)\rangle = G_n e^{-ip\cdot x}, \tag{2.19}$$

with $F_n \neq 0$ and $G_n \neq 0$ when n corresponds to the Nambu-Goldstone mode. Namely, since we identify eq.(2.18) with the current sandwiching between the vacuum and the one Nambu-Goldstone boson state $|\pi^a\rangle$,

$$\langle 0|j^a_\mu(x)|\pi^b(p)\rangle = i\delta^{ab}p_\mu f_\pi e^{-ip\cdot x}, \tag{2.20}$$

where f_π is the pion decay constant, the current can be expressed as

$$j^a_\mu(x) = -f_\pi \partial_\mu \pi^a(x) + \cdots , \tag{2.21}$$

where the dots stand for the continuous spectrum parts. The current conservation implies the masslessness of the Nambu-Goldstone boson π^a. In the soft limit eq.(2.16) requires the existence of massless Nambu-Goldstone bosons coupled to the current j_μ (the Goldstone theorem [29]). We can easily find that the number of the independent Nambu-Goldstone bosons is given by the number of independent broken generators.

2.3 Nonlinear realization

In a system realizing the symmetry G which is spontaneously broken down to the subgroup H, we show the procedure for constructing a low energy effective Lagrangian, the CCWZ Lagrangian, which has been introduced in [30] (and see ref.[31] for review). The effective Lagrangian is constructed in terms of the nonlinearly transforming Nambu-Goldstone bosons and the terms of the lowest order in derivatives on the Nambu-Goldstone bosons are uniquely determined without any parameter.

We consider the case that the symmetry group G is spontaneously broken down to the subgroup H. Here we assume that G and H are compact simple groups.

The set of the generators T^A of G is divided into the generators $S^\alpha \in \mathcal{H}$ of the unbroken subgroup H and the rest $X^a \in \mathcal{G} - \mathcal{H}$ as

$$\{T^A\} = \{S^\alpha \in \mathcal{H}, X^a \in \mathcal{G} - \mathcal{H}\}. \tag{2.22}$$

We employ the normalization and orthogonality of generators as

$$\mathrm{tr}(T^A T^B) = \frac{1}{2}\delta^{AB}, \quad \mathrm{tr}(S^\alpha X^a) = 0, \tag{2.23}$$

where the second equation implies

$$\mathrm{tr}(S^\alpha[S^\beta, X^a]) = \mathrm{tr}([S^\alpha, S^\beta]X^a) = 0, \tag{2.24}$$

so that the element $[S^\alpha, X^a]$ always lies in $\mathcal{G} - \mathcal{H}$,

$$[\mathcal{H}, \mathcal{G} - \mathcal{H}] \subset \mathcal{G} - \mathcal{H}. \tag{2.25}$$

The Nambu-Goldstone bosons $\pi(x)$, whose number is equal to the dimension of the (right) coset space G/H, $\dim G - \dim H$, are transformed under H, so that $\pi(x)$ can be identified with the coordinates in coset space G/H. The Nambu-Goldstone bosons are not linearly transformed under G. To construct a G-invariant nonlinear Lagrangian with such Nambu-Goldstone bosons, we see the non-trivial transformation property of $\pi(x)$. Let $\xi(\pi)$ be "representatives" of the coset space G/H, which is parameterized by the Nambu-Goldstone bosons $\pi(x)$ as

$$\xi(\pi) = e^{i\pi(x)/f}, \quad \pi(x) \equiv \sum_{a \in \mathcal{G}-\mathcal{H}} \pi^a(x) X^a, \tag{2.26}$$

where f is a scale parameter or the decay constant at the tree level with a mass dimension. An element $g\xi(\pi)$ yielded by the left multiplication of $g \in G$ is in G. There exists the representative $\xi(\pi')$ corresponding to the element $g\xi(\pi)$ (see Fig.(2.1)). We find that the element can be decomposed into the coset part and unbroken part as

$$g\xi(\pi) = \xi(\pi')h(\pi, g), \quad h(\pi, g) \in H. \tag{2.27}$$

Note that this element h depends on $\pi(x)$ as well as on g. Therefore, we define the transformation of the Nambu-Goldstone bosons $\pi(x)$ under the G-transformation as

$$\xi(\pi') = g\xi(\pi)h^{-1}(\pi, g), \quad g \in G. \tag{2.28}$$

As we expected, the transformation becomes linear as

$$\xi(\pi') = h\xi(\pi)h^{-1}(\pi, h), \Rightarrow \pi'(x) = h\pi(x)h^{-1}, \quad h \in H, \tag{2.29}$$

when the left multiplication is an element h belonging to subgroup H in eq.(2.28).

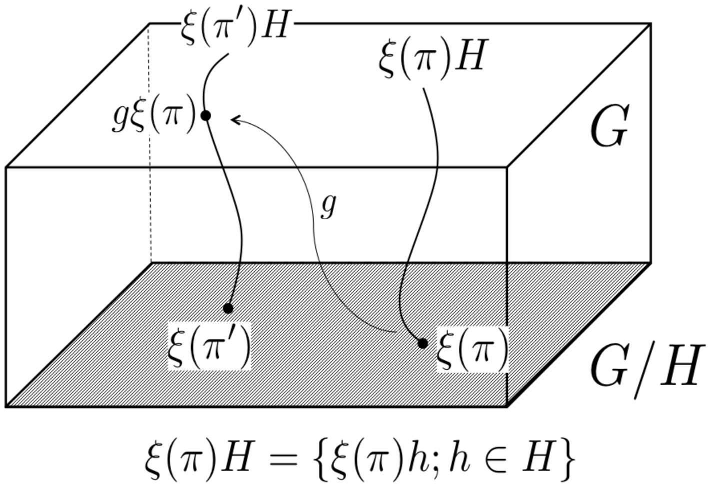

Figure 2.1: Image of the decomposition of the element $g\xi(\pi)$. A box implies a set of elements of G and the bottom of box (shaded area) is a set of elements of G/H. The two carves represent the equivalence classes $\xi(\pi)H$.

We now consider the case when G is a simple group. We introduce a 1-form as

$$\alpha(\pi) = \frac{1}{i}\xi^{-1}d\xi, \quad \xi \in G/H, \tag{2.30}$$

or more explicitly as

$$\alpha_\mu(\pi)dx^\mu = \frac{1}{i}\xi^{-1}(\pi)\frac{\partial\xi}{\partial x^\mu}dx^\mu \Rightarrow \alpha_\mu(\pi) = \frac{1}{i}\xi^{-1}\partial_\mu\xi, \tag{2.31}$$

which is well-known as the Maurer-Cartan 1-form. Since the 1-form $\alpha(\pi)$ belongs to the Lie algebra $\mathcal{G}$ and can be expanded with its generators $\{T^A\} = \{S^\alpha \in \mathcal{H}, X^a \in \mathcal{G} - \mathcal{H}\}$, we can define the parallel and perpendicular components of $\alpha_\mu(\pi)$ to $\mathcal{H}$ as

$$\begin{aligned} \alpha_{\mu\|}(\pi) &\equiv \alpha_\mu^\alpha(\pi)S^\alpha = 2\mathrm{tr}(S^\alpha\alpha_\mu(\pi)) \cdot S^\alpha \in \mathcal{H}, \\ \alpha_{\mu\perp}(\pi) &\equiv \alpha_\mu^a(\pi)X^a = 2\mathrm{tr}(X^a\alpha_\mu(\pi)) \cdot X^a \in \mathcal{G} - \mathcal{H}. \end{aligned} \tag{2.32}$$

From eq.(2.28), we find that the transformation law of $\alpha_\mu(\pi)$ is

$$\alpha_\mu(\pi) \to \alpha_\mu(\pi') = h(\pi,g)\alpha_\mu(\pi)h^{-1}(\pi,g) + \frac{1}{i}h(\pi,g)\partial_\mu h^{-1}(\pi,g). \tag{2.33}$$

The second term in the above equation comes from the transformation of the parallel component of $\alpha_\mu(\pi)$, since $h(\pi, g)\partial_\mu h^{-1}(\pi, g)$ is in $\mathcal{H}$. That is to say, each component is transformed under the G-transformation as

$$\begin{aligned}
\alpha_{\mu\parallel}(\pi) &\to \alpha_{\mu\parallel}(\pi') = h(\pi, g)\alpha_{\mu\parallel}(\pi)h^{-1}(\pi, g) + \frac{1}{i}h(\pi, g)\partial_\mu h^{-1}(\pi, g) \\
\alpha_{\mu\perp}(\pi) &\to \alpha_{\mu\perp}(\pi') = h(\pi, g)\alpha_{\mu\perp}(\pi)h^{-1}(\pi, g).
\end{aligned} \tag{2.34}$$

We see that only the perpendicular component $\alpha_{\mu\perp}$ transforms homogeneously, and the G-invariant Lagrangian can be constructed in terms of $\mathrm{tr}(\alpha_{\mu\perp}(\pi))^2$. The most general Lagrangian with the lowest order in derivatives is given by

$$\mathcal{L} = f^2 \mathrm{tr}(\alpha_{\mu\perp}(\pi))^2 \tag{2.35}$$

where the square of a factor f is multiplied in order to normalize the kinetic terms of the $\pi(x)$ fields.

In QCD, the Lagrangian has the approximate symmetry, the chiral symmetry, which is the global $\mathrm{U}(\mathrm{N_f})\times\mathrm{U}(\mathrm{N_f})$ symmetry with the number of the quark flavors $\mathrm{N_f}$. The chiral symmetry is spontaneously broken by the quark condensate. Therefore, we can construct the chiral effective theory using the procedure for constructing CCWZ Lagrangian.

Chapter 3

Non-trivial vacuum structure in QCD

It is widely believed that a non-trivial vacuum structure exists in QCD. The non-trivial vacuum structure is described by the quantum mechanical superposition of equivalent vacua classified by an integer called “a winding number”. In the quantum mechanics, tunneling effects are transitions from one vacuum to another vacuum. The effects can be described by the classical solutions to the equation of motion in the semi-classical approximation and the classical solutions are called “instanton solutions”. Tunneling effects in QCD could be described by the instanton solutions which are classical solutions (or often referred to as gauge field configurations) formulated in Euclidean space-time.

In QCD, the axial current, which is related to the chiral symmetry, is not conserved at the quantum level and this is known as the chiral anomaly [12, 13]. It could be interpreted that the tunneling effect, namely instanton effect, causes non-conserving of the chiral charges. An effective interaction, the so-called 't Hooft vertex, is induced by the instanton effect, which changes the axial charge by twice of the number of flavors in QCD.

In this chapter, we start by introducing the path integral in Euclidean space-time and discuss a tunneling effect in a system with a double-well potential. Then, we overview the non-trivial vacuum structure and the instanton effect in QCD, which induce the 't Hooft vertex.

3.1 Instanton solutions in quantum mechanics

Before we discuss QCD, let us review tunneling effects in quantum mechanics using the method of pass integral as a simple example. We consider the theory of a particle with mass m moving in a one-dimensional potential $V(x)$ in Minkowski space-time. The action and Lagrangian are

$$S = \int dt L, \quad L = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x). \tag{3.1}$$

The Hamiltonian is

$$H = p\frac{dx}{dt} - L = \frac{p^2}{2m} + V(x) = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + V(x) \tag{3.2}$$

with canonical momentum

$$p \equiv \frac{\partial L}{\partial \dot{x}} = m\frac{dx}{dt}. \tag{3.3}$$

The transition amplitude from $x = x_i$ at $t = -T/2$ to $x = x_f$ at $t = T/2$ is given by

$$\langle x_f | e^{iHT/\hbar} | x_i \rangle = \mathcal{N} \int \mathcal{D}x e^{iS/\hbar} \tag{3.4}$$

in the path integral representation. On the left-hand side, $|x_i\rangle$ and $|x_f\rangle$ are the position eigenstates. On the right-hand side, $\mathcal{N}$ is a normalization factor and $\mathcal{D}x$ donotes integration over all functions $x(t)$, satisfying the boundary conditions, $x(-T/2) = x_i$ and $x(T/2) = x_f$.

The action in Euclidean space-time is given by the analytic continuation in the time coordinate. Euclidean coordinates are denoted as

$$x_{\mathrm{E}}^{\mu} = (x^1, x^2, x^3, x^4) = (x^1, x^2, x^3, -ix^0), \tag{3.5}$$

and the metric is

$$g_{\mu\nu}^{\mathrm{E}} = \mathrm{diag}(-1, -1, -1, -1). \tag{3.6}$$

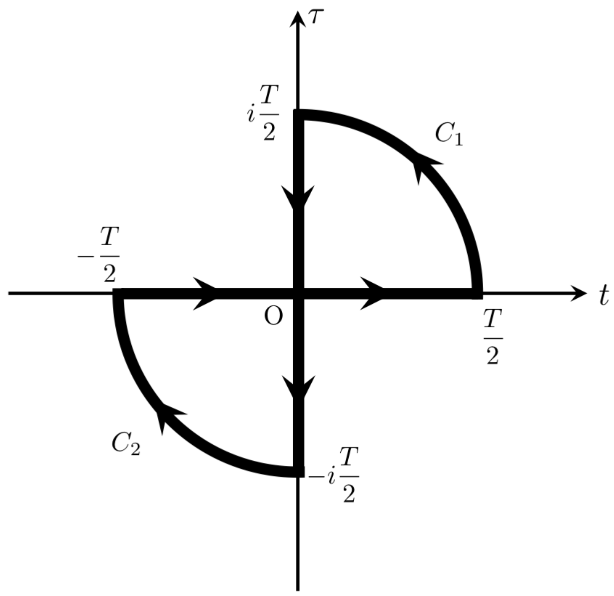

Figure 3.1: Integration along the path C.

We take a closed path C in the complex-plane shown as Fig.3.1, and decompose the integral into four parts of the path as

$$\oint_C dtL = \int_{C_1} dtL + \int_{C_2} dtL + \int_{-T/2}^{T/2} dtL + \int_{-iT/2}^{iT/2} dtL. \tag{3.7}$$

Assuming that there are no singularities inside the closed path and that the contribution from two integrals on the contours C_1 and C_2 vanish in the limit $T \to \infty$, we have

$$\begin{aligned} S = \int_{-T/2}^{T/2} dtL &= -\int_{-iT/2}^{iT/2} dtL \\ &= -\int_{-T/2}^{T/2} id\tau \left(-\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 - V(x) \right) \\ &= i \int_{T/2}^{T/2} d\tau L_{\mathrm{E}} \\ &\equiv iS_{\mathrm{E}}, \end{aligned} \tag{3.8}$$

where $\tau \equiv -ix^0$ and the Lagrangian in Euclidean space-time is

$$L_{\rm E} \equiv \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x). \tag{3.9}$$

We can naively see that the Euclidean action is defined as $-i$ times the Minkowskian action. Since we consider QCD in Euclidean space-time later, we use a subscript E.

The Hamiltonian is

$$H_{\rm E} = p_{\rm E}\frac{dx}{d\tau} - L_{\rm E} = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x) = -H\bigg|_{t=i\tau}, \tag{3.10}$$

where the canonical momentum in Euclidean space-time is

$$p_{\rm E} \equiv \frac{\partial L_{\rm E}}{\partial(dx/d\tau)} = m\frac{dx}{d\tau}. \tag{3.11}$$

The amplitude of transition from $x = x_i$ at $\tau = -T/2$ to $x = x_f$ at $\tau = T/2$ is given by

$$\begin{aligned} \langle x_f|e^{-iHT/\hbar}|x_i\rangle &= \langle x_f|e^{-i(-H)(-iT)/\hbar}|x_i\rangle \\ &= \langle x_f|e^{-H_{\rm E}T/\hbar}|x_i\rangle \\ &= \mathcal{N}\int \mathcal{D}x e^{-S_{\rm E}/\hbar} \end{aligned} \tag{3.12}$$

in the path integral representation. If we expand the left-hand side in a complete set of energy eigenstates,

$$H_{\rm E}|n\rangle = E_n|n\rangle, \tag{3.13}$$

then

$$\langle x_f|e^{-H_{\rm E}T/\hbar}|x_i\rangle = \sum_n e^{-E_nT/\hbar}\langle x_f|n\rangle\langle n|x_i\rangle. \tag{3.14}$$

The leading term in this expression for large T is saturated by the energy and the wave-function of the lowest-lying energy eigenstate.

On the right-hand side of eq.(3.12), the integration parameter can be written as

$$x(\tau) = x^{\rm cl}(\tau) + \sum_n c_n x_n(\tau), \tag{3.15}$$

where $x^{\rm cl}(\tau)$ is the classical solution to the equation of motion

$$m\frac{d^2x^{\rm cl}}{d\tau^2}-\frac{dV(x^{\rm cl})}{dx}=0, \tag{3.16}$$

and x_n are a complete set of real orthonormal function satisfying the boundary conditions,

$$\begin{aligned}\int_{-T/2}^{T/2}d\tau x_n(\tau)x_m(\tau)&=\delta_{mn},\\ x_n(\tau=\pm T/2)&=0.\end{aligned} \tag{3.17}$$

The measure is defined by

$$\mathcal{D}x\equiv\prod_n\frac{dc_n}{\sqrt{2\pi\hbar}}. \tag{3.18}$$

We can readily evaluate the path integral in eq.(3.12). The Lagrangian becomes

$$\begin{aligned}L_{\rm E}=&\frac{m}{2}\left(\frac{dx^{\rm cl}}{d\tau}\right)^2+V(x^{\rm cl})\\ &+\frac{1}{2}\sum_n\sum_m c_nc_mx_m\left(-m\frac{d^2}{d\tau^2}+\frac{d^2V(x^{\rm cl})}{dx^2}\right)x_n+\mathcal{O}(\hbar).\end{aligned} \tag{3.19}$$

Choosing x_n to be the eigenfunctions of the second derivative of $S_{\rm E}$ at $x^{\rm cl}$,

$$\left(-m\frac{d^2}{d\tau^2}+\frac{d^2V(x^{\rm cl})}{dx^2}\right)x_n=\lambda_nx_n(\tau), \tag{3.20}$$

we obtain

$$L_{\rm E}=\frac{m}{2}\left(\frac{dx^{\rm cl}}{d\tau}\right)^2+V(x^{\rm cl})+\frac{1}{2}\sum_n\sum_m c_nc_mx_m\lambda_nx_n+\mathcal{O}(\hbar). \tag{3.21}$$

We can carry out the integral of the action

$$S_{\rm E}=S_{\rm E}(x^{\rm cl})+\frac{1}{2}\sum_n c_n^2\lambda_n+\mathcal{O}(\hbar), \tag{3.22}$$

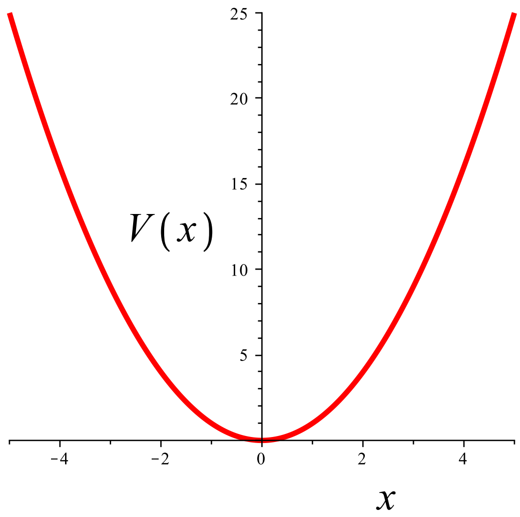

Figure 3.2: The shape of the potential as a simple example.

and the amplitude is

$$\begin{aligned}\langle x_f|e^{-H_{\rm E}T/\hbar}|x_i\rangle &= \mathcal{N}e^{-S_{\rm E}(x^{\rm cl})/\hbar}\left(\prod_n \lambda_n\right)^{-\frac{1}{2}}(1+\mathcal{O}(\hbar))\\ &= \mathcal{N}e^{-S_{\rm E}(x^{\rm cl})/\hbar}\left[\det\left(-m\frac{d^2}{d\tau^2}+V''(x^{\rm cl})\right)\right]^{-\frac{1}{2}}(1+\mathcal{O}(\hbar))\,,\end{aligned} \tag{3.23}$$

where the prime denotes differentiation with respect to x. We find that the transition amplitude is proportional to $\exp(-S_{\rm E}(x^{\rm cl})/\hbar)$. When $S_{\rm E}$ is much greater than $\hbar$, this expansion on $\hbar$ is a good approximation. The expansion is known as a semi-classical approximation or the WKB approximation.

As a simple example of applying a semi-classical approximation, consider the parabola potential shown in Fig.(3.2) with the boundary condition that both the initial and final states are at the origin, namely $x_i = x_f = 0$. We expect that the vacuum energy of the system is that of a harmonic oscillator. The only solution to the classical equation of motion is

$$x^{\rm cl}(\tau) = 0, \quad \dot{x}^{\rm cl} = 0, \quad S_{\rm E}(x^{\rm cl}) = 0. \tag{3.24}$$

The transition amplitude from the initial state to the final state after (Euclidean) time T is

$$\langle 0|e^{-H_{\mathrm{E}}T/\hbar}|0\rangle = \mathcal{N}\left[\det\left(-m\frac{d^2}{d\tau^2}+m\omega^2\right)\right]^{-\frac{1}{2}}, \tag{3.25}$$

where

$$\omega^2 \equiv \frac{V''(0)}{m}. \tag{3.26}$$

To calculate the determinant, we consider the equation

$$\left(-m\frac{d^2}{d\tau^2}+m\omega^2\right)f_n = \lambda_n f_n, \quad n = 1,2,3,\cdots, \tag{3.27}$$

where the eigenfunctions $f_n(\tau)$ satisfies the conditions

$$f_n\left(-\frac{T}{2}\right) = f_n\left(\frac{T}{2}\right) = 0. \tag{3.28}$$

The eigenfunctions can be taken both symmetric and anti-symmetric function as

$$\begin{aligned} f_k^{\mathrm{sym}} &= \cos\left((2k+1)\pi\frac{\tau}{T}\right), \quad k = 0,1,2,\cdots, \\ f_k^{\mathrm{anti-sym}} &= \sin\left(2k\pi\frac{\tau}{T}\right), \quad k = 1,2,3,\cdots, \end{aligned} \tag{3.29}$$

with each of the eigenvalues

$$\begin{aligned} m\left\{\frac{(2k+1)\pi}{T}\right\} + m\omega^2, &\quad k = 0,1,2,\cdots, \\ m\left\{\frac{2k\pi}{T}\right\} + m\omega^2, &\quad k = 1,2,3,\cdots. \end{aligned} \tag{3.30}$$

This results in

$$\lambda_n = m\left\{\left(\frac{n\pi}{T}\right)^2 + \omega^2\right\}, \quad n = 1,2,3,\cdots, \tag{3.31}$$

and

$$\det\left(-m\frac{d^2}{d\tau^2}+m\omega^2\right)=\prod_{n=1}^{\infty}m\left\{\left(\frac{n\pi}{T}\right)^2+\omega^2\right\}$$

$$\begin{aligned}\Rightarrow \ln\det\left(-m\frac{d^2}{d\tau^2}+m\omega^2\right)&=\ln\prod_{n=1}^{\infty}m\left\{\left(\frac{n\pi}{T}\right)^2+\omega^2\right\}\\&=\ln\prod_{n=1}^{\infty}m\left(\frac{n\pi}{T}\right)^2+\ln\prod_{n=1}^{\infty}\left\{1+\left(\frac{n\pi}{\omega T}\right)^{-2}\right\}\\&=\ln\prod_{n=1}^{\infty}m\left(\frac{n\pi}{T}\right)^2+\ln\frac{\sinh\omega T}{\omega T},\end{aligned}\tag{3.32}$$

where we used the formula

$$\prod_{n=1}^{\infty}\left\{1+\frac{1}{(n/z)^2}\right\}=\frac{\sinh\pi z}{\pi z}\tag{3.33}$$

in the last equality. Therefore, we obtain

$$\det\left(-m\frac{d^2}{d\tau^2}+m\omega^2\right)=\left(\prod_{n=1}^{\infty}\frac{m\pi^2}{T^2}n^2\right)\frac{\sinh\omega T}{\omega T}\tag{3.34}$$

and the transition amplitude

$$\begin{aligned}\langle 0|e^{-H_{\mathrm{E}}T/\hbar}|0\rangle&=\mathcal{N}\left(\prod_{n=1}^{\infty}\frac{m\pi^2}{T^2}n^2\right)^{-\frac{1}{2}}\sqrt{\omega T}(\sinh\omega T)^{-\frac{1}{2}}\left(1+\mathcal{O}(\hbar)\right)\\&\to\mathcal{N}\left(\prod_{n=1}^{\infty}\frac{m\pi^2}{T^2}n^2\right)^{-\frac{1}{2}}\sqrt{2\omega T}e^{-\omega T/2},\end{aligned}\tag{3.35}$$

for large T. We immediately find that the exponential behavior in this equation gives vacuum energy

$$E_0=\frac{1}{2}\hbar\omega,\tag{3.36}$$

which coincides with the energy of a harmonic oscillator.

On the other hand, eq.(3.35) can be represented as

$$\begin{aligned}\langle 0|e^{-H_{\mathrm{E}}T/\hbar}|0\rangle&=\langle 0|e^{-H_{\mathrm{E}}T/\hbar}\sum_n|n\rangle\langle n|0\rangle\\&\to e^{-E_0T/\hbar}|\langle 0|n=0\rangle|^2,\end{aligned}\tag{3.37}$$

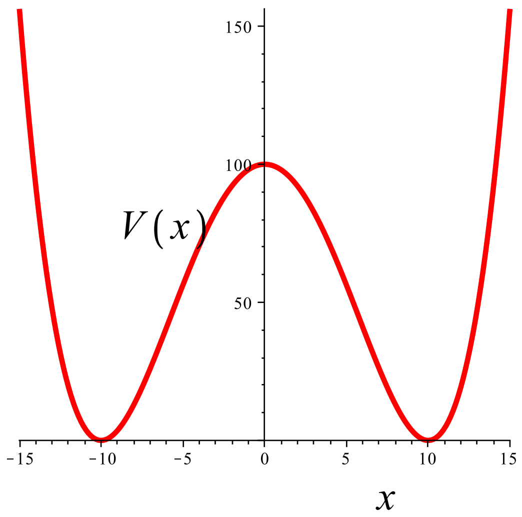

Figure 3.3: The shape of double-well potential.

for large T. Here the coefficient $|\langle 0|n=0\rangle|^2$ is the probability that the particle stays at the origin in the ground state and we find

$$|\langle 0|n=0\rangle|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}. \tag{3.38}$$

Therefore, we can determine the normalization factor

$$\mathcal{N} = \sqrt{\frac{m}{2\pi\hbar T}} \left(\prod_{n=1}^{\infty} \frac{m\pi^2}{T^2} n^2\right)^{\frac{1}{2}}. \tag{3.39}$$

To see a system with the tunneling effect, we consider the double-well potential (see Fig.3.3) given by

$$V(x) \equiv \frac{m\omega^2}{8a^2}(x^2-a^2)^2, \quad a>0, \tag{3.40}$$

where an ω is corresponding to the height of the barrier between minimum points, $x=\pm a$ and we consider in the case of a large ω. We can find that $x^{\mathrm{cl}} = \pm a$ can be classical solutions to the equation of motion (3.16) since the potential is a downward convex around $x=\pm a$ so that

$$-m\frac{d^2x^{\mathrm{cl}}}{d\tau^2} - \frac{m\omega^2}{2}x^{\mathrm{cl}} + \frac{m\omega^2}{2a^2}(x^{\mathrm{cl}})^3 = 0. \tag{3.41}$$

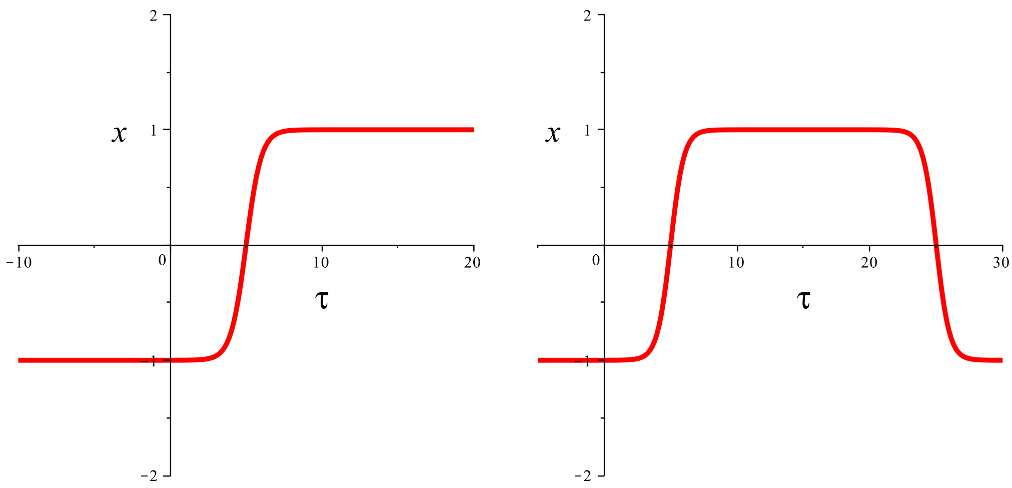

Figure 3.4: A shape of classical solution with positive sign in eq.(3.44) (left-side panel). A shape of the solution which is contained a single instanton and a single anti-instanton (right-side panel).

The transition amplitude from $x_i = a$ in the initial state to $x_f = a$ in the final state or from $x_i = -a$ in the initial state to $x_f = -a$ in the final state after time T is given by

$$z_0 = \langle a|e^{-H_{\mathrm{E}}T/\hbar}|a\rangle_0 = \langle -a|e^{-H_{\mathrm{E}}T/\hbar}|-a\rangle_0, \tag{3.42}$$

where the index 0 stands for a number of instantons and its approximate energy of the ground state is

$$E_0 = \frac{\hbar\omega}{2}, \tag{3.43}$$

in the limit $\omega \to \infty$. This coincides with the energy of harmonic oscillator. As we see later, the energy decreases due to the tunneling effect which is a transition effect between the two minimum points.

Actually

$$x^{\mathrm{cl}}(\tau) = \pm a \tanh\frac{\omega(\tau-\tau_0)}{2} \tag{3.44}$$

is also the solution to the equation of motion (3.16). Here the constant of integration τ_0 is a position of the solution. The solution with its signature

$+$ in eq.(3.44) is called an "instanton" solution and the other is called an "anti-instanton" solution because these solutions describe the instantaneous transition at $\tau = \tau_0$ (see the left-side panel in Fig.3.4). This solution represents the transition from $x = -a$ at $\tau = -\infty$ to $x = a$ at $\tau = \infty$. The instanton solution would be interpreted as the tunneling effect in quantum dynamics.

We define the contribution from instanton or anti-instanton as

$$K \equiv \frac{z_1}{z_0} = \frac{\langle a|e^{-H_{\rm E}T/\hbar}|-a\rangle_1}{\langle a|e^{-H_{\rm E}T/\hbar}|a\rangle_0} = \frac{\langle -a|e^{-H_{\rm E}T/\hbar}|a\rangle_1}{\langle a|e^{-H_{\rm E}T/\hbar}|a\rangle_0}. \tag{3.45}$$

In the following, we employ a "dilute gas approximation" where instantons and anti-instantons are sufficiently apart from each other and each instanton independently contributes to the amplitude. At first, we consider a process contained a single instanton and a single anti-instanton shown as the right-hand panel in Fig.(3.4). This transition is described by the solution

$$x^{\rm cl}(\tau) = \pm a \tanh\frac{\omega(\tau-\bar\tau_0)}{2}\tanh\frac{\omega(\tau-\tau_0)}{2} \tag{3.46}$$

for large ω. Dividing instanton and anti-instanton by $\tau = \tau_1$ with $\bar\tau_0 < \tau_1 < \tau_0$, the amplitude is

$$\begin{aligned} z_2 &= \langle a|e^{-H_{\rm E}T/\hbar}|a\rangle_2 \quad (= \langle -a|e^{-H_{\rm E}T/\hbar}|-a\rangle_2) \\ &\to \langle a|e^{-H_{\rm E}(T/2-\tau_1)/\hbar}|-a\rangle_1\langle -a|e^{-H_{\rm E}(T/2+\tau_1)/\hbar}|a\rangle_1 \end{aligned} \tag{3.47}$$

for large T. On the other hand, we obtain

$$z_0 \to \left\{ \begin{array}{c} \langle -a|e^{-H_{\rm E}(T/2-\tau_1)/\hbar}|-a\rangle_0\langle -a|e^{-H_{\rm E}(T/2+\tau_1)/\hbar}|-a\rangle_0 \\ \langle a|e^{-H_{\rm E}(T/2-\tau_1)/\hbar}|a\rangle_0\langle a|e^{-H_{\rm E}(T/2+\tau_1)/\hbar}|a\rangle_0 \end{array}\right. ,$$

$$z_1 \to \left\{ \begin{array}{c} \langle a|e^{-H_{\rm E}(T/2-\tau_1)/\hbar}|-a\rangle_1\langle -a|e^{-H_{\rm E}(T/2+\tau_1)/\hbar}|-a\rangle_0 \\ \langle -a|e^{-H_{\rm E}(T/2-\tau_1)/\hbar}|-a\rangle_0\langle -a|e^{-H_{\rm E}(T/2+\tau_1)/\hbar}|a\rangle_1 \end{array}\right. , \tag{3.48}$$

for large T. Therefore eq.(3.47) can be represented as

$$\begin{aligned} z_2 &\to \langle a|e^{H_{\rm E}(T/2-\tau_1)/\hbar}|-a\rangle_1\langle -a|e^{-H_{\rm E}(T/2+\tau_1)/\hbar}|a\rangle_1 \\ &\quad\times\frac{\langle -a|e^{-H_{\rm E}(T/2-\tau_1)/\hbar}|-a\rangle_0\langle -a|e^{-H_{\rm E}(T/2+\tau_1)/\hbar}|-a\rangle_0}{z_0} \\ &= \frac{z_1^2}{z_0} = K^2 z_0 \end{aligned} \tag{3.49}$$

Note that the integral over the location of instantons is given by

$$\int_{-T/2}^{T/2} d\tau_0 \int_{-T/2}^{\tau} d\bar{\tau}_0 = \frac{1}{2}\int_{-T/2}^{T/2} d\tau_0 d\bar{\tau}_0, \tag{3.50}$$

and the amplitude for transition from a to a with a single instanton and a single anti-instanton can be written in terms of the amplitude without instantons as

$$z_2 \to \frac{1}{2!}K^2 z_0. \tag{3.51}$$

It is easy that the amplitude is extended to

$$z_{2n} \to \frac{1}{2n!}K^{2n} z_0, \tag{3.52}$$

which is the amplitude with $2n$ transitions. The amplitude with all the general classical solutions is given by summation for even number of instantons and anti-instantons as

$$\langle a|e^{-H_{\mathrm{E}}T/\hbar}|a\rangle = \sum_{n=0}^{\infty} z_{2n} = (\cosh K) z_0. \tag{3.53}$$

Now we consider the calculation of the Van Vleck determinant

$$\left[\det\left(-m\frac{d^2}{d\tau^2} + V''(x^{\mathrm{cl}})\right)\right]^{-\frac{1}{2}}. \tag{3.54}$$

Since the instanton solution is time-dependent, time derivative of eq.(3.16)

$$\left[-m\frac{d^2}{d\tau^2} + V''(x^{\mathrm{cl}})\right]\dot{x}^{\mathrm{cl}} = 0 \tag{3.55}$$

supply zero eigenvalues (or zero-eigenmode) and the Van Vleck determinant divergences (zero-mode problem). For large T, what x^{cl} is translated on τ becomes also a solution and the value of the action does not change for the time-translated solutions. The integral over c_0 which is the zero eigenvalue for $\lambda_0 = 0$ is not Gaussian integral in the formula of determinant. We should remove this zero eigenvalue in the definition of the determinant by the treatment shown as follows.

The eigenfunction in eq.(3.20) for zero eigenvalue is given by

$$x_0 = \sqrt{\frac{m}{S_{\mathrm{E}}(x^{\mathrm{cl}})}}\frac{dx^{\mathrm{cl}}}{d\tau}, \tag{3.56}$$

and we can find

$$\int_{-T/2}^{T/2} d\tau (x_0)^2 = 1, \tag{3.57}$$

where we use the formula

$$\frac{dx^{\mathrm{cl}}}{d\tau} = \sqrt{\pm\frac{2V}{m}} \tag{3.58}$$

for large T. The relation between integration measure dc_0 and $d\tau$ is given by

$$\frac{dx^{\mathrm{cl}}}{d\tau}d\tau = dc_0 x_0 = dc_0\sqrt{\frac{m}{S_{\mathrm{E}}(x^{\mathrm{cl}})}}\frac{dx^{\mathrm{cl}}}{d\tau}, \tag{3.59}$$

so that

$$\frac{dc_0}{\sqrt{2\pi\hbar}} = \sqrt{\frac{S_{\mathrm{E}}(x^{\mathrm{cl}})}{2\pi\hbar m}}d\tau. \tag{3.60}$$

We can remove zero eigenvalue from the Gaussian integral as

$$\begin{aligned}
&\int\prod_n \frac{dc_n}{\sqrt{2\pi\hbar}}\exp\left\{-\frac{1}{2\hbar}\sum_{m,n}c_m c_n\left(-m\frac{d^2}{d\tau^2}+V''(x^{\mathrm{cl}})\right)x_n\right\}\\
&=\int_{-T/2}^{T/2}d\tau\sqrt{\frac{S_{\mathrm{E}}(x^{\mathrm{cl}})}{2\pi\hbar m}}\int\prod_{n\neq 0}\frac{dc_n}{\sqrt{2\pi\hbar}}\exp\left\{-\frac{1}{2\hbar}\sum_{m,n}c_m c_n\left(-m\frac{d^2}{d\tau^2}+V''(x^{\mathrm{cl}})\right)x_n\right\}\\
&\equiv T\sqrt{\frac{S_{\mathrm{E}}(x^{\mathrm{cl}})}{2\pi\hbar m}}\left[\det{}'\left(-m\frac{d^2}{d\tau^2}+V''(x^{\mathrm{cl}})\right)\right]^{-\frac{1}{2}},
\end{aligned} \tag{3.61}$$

where $\det'$ means that zero eigenvalue is not included in the integral measure. Eq.(3.45) can be represented as

$$K = e^{-S_{\mathrm{E}}(x^{\mathrm{cl}})/\hbar}T\sqrt{\frac{S_{\mathrm{E}}(x^{\mathrm{cl}})}{2\pi\hbar m}}\left[\frac{\det'\left(-m\frac{d^2}{d\tau^2}+V''(x^{\mathrm{cl}})\right)}{\det\left(-m\frac{d^2}{d\tau^2}+m\omega^2\right)}\right]^{\frac{1}{2}}. \tag{3.62}$$

Substituting this result for $\langle a|e^{-H_{\mathrm{E}}T/\hbar}|a\rangle$ gives the energy of the ground state as

$$E_0 = \frac{\hbar\omega}{2} - e^{-S_{\mathrm{E}}(x^{\mathrm{cl}})/\hbar}\sqrt{\frac{\hbar S_{\mathrm{E}}(x^{\mathrm{cl}})}{2\pi m}}\left[\frac{\det'\left(-m\frac{d^2}{d\tau^2}+V''(x^{\mathrm{cl}})\right)}{\det\left(-m\frac{d^2}{d\tau^2}+m\omega^2\right)}\right]^{\frac{1}{2}}. \quad (3.63)$$

This consequence states that the energy of system decreases due to the instanton effect, namely the tunneling effect.

We give the representation of the polar coordinates in Euclidean space-time. The transformation from the cartesian coordinates to the polar coordinates in Euclidean space-time is

$$\begin{aligned} x^1 &= r\cos\alpha_1, \\ x^2 &= r\sin\alpha_1\cos\alpha_2, \\ x^3 &= r\sin\alpha_1\sin\alpha_2\cos\alpha_3, \\ x^4 &= r\sin\alpha_1\sin\alpha_2\sin\alpha_3, \end{aligned} \quad (3.64)$$

with $0 \le r < \infty$, $0 \le \alpha_1 < \pi$, $0 \le \alpha_2 < \pi$, $0 \le \alpha_3 < 2\pi$ and $r^2 = (x^4)^2 + (\boldsymbol{x})^2 = |x_{\mathrm{E}}|^2$. From the Jacobian

$$\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(r,\alpha_1,\alpha_2,\alpha_3)} = r^3\sin^2\alpha_1\alpha_2, \quad (3.65)$$

the volume element is obtained as

$$d^4x_{\mathrm{E}} = r^3\sin^2\alpha_1\sin\alpha_2 dr d\alpha_1 d\alpha_2 d\alpha_3 \equiv r^3 dr d\Omega, \quad (3.66)$$

where $d\Omega$ is the solid angle in the polar coordinates with

$$\int d\Omega = 2\pi^2. \quad (3.67)$$

The surface element $d\sigma_\mu$ is

$$d\sigma_\mu = |x_{\mathrm{E}}|^3 d\Omega \frac{x_\mu^{\mathrm{E}}}{|x_{\mathrm{E}}|} \quad (3.68)$$

on the hypersphere with the radius r.

We give a remark on the validity of analytic continuation. For example, in the free theory the 2-point function is

$$\langle 0|\mathrm{T}\phi(x)\phi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik(x-y)}, \tag{3.69}$$

and no singularities in the area enclosed by the integral path ensure that the analytic continuation is well-defined. However, the validity of analytic continuation becomes non-trivial because there are no assurance in the case that the mass has some correction in the full theory. In the following we will discuss under the assumption that the analytic continuation can be done.

3.2 QCD instantons

The Lagrangian of SU(N) Yang-Mills theory with the fermions is represented by

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D}-m)\psi, \tag{3.70}$$

where the gauge field and the field strength are

$$A_\mu = A_\mu^a\frac{T^a}{2}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{3.71}$$

and T^as are the generators associated with the SU(N) gauge group and satisfies the normalization as

$$\mathrm{tr}\left[\frac{T^a}{2}\frac{T^b}{2}\right] = \frac{1}{2}\delta^{ab}, \tag{3.72}$$

and ψ denote fermions. The gauge field $A_\mu(x)$ is transformed under the SU(N) gauge transformation as

$$A_\mu(x) \to \Omega(x)A_\mu(x)\Omega(x)^\dagger + \frac{i}{g}\Omega(x)\partial_\mu\Omega(x)^\dagger, \tag{3.73}$$

where a unitary matrix

$$\Omega(x) = e^{i\theta^a(x)T^a/2} \tag{3.74}$$

is an element of SU(N). The Yang-Mills action is

$$S_{\rm YM} = \int d^4x_{\rm E} \left[-\frac{1}{2} {\rm Tr} F_{\mu\nu} F_{\mu\nu} \right] \tag{3.75}$$

and the field equation is

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} - ig[A_\mu, F_{\mu\nu}] = 0 \tag{3.76}$$

in Euclidean space-time. Here, $A_\mu = 0$, which leads $F_{\mu\nu} = 0$, is the trivial solution. The trivial solution is transformed into "pure gauge" as

$$A_\mu(x) = \frac{i}{g} \Omega(x) \partial_\mu \Omega(x)^\dagger \tag{3.77}$$

and it also leads to $F_{\mu\nu} = 0$ and satisfies the field equation (3.76).

It is convenient to study the theory in the gauge $A_4 = 0$. In this gauge we can choose the gauge transformation as a time independent, so that the spatial pure gauge fields are

$$A_i(\vec{x}) = \frac{i}{g} \Omega(\vec{x}) \partial_i \Omega^\dagger(\vec{x}). \tag{3.78}$$

The gauge fields are required to vanish at spatial infinity. This can be accomplished by a restriction as

$$\Omega(\vec{x}) \to 1, \quad \text{for } |\vec{x}| \to \infty. \tag{3.79}$$

QCD is the SU(3) Yang-Mills theory. For simplicity, we concentrate on the SU(2) subgroup in QCD. Any elements of SU(2) can be represented as

$$\begin{aligned} \Omega(x) &= a(x) + i\vec{b}(x) \cdot \vec{\sigma}, \\ \Omega(x)\Omega(x)^\dagger &= a(x)^2 + (\vec{b}(x))^2 = 1. \end{aligned} \tag{3.80}$$

This means that the elements are identified with a unit hypersphere, S^3, in a four-dimensional space spanned by linear combination of a and $\vec{b}$, namely, topologically SU(2) is S^3. We can define an integer n called "winding number", which measures the number of windings of S^3 on the gauge space. The winding number can be represented in terms of an integral over the gauge fields as

$$n = \frac{ig^3}{24\pi^2} \int d^3x \epsilon^{ijk} {\rm Tr} \left[A_i^{(n)}(\vec{x}) A_j^{(n)}(\vec{x}) A_k^{(n)}(\vec{x}) \right], \tag{3.81}$$

where the gauge fields are

$$A_i^{(n)}(\vec{x}) = \frac{i}{g}\Omega_n(\vec{x})\partial_i\Omega_n^\dagger(\vec{x}). \tag{3.82}$$

We can construct the representative for Ω_n. The representatives of gauge transformations giving $n = 1$ is

$$\Omega_1(\vec{x}) = \frac{\vec{x}^2 - \lambda^2}{\vec{x}^2 + \lambda^2} + \frac{2i\lambda\vec{x}\cdot\vec{\sigma}}{\vec{x}^2 + \lambda^2} \tag{3.83}$$

with $\lambda > 0$. Indeed, the insertion of the gauge fields $A_i^{(1)}(\vec{x})$ with $\Omega_1(\vec{x})$ into the formula of winding number gives unity. Then, we consider the gauge transformation of the pure gauge fields as

$$\begin{aligned} A_i^{(1)} &\to \Omega_1 A_i^{(1)}\Omega^\dagger + \frac{i}{g}\Omega_1\partial_i\Omega_1^\dagger \\ &= \Omega_1\left(\frac{i}{g}\Omega_1\partial_i\Omega_1^\dagger\right)\Omega_1^\dagger + \frac{i}{g}\Omega_1\Omega_1\Omega_1^\dagger\partial_i\Omega_1^\dagger \\ &= \frac{i}{g}\Omega_1^2\partial_i(\Omega_1^2)^\dagger \\ &\equiv A_i^{(2)} \end{aligned} \tag{3.84}$$

with $\Omega_2 \equiv (\Omega_1)^2$. Again the insertion of the gauge fields $A_i^{(2)}(\vec{x})$ into the formula of winding number gives

$$\begin{aligned} &\frac{ig^3}{24\pi^2}\int d^3x\epsilon^{ijk}\mathrm{Tr}\left[\frac{i}{g}\Omega_1^2\partial_i(\Omega_1^2)^\dagger\frac{i}{g}\Omega_1^2\partial_j(\Omega_1^2)^\dagger\frac{i}{g}\Omega_1^2\partial_k(\Omega_1^2)^\dagger\right] \\ &= \frac{ig^3}{24\pi^2}\int d^3x\epsilon^{ijk}\mathrm{Tr}\left[\frac{i}{g}\Omega_1\partial_i\Omega_1^\dagger\frac{i}{g}\Omega_1\partial_j\Omega_1^\dagger\frac{i}{g}\Omega_1\partial_k\Omega_1^\dagger\right]\times 2 \\ &= 2. \end{aligned} \tag{3.85}$$

We can immediately find that substituting the gauge fields $A_i^{(n)}(\vec{x})$ with

$$\Omega_n(\vec{x}) \equiv (\Omega_1(\vec{x}))^n \tag{3.86}$$

for eq.(3.81) gives the winding number n.

A "vacuum state" associated with each of the gauge configurations is classified by the winding number. To introduce vacuum states, so-called "n-vacua", we consider the path integral quantization. The gauge fields can be decomposed as

$$A_\mu^{(n)}(x) = A_\mu^{\mathrm{cl}(n)}(x) + \hat{A}_\mu^{(n)}(x), \tag{3.87}$$

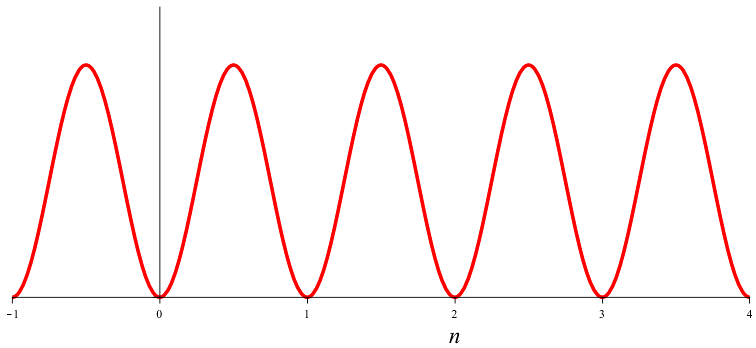

Figure 3.5: The n-vacua are described as a periodic potential by analogy with the Bloch potential.

where $A_\mu^{\mathrm{cl}(n)}(x)$ is the classical solution to the field equation and $\hat{A}_\mu^{(n)}(x)$ is the fluctuation around the classical solution. In the procedure of path integral quantization, fluctuations are treated as the measures in the path integral (see eq.(3.18)). The vacuum states on each of the gauge fields are defined as

$$\hat{A}_\mu^{(n)}(x)|n\rangle = 0. \tag{3.88}$$

Actually, the "true vacuum state" of QCD cannot be any one of n-vacua because state $|n\rangle$ transforms under the gauge transformation as

$$\Omega_1|n\rangle = |n+1\rangle. \tag{3.89}$$

The tunneling effect from the state $|n\rangle$ to $|n+1\rangle$ is described by the instanton solution. As we will see, the Pontryagin index is given by the difference in the winding numbers between $\tau = -\infty$ and $\tau = \infty$, and the transition amplitude is proportional to $\exp(-8\pi^2/g^2)$. We see that the instanton effect is a non-perturbative effect. Due to the tunneling effect, the situation is similar to the one in which a particle is in a periodic potential as shown in Fig.3.5. There is an *analogy* between the n-vacua and the Bloch potential. We must define the "true vacuum state" as the superposition of the n-vacua, called the Θ-vacuum,

$$|\Theta\rangle \equiv \sum_{n\in\mathbb{Z}} e^{-in\Theta}|n\rangle, \tag{3.90}$$

which is clearly gauge invariant up to a phase, namely

$$\Omega_1|\Theta\rangle = \sum_n e^{-in\Theta}|n+1\rangle = e^{i\Theta}|\Theta\rangle. \tag{3.91}$$

Interestingly, each value of Θ represents a different theory. To see that, let us consider the transition amplitude by a gauge invariant operator $\mathcal{O}$. Since the amplitude

$$\begin{aligned}\langle m|\mathcal{O}|n\rangle &= \langle m|\Omega_1^\dagger\Omega_1\mathcal{O}\Omega_1^\dagger\Omega_1|n\rangle \\ &= \langle m+1|\mathcal{O}|n+1\rangle = F(\nu)\end{aligned} \tag{3.92}$$

only depends upon the difference $\nu = m - n$ in the winding number, we find

$$\begin{aligned}\langle\Theta'|\mathcal{O}|\Theta\rangle &= \sum_m\sum_n e^{im\Theta'}e^{-in\Theta}\langle m|\mathcal{O}|n\rangle \\ &= \sum_\nu\sum_n e^{i(n+\nu)\Theta'}e^{-in\Theta}F(\nu) \\ &= \sum_\nu\sum_n e^{in(\Theta'-\Theta)}e^{i\nu\Theta}F(\nu) \\ &= 2\pi\delta(\Theta'-\Theta)e^{i\nu\Theta}F(\nu).\end{aligned} \tag{3.93}$$

This implies that the state $|\Theta\rangle$ cannot be changed to another state $|\Theta'\rangle$ by gauge invariant operators.

A path integral of the vacuum to vacuum transition amplitude with sources is given by integrating over the all field configurations weighted by the action (3.75) with source terms. The gauge field configurations in the path integral cause the transition of changing the winding number. The net charge in the winding number between $\tau = \infty$ and $\tau = -\infty$ is given by ν which is related to an integral over all space-time of $F_{\mu\nu}\tilde{F}_{\mu\nu}$. We see it as follows.

First, we introduce the dual of $F_{\mu\nu}$ as

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{3.94}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is an antisymmetric tensor with $\epsilon_{1234} = \epsilon^{1234} = -1$. From the formula

$$\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu\kappa\lambda} = 2(\delta_{\rho\kappa}\delta_{\sigma\lambda} - \delta_{\rho\lambda}\delta_{\kappa\sigma}), \tag{3.95}$$

we can find

$$F_{\mu\nu}F_{\mu\nu} = \tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{3.96}$$

Substituting the classical solution eq.(3.82) for $F_{\mu\nu}$ gives

$$F_{\mu\nu} = \tilde{F}_{\mu\nu}. \tag{3.97}$$

Therefore we have

$$\int \mathrm{Tr} F_{\mu\nu}F_{\mu\nu} d^4x_{\mathrm{E}} = \int \mathrm{Tr} F_{\mu\nu}\tilde{F}_{\mu\nu} d^4x_{\mathrm{E}}. \tag{3.98}$$

This can be represented by a total divergence as

$$\frac{1}{4}\mathrm{Tr} F_{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K_\mu, \tag{3.99}$$

where , for SU(2),

$$K_\mu = \epsilon_{\mu\nu\kappa\lambda}\mathrm{Tr}\left[\frac{1}{2}A_\nu\partial_\kappa A_\lambda - \frac{ig}{3}A_\nu A_\kappa A_\lambda\right]. \tag{3.100}$$

Only K_4 is nonvanishing for the classical solution in the $A_4 = 0$ gauge and we obtain

$$K_4 = \frac{ig}{6}\epsilon^{ijk}\mathrm{Tr} A_i A_j A_k. \tag{3.101}$$

Therefore,

$$\begin{aligned}
\frac{g^2}{16\pi^2}\int d^4x_{\mathrm{E}} \mathrm{Tr} F_{\mu\nu}\tilde{F}_{\mu\nu} &= \frac{g^2}{4\pi^2}\int d^4x_{\mathrm{E}}\partial_\mu K_\mu \\
&= \frac{g^2}{4\pi^2}\int d^3x K^0\Big|_{\tau=-\infty}^{\tau=\infty} \\
&= n_+ - n_- = \nu,
\end{aligned} \tag{3.102}$$

which is called the Pontryagin index.

Returning to the path integral, we have

$$\begin{aligned}
\langle\Theta_+|\Theta_-\rangle^J &= \sum_m\sum_n e^{im\Theta}e^{-in\Theta}\langle m_+|n_-\rangle^J \\
&= \sum_\nu e^{i\nu\Theta}\left\{\sum_n \langle n_+ + \nu|n_-\rangle^J\right\},
\end{aligned} \tag{3.103}$$

where the transition amplitudes $\langle n_+ + \nu | n_- \rangle$ are given by a path integral with ν fixed by eq.(3.102). Hence,

$$\begin{aligned} \langle \Theta_+ | \Theta_- \rangle^J &= \sum_\nu e^{i\nu\Theta} \int d\mu_{\rm fields} e^{iS^J} \delta(\nu - \frac{g^2}{16\pi^2} \int d^4x_{\rm E} {\rm Tr} F_{\mu\nu} \tilde{F}_{\mu\nu}) \\ &= \int d\mu'_{\rm fields} e^{iS^J_{\rm eff}}, \end{aligned} \tag{3.104}$$

where

$$\begin{aligned} S^J_{\rm eff} &\equiv S^J_{\rm YM} + \Theta \frac{g^2}{16\pi^2} \int d^4x_{\rm E} {\rm Tr} F_{\mu\nu} \tilde{F}_{\mu\nu} \\ d\mu_{\rm fields} &\equiv \sum_\nu d\mu_{\rm fields} \delta(\nu - \frac{g^2}{16\pi^2} \int d^4x_{\rm E} {\rm Tr} F_{\mu\nu} \tilde{F}_{\mu\nu}), \end{aligned} \tag{3.105}$$

with Yang-Mills action $S^J_{\rm YM}$ with sources.

We see that the non-trivial vacuum structure would lead the additional term

$$\Theta \frac{g^2}{16\pi^2} {\rm Tr} F_{\mu\nu} \tilde{F}_{\mu\nu} \tag{3.106}$$

in the effective Lagrangian, which violates P, T and CP invariance in QCD and is called the "Θ-term".

3.3 The 't Hooft vertex

The additional CP violating term (3.106) is related with the chiral anomaly. To define the U(1) axial current,

$$J_5^\mu = -\sum_{i=1}^{{\rm N}_f} \bar{\psi}_i \gamma^\mu \gamma_5 \psi_i, \tag{3.107}$$

we consider the Lagrangian eq.(3.70) in the fermion massless limit. In this limit, the Lagrangian is invariant under the transformations as

$$\psi_L \to e^{i\varphi_L} U_L \psi_L = e^{i\varphi_L} \exp\left(i\theta^a_L \frac{\lambda^a}{2} \right) \psi_L \tag{3.108}$$

$$\psi_R \to e^{i\varphi_R} U_R \psi_R = e^{i\varphi_R} \exp\left(i\theta^a_R \frac{\lambda^a}{2} \right) \psi_R \tag{3.109}$$

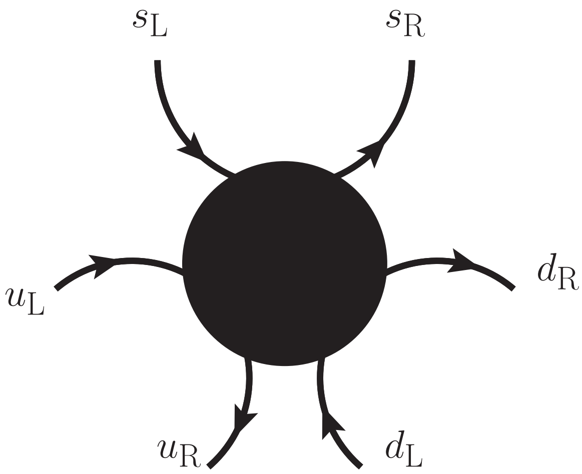

Figure 3.6: The diagram of the effective interaction induced by the instanton effect in the case of three flavors.

where $e^{i\varphi_L}$ and $e^{i\varphi_R}$ are the elements of $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_R$ and the elements U_l and U_R are in $\mathrm{SU}(N)_L$ and $\mathrm{SU}(N)_R$, respectively, and λ^a denotes the generators associated with the $\mathrm{SU}(N)_{L,R}$. Noether currents corresponding to $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_R$ transformations are

$$L^\mu = -\sum_i \bar{\psi}^i_L \gamma^\mu \psi^i_L, \tag{3.110}$$

$$R^\mu = -\sum_i \bar{\psi}^i_R \gamma^\mu \psi^i_R \tag{3.111}$$

The U(1) axial current defined as

$$J^\mu_5 = -L^\mu + R^\mu. \tag{3.112}$$

The U(1) axial current is classically conserved in QCD, in the quark massless limit. However, there is an anomaly

$$\partial^\mu J^\mu_5 = -2\mathrm{N}_f \frac{g^2}{16\pi^2} \mathrm{Tr} F^{\mu\nu} \tilde{F}^{\mu\nu}, \tag{3.113}$$

at the quantum level.

We can redefine a conserved current as

$$\tilde{J}^\mu_5 = J^\mu_5 + 2\mathrm{N}_f \frac{g^2}{4\pi^2} K^\mu, \tag{3.114}$$

whose associated charge

$$\tilde{Q}_5 = -\int d^3x \tilde{J}^0_5 \tag{3.115}$$

is time independent. Although $\tilde{Q}_5$ is a generator for the chiral $\mathrm{U}(1)_\mathrm{A}$ transformations, it is not invariant under the gauge transformations. In fact, since under the gauge transformations the charge is transformed as

$$\Omega_1 \left(\frac{g^2}{4\pi^2} \int d^3x K^0 \right) \Omega_1^\dagger = \frac{g^2}{4\pi^2} \int d^3x K^0 - 1, \tag{3.116}$$

it follows that

$$\Delta Q_5 = 2\mathrm{N}_f. \tag{3.117}$$

This consequence could be understood that an effective interaction which changes the chiral charge by $2\mathrm{N}_f$ is generated in transition of changing the winding number by the amount of $\nu = 1$. It is called the 't Hooft vertex and is diagrammatically represented for $\mathrm{N}_f = 3$ in Fig.3.6.

When fermion zero-modes, ψ^0 which satisfy $\not{D}\psi^0 = 0$, are exist, the action for fermion zero-mode vanishes. Therefore the integration corresponding to the fermion zero-modes vanishes in the path integral representation, because the fermion integrals are the Grassmannian integrals. However, this consequence contradicts the fact that the instanton effects give the finite transition amplitudes between vacua with different winding numbers. In order for the integral to be non-vanishing, fermion fields with a number of fermion zero-modes must be inserted in front of exponential in the path integral. This is equivalent to introducing an effective interaction of fermions with a number of fermion zero-modes. This interaction is known as the 't Hooft vertex. A number of fermion zero-modes coincides with the change of chiral charge, ΔQ_5, in transition between n-vacua, and when $\nu = 1$, $\Delta Q_5 = 2\mathrm{N}_f$ with N_f flavor in the theory.

In [26, 27], it has been pointed that this effective interaction could provide the way to avoid the strong CP problem. When $m_u = 0$, the strong CP phase becomes unphysical. Although the situation with $m_u = 0$ is apparently ruled out by, for example, the current algebra estimation of $m_u/m_d \neq 0$, in fact, $m_u = 0$ in the sense of current algebra does not necessarily require $m_u = 0$

in the current mass in QCD Lagrangian. The interaction leads to the light quark mass corrections at the loop level. Therefore, if $m_u = 0$ in the sense of bare mass, the effective up quark mass induced by the correction could be large enough to satisfy the current algebra.

Chapter 4

Light meson sector

QCD has the typical energy scale denoted as Λ_{QCD}. This energy scale roughly separates the regions of large and small coupling constant. When the coupling is large, non-perturbative effects are dominant. When the mass of quark q is much smaller than Λ_{QCD}, q is called a light quark. On the other hand, when the mass of quark Q is much larger than the scale, Q is called a heavy quark.

In the light quark massless limit, $m_q \to 0$, the QCD Lagrangian has the chiral symmetry. The chiral symmetry is defined as the invariance under independent transformations of left- and right-handed fermions.

In this chapter, we review the construction of the light meson effective theory with manifest chiral symmetry. We also introduce the instanton transformation and constrain the size of the instanton-induced effect using the light meson mass spectra and the meson decay constants.

4.1 Construction of light meson effective Lagrangian

The QCD Lagrangian is given by

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi, \tag{4.1}$$

where the covariant derivative and field strength are

$$
\begin{aligned}
D_\mu &= \partial_\mu - igA_\mu \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].
\end{aligned} \tag{4.2}
$$

The gauge fields and fermion fields are transformed under the $\mathrm{SU}(3)_\mathrm{C}$ gauge transformation as

$$
\begin{aligned}
A_\mu(x) &\to U(x)A_\mu(x)U(x)^\dagger + \frac{i}{g}U(x)\partial_\mu U(x)^\dagger, \\
\psi(x) &\to U(x)\psi(x),
\end{aligned} \tag{4.3}
$$

where the unitary matrix is

$$
U(x) = e^{ig\theta^a(x)T^a} \in \mathrm{SU}(3)_\mathrm{C} \tag{4.4}
$$

in terms of the continuous parameter $\theta^a(x)$ and the generator T^a of $\mathrm{SU}(3)_\mathrm{C}$ algebra.

In the system with light quarks, u, d and s, we consider the chiral limit where the typical QCD scale Λ_QCD is much greater than light quark masses m_q, namely $m_q/\Lambda_\mathrm{QCD} \to 0$. The Lagrangian becomes

$$
\begin{aligned}
\mathcal{L}_\mathrm{QCD}\Big|_{m=0} &= -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi \\
&= -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \overline{\psi_\mathrm{L}}i\not{D}\psi_\mathrm{L} + \overline{\psi_\mathrm{R}}i\not{D}\psi_\mathrm{R},
\end{aligned} \tag{4.5}
$$

where a left-handed fermion, ψ_L, and a right-handed, ψ_R, are defined with γ_5-matrix as

$$
\psi_\mathrm{L} \equiv \frac{1-\gamma_5}{2}\psi, \quad \psi_\mathrm{R} \equiv \frac{1+\gamma_5}{2}\psi. \tag{4.6}
$$

The invariance arises under the independent transformations with respect to the left- and right-handed fermions. It is so-called "chiral transformation"

$$
\begin{aligned}
\psi_\mathrm{L} &\to e^{i\varphi_\mathrm{L}}U_\mathrm{L}\psi_\mathrm{L} = e^{i\varphi_\mathrm{L}}\exp(i\theta^a_\mathrm{L}\frac{\lambda^a}{2})\psi_\mathrm{L}, \\
\psi_\mathrm{R} &\to e^{i\varphi_\mathrm{R}}U_\mathrm{R}\psi_\mathrm{R} = e^{i\varphi_\mathrm{R}}\exp(i\theta^a_\mathrm{R}\frac{\lambda^a}{2})\psi_\mathrm{R},
\end{aligned} \tag{4.7}
$$

where $e^{i\varphi_\mathrm{L}}$ and $e^{i\varphi_\mathrm{R}}$ are the elements of $\mathrm{U}(1)_\mathrm{L}$ and $\mathrm{U}(1)_\mathrm{R}$, and the elements U_L and U_R are in $\mathrm{SU}(3)_\mathrm{L}$ and $\mathrm{SU}(3)_\mathrm{R}$, respectively and λ^a denotes the Gell-Mann matrix. In this limit, the system has the $\mathrm{U}(3)_\mathrm{L}\times\mathrm{U}(3)_\mathrm{R}$ chiral symmetry. This symmetry is not exact however, because anomaly, which is a phenomenon at the quantum level, breaks the $\mathrm{U}(1)_A$ symmetry defined as $\mathrm{U}(1)_\mathrm{L}-\mathrm{U}(1)_\mathrm{R}$.

The effect of quark condensate causes spontaneous breaking of chiral symmetry down to $\mathrm{SU}(3)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{V}$. We construct the effective Lagrangian following the procedure in chap.2 as follows. There are 8 Nambu-Goldstone bosons and they are identified as light mesons π, K and η in the real world.

We consider the case where the symmetry group $G = \mathrm{SU}(3)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{R}$ is spontaneously broken down to the subgroup $H = \mathrm{SU}(3)_V$. An element $g \in G$ is given by $(\mathrm{U_L}, \mathrm{U_R})$, where $\mathrm{U_L} \in \mathrm{SU}(3)_\mathrm{L}$ and $\mathrm{U_R} \in \mathrm{SU}(3)_\mathrm{R}$. In the language of chap.2, the perpendicular component of 1-form is given by

$$\begin{aligned}\alpha_{\mu\perp}(\pi) &= \frac{1}{2i}\{\xi^{-1}(\pi)\partial_\mu\xi(\pi) - \xi(\pi)\partial_\mu\xi^{-1}(\pi)\} \\ &= \frac{1}{2i}\xi(U^{-1}\partial_\mu U)\xi^{-1},\end{aligned} \tag{4.8}$$

where ξ is transformed under the G-transformation as

$$\xi \to \mathrm{U_L}\xi h(\pi, \mathrm{U_L}, \mathrm{U_R})^\dagger = h(\pi, \mathrm{U_L}, \mathrm{U_R})\xi\mathrm{U_R^\dagger}, \tag{4.9}$$

and we define a unitary matrix

$$U \equiv \xi^2 = e^{2i\pi(x)/f}, \tag{4.10}$$

so that the transformation property under the G-transformation is given by

$$U \to \mathrm{U_L}U\mathrm{U_R^\dagger}. \tag{4.11}$$

The field π includes the Nambu-Goldstone bosons as

$$\pi = \frac{1}{\sqrt{2}}\begin{pmatrix} \dfrac{\pi^0}{\sqrt{2}} + \dfrac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\dfrac{\pi^0}{\sqrt{2}} + \dfrac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\dfrac{2}{\sqrt{6}}\eta \end{pmatrix}. \tag{4.12}$$

The effective field containing η' is

$$\Sigma = U\exp\left(\frac{2i}{\sqrt{6}f'}\eta'\right). \tag{4.13}$$

The first order G-invariant effective Lagrangian, which includes only two derivatives, that is to say an $\mathcal{O}(p^2)$ term, is obtained by

$$\mathcal{L}_{\rm chi}^{\rm LO} = \frac{f^2}{4}\mathrm{tr}(\partial_\mu \Sigma \partial^\mu \Sigma^\dagger), \tag{4.14}$$

and this term gives kinetic the term of the Nambu-Goldstone bosons.

Chiral symmetry is not an exact symmetry such as the color gauge symmetry $\mathrm{SU}(3)_{\rm C}$ but an approximate symmetry in the light quark massless limit. To make the effective theory more realistic we perturbatively include the effect of finite light quark masses. We introduce the chiral breaking term

$$\chi = 2B_0\mathcal{M}, \quad \mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \tag{4.15}$$

where B_0 is a constant of mass dimension one, and is related to the quark condensate. The QCD Lagrangian has an invariance under the $\mathrm{U}(3)_{\rm L}\times\mathrm{U}(3)_{\rm R}$ chiral transformation, if the mass matrix $\mathcal{M}$ transforms appropriately under the chiral transformation

$$\chi \to e^{i\theta_{\rm A}}\mathrm{U}_{\rm L}\chi\mathrm{U}_{\rm R}^\dagger. \tag{4.16}$$

This χ is considered as a quantity of $\mathcal{O}(p^2)$ in the effective Lagrangian.

Since $\mathcal{M}$ is just an expansion parameter in the chiral perturbation theory, we can use

$$\mathcal{M}^{\rm eff} = \mathcal{M} + \frac{2B_0\omega}{(4\pi f)^2}(\det\mathcal{M}^\dagger)(\mathcal{M}^\dagger)^{-1} \tag{4.17}$$

instead of the original $\mathcal{M}$, where $\mathcal{M}^{\rm eff}$ has the same transformation property as that of $\mathcal{M}$ and ω is a parameter..

The $\mathcal{O}(p^4)$ effective Lagrangian with quark masses is

$$\begin{aligned}\mathcal{L}_{\rm chi}^{\rm NLO} &= \frac{f^2}{4}\Big\langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \Big\rangle + \frac{f^2}{4}\Big\langle \chi\Sigma^\dagger + \chi^\dagger\Sigma\Big\rangle \\ &+L_1\Big\langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \Big\rangle^2 + L_2\Big\langle \partial_\mu \Sigma^\dagger \partial_\nu \Sigma \Big\rangle\Big\langle \partial^\mu \Sigma^\dagger \partial^\nu \Sigma \Big\rangle \\ &+L_3\Big\langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma \Big\rangle + L_4\Big\langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \Big\rangle\Big\langle \chi\Sigma^\dagger + \chi^\dagger\Sigma\Big\rangle \\ &+L_5\Big\langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma (\chi\Sigma^\dagger + \chi^\dagger\Sigma)\Big\rangle + L_6\Big\langle \chi\Sigma^\dagger + \chi^\dagger\Sigma\Big\rangle^2 \\ &+L_7\Big\langle \chi\Sigma^\dagger - \chi^\dagger\Sigma\Big\rangle^2 + L_8\Big\langle \chi\Sigma^\dagger\chi\Sigma^\dagger + \chi^\dagger\Sigma\chi^\dagger\Sigma\Big\rangle, \end{aligned} \tag{4.18}$$

where $\langle A\rangle$ denotes the trace of the matrix A over light flavor indices and the low-energy coupling constants, L_i, are dimensionless parameters [32].

4.2 Instanton transformation

The chiral effective Lagrangian has the following symmetries which have well-defined dynamical meaning in QCD. The Lagrangian of eq.(4.18) is invariant under

$$\mathcal{M} \to \gamma\mathcal{M}\ ,\ \ B_0 \to \gamma^{-1}B_0, \tag{4.19}$$

where γ is a multiplicative renormalization factor. This symmetry is the consequence of the fact that the physics is independent of the scale of multiplicative renormalization of quark masses $\mathcal{M}$.

The Lagrangian of eq.(4.18) is also invariant under the transformation, which is known as the instanton transformation

$$\chi \to \chi + \frac{\omega}{(4\pi f)^2}(\det\chi^\dagger)(\chi^\dagger)^{-1}, \tag{4.20}$$

and

$$\begin{aligned} L_6 &\to L_6 - \frac{\omega}{(16\pi)^2}, \\ L_7 &\to L_7 - \frac{\omega}{(16\pi)^2}, \\ L_8 &\to L_8 + 2\frac{\omega}{(16\pi)^2}. \end{aligned} \tag{4.21}$$

Note that the parameter ω is invariant under the multiplicative renormalization of quark masses. It is easy to check the invariance using the following matrix identity

$$\det M = M^3 - M^2 \mathrm{tr} M - \frac{1}{2} M[\mathrm{tr}(M^2) - (\mathrm{tr} M)^2], \tag{4.22}$$

where M is any 3×3 complex matrix. This identity can be proved as follow. A characteristic polynomial for M is given by

$$\begin{aligned} f_M(x) &\equiv \det(M - xI) \\ &= -x^3 + \mathrm{tr} M x^2 - \frac{1}{2}[(\mathrm{tr} M)^2 - \mathrm{tr} M^2]x + \det M. \end{aligned} \tag{4.23}$$

Cayley-Hamilton theorem states that substituting M for x in this polynomial results in a zero matrix, namely

$$f_M(M) = 0. \tag{4.24}$$

We obtain the matrix identity from eq.(4.24).

The symmetry under the transformations (4.20) and (4.21) is related to the instanton-induced quark mass corrections. The instanton-induced mass is proportional to the product of different quark flavor masses (see Fig.4.1). The instanton effect gives a six-quark interaction, known as the 't Hooft vertex, which induces corrections to the light quark masses. The physics should be independent of whether or not the instanton correction is included in χ or L_i [26, 28, 33]. Notice that the couplings L_6, L_7 and L_8 are transformed under the instanton transformation. We do not pay attention to L_6 since we can not extract it by meson masses (we see below that L_6 enters in the same form in each of the mass formula). We expect that particularly L_7, which gives the contribution of the type in Fig.4.1 in meson mass formulae, is produced dominantly by the instanton dynamics, though the other couplings, L_6 and L_8, should be also sensitive to the instanton effect. We estimate the value of the coupling L_7 in the following.

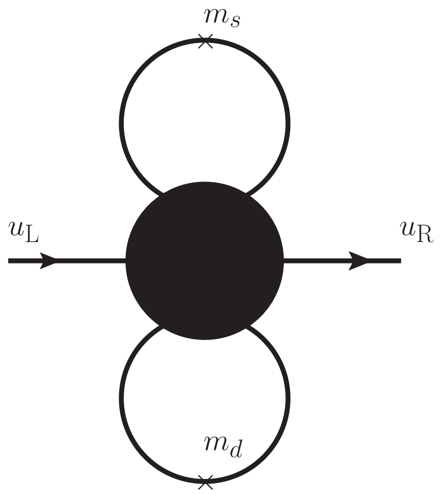

Figure 4.1: The instanton mass correction to up-quark mass by the 't Hooft vertex.

4.3 Constraints in light meson system

From the effective Lagrangian (4.18), we have the meson mass formulae of $\mathcal{O}(p^4)$:

$$m_{\pi^0}^2 = B_0(m_u+m_d)\left[1+\left(\hat{L}_6-\frac{\hat{L}_4}{2}\right)(m_u+m_d+m_s)-\frac{\hat{L}_5}{4}(m_u+m_d) + \hat{L}_7\frac{(m_u-m_d)^2}{m_u+m_d}+\hat{L}_8\frac{m_u^2+m_d^2}{m_u+m_d}\right], \tag{4.25}$$

$$m_{\pi^\pm}^2 = B_0(m_u+m_d)\left[1+\left(\hat{L}_6-\frac{\hat{L}_4}{2}\right)(m_u+m_d+m_s) + \left(\frac{\hat{L}_8}{2}-\frac{\hat{L}_5}{4}\right)(m_u+m_d)\right], \tag{4.26}$$

$$m_{K^0}^2 = B_0(m_d+m_s)\left[1+\left(\hat{L}_6-\frac{\hat{L}_4}{2}\right)(m_u+m_d+m_s) + \left(\frac{\hat{L}_8}{2}-\frac{\hat{L}_5}{4}\right)(m_d+m_s)\right], \tag{4.27}$$

$$m_{K^\pm}^2 = B_0(m_u+m_s)\Bigg[1+\left(\hat{L}_6-\frac{\hat{L}_4}{2}\right)(m_u+m_d+m_s) \\ +\left(\frac{\hat{L}_8}{2}-\frac{\hat{L}_5}{4}\right)(m_u+m_s)\Bigg], \quad (4.28)$$

$$m_\eta^2 = B_0\frac{m_u+m_d+4m_s}{3} \\ \times\Bigg[1+\left(\hat{L}_6-\frac{\hat{L}_4}{2}\right)(m_u+m_d+m_s)-\frac{\hat{L}_5}{12}(m_u+m_d+4m_s) \\ +\hat{L}_7\frac{(m_u+m_d-2m_s)^2}{m_u+m_d+4m_s}+\hat{L}_8\frac{m_u^2+m_d^2+4m_s^2}{m_u+m_d+4m_s}\Bigg], \quad (4.29)$$

$$m_{\eta'}^2 = \frac{2}{3}\frac{f^2}{f'^2}B_0(m_u+m_d+m_s) \\ \times\left[1+\left(\hat{L}_7+\hat{L}_6-\frac{\hat{L}_5}{6}-\frac{\hat{L}_4}{2}\right)(m_u+m_d+m_s)+\hat{L}_8\frac{m_u^2+m_d^2+m_s^2}{m_u+m_d+m_s}\right], \quad (4.30)$$

where $\hat{L}_i \equiv 32B_0L_i/f^2$ and we neglect the mass mixing between neutral mesons, π^0, η and η'. In fact, there are the pion loop effects in the mass formulae. However, we attempt to investigate with the heavy meson effective theory later and need to obtain the method which is simpler than and is consistent with the previous research [34].

The quantum electrodynamics (QED) correction to the mass squared of the meson P is proportional to the square of its charge Q_P as [35]

$$\hat{m}_P^2 = m_P^2 + e^2Q_P^2C, \quad (4.31)$$

where C is a constant and $\hat{m}$ means the observable mass, which is measured by experiments. If the QED correction is turned off, π^+ and π^0 become degenerate in the leading order, $\mathcal{O}(p^2)$. Therefore the quantity e^2C can be determined to good approximation from the observed value as $e^2C \equiv \hat{m}_{\pi^\pm}^2 - \hat{m}_{\pi^0}^2$.

To obtain the values of light quark masses at the energy scale where the effective theory is applicable, we fit the light quark masses with the mass formulae of mesons *in the leading order* including the QED corrections and with the value of B_0 given by the lattice calculations under the non-perturbative RI-MOM renormalization scheme in [36][1]:

$$\begin{aligned} m_u &= 2.78 \pm 0.19 \text{ MeV}, \\ m_d &= 4.97 \pm 0.34 \text{ MeV}, \\ m_s &= 100.4 \pm 6.8 \text{ MeV}. \end{aligned} \tag{4.32}$$

These results are consistent with the quark mass ratio m_u/m_d given by the lattice calculation in [37]. The parameters are fitted order by order in a spirit of chiral expansion theory. Once the light quark masses are determined at the leading order, the freedom of the instanton transformation is fixed, since the effective theory up to the leading order does not have an invariance under the instanton transformation. The instanton corrections are included in the next-to-leading order terms.

Now we are going to determine the values of L_5, L_8, and L_7 in order by using the observable masses and the decay constants. The quantity L_5/f^2 can be fixed by using the formulae of decay constants in the next-to-leading order, $\mathcal{O}(p^4)$, as

$$\frac{L_5}{f^2} = \frac{f_{K^\pm} - f_{\pi^\pm}}{f_{\pi^\pm}} \frac{1}{4(\hat{m}_{K^\pm}^2 - \hat{m}_{\pi^\pm}^2)}, \tag{4.33}$$

where

$$\begin{aligned} f_{\pi^\pm} &= f\left[1 + \frac{8B_0L_4}{f^2}(m_u + m_d + m_s) + \frac{4B_0L_5}{f^2}(m_u + m_d)\right], \\ f_{K^\pm} &= f\left[1 + \frac{8B_0L_4}{f^2}(m_u + m_d + m_s) + \frac{4B_0L_5}{f^2}(m_u + m_s)\right]. \end{aligned} \tag{4.34}$$

The experimental values

$$\begin{aligned} f_{\pi^\pm} &= 92.4 \pm 0.2 \text{ MeV}, \\ f_{K^\pm} &= 113.0 \pm 1.0 \text{ MeV}, \end{aligned} \tag{4.35}$$

[1]The quark masses are usually given in the $\overline{\text{MS}}$ renormalization scheme. In our present work we do not need to take the $\overline{\text{MS}}$ scheme, since we do not use the values in eq.(4.32) to compare with other determinations of the quark masses.

are obtained from the decay processes $\pi^+ \to \mu^+\nu_\mu,\ \mu^+\nu_\mu\gamma$ and $K^+ \to \mu^+\nu_\mu,\ \mu^+\nu_\mu\gamma$, respectively [38], and we obtain

$$\frac{L_5}{f^2} = (2.5 \pm 0.1) \times 10^{-7}\ \mathrm{MeV}^{-2}. \tag{4.36}$$

We can determine L_8/f^2 from the $\mathcal{O}(p^4)$ relation

$$\begin{aligned}
&\frac{m_{K^0}^2 - m_{K^\pm}^2}{m_{\pi^\pm}^2} \\
&= \frac{\hat{m}_{K^0}^2 - \hat{m}_{K^\pm}^2 + \hat{m}_{\pi^\pm}^2 - \hat{m}_{\pi^0}^2}{\hat{m}_{\pi^0}^2} \\
&= \frac{m_d - m_u}{m_d + m_u}\left[1 + \left(\frac{16L_8}{f^2} - \frac{8L_5}{f^2}\right)(\hat{m}_{K^\pm}^2 + \hat{m}_{K^0}^2 - \hat{m}_{\pi^\pm}^2)\right]
\end{aligned} \tag{4.37}$$

as

$$\frac{L_8}{f^2} = (1.24 \pm 0.06) \times 10^{-7}\ \mathrm{MeV}^{-2}, \tag{4.38}$$

with the quantity

$$R \equiv \frac{m_d + m_u}{m_d - m_u} = 3.53 \pm 0.01 \tag{4.39}$$

determined at the leading order. At the next-to-leading order, we can derive the relation

$$\begin{aligned}
&\frac{2m_{K^\pm}^2 + 2m_{K^0}^2 - m_{\pi^\pm}^2 - 3m_\eta^2}{m_\eta^2 - m_{\pi^0}^2} \\
&= \frac{2\hat{m}_{K^\pm}^2 + 2\hat{m}_{K^0}^2 - 2\hat{m}_{\pi^\pm}^2 + \hat{m}_{\pi^0}^2 - 3\hat{m}_\eta^2}{\hat{m}_\eta^2 - \hat{m}_{\pi^0}^2} \\
&= \frac{8}{f^2}(L_5 + 6L_7 + 3L_8)(3\hat{m}_{\pi^0}^2 - \hat{m}_{K^\pm}^2 - \hat{m}_{\pi^0}^2 - \hat{m}_{K^0}^2) \\
&\quad - \frac{48}{f^2}L_5\frac{(\hat{m}_{\pi^0}^2 - \hat{m}_{K^0}^2)(\hat{m}_{\pi^\pm}^2 - \hat{m}_{K^\pm}^2)}{3\hat{m}_{\pi^0}^2 - \hat{m}_{K^\pm}^2 - \hat{m}_{\pi^0}^2 - \hat{m}_{K^0}^2}
\end{aligned} \tag{4.40}$$

neglecting mass mixing of neutral mesons and we finally obtain

$$\frac{L_7}{f^2} = (-5.1 \pm 0.3) \times 10^{-8}\ \mathrm{MeV}^{-2}. \tag{4.41}$$

On the assumption, which is conservative in extracting the possible magnitude of the instanton effect, that all the value of L_7 is produced by the instanton dynamics, we have

$$\omega_{\max} = 0.4 \pm 0.1, \tag{4.42}$$

using eq.(4.21) with the result of the lattice calculation of $f = 54.1 \pm 4.0$ MeV [36]. The error of the omega parameter is mainly from the error of quark condensate given by the lattice calculations. This conservative maximum value of ω can not reproduce the situation $m_u = 0$ with a sufficiently large value of m_u^{eff} in $\mathcal{M}^{\text{eff}}$ in eq.(4.17), and, thus, be a solution to the strong CP problem. In fact, the instanton effect can generate

$$m_u^{\text{eff}} = 1.93 \pm 0.18 \text{ MeV}, \tag{4.43}$$

when $m_u = 0$ in eq.(4.17). This value differs from the value in eq.(4.32) by about 4.7σ, and $m_u = 0$ is not the solution to the strong CP problem.

Since L_8 does not seem to directly represent the effect of the 't Hooft vertex, the coupling L_8 would not be produced dominantly by the instanton effect. However, it is transformed under the instanton transformation, and therefore an evaluation of the maximum omega parameter is possible with L_8. We obtain

$$\omega_{\max} = 0.5 \pm 0.1. \tag{4.44}$$

Note that in this case we do not need to neglect neutral meson mixing which is necessary for L_7. Since L_8 can have this the contribution without instanton dynamics, eq.(4.44) gives a "weak constraint" on the maximum omega parameter. The instanton-induced mass correction gives

$$m_u^{\text{eff}} = 2.33 \pm 0.20 \text{ MeV}, \tag{4.45}$$

when $m_u = 0$. This value differs from the value in eq.(4.32) by about 2.3σ, and $m_u = 0$ is not favored as a solution to the strong CP problem. We have confirmed this known result in a simple way without the loop effects of the pseudo-Nambu-Goldstone bosons, which could be large contributions in chiral perturbation theory. These results [39] are consistent with the result

Table 4.1: The results determined using two the different renormalization schemes. The errors of results, which are determined with the $\overline{\mathrm{MS}}$ scheme at the energy scale 2 GeV, is larger than ones with RI-MOM scheme due to error propagation from the conversion factor.

the coupling	RI-MOM scheme		$\overline{\mathrm{MS}}$ scheme	
	ω_{max}	m_u^{eff}[MeV]	ω_{max}	m_u^{eff}[MeV]
using L_7	0.4 ± 0.1	1.93 ± 0.18	1.7 ± 0.3	1.80 ± 0.23
using L_8	0.5 ± 0.1	2.33 ± 0.20	2.0 ± 0.3	2.20 ± 0.27
m_u [MeV]	2.78 ± 0.19		2.60 ± 0.29	

$m_u/m_d \neq 0$ by Leutwyler in [34], which supports the validity of our simple estimate. These our results indicate that the instanton effect is small.

Finally, we make a comment on the conversion from the result using the RI-MOM renormalization scheme to the result using the $\overline{\mathrm{MS}}$ renormalization scheme. As we mentioned, the value of B_0 is determined with the RI-MOM renormalization scheme. The quantity with the $\overline{\mathrm{MS}}$ renormalization scheme is obtained by multiplying the conversion factor. Since the conversion factor has an error, it propagates to the results with the $\overline{\mathrm{MS}}$ scheme so that our claim becomes milder as shown in Table 4.1.

Chapter 5

Heavy meson sector

In a system with a single heavy quark and light constituents at low energy, the properties of the system cannot be calculated analytically in a perturbative way from the first principles because of the asymptotic freedom in QCD. In the heavy quark mass limit $M_Q \to \infty$, the exact symmetries arise. In fact, these symmetries are approximate symmetries because the heavy quark masses are finite and are much larger than the typical energy scale of strong interaction in the system.

On the basis of a concept of the symmetries related to invariance under the heavy quark spin and flavor transformations and of the chiral symmetry related to the light constituents, we can construct an effective theory which describes the heavy meson system [40, 41, 42]. In fact, the symmetries is broken due to the finiteness of heavy quark mass.

On the other hand, the instanton-induced effect gives a correction of next-to-leading order in the expansion in the light quark mass matrix, $\mathcal{M}$. In order to discuss the size of the instanton-induced effect as shown in the light meson sector, we construct the effective Lagrangian up to $\mathcal{O}(p^4)$ in the chiral expansion in the heavy meson effective theory and we use only heavy meson mass spectrum.

For simplicity, the subscript QCD in $\Lambda_{\rm QCD}$ is sometimes omitted in the following formulae.

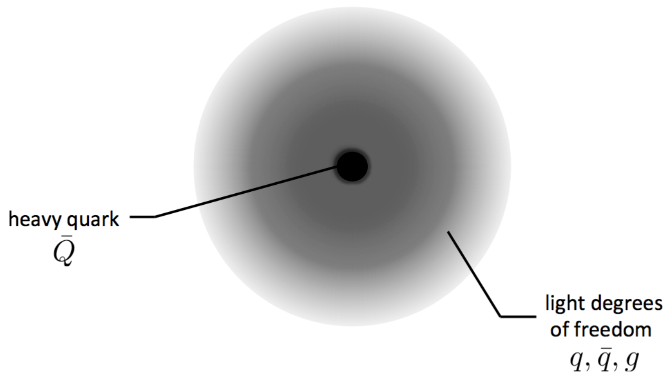

Figure 5.1: A hadronic system containing a single heavy quark and light degrees of freedom. Since the quark is sufficiently heavy, the heavy quark behaves as a static source of color localized at the origin and the light quark and gluonic degrees of freedom are distributed around the source.

5.1 Realization of heavy quark symmetry

We consider systems of hadrons contained a single heavy quark Q and light constituents at low energy. A physical picture of the system can be expressed as the heavy quark surrounded by strongly interacting "clouds" of light quarks, anti-quarks and gluons. These clouds are referred to as "light degrees of freedom" (see Fig.5.1).

The typical momentum transfer between the heavy and the light degrees of freedom are of order $\Lambda_{\rm QCD}$. Namely,

$$\Delta p = \Delta(M_Q v) \sim \Lambda_{\rm QCD}, \tag{5.1}$$

where Δp is the transfer momentum and v is the four-velocity of the heavy quark normalized by the on-shell condition

$$v_\mu v^\mu = 1. \tag{5.2}$$

Since the heavy quark mass M_Q is much greater than the scale $\Lambda_{\rm QCD}$ of the strong interaction, the change of the four-velocity is small so that the heavy quark behaves as a static color triplet and light degrees of freedom follow the equations of QCD with the boundary condition in the system. The strong interaction cannot distinguish quark flavors. When the heavy quark flavor is replaced with one another, the solutions for light degrees of freedom are the same. Therefore, the light degrees of freedom are symmetric under an isospin-like rotation of heavy quark flavor.

The current can be decomposed with the Gordon identity as

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2M_Q}\bar{u}(p')\left[(p'+p)^\mu + i\sigma^{\mu\nu}(p'-p)_\mu\right]u(p), \tag{5.3}$$

where u denotes the heavy quark spinor and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. In case of $M_Q \gg \Lambda_{\rm QCD}$, this current becomes the spin-independent interaction since the first term in the right-handed side of eq.(5.3) is of order 1, $p' + p \sim M_Q$, and the second term is of order of $\Lambda_{\rm QCD}/M_Q$, $p' - p \sim \Lambda_{\rm QCD}$, and vanishes in the limit. That is to say, the systems which contain heavy quark with difference of spin are degenerate.

The heavy quark is affected by surrounding light degrees of freedom so that the heavy quark becomes off-shell and its momentum can be expressed

$$p_Q^\mu = M_Q v^\mu + k^\mu, \tag{5.4}$$

where the residual momentum k^μ is of the order of $\Lambda_{\rm QCD}$. The quark propagator becomes

$$\frac{i}{\not{p}_Q - M_Q} = \frac{(M_Q\not{v} + \not{k} + M_Q)}{M_Q^2 + 2M_Q v\cdot k + k^2 - M_Q} \to \frac{i}{v\cdot k}\frac{1+\not{v}}{2} \tag{5.5}$$

in the leading order of the $1/M_Q$ expansion. A velocity-dependent projection operator in the propagator

$$\frac{1+\not{v}}{2} \tag{5.6}$$

projects onto the particle components of the four-component Dirac spinor in the rest frame of the heavy quark. To obtain the Feynman rule for the heavy quark-gluon vertex, we see the vertex sandwiched between quark propagators. Since the propagator is proportional to $(1+\not{v})/2$, the factor of γ_μ in the vertex can be replaced by v^μ as

$$\begin{aligned} &\frac{i}{v\cdot k}\frac{1+\not{v}}{2}ig\gamma_\mu T^a\frac{i}{v\cdot k}\frac{1+\not{v}}{2} \\ =&\frac{i}{v\cdot k}\frac{1+\not{v}}{2}igv_\mu T^a\frac{i}{v\cdot k}\frac{1+\not{v}}{2} \end{aligned} \tag{5.7}$$

where T^as are the generators of the color gauge group. We also see the vertex sandwiched between Dirac spinors of quarks. From Dirac equation,

$$(\not{p}_Q - M_Q)u(p_Q) = 0 \rightarrow (\not{v} + \frac{\not{k}}{M_Q} - 1)u(p_Q) = 0 \tag{5.8}$$

therefore,

$$\not{v}u(p_Q) = u(p_Q) + \mathcal{O}(\Lambda_{\mathrm{QCD}}/M_Q), \tag{5.9}$$

namely, in the leading order, the heavy quark spinor satisfies

$$\frac{1+\not{v}}{2}u(p_Q) = u(p_Q). \tag{5.10}$$

It follows that

$$\begin{aligned} \bar{u}(p'_Q)ig\gamma_\mu T^a u(p_Q) &= \bar{u}(p'_Q)\frac{1+\not{v}}{2}ig\gamma_\mu T^a\frac{1+\not{v}}{2}u(p_Q) \\ &= \bar{u}(p'_Q)\frac{1+\not{v}}{2}igv_\mu T^a\frac{1+\not{v}}{2}u(p_Q) \\ &= \bar{u}(p'_Q)igv_\mu T^a u(p_Q) \end{aligned} \tag{5.11}$$

due to the factor of $(1+\not{v})/2$ and the case that the vertex is sandwiched between a quark propagator and a spinor results in the same conclusion. The Feynman rule for the vertex is given by

$$igv_\mu T^a \tag{5.12}$$

and for the propagator simplifies to

$$\frac{i}{v\cdot k} \tag{5.13}$$

since the factor of $(1+\not{v})/2$ can be absorbed into the heavy quark spinor.

We introduce an effective field $h_v^{(Q)}(x)$ as

$$\psi(x) = e^{-iM_Q v\cdot x}h_v^{(Q)}(x) + \mathcal{O}(\Lambda_{\mathrm{QCD}}/M_Q) \tag{5.14}$$

and substituting it for QCD Lagrangian gives an effective Lagragian which derives the Feynman rules as we saw above. In the leading order, the effective Lagrangian is

$$\begin{aligned}\mathcal{L}_v^{\text{eff}} &= \bar{h}_v^{(Q)}(x)e^{iM_Q v\cdot x}(i\not{D} - M_Q)e^{-iM_Q v\cdot x}h_v^{(Q)}(x) \\ &= \bar{h}_v^{(Q)}(x)[M_Q(\not{v} - 1) + i\not{D}]h_v^{(Q)}(x) \end{aligned} \tag{5.15}$$

so that the equation of motion for the heavy quark is

$$[M_Q(\not{v} - 1) + i\not{\partial}]h_v^{(Q)} = 0. \tag{5.16}$$

In the leading order of the $1/M_Q$ expansion, the effective field satisfies

$$\frac{1+\not{v}}{2}h_v^{(Q)} = h_v^{(Q)}, \tag{5.17}$$

and then the effective Lagrangian becomes independent of heavy quark masses M_Q. Furthermore, with eq.(5.17), the effective Lagrangian can be reduced to

$$\begin{aligned}\mathcal{L}_v^{\text{eff}} &= \bar{h}_v^{(Q)}(x)\frac{1+\not{v}}{2}iD^\mu\gamma_\mu\frac{1+\not{v}}{2}h^{(Q)}v(x) \\ &= \bar{h}_v^{(Q)}(x)\frac{1+\not{v}}{2}iD^\mu v_\mu\frac{1+\not{v}}{2}h^{(Q)}v(x) \\ &= \bar{h}_v^{(Q)}(iD\cdot v)h_v^{(Q)}(x). \end{aligned} \tag{5.18}$$

We can verify that this effective Lagrangian reproduces the Feynman rules eq.(5.12) and eq.(5.13). If there is more than one heavy quark flavor, the effective Lagrangian at the leading order in Λ_{QCD}/M_Q is

$$\mathcal{L}^{\text{eff}} = \sum_{i=1}^{\mathrm{N}_f}\bar{h}_v^{(i)}(x)(iD\cdot v)h_v^{(i)}(x), \tag{5.19}$$

where N_f is the number of heavy quark flavors and the heavy quarks have the same four-velocity v. The effective Lagrangian in (5.19) is independent of the masses and spins of the heavy quark, that is, has a spin-flavor symmetry.

5.2 Construction of heavy meson effective Lagrangian

The total angular momentum of the hadron $\mathbf{J}$ is conserved. In the heavy quark mass limit, since the spin of heavy quark $\mathbf{S}_Q$ is conserved, the spin of the light degrees of freedom defined by

$$\mathbf{S}_l \equiv \mathbf{J} - \mathbf{S}_Q \tag{5.20}$$

is also conserved when the orbital angular momentum $L = 0$. The total spin of light degrees of freedom is good quantum number in heavy hadrons. We define the quantum numbers j, s_Q and s_l as the eigenvalues $j(j+1)$, $s_Q(s_Q+1)$ and $s_l(s_l+1)$ for $\mathbf{J}^2$, $\mathbf{S}_Q^2$ and $\mathbf{S}_l^2$ in the heavy hadron state, respectively. In the heavy $\bar{Q}q$ meson systems, the light degrees of freedom must have the quantum numbers of a single quark q. Since the ground state mesons consist of a heavy quark with $s_Q = 1/2$ and light degrees of freedom with $s_l = 1/2$, they come in doublets with spin

$$j = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1, \tag{5.21}$$

which have negative parity because the $\bar{Q}$ and q have opposite intrinsic parity. The states with $j = 0$ are pseudoscalar mesons and those with $j = 1$ are vector mesons. These doublets are degenerate in the heavy quark mass limit. When $\bar{Q}$ is an anti-charm quark, these states are $\bar{D}$ and $\bar{D}^*$ mesons, and when $\bar{Q}$ is an anti-bottom quark, these states are B and B^* mesons.

In the heavy meson effective theory, a formalism is employed, in which the entire multiplet of degenerate states is treated as a single object, H_v, that transforms linearly under the heavy quark spin-flavor symmetry. Heavy mesons are described by the effective field

$$H_v(x) = \frac{1 + \not{v}}{2}[P^*_{v\mu}(x)\gamma^\mu + iP_v(x)\gamma_5], \tag{5.22}$$

where v_μ is the velocity of the heavy meson, and H_v has mass dimension 3/2. With two heavy flavors, c and b, and three light flavors, u, d and s, the fields P^*_v and P_v include six heavy-light vector and six heavy-light pseudoscalar mesons

$$P_v^{(*)} = \begin{pmatrix} D_u^{(*)} & D_d^{(*)} & D_s^{(*)} \\ B_u^{(*)} & B_d^{(*)} & B_s^{(*)} \end{pmatrix}. \tag{5.23}$$

The fields B_u, B_d and B_s correspond to the mesons B^+, B^0 and B_s^0, respectively. The field H_v is transformed under the spin-flavor SU(4) transformation, which can be decomposed by spin $\mathrm{SU(2)_s}$ and heavy flavor $\mathrm{SU(2)_f}$

transformations, as

$$H_v \to S H_v, \tag{5.24}$$

$$H_v \to z_H H_v, \tag{5.25}$$

where $S \in \mathrm{SU}(2)_\mathrm{s}$ act on the spinor index and $z_H \in \mathrm{SU}(2)_\mathrm{f}$ act on the heavy flavor index. The field is also transformed under the chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_A$ transformation as

$$H_v \to H_v h(\Pi, V_L, V_R)^\dagger. \tag{5.26}$$

The parity transformation and charge conjugation are defined as follows,

$$\mathcal{P} H_v(x) \mathcal{P}^\dagger = \gamma^0 H_{v_P}(x_P) \gamma^0, \tag{5.27}$$

$$\mathcal{C} H_v(x) \mathcal{C}^\dagger = C(\bar{H}_{-v}(x))^T C^\dagger, \tag{5.28}$$

where $v_P = (v^0, -\mathbf{v})$, $x_P = (x^0, -\mathbf{x})$ and $C = i\gamma^2\gamma^0$. We need to introduce

$$M(x) = \xi^\dagger(x) \exp\left(\frac{-i\eta'}{\sqrt{6} f'}\right) \chi \xi^\dagger(x) \exp\left(\frac{-i\eta'}{\sqrt{6} f'}\right) \tag{5.29}$$

to describe chiral symmetry breaking by the masses of light quarks and it is transformed

$$M(x) \to h(\Pi, V_L, V_R) M(x) h(\Pi, V_L, V_R)^\dagger \tag{5.30}$$

under the chiral transformation, if light quark masses transform appropriately. This is also transformed under the parity transformation and charge conjugation as

$$\mathcal{P} M(x) \mathcal{P}^\dagger = M^\dagger(x_P), \tag{5.31}$$

$$\mathcal{C} M(x) \mathcal{C}^\dagger = M^T(x). \tag{5.32}$$

The effective Lagrangian can be obtained by imposing the chiral symmetry, spin-flavor symmetry, and invariance under the parity transformation, charge conjugation and Hermite conjugation. The effect of spin-flavor symmetry breaking is included by introducing the matrix

$$\frac{1}{M_Q} \equiv \begin{pmatrix} \frac{1}{M_c} & 0 \\ 0 & \frac{1}{M_b} \end{pmatrix}, \tag{5.33}$$

where M_c and M_b are charm and bottom quark masses, respectively. Since we consider the situation that the heavy quark masses are much larger than the typical QCD scale, Λ, the effective Lagrangian is expanded in powers of Λ/M_Q. We do not consider the terms which includes derivatives on H_v, since we are going to analyse only the masses of heavy mesons. We write down non-derivative part of the effective Lagrangian up to $\mathcal{O}(\Lambda/M_Q)$ in the Λ/M_Q expansion and $\mathcal{O}((m_q/\Lambda)^2)$ in the chiral expansion

$$\begin{aligned}\mathcal{L}_v^{\text{mass}} &= \Lambda\Big\langle \mathrm{tr}[\bar{H}_v H_v]\Big\rangle + \mathcal{L}^{\text{mass}}_{(\Lambda/M)} + \mathcal{L}^{\text{mass}}_{(m/\Lambda)} + \mathcal{L}^{\text{mass}}_{(\Lambda/M)(m/\Lambda)} + \mathcal{L}^{\text{mass}}_{(\Lambda/M)^2} \\ &\quad + \mathcal{O}\left(\left(\frac{\Lambda}{M_Q}\right)^2\right), \end{aligned} \tag{5.34}$$

where

$$\mathcal{L}^{\text{mass}}_{(\Lambda/M)} = \kappa'\Lambda\Big\langle \mathrm{tr}[\bar{H}_v \frac{\Lambda}{M_Q} H_v]\Big\rangle + \kappa\Lambda\Big\langle \mathrm{tr}[\bar{H}_v \frac{\Lambda}{M_Q}\sigma_{\rho\sigma} H_v \sigma^{\rho\sigma}]\Big\rangle, \tag{5.35}$$

$$\mathcal{L}^{\text{mass}}_{(m/\Lambda)} = \frac{\chi_1}{\Lambda}\Big\langle \mathrm{tr}[\bar{H}_v H_v]\Big\rangle\Big\langle M + M^\dagger\Big\rangle + \frac{\chi_2}{\Lambda}\Big\langle \mathrm{tr}[\bar{H}_v H_v](M + M^\dagger)\Big\rangle, \tag{5.36}$$

$$\begin{aligned}\mathcal{L}^{\text{mass}}_{(\Lambda/M)(m/\Lambda)} &= \frac{a_1}{\Lambda}\Big\langle \mathrm{tr}[\bar{H}_v \frac{\Lambda}{M_Q} H_v]\Big\rangle\Big\langle M + M^\dagger\Big\rangle + \frac{a_2}{\Lambda}\Big\langle \mathrm{tr}[\bar{H}_v \frac{\Lambda}{M_Q} H_v](M + M^\dagger)\Big\rangle \\ &\quad + \frac{b_1}{\Lambda}\Big\langle \mathrm{tr}[\bar{H}_v \frac{\Lambda}{M_Q}\sigma_{\rho\sigma} H_v \sigma^{\rho\sigma}]\Big\rangle\Big\langle M + M^\dagger\Big\rangle \\ &\quad + \frac{b_2}{\Lambda}\Big\langle \mathrm{tr}[\bar{H}_v \frac{\Lambda}{M_Q}\sigma_{\rho\sigma} H_v \sigma^{\rho\sigma}](M + M^\dagger)\Big\rangle, \end{aligned} \tag{5.37}$$

and

$$
\begin{aligned}
\mathcal{L}^{\rm mass}_{(\Lambda/M)^2} = & \frac{K_1}{\Lambda^3}\Big\langle {\rm tr}[\bar{H}_v H_v]\Big\rangle\Big\langle M\Big\rangle\Big\langle M^\dagger\Big\rangle \\
& +\frac{K_2}{\Lambda^3}\left\{\Big\langle {\rm tr}[\bar{H}_v H_v]M\Big\rangle\Big\langle M^\dagger\Big\rangle + \Big\langle {\rm tr}[\bar{H}_v H_v]M^\dagger\Big\rangle\Big\langle M\Big\rangle\right\} \\
& +\frac{K_3}{\Lambda^3}\left\{\Big\langle {\rm tr}[\bar{H}_v H_v]MM^\dagger\Big\rangle + \Big\langle {\rm tr}[\bar{H}_v H_v]M^\dagger M\Big\rangle\right\} \\
& +\frac{K_4}{\Lambda^3}\Big\langle {\rm tr}[\bar{H}_v H_v]\Big\rangle\left\{\Big\langle MM + M^\dagger M^\dagger\Big\rangle\right\} \\
& +\frac{K_5}{\Lambda^3}\Big\langle {\rm tr}[\bar{H}_v H_v]\Big\rangle\left\{\Big\langle M\Big\rangle\Big\langle M\Big\rangle + \Big\langle M^\dagger\Big\rangle\Big\langle M^\dagger\Big\rangle\right\} \\
& +\frac{K_6}{\Lambda^3}\left\{\Big\langle {\rm tr}[\bar{H}_v H_v]M\Big\rangle\Big\langle M\Big\rangle + \Big\langle {\rm tr}[\bar{H}_v H_v]M^\dagger\Big\rangle\Big\langle M^\dagger\Big\rangle\right\} \\
& +\frac{K_7}{\Lambda^3}\left\{\Big\langle {\rm tr}[\bar{H}_v H_v]MM\Big\rangle + \Big\langle {\rm tr}[\bar{H}_v H_v]M^\dagger M^\dagger\Big\rangle\right\}. \qquad (5.38)
\end{aligned}
$$

Here $\sigma^{\rho\sigma} = i[\gamma^\rho, \gamma^\sigma]/2$ and the couplings κ', κ, χ_i, a_i, b_i and the seven couplings K_i are dimensionless parameters. A possible term

$$\frac{K_0}{\Lambda}\Big\langle {\rm tr}[\bar{H}_v H_v]\Big\rangle\Big\langle M^\dagger M + MM^\dagger\Big\rangle \qquad (5.39)$$

can be absorbed into the first term in $\mathcal{L}^{\rm mass}_v$ due to unitarity of ξ. The terms with coupling K_i in $\mathcal{L}^{\rm mass}_{(\Lambda/M)^2}$ gives a *complete* collection of terms of $\mathcal{O}(p^4)$ without derivatives [39], though some of the terms have already been given by [43]. The order $(\Lambda/M_Q)^2$ terms, which could give the contribution of the same order of $\mathcal{O}(p^4)$ terms, are irrelevant to our present analysis, since we are not going to consider the mass differences between heavy pseudoscalar mesons and heavy vector mesons.

The heavy meson effective Lagrangian has also the invariance under the instanton transformation of eq.(4.20) with

$$
\begin{aligned}
& K_4 \to K_4 + \frac{\chi_1+\chi_2}{2}\omega, \\
& K_5 \to K_5 - \frac{\chi_1+\chi_2}{2}\omega, \\
& K_6 \to K_6 + \chi_2\omega, \\
& K_7 \to K_7 - \chi_2\omega.
\end{aligned} \qquad (5.40)
$$

This invariance can be shown by using eq.(4.22). The terms with couplings K_4, K_5, K_6 and K_7 break $\mathrm{U}(1)_A$ symmetry, because M is transformed under the $\mathrm{U}(1)_A$ axial transformation (note that in eq.(5.29) η' transforms as $\eta' \to \eta' + \sqrt{6} f' \theta_A/2$ and χ is invariant since it is essentially the mass matrix). Remember that the instanton effect breaks $\mathrm{U}(1)_A$ symmetry, and in fact, these couplings are sensitive to the instanton transformation. On the other hand, the other terms of K_1, K_2 and K_3 are invariant under $\mathrm{U}(1)_A$ axial transformation, and they are insensitive to the instanton transformation.

In the following, we fit a combination of couplings, $K_3 + K_7$, which is sensitive to the instanton correction using the well-known masses of pseudoscalar B mesons only.

5.3 Constraints in heavy meson system

We obtain the pseudoscalar B meson mass formulae from eq.(5.34)

$$
\begin{aligned}
M_{B_q}^2 = M_b^2 \Bigg[1 &+ \frac{\Lambda}{M_b} \Bigg\{ 2 + 2\left(\kappa' + 6\kappa\right) \frac{\Lambda}{M_b} \\
&\quad + 8 \left(\chi_1 + a_1 \frac{\Lambda}{M_b} + 6 b_1 \frac{\Lambda}{M_b} \right) \frac{B_0}{\Lambda} \frac{m_u + m_d + m_s}{\Lambda} \\
&\quad + 8 \left(K_1 + 2K_5 \right) \frac{B_0^2}{\Lambda^2} \frac{(m_u + m_d + m_s)^2}{\Lambda^2} + 16 K_4 \frac{B_0^2}{\Lambda^2} \frac{m_u^2 + m_d^2 + m_s^2}{\Lambda^2} \Bigg\} \\
&+ \frac{\Lambda}{M_b} \Bigg\{ 8 \left(\chi_2 + a_2 \frac{\Lambda}{M_b} + 6 b_2 \frac{\Lambda}{M_b} \right) \frac{B_0}{\Lambda} \\
&\qquad + 16 \left(K_2 + K_6 \right) \frac{B_0^2}{\Lambda^2} \frac{m_u + m_d + m_s}{\Lambda} \Bigg\} \frac{m_q}{\Lambda} \\
&+ \frac{\Lambda}{M_b} 16 \left(K_3 + K_7 \right) \frac{B_0^2}{\Lambda^2} \frac{m_q^2}{\Lambda^2} \Bigg] + \mathcal{O}\left(\left(\frac{\Lambda}{M_Q} \right)^2 \right), \qquad (5.41)
\end{aligned}
$$

where q is the light flavor index, u, d and s [39]. In this heavy meson mass formulae, we see that the couplings K_5 and K_6 describe the direct contribution of the instanton-induced mass like Fig.4.1, though the couplings K_4 and K_7 should be also sensitive to the instanton transformation. This is the same argument as that on L_6, L_7 and L_8 in previous chapter. We can fit

$$\chi_2 = \frac{\Lambda}{4}\frac{\hat{M}_{B_s} - \hat{M}_{B_d}}{\hat{m}^2_{K^\pm} - \hat{m}^2_{\pi^\pm}} = 0.065 \pm 0.004, \tag{5.42}$$

at $\mathcal{O}(p^2)$. Up to $\mathcal{O}(p^4)$ the meson mass differences are obtained as follows

$$\begin{aligned} M_{B_s} - M_{B_d} = \frac{B_0(m_s - m_d)}{\Lambda} & \left[4\chi_2 + (4a_2 + 24b_2)\frac{\Lambda}{M_b} \right. \\ & \left. + 8(K_2 + K_6)\frac{B_0(m_u + m_d + m_s)}{\Lambda^2} + 8(K_3 + K_7)\frac{B_0(m_s + m_d)}{\Lambda^2} \right], \end{aligned} \tag{5.43}$$

and

$$\begin{aligned} M_{B_d} - M_{B_u} = \frac{B_0(m_d - m_u)}{\Lambda} & \left[4\chi_2 + (4a_2 + 24b_2)\frac{\Lambda}{M_b} \right. \\ & \left. + 8(K_2 + K_6)\frac{B_0(m_u + m_d + m_s)}{\Lambda^2} + 8(K_3 + K_7)\frac{B_0(m_d + m_u)}{\Lambda^2} \right]. \end{aligned} \tag{5.44}$$

Therefore, we can extract only one combination

$$\begin{aligned} & K_3 + K_7 \\ = & \frac{\Lambda^3}{8(\hat{m}^2_{K^0} - \hat{m}^2_{\pi^0})} \left\{ \frac{\hat{M}_{B_s} - \hat{M}_{B_d}}{\hat{m}^2_{K^\pm} - \hat{m}^2_{\pi^\pm}} - \frac{\hat{M}_{B_d} - \hat{M}_{B_u} + (\hat{M}_{B_u} - \hat{M}_{B_d})_{\mathrm{EM}}}{\hat{m}^2_{K^0} - \hat{m}^2_{K^\pm} + \hat{m}^2_{\pi^\pm} - \hat{m}^2_{\pi^0}} \right\} \\ & -\frac{\chi_2}{2}\frac{\Lambda^2}{\hat{m}^2_{K^\pm} + \hat{m}^2_{K^0} - \hat{m}^2_{\pi^\pm}} \left\{ \frac{\hat{m}^2_{K^0} - \hat{m}^2_{K^\pm} + \hat{m}^2_{\pi^\pm} - \hat{m}^2_{\pi^0}}{\hat{m}^2_{\pi^0}} \frac{m_d + m_u}{m_d - m_u} - 1 \right\}, \end{aligned} \tag{5.45}$$

where we have considered the QED correction to the masses of the charged mesons. Substituting

$$(M_{B_u} - M_{B_d})_{\mathrm{EM}} = 2.09 \pm 0.18 \text{ MeV}, \tag{5.46}$$

which is given by a theoretical calculation in [44], and the observed value of the mass differences in [38], we obtain

$$K_3 + K_7 = -0.013 \pm 0.007, \tag{5.47}$$

Since K_7 does not seem to describe direct the contribution of the instanton effect in the mass formulae of eq.(5.41), and since K_3 does not transform under the instanton transformation, we are going to discuss the "weak constraint" on the possible maximal value of the parameter ω. This should be equivalent what we have obtained from L_8 in the previous chapter. It can be obtained as

$$\omega_{\max} = 0.2 \pm 0.1, \tag{5.48}$$

by using eqs.(5.40) and (5.42) and by assuming that all the values in eq.(5.47) are produced by the instanton dynamics.

The constraint on the possible maximal value of the omega parameter can be also evaluated *in the heavy meson system*. It is found that the constraint on the instanton effect in the heavy meson system is as strong as that in the light meson system. In this case, the instanton-induced effective up-quark mass is given as

$$m_u^{\text{eff}} = 1.06 \pm 0.50 \text{ MeV}, \tag{5.49}$$

when $m_u = 0$ by eq.(4.17). This value differs from the value in eq.(4.32) by about 3.4σ and the solution to the strong CP problem by $m_u = 0$ is disfavored [39]. Note that, if the sign of non-instanton K_3 is opposite of the sign of K_7, we are overconstraining the instanton effect. To avoid this, we must determine the value of K_3, which should be future work. The quantitative investigation of the contribution of instanton effects to K_7 is a future task. We believe that the value of K_3 and K_7 will be independently extracted from the data by future experiments. Also note that, there would be room for improvement to take into account some possible loop effects of the pseudo-Nambu-Goldstone bosons as we have noted in previous chapter. Our analysis using the heavy meson effective theory indicates that the instanton-induced mass correction does not seem to be large enough to solve the $\mathrm{U}(1)_A$ problem.

Chapter 6

Conclusion

We have investigated the light meson effective Lagrangian of the system with pseudo Nambu-Goldstone bosons at low energies up to $\mathcal{O}(p^4)$, and we have confirmed that the effective Lagrangian is invariant under the instanton transformation. This symmetry is the consequence of the fact that the physics is independent of whether the instanton correction is included in χ or L_6, L_7 and L_8. We have evaluated the value of the coupling L_7, which is expected to be dominantly produced by the instanton dynamics. The maximum value of the omega parameter given in eq.(4.42) was calculated under the assumption that the whole value of L_7 is produced by the instanton dynamics. The result $\omega_{\max} = 0.4 \pm 0.1$ cannot create the situation that $m_u = 0$ could be a solution to the strong CP problem. In other words, the instanton corrections to quark masses are too small to explain the strong CP problem. The "weakly constrained" maximum omega parameter has been also calculated with the coupling L_8 though the coupling might have the contribution from the other non-instanton dynamics in QCD. The result in this case also indicates that the instanton effect is small. We have confirmed this known result in a simple way.

We have shown that the same analysis is possible for the heavy meson systems in Sec.5. We have constructed the heavy meson effective Lagrangian with chiral symmetry breaking up to $\mathcal{O}(p^4)$ in the chiral expansion (eq.(5.38)). This effective Lagrangian also is invariant under the instanton transformation and we have identified the couplings which are non-trivially transformed under the instanton transformation. The bottomed pseudoscalar meson mass formulae up to $\mathcal{O}(p^4)$ have been derived in eq.(eq:heavymassformulae). This was possible because their masses have already been measured well by experiments and Λ/M_b expansion is more reliable than Λ/M_c expansion in the heavy meson effective theory. Although we could only determine a combination of K_3+K_7 (K_3 is insensitive and K_7 is sensitive to the instanton transformation), the possible maximal value ("weak constraint") $\omega_{\rm max} = 0.2 \pm 0.1$ is obtained from the meson mass differences. It has been found that the constraint on the instanton parameter in the heavy meson system is as strong as that in the light meson system. The development of the lattice calculations is going to give a more precise omega parameter, since the error of the omega parameter is mainly from the error of quark condensate given by the lattice calculations.

Our results are obtained following the procedure based on a spirit of the chiral expansion theory, where the parameters are fitted order by order. When the light quark masses are fitted in the leading order, the freedom of the instanton transformation is fixed. The instanton corrections are included in the higher order terms. The instanton correction can be absorbed into the leading order quark masses with a special instanton transformation by which higher order couplings vanish under a conservative assumption of dominance of the instanton contributions to the higher order couplings. The value of $\omega_{\rm max}$ corresponds to such an instanton transformation, and it can be a measure of the magnitude of the instanton effect as well as $m_u^{\rm eff}$.

The non-trivial vacuum structure in QCD is in the theory not in Minkowski space-time but in Euclidean space-time. Whether the analytic continuation can be done or not is an assumption. The proof of the validity seems to be difficult. On the other hand, there are some attempts to investigate the phenomenon induced by the non-trivial vacuum structure. No attempt has succeeded in supporting the existence of the non-trivial vacuum structure in QCD. Our analysis using heavy meson effective theory is not in favor of the existence by showing that the instanton-induced effect cannot produce the sufficiently large value of $m_u^{\rm eff}$ to solve the strong CP problem.

In the following, we point out the problems and future subjects of our present analysis in order.

We note that both the results in the light and heavy meson systems were obtained under the assumption that the whole values of the corresponding couplings were generated by the instanton dynamics only. If the contribution of the non-instanton dynamics in QCD to the couplings in the effective theory is large, especially if it cancels the contribution of the instanton dynamics, our constraints do not apply.

In order to make the constraint more precise for heavy meson system, we must determine the value of K_3 independently. In the future, the information of b-flavored vector meson masses given by experiments will enable us to fit the couplings K_i more precisely and systematically. When we consider the mass differences between the vector mesons and pseudoscalar mesons, we need to include the $\mathcal{O}((\Lambda/M_b)^2)$ terms in the heavy meson effective Lagrangian.

As a first step of the analysis, we have neglected, for simplicity, the chiral symmetry breaking terms in the leading order in derivative of H_v, $\langle \mathrm{tr}(Hv \cdot \partial H)\rangle\langle M\rangle$ and $\langle \mathrm{tr}(Hv \cdot \partial H)M\rangle$, which require the field redefinitions, and the mass formulae should obtain corrections. Again, we hope that the results of the future B factory experiments will enable us to include these terms.

We also have neglected the loop effects of the pseudo-Nambu-Goldstone bosons in chiral perturbation theory, because we have attempted to investigate with the heavy meson and have needed to obtain the consistent method which is simpler than [34]. We leave this problem as a future work.

According to [44], the masses of B_d and B_u are almost degenerate as a result of the cancellation of two sources of isospin breaking: mass difference of up- and down-quarks and the QED effect. This situation is very different from that in light mesons or charmed mesons, π, K and D. The phenomena should be studied more to obtain more precise input of the theoretical QED effect.

In this thesis a new quantitative constraint on the instanton effect in the heavy meson system has been given. We hope that future development of this approach will help to verify the instanton effect experimentally.

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