テンソルネットワークの進展

武田　真滋
（金沢大学）

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Contents

- Introduction of tensor networks
  - Why/What’s tensor network (TN)
  - Lagrangian/path integral approach
  - Tensor renormalization group (TRG)
  - Recent progress of TN study
- Topical Topics
  - Spectroscopy
  - Entanglement entropy

using TRG method

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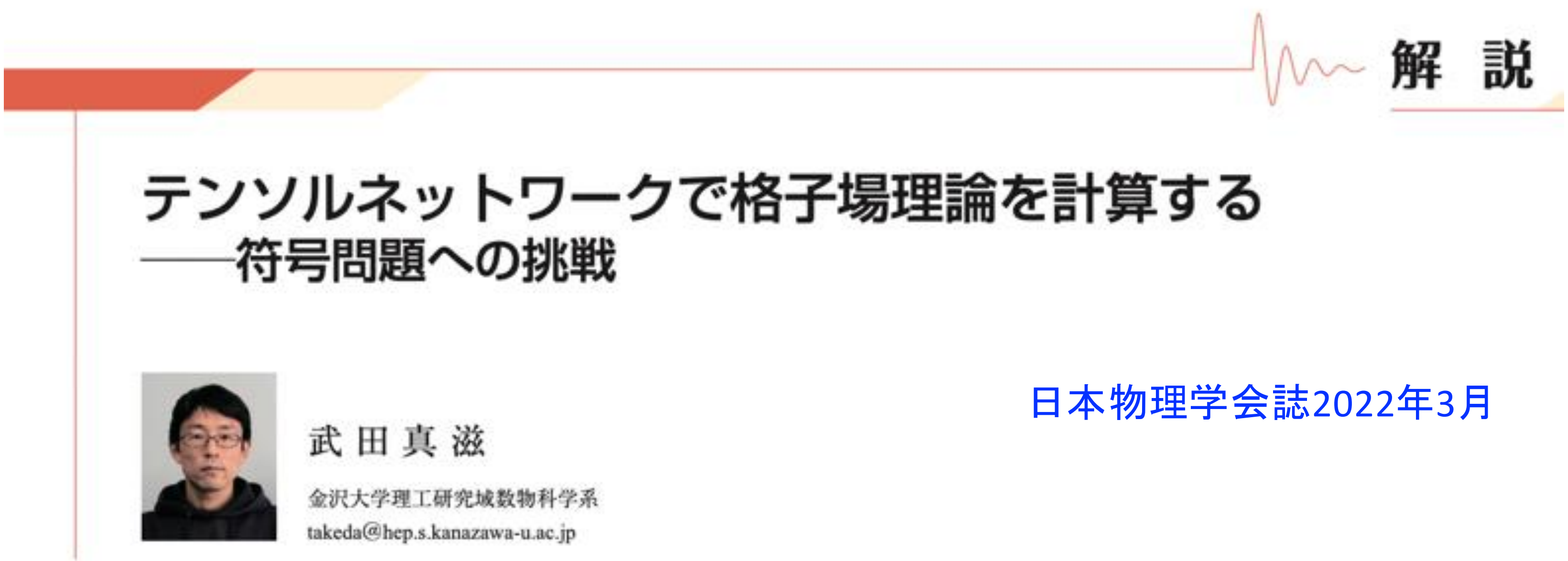
解 説

テンソルネットワークで格子場理論を計算する
——符号問題への挑戦

武 田 真 滋
金沢大学理工研究域数物科学系
takeda@hep.s.kanazawa-u.ac.jp

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Why tensor networks?

- Good
  - Applicable to any models even for complex action

    ≈ no sign problem

  - Extremely large volume (thermodynamic limit)
  - High-precision is attainable in 2D system with simple internal d.o.f.
- Bad
  - Expensive for higher dimensional system

Why tensor networks?

- **Good**
  - Applicable to any models even for complex action

    ≈ no sign problem

Challenge to

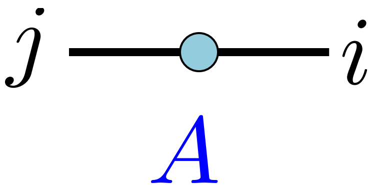
- QCD + μ
- θ-term
- Lattice SUSY
- Real-time dynamics
- Chiral gauge theory

Other methos to overcome sign problem

- Complex Langevin
- Lefshetz thimble
- Path optimization
- Quantum computing
- …

Notational rules

Rank 2 tensor (matrix)

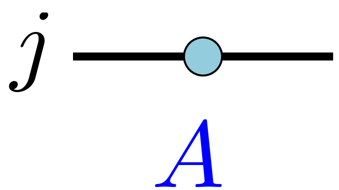


$= A_{ij}$

Tensor : vertex
index : link

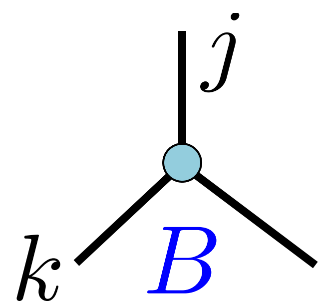
Notational rules

Rank 2 tensor (matrix)



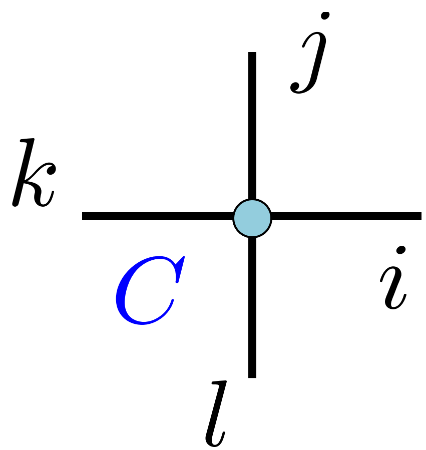
$$= A_{ij}$$

Rank 3 tensor



$$= B_{ijk}$$

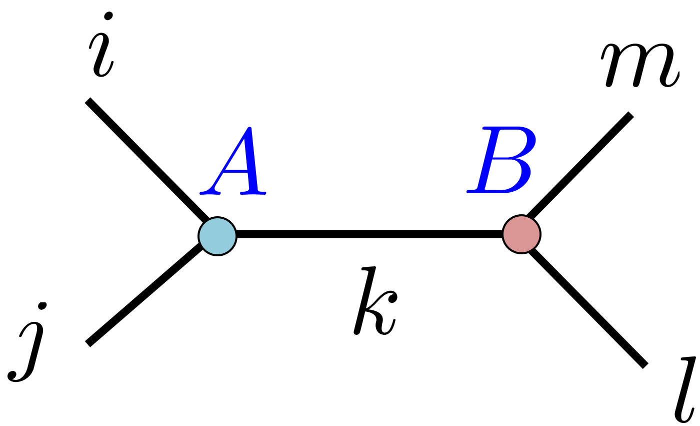
Rank 4 tensor



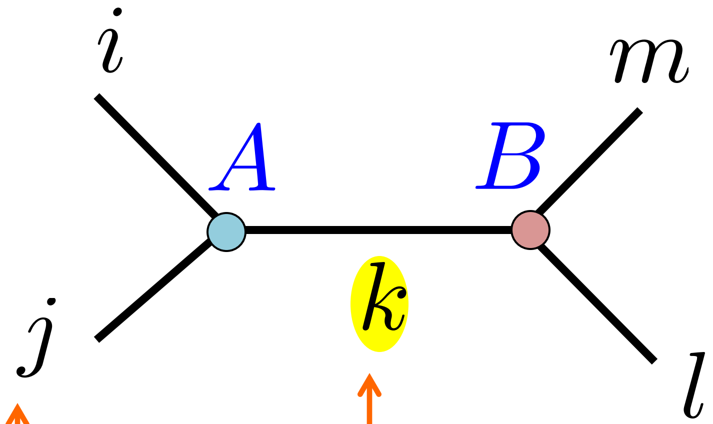
$$= C_{ijkl}$$

Tensor : vertex
index : link

Contraction (summation) rule



Contraction (summation) rule



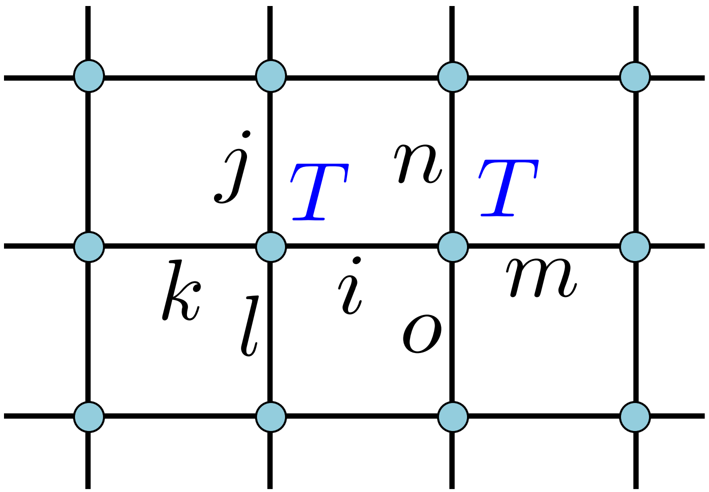
$$= \sum_{k} A_{ijk} B_{klm}$$

index shared by two tensors : summation

index connecting with a single tensor : no summation

What's tensor network?

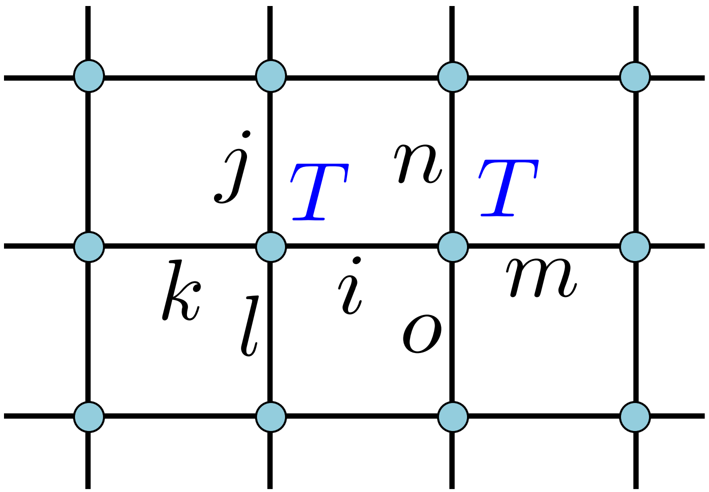
Example: TN for square lattice



$$= \sum_{\cdots, i, j, k \cdots} \cdots T_{ijkl} T_{mnio} \cdots$$

What’s tensor network?

Example: TN for square lattice



$$= \sum_{\cdots,i,j,k\cdots} \cdots T_{ijkl} T_{mnio} \cdots$$

A target quantity (wave function/partition function) is represented by tensor network

Two approaches in TN

	Hamiltonian approach	**Lagrangian approach**
TN is used to express	wave function	partition function, path integral
target system	quantum many-body system	Classical statistical system, Path-integral rep. of quantum field theory
combining with	variational method	coarse-graining (real-space renormalization group)

This talk

Working flow of Lagrangian approach

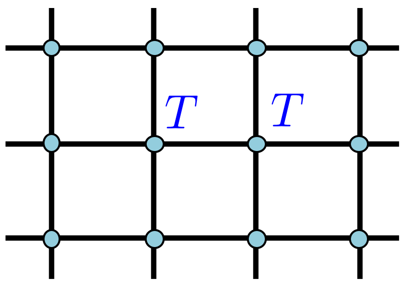
Target model in continuum space-time

Lattice regularization

Lattice model

MC if no sign problem

Tensor network representation of Z



$$Z \equiv \int [d\phi] e^{-S[\phi]} \overset{?}{=} \sum_{..,i,j,k,l,..} \cdots T_{ijkl} T_{mnio} \cdots$$

Working flow of Lagrangian approach

Target model in continuum space-time

↓ Lattice regularization

Lattice model → MC if no sign problem

↓

TN rep. of Z

- Non-trivial step, but
- OK for scalar, gauge, and fermion fields as long as the interaction is local

$$Z \equiv \int [d\phi] e^{-S[\phi]} \stackrel{?}{=} \sum_{..,i,j,k,l,..} \cdots T_{ijkl} T_{mnio} \cdots$$

Working flow of Lagrangian approach

Target model in continuum space-time

Lattice regularization

Lattice model

MC if no sign problem

TN rep. of Z

For each model, tensor is prepared

Bond dimension

cost $\propto \chi^{2V}$ for $1 \leq i, j, \ldots \leq \chi$

$$Z \equiv \int [d\phi] e^{-S[\phi]} = \sum_{\ldots,i,j,k,l,\ldots} \cdots T_{ijkl} T_{mnio} \cdots$$

Working flow of Lagrangian approach

Target model in continuum space-time

Lattice regularization

Lattice model

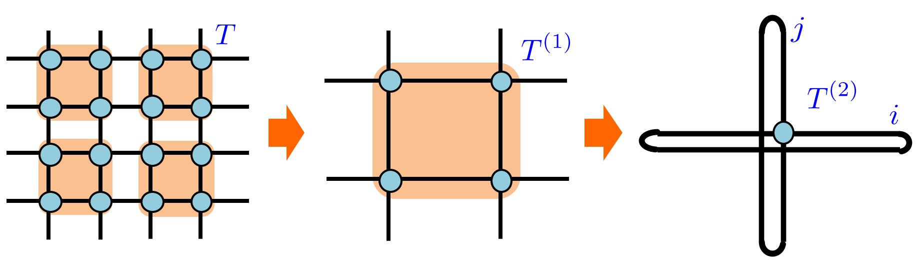
MC if no sign problem

TN rep. of Z

For each model, tensor is prepared

Coarse-graining

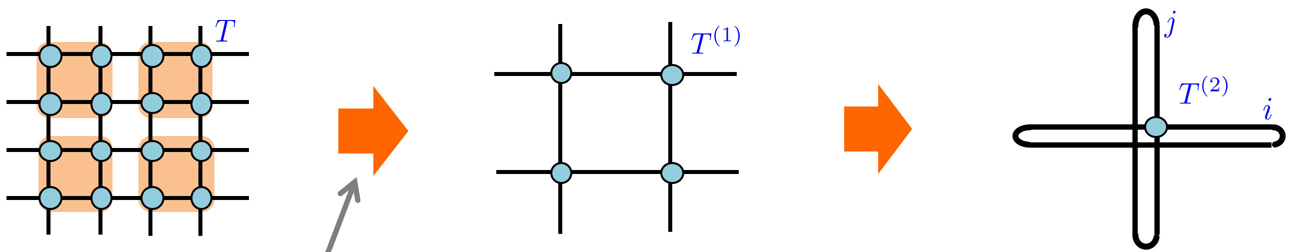
like spin-blocking/real space renormalization group



$$Z \approx \sum_{i,j} T^{(n)}_{ijij}$$

Coarse-graining

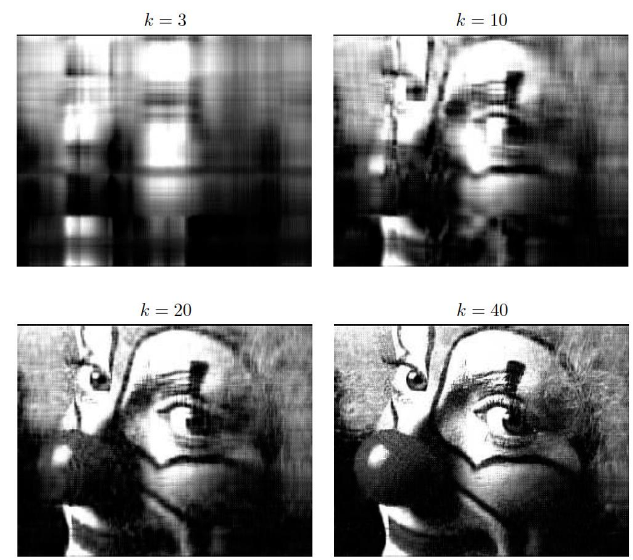
Tensor renormalization group (TRG)
PRL99,120601(2007)



$$Z \approx \sum_{i,j} T^{(n)}_{ijij}$$

Information compression using singular value decomposition (SVD)

less modes



many modes

original

truncated SVD

SVD: $$M_{ab} = \sum_m u_{am}\Lambda_m (v^\dagger)_{mb} \approx \sum_{m=1}^{k} u_{am}\Lambda_m (v^\dagger)_{mb}$$

http://www.na.scitec.kobe-u.ac.jp/~yamamoto/lectures/cse-introduction2009/cse-introduction090512.PPT

Progress of coarse-graining algorithm

Better precision

Tensor Network Renormalization (TNR)
Evenbly & Vidal, PRL115,180405(2015)

Loop-TNR Yang et al., PRL118,110504(2017)

Graph Independent Local Truncations (GILT)
Hauru et al., PRB97,045111(2018)

Tensor Renormalization Group (TRG)
Levin and Nave, PRL99,120601(2007)

d : dimensionality of system

Higher dimensional System and cost reduction

Higher Order Tensor Renormalization Group (HOTRG) $O(\chi^{4d-1})$
Xie et al., PRB86,045139(2012)

Anisotropic TRG (ATRG) Adachi et al., PRB102,054432(2020) $O(\chi^{2d+1})$

Triad TRG Kadoh & Nakayama, arXiv:1912:02414 $O(\chi^{d+3})$

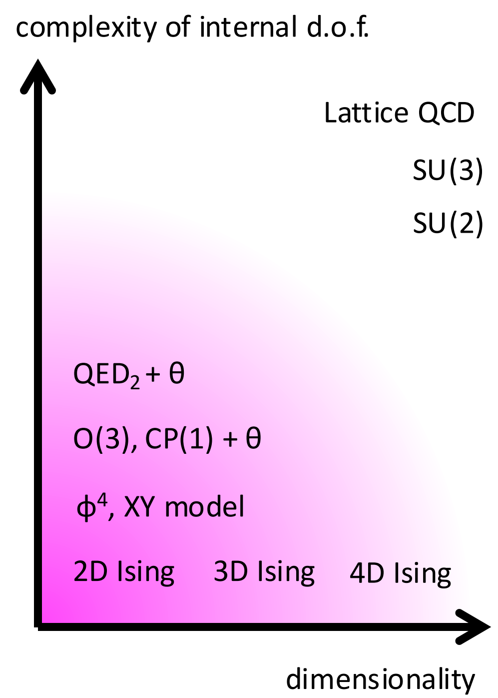
Minimally decomposed TRG(MDTRG) $O(\chi^{2d+1})$
Triad MDTRG $O(\chi^{d+3})$ Nakayama, arXiv:2307:14191

No truncation error but introducing statistical error

All-mode renormalization Ohki+ PRD107,114515(2023)

Studies of Lagrangian approach

- 2D system
  - Spin model : Ising model Levin & Nave PRL99,120601(2007), Aoki et al. Int. Jour. Mod. Phys. B23,18(2009) , X-Y model Meurice et al. PRE89,013308(2014), X-Y model with Fisher zero Meurice et al. PRD89,016008(2014), O(3) model Unmuth-Yockey et al. LATTICE2014, X-Y model + μ Meurice et al. PRE93,012138(2016)
  - Abelian-Higgs Bazavov et al. LATTICE2015
  - ϕ^4 theory Shimizu Mod.Phys.Lett.A27,1250035(2012), Sakai et al., JHEP05(2019)184
  - QED_2 Shimizu & Kuramashi PRD90,014508(2014) & PRD90,034502(2018)
  - $QED_2 + \theta$ Shimizu & Kuramashi PRD90,074503(2014), Kanno+ Lattice 2024
  - Gross-Neveu model + μ ST & Yoshimura PTEP043B01(2015)
  - CP(N-1) + θ Kawauchi & ST PRD93,114503(2016)
  - Towards Quantum simulation of O(2) model Zou et al, PRA90,063603
  - N=1 Wess-Zumino model (SUSY model) Sakai et al., JHEP03(2018)141
  - Hubbard model Akiyama et al., PRD104(2021)014504
  - U(N) and SU(N) gauge theory Nishimura et al., JHEP12(2021)011
- 3D system Higher order TRG(HOTRG) : Xie et al. PRB86,045139(2012)
  - Ising, Potts model Wan et al. CPL31,070503(2014)
  - Pure fermion system Sakai et al.,PTEP063B07(2017)
  - Gauge Ising Yoshimura et al.,JHEP08(2019)023
  - Hubbard model Akiyama et al., PTEP(2022)023I01
  - SU(2) gauge Tsuchiya et al., arXiv:2205.08883
  - SU(2) principle chiral + μ Akiyama et al., arXiv:2312.11649
- 4D system
  - Ising model Akiyama et al., PRD100,054510(2019), Sasaki+Sugimoto Lattice 2024
  - Complex ϕ^4 + μ Akiyama et al., JHEP09(2020)177
  - NJL model Akiyama et al., JHEP01(2021)121
  - Real ϕ^4 theory Akiyama et al., PRD104(2021)034507
  - Z_2 gauge-Higgs + μ Akiyama et al., JHEP05(2022)102
  - Z_3 gauge-Higgs + μ Akiyama et al., JHEP05(2023)077



Spectroscopy using TRG method

arXiv:2404.15666 (to be published in PRD)

共同研究者
Fathiyya Az-zahra（金沢大）
山崎剛（筑波大）

Energy spectrum

- Schrödinger eq.

(e.g., 0^{-+} for pion)

quantum number : J^{PC}, flavors, $\cdots$

$$\hat{H}|n,q\rangle = E_{n,q}|n,q\rangle$$

QCD Hamiltonian

$$(n = 0, 1, 2, \cdots)$$

$\hat{H}|\Omega\rangle = 0$: vacuum

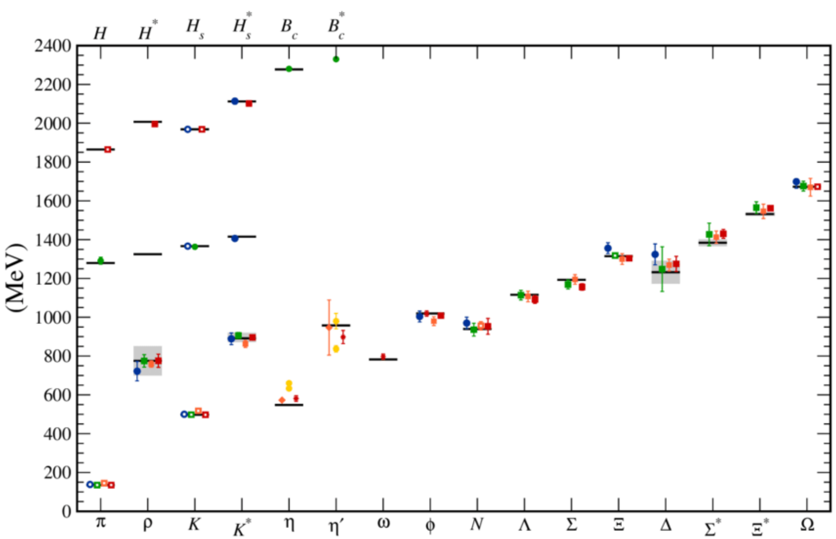
- Two-point function (Euclidean time)

$$\lim_{\beta\to\infty} \mathrm{Tr}\left[\hat{\mathcal{O}}_q^\dagger(\tau)\hat{\mathcal{O}}_q(0)e^{-\beta\hat{H}}\right] = \sum_{n=0}^{\infty} |\langle\Omega|\hat{\mathcal{O}}_q(0)|n,q\rangle|^2 e^{-\tau E_{n,q}}$$

$$\hat{1} = \sum_{n,q'} |n,q'\rangle\langle n,q'|$$

Hadron spectroscopy with MC

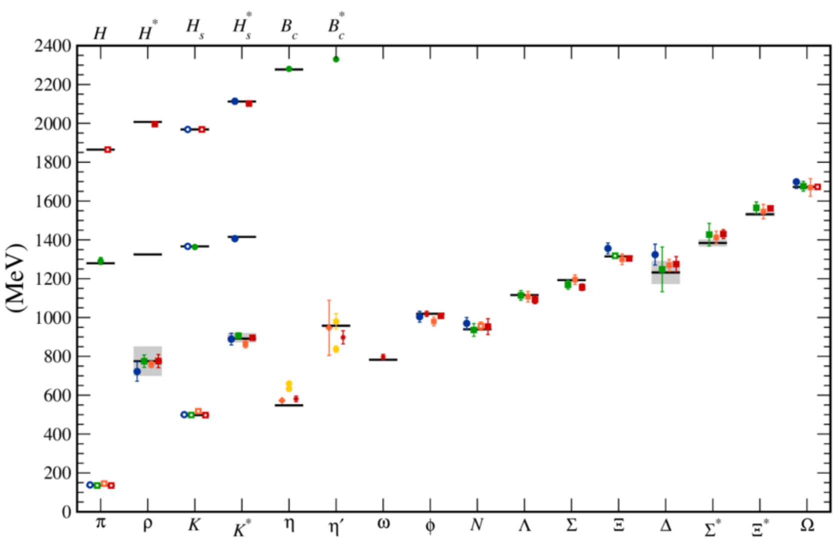
$$\lim_{\beta\to\infty} \mathrm{Tr}\left[\hat{\mathcal{O}}_q^\dagger(\tau)\hat{\mathcal{O}}_q(0)e^{-\beta\hat{H}}\right] = \sum_{n=0}^{\infty} |\langle\Omega|\hat{\mathcal{O}}_q(0)|n,q\rangle|^2 e^{-\tau E_{n,q}}$$



2013 snowmass report

Hadron spectroscopy with MC

$$\lim_{\beta\to\infty} \mathrm{Tr}\left[\hat{\mathcal{O}}_q^\dagger(\tau)\hat{\mathcal{O}}_q(0)e^{-\beta\hat{H}}\right] = \sum_{n=0}^{\infty} |\langle\Omega|\hat{\mathcal{O}}_q(0)|n,q\rangle|^2 e^{-\tau E_{n,q}}$$



2013 snowmass report

Problems:

- Need large time extent β and time separation τ
- Need large statistics to extract higher excited states

How to get spectrum by TN?

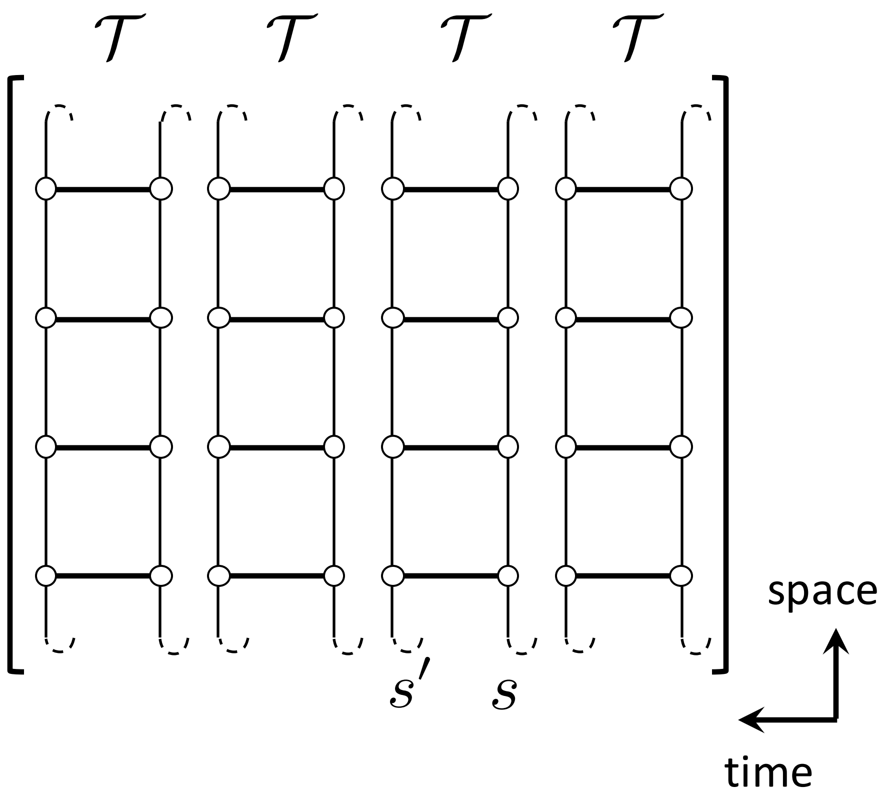
- Hamiltonian formalism

  ⇒ Matsumoto+Itou+Tanizaki 2023, 2024

- Lagrangian formalism
  - Two-point function
    - Large time extent and separation are easily realized
    - nothing new! (just do it)
  - Transfer matrix $\boxed{\text{We use here!}}$
    - No need to extrapolate time extent and separation

Transfer matrix

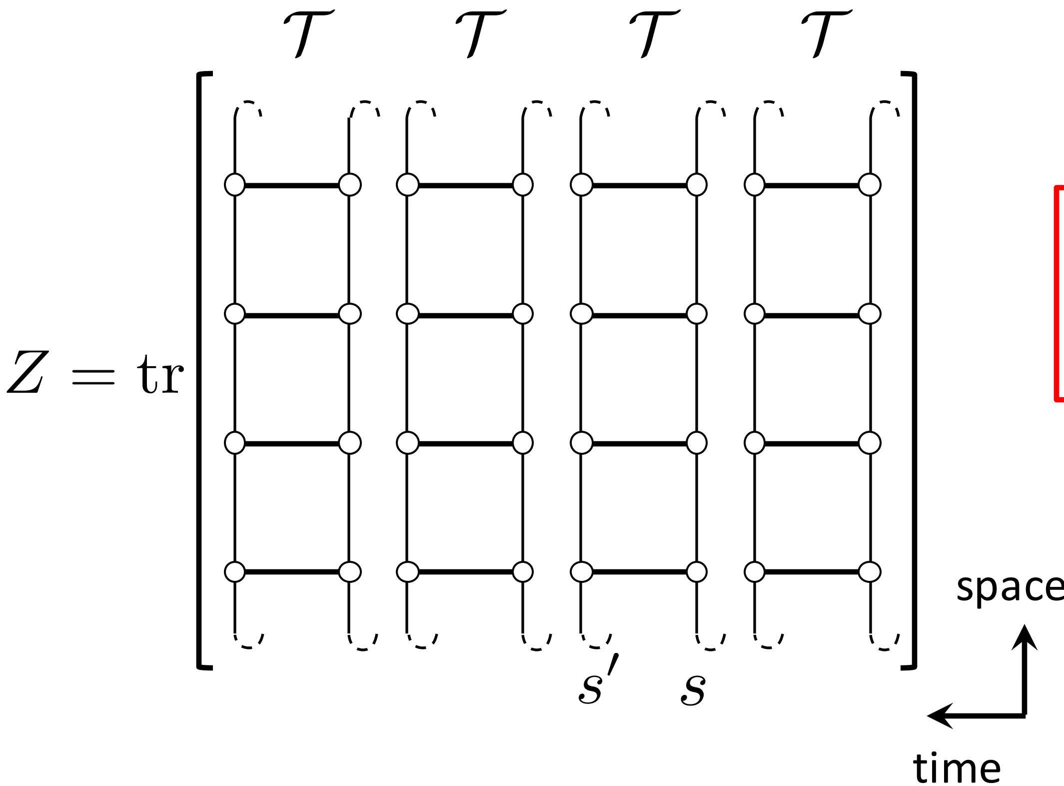
$Z = \mathrm{tr}$



$$\left(\mathcal{T}|a\rangle = e^{-\omega_a}|a\rangle \ \text{ for } \ a = 0, 1, 2\cdots\right)$$

$$\mathcal{T} \leftrightarrow e^{-\hat{H}} \quad \Rightarrow \quad \omega_a \leftrightarrow E_{n,q}$$

Transfer matrix

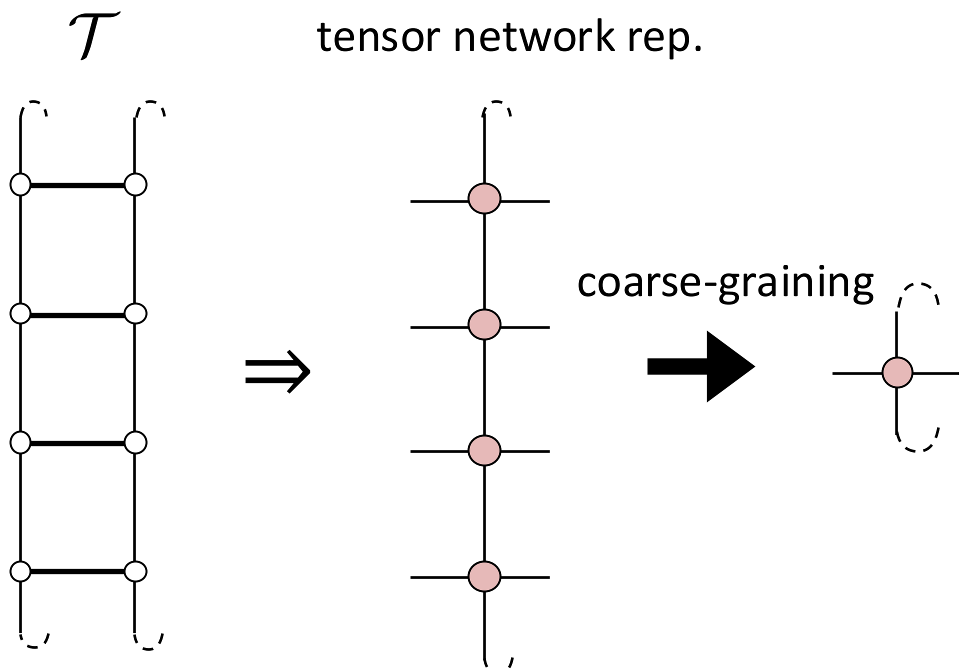


But, the size of the transfer matrix increases exponentially with the volume of the system

$$\left(\mathcal{T}|a\rangle = e^{-\omega_a}|a\rangle \ \text{ for } \ a = 0, 1, 2\cdots\right)$$

$$\mathcal{T} \leftrightarrow e^{-\hat{H}} \quad \Rightarrow \quad \omega_a \leftrightarrow E_{n,q}$$

Transfer matrix + Tensor network



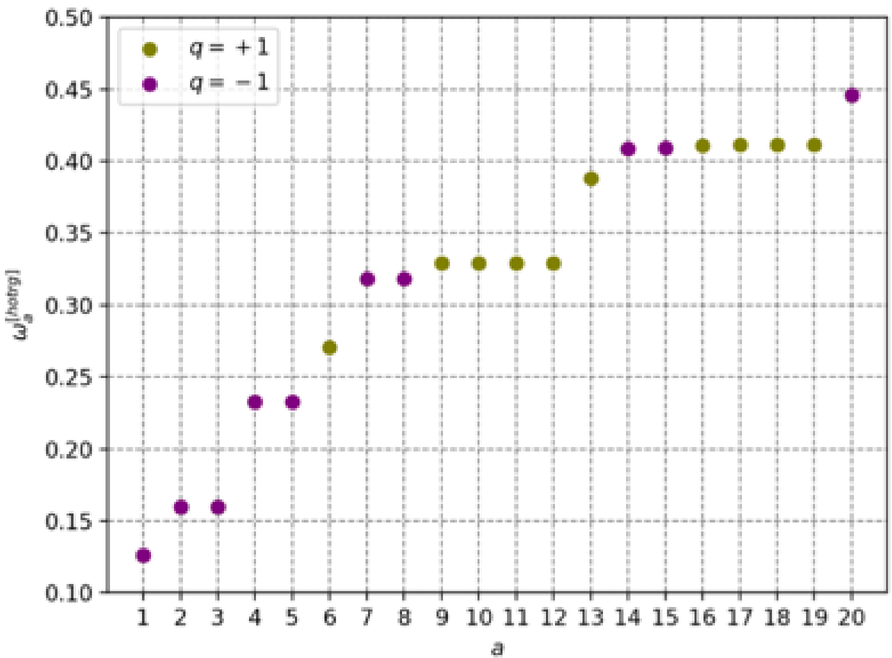
EVD

$$= Ue^{-\omega_a}U^\dagger$$

Get spectrum!

Spectroscopy for 1+1dim Ising

$T = 2.44 > T_{\mathrm{c}} \quad L = 64 \quad \chi = 80$

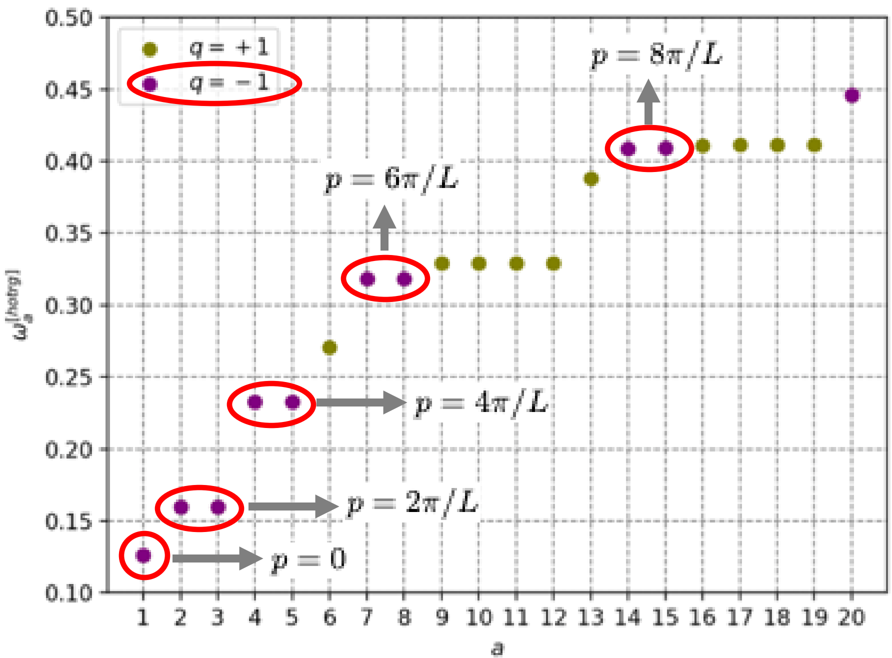


Quantum number $q = \pm 1$ is determined by selection rule

$$\langle \Omega | s | a \rangle \neq 0 \Longrightarrow q_a = -1$$

Spectroscopy for 1+1dim Ising

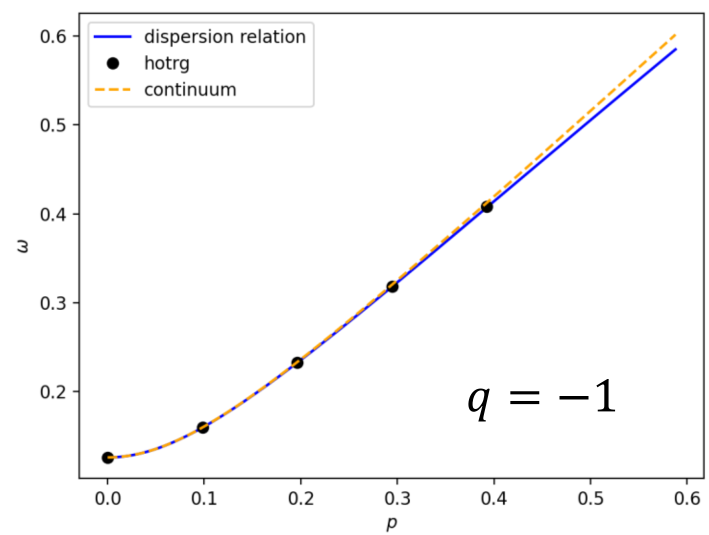
$T = 2.44 > T_c \quad L = 64 \quad \chi = 80$



Quantum number $q = \pm 1$ is determined by selection rule

$$\langle \Omega | s | a \rangle \neq 0 \Longrightarrow q_a = -1$$

The momentum is also determined by selection rule

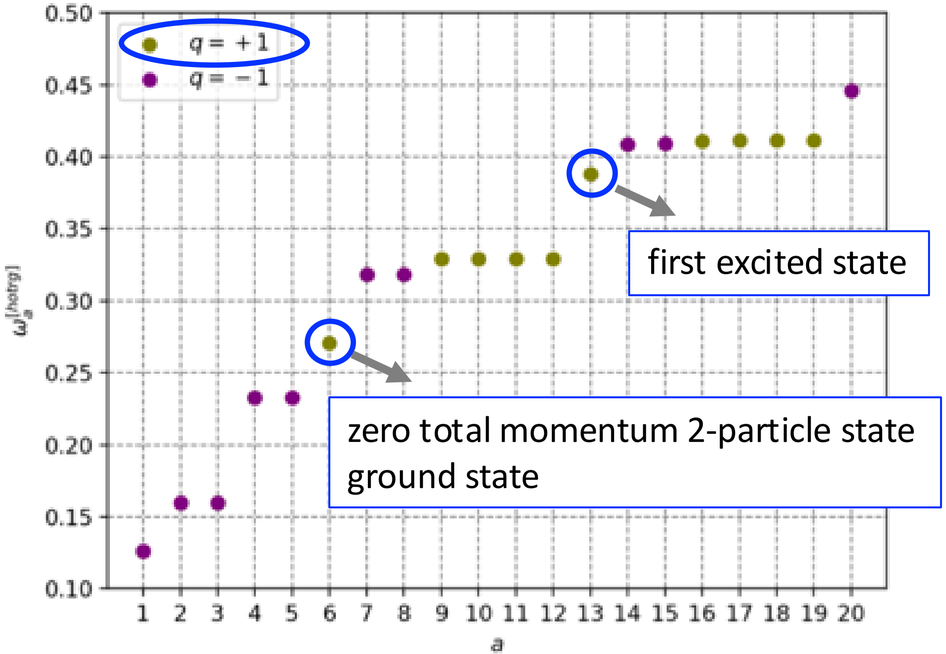


lattice: $\omega(p) = \cosh^{-1}(1 - \cos p + \cosh m)$

continuum: $\omega(p) = \sqrt{m^2 + p^2}$

Scattering phase shift

$T = 2.44 > T_c \quad L = 64 \quad \chi = 80$



Scattering phase shift

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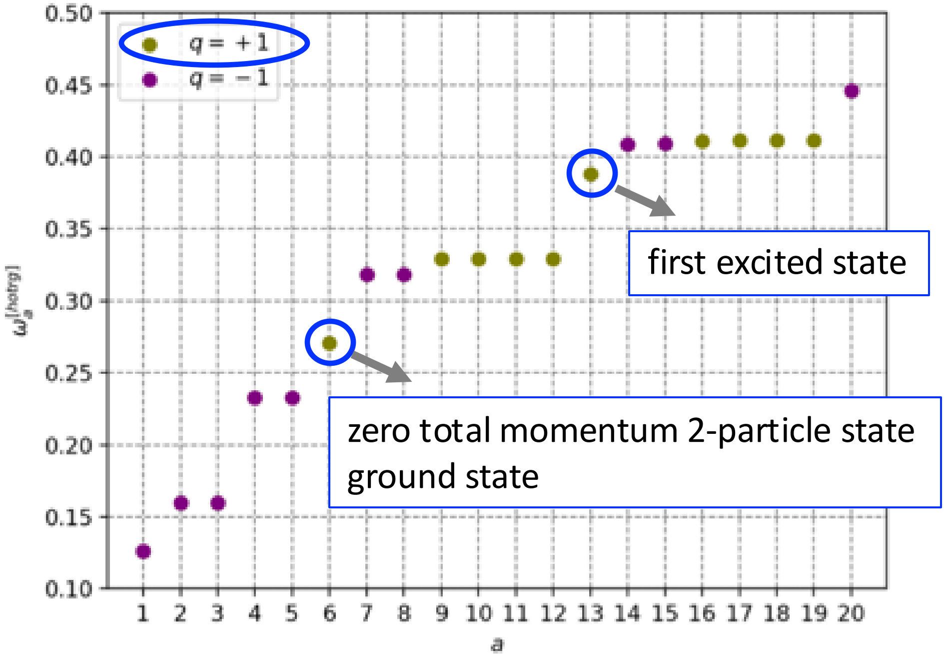


Get relative momentum from 2-particle energy

$$\omega = 2\sqrt{k^2 + m^2}$$

relative momentum

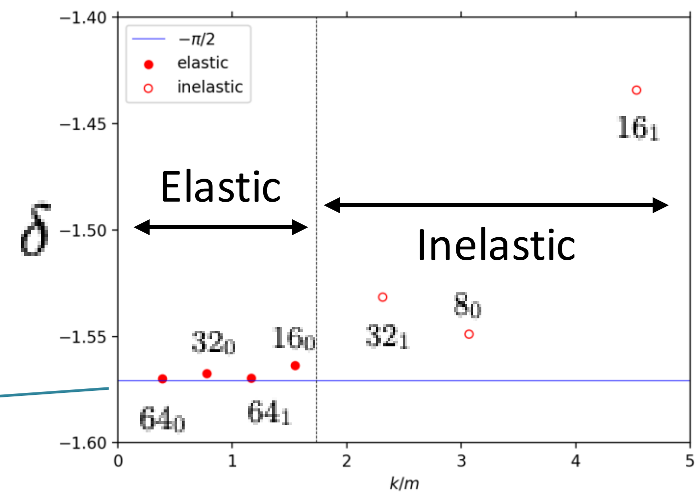
1-particle rest mass

Luescher's formula for 2dim

$$e^{i2\delta(k)} = e^{-ikL}$$

Luescher+Wolff 1990

phase shift



Elastic region

$$2m \leq \omega < 4m$$

$$\delta_{\text{Ising}} = -\frac{\pi}{2}$$

Gattringer+Lang 1993

Entanglement Entropy with TN

will appear in arXiv soon

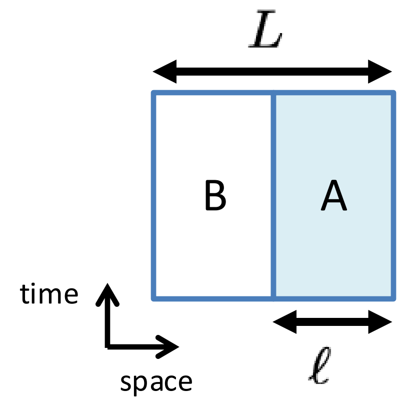
共同研究者
加堂大輔（明治学院大）
田中豪太（明治学院大）
早崎貴大（金沢大）

What and Why EE?

- Def : measure of quantum correlation between two subsystems A and B

$$S_A = -\mathrm{Tr}_A \rho_A \log \rho_A$$

$$\rho_A = \mathrm{Tr}_B \rho_{A+B}$$



- Quantum information
- A key quantity to understand BH Suskind+Uglum 1996
- EE can also be used as “order parameter” to study quantum phase transition Calabrese+Cardy 2004

Monte Carlo study

- 3+1dim pure SU(3) gauge theory Nakagawa+ 2009, Itou+ 2016
  - Order parameter of confinement
- 1+1, 2+1dim Ising Bulgarelli+Panero 2023
  - Computation of c-function

Renyi Entropy

replica trick

$$S_A^{(n)} = \frac{1}{1-n}\mathrm{Tr}_A(\rho_A)^n \xrightarrow{n \longrightarrow 1} S_A$$

Difficulties in MC

- extrapolation $n \rightarrow 1$ OK?
- To realize zero temperature, one needs large Euclidean time

$$\rho_{A+B} \longrightarrow |0\rangle\langle 0| \quad \text{for} \quad T \longrightarrow 0$$

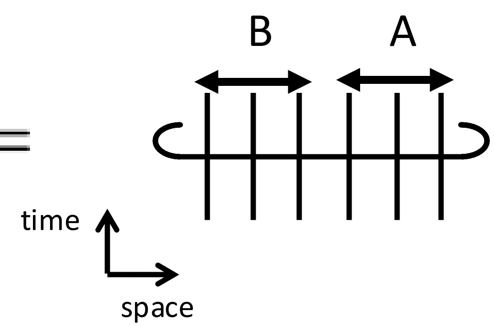
(previous) Tensor network study

EE can be directly computed without relying on replica method

- TN representation of ρ_A

Bazavov+ 2016

Tensor network rep. of
Transfer matrix $=$



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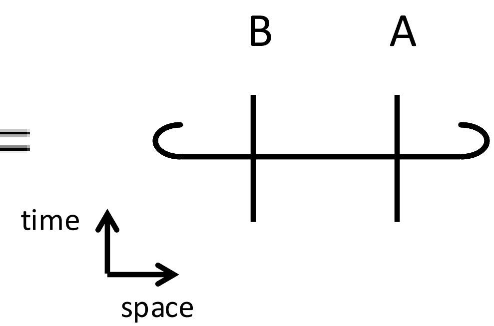
Coarse-grained Transfer matrix $=$



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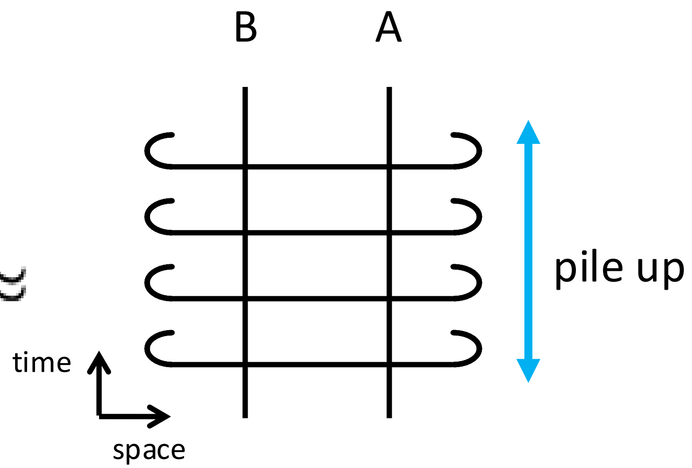
$$\rho_{A+B} \approx$$



(previous) Tensor network study

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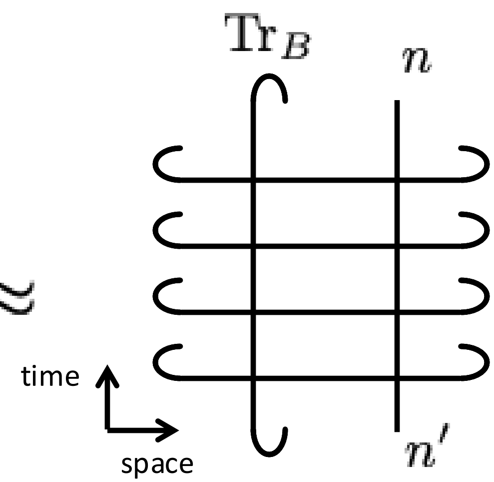
Bazavov+ 2016

$$(\rho_A)_{n,n'} = \mathrm{Tr}_B \rho_{A+B} \approx$$



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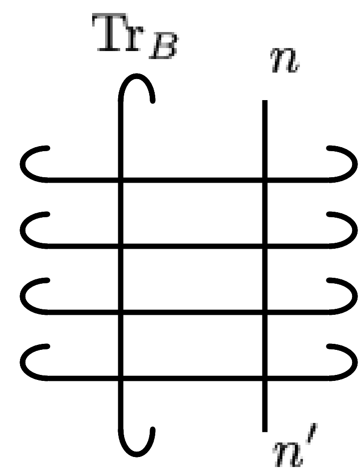

$$\Rightarrow \quad S_A = -\mathrm{Tr}_A \rho_A \log \rho_A \approx -\sum_i \lambda_i \log \lambda_i$$

(previous) Tensor network study

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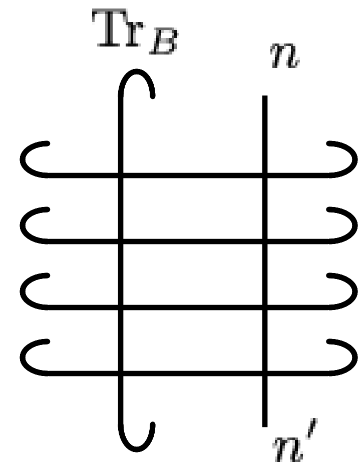

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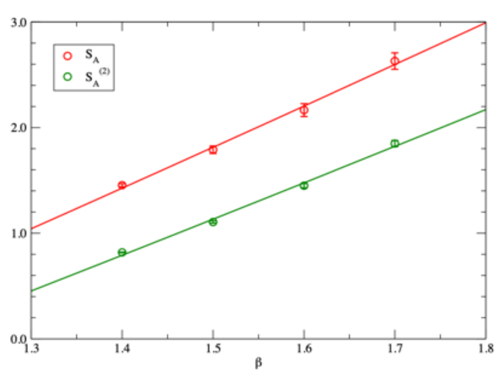
- 1+1dim O(3) Luo+Kuramashi 2023



for off-critical

central charge

$$S_A = \frac{c}{3} \log \xi$$

correlation length

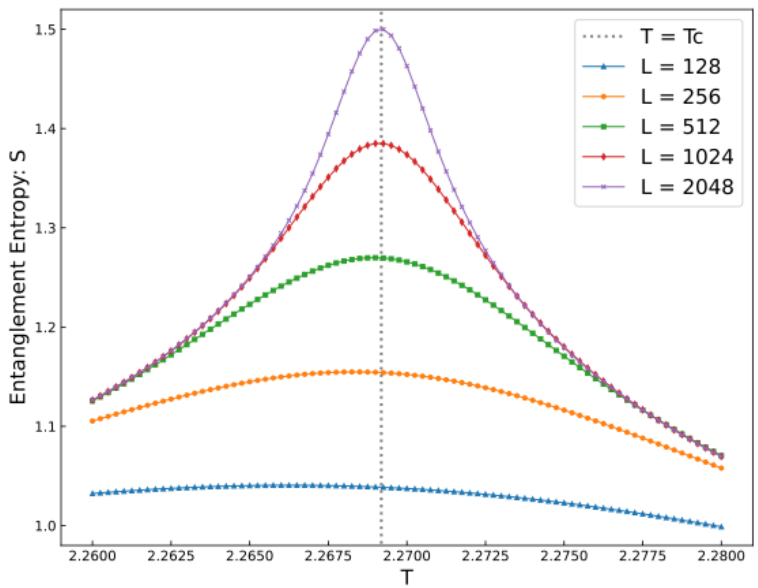
Hasenfratz+1990

$$\xi^{-1} \propto \beta e^{-2\pi\beta}$$

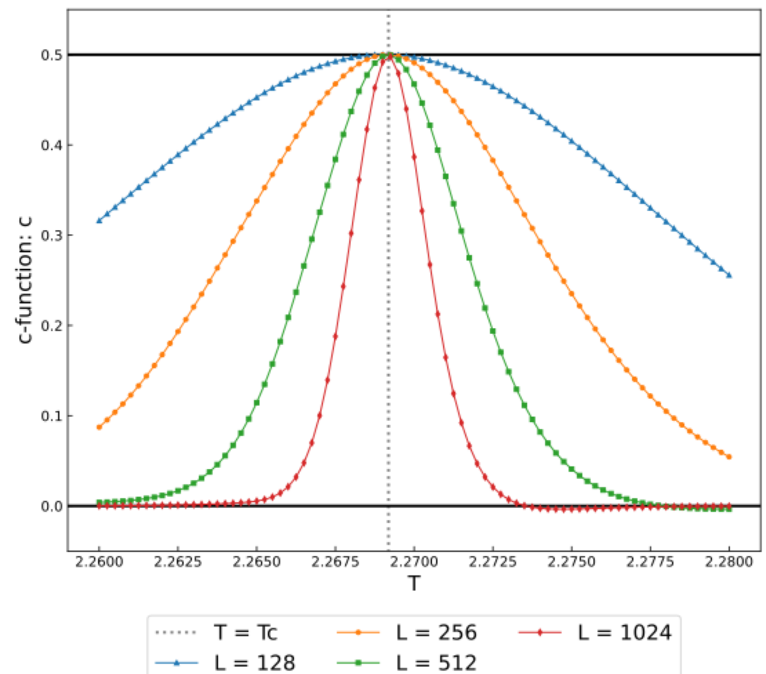
$\Rightarrow$ c=1.97(9)

(Our) Tensor network study

1+1D Ising
$\chi = 96$
$\ell/L = 1/2$
time/space=16



$$c = 3\frac{\partial S_A}{\partial(\log L)}$$



$$S_A = \frac{c}{3} \times \begin{cases} \log \xi & \text{for } T \neq T_c \\ \log L & \text{for } T = T_c \end{cases}$$

Transition point and value of central charge can be precisely estimated

Summary

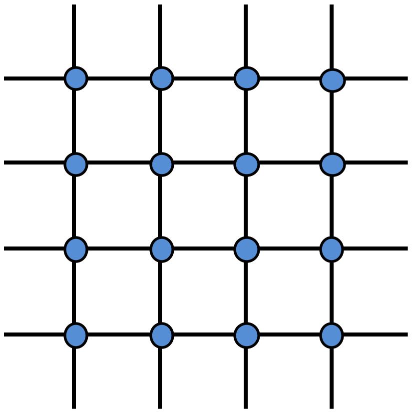
- Tensor network is free of sign problem
- For Lagrangian approach, key point of coarse-graining scheme is information compression based on singular value decomposition
- Improvement of coarse-graining algorithm is essentially important and currently developing
- 4D system with simple d.o.f. is now feasible, but the road to lattice QCD is still long

- As topical topics, we demonstrate the spectroscopy and computation of EE by TRG method
- TN can explore areas where MC cannot

Backup

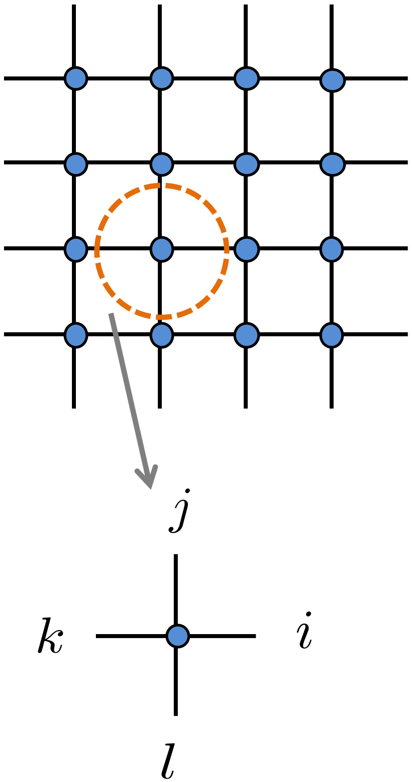
Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)



Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)



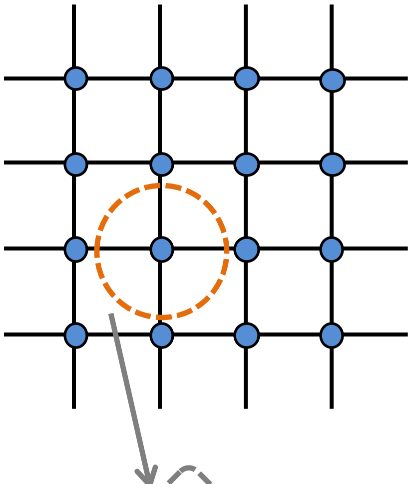
Bond dimension

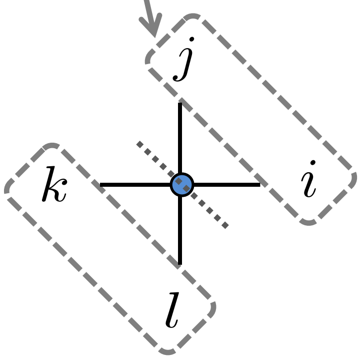
$$1 \leq i, j, \ldots \leq \chi$$

$$\Leftrightarrow \quad T_{ijkl}$$

Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)





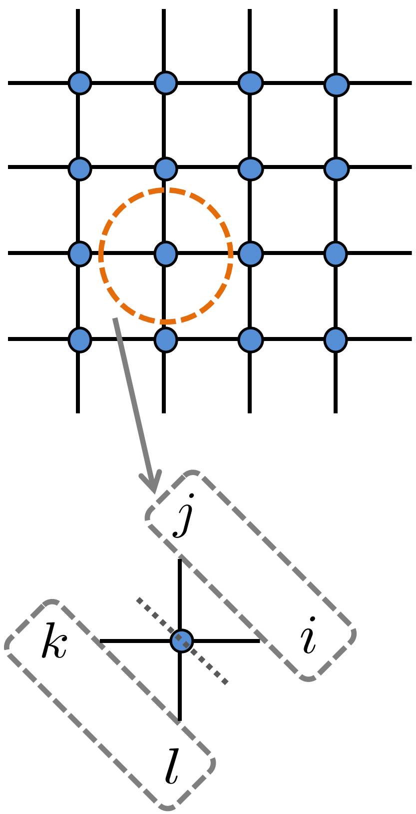
$$1 \leq i, j, \ldots \leq \chi$$

$\chi^2 \times \chi^2\,\mathrm{matrix}$

$$\Leftrightarrow \quad T_{ijkl} = M_{(ij)(kl)}$$

Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)



$$M \in \mathbb{C}^{\chi^2 \times \chi^2}$$

Singular Value Decomposition(SVD)

$$M_{ab} = \sum_m U_{am} \sigma_m (V^\dagger)_{mb}$$

unitary matrix

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$$

: singular value (non-negative)

$$\Leftrightarrow \quad T_{ijkl} = M_{(ij)(kl)}$$

Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)

$M \in \mathbb{C}^{\chi^2 \times \chi^2}$ ⇒ TN is sign-problem-free

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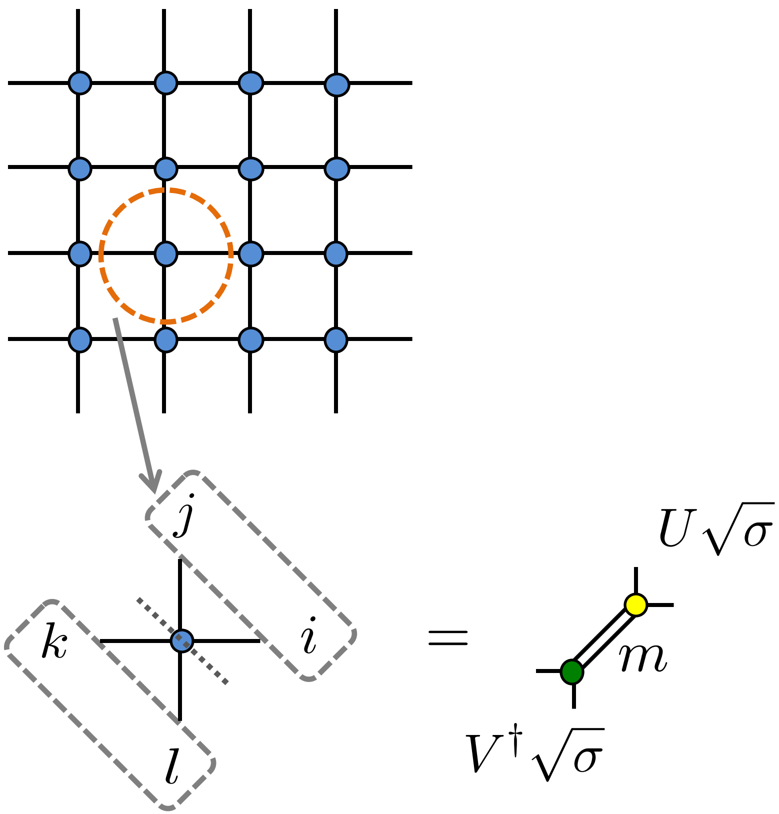


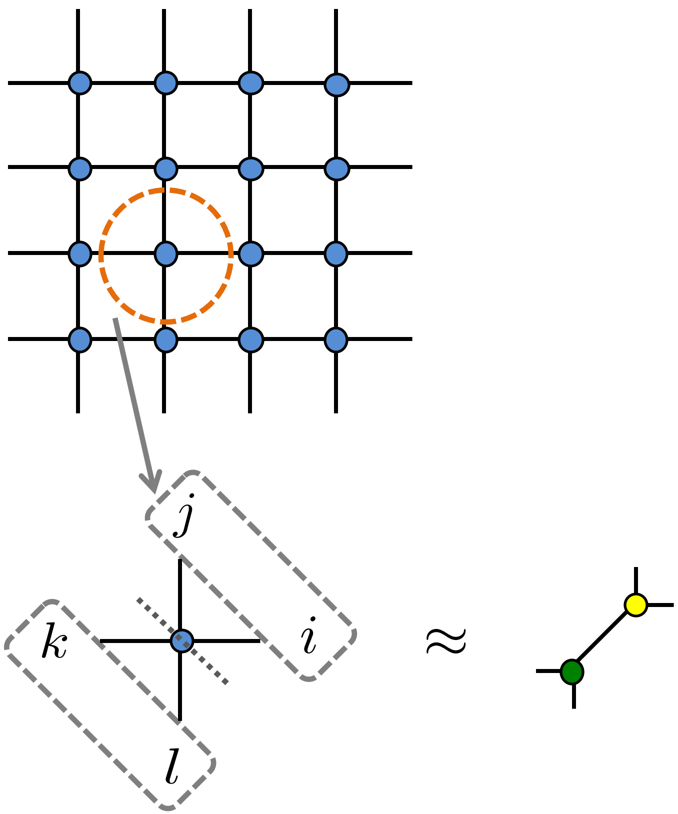
$$\Leftrightarrow \quad T_{ijkl} = M_{(ij)(kl)}$$

$$\overset{\text{SVD}}{=} \sum_{m=1}^{\chi^2} U_{(ij)m}\sigma_m V^\dagger_{m(kl)}$$

Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)

$M \in \mathbb{C}^{\chi^2 \times \chi^2}$ ⇒ TN is sign-problem-free



Singular Value Decomposition(SVD)

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unitary matrix

$$\sigma_1 \geq \sigma_2 \geq \dots \geq 0$$

: singular value (non-negative)

$$\Leftrightarrow \quad T_{ijkl} = M_{(ij)(kl)}$$

truncation

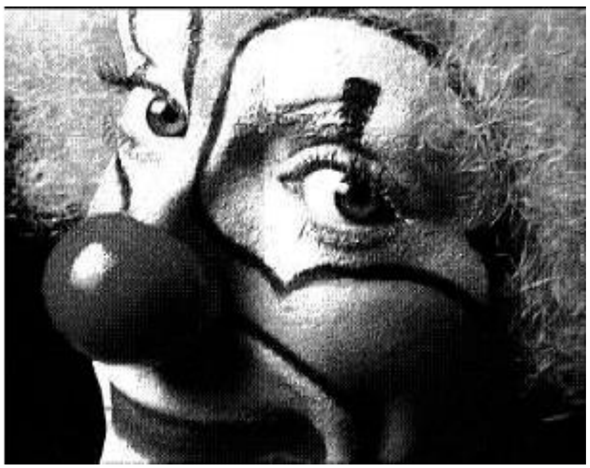
$$\approx \sum_{m=1}^{\chi} U_{(ij)m} \sigma_m V^\dagger_{m(kl)}$$

truncation of SVD = information compression

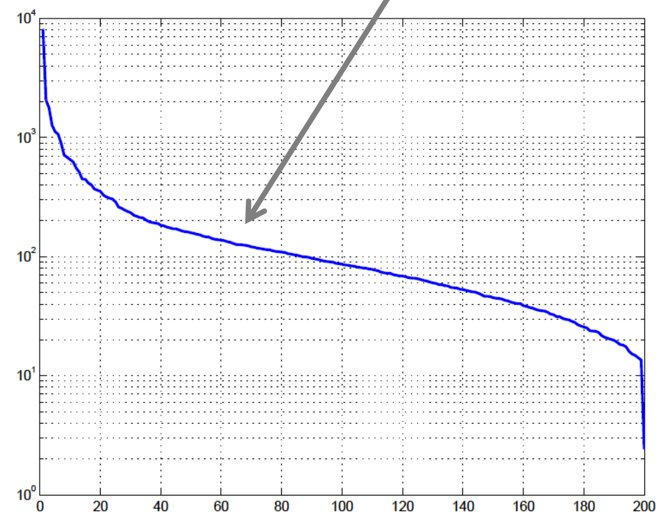
Image compression

$$|U_{am}| \leq 1$$

200 x 320 pixels B&W photograph
= 200 x 320 real matrix → $M_{ab} = \sum_m U_{am} \underline{\sigma_m} (V^\dagger)_{mb}$



σ_m

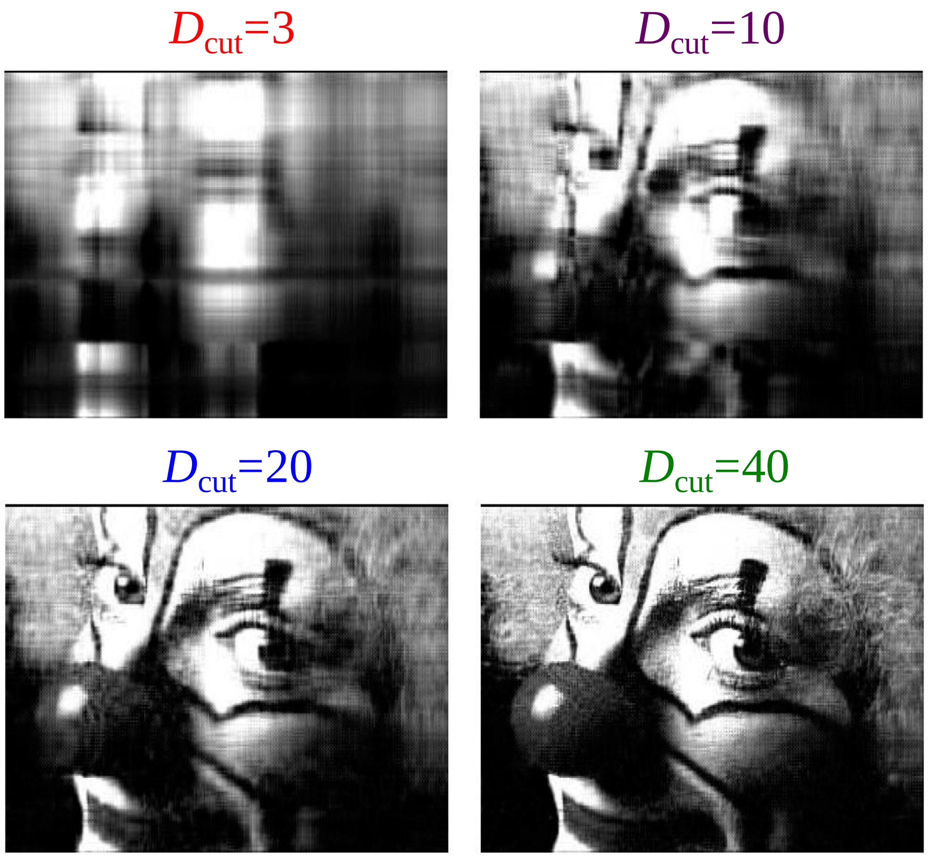


numbering m

http://www.na.scitec.kobe-u.ac.jp/~yamamoto/lectures/cse-introduction2009/cse-introduction090512.PPT

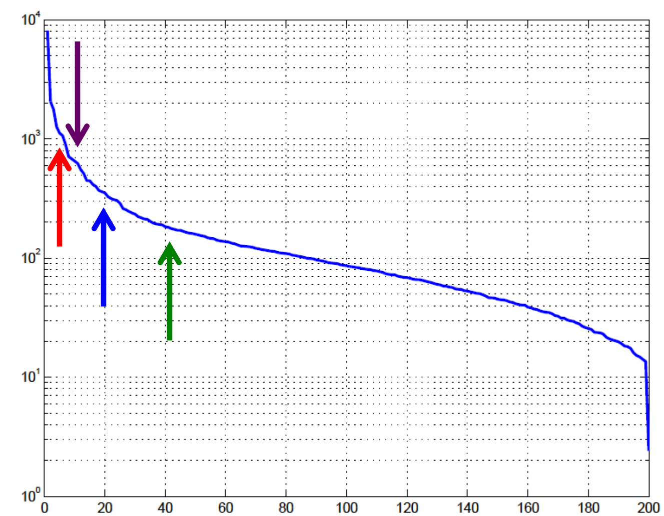
Image compression

$$D_{\mathrm{cut}} = \chi$$



$$M_{ab} \approx \sum_{m=1}^{D_{\mathrm{cut}}} U_{am}\sigma_m (V^\dagger)_{mb}$$

σ_m



numbering m

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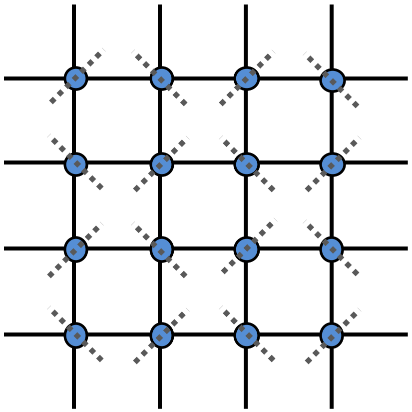
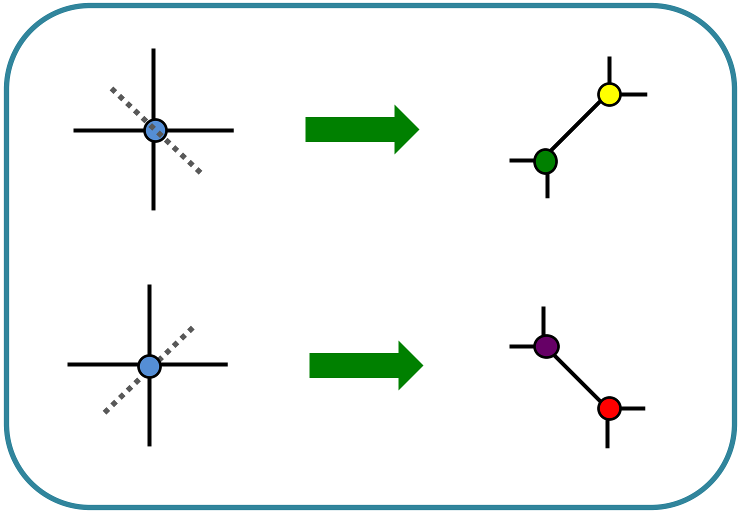
Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)





truncated SVD

Coarse-graining

Tensor renormalization group (TRG)
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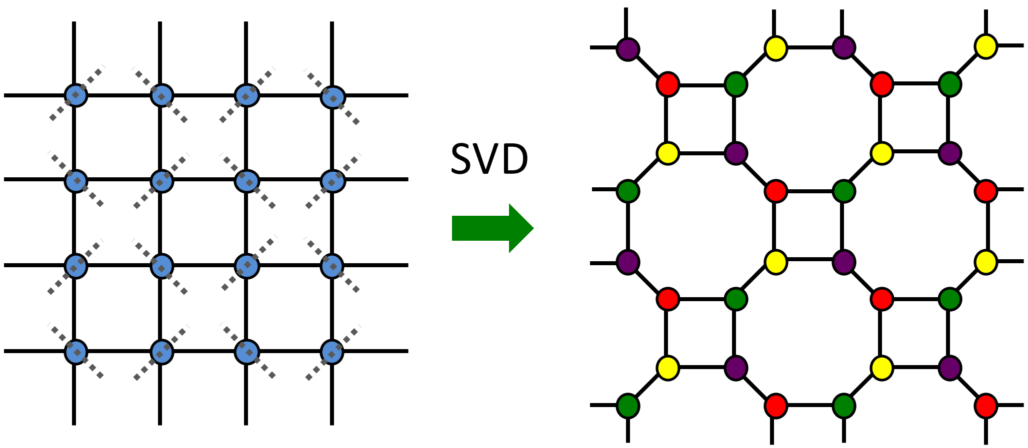

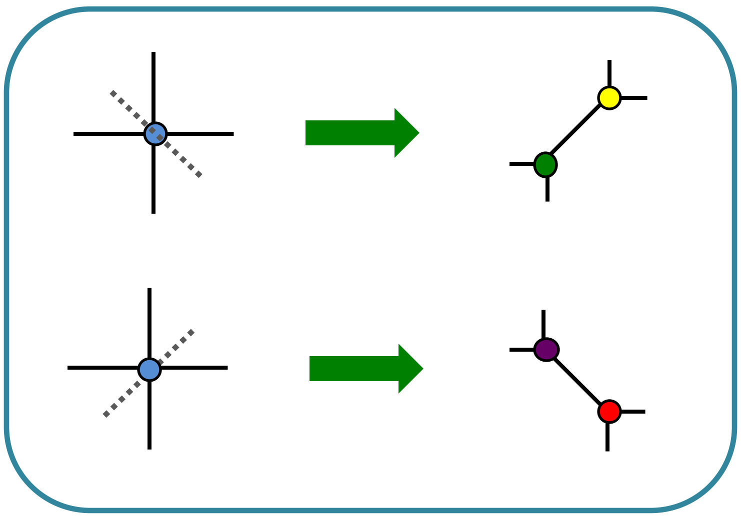

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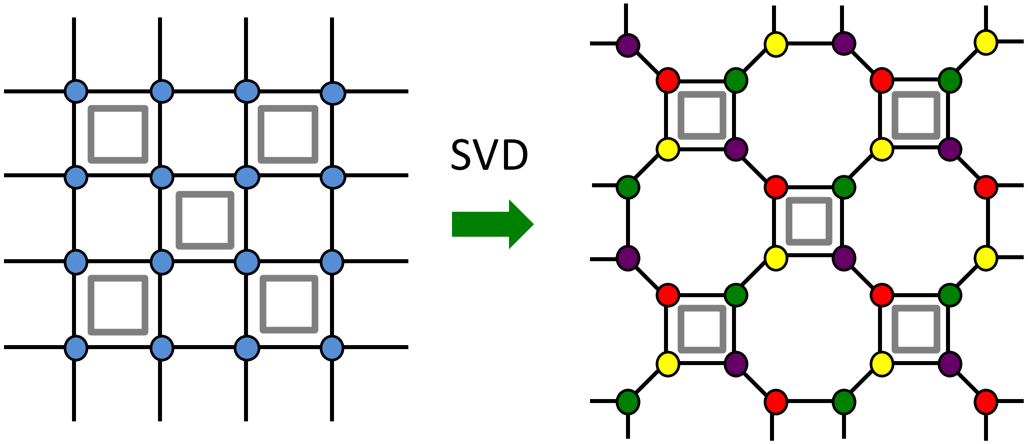

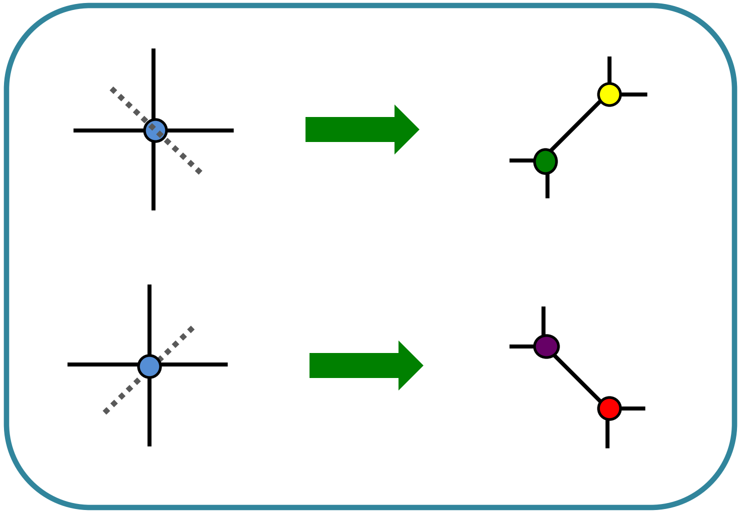

truncated SVD

Coarse-graining

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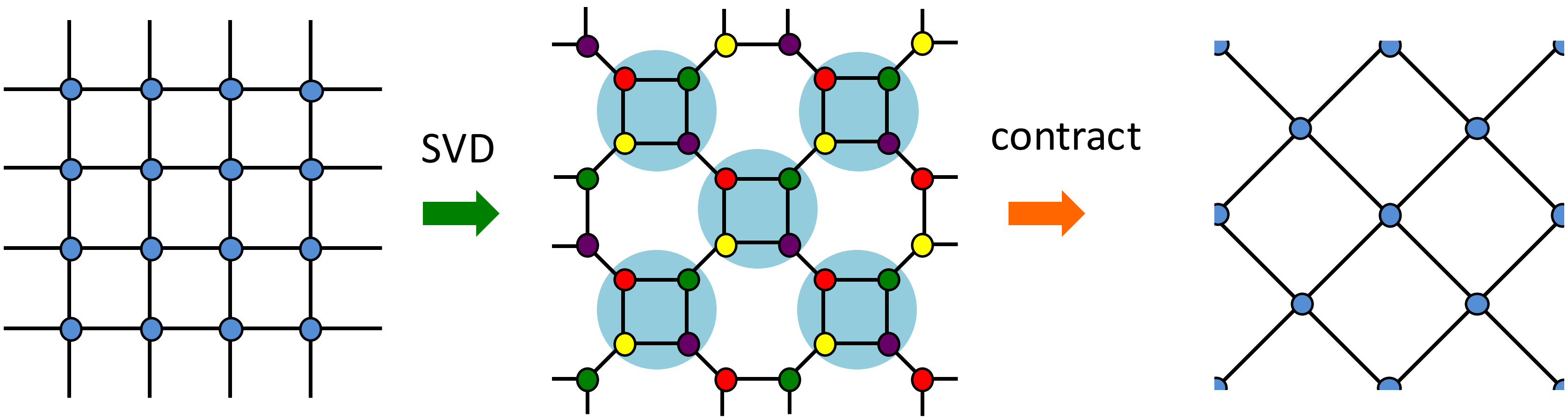
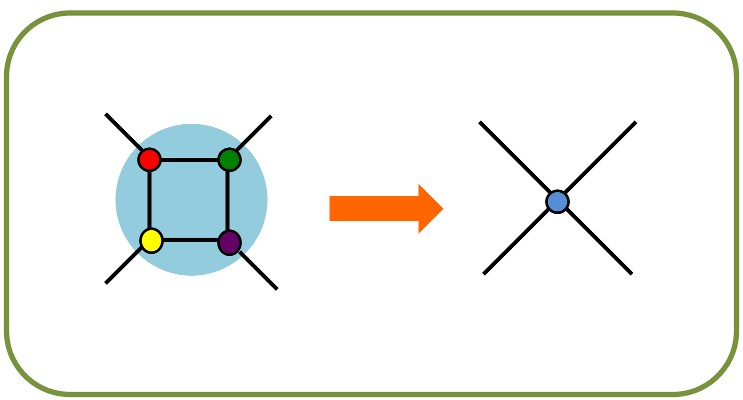


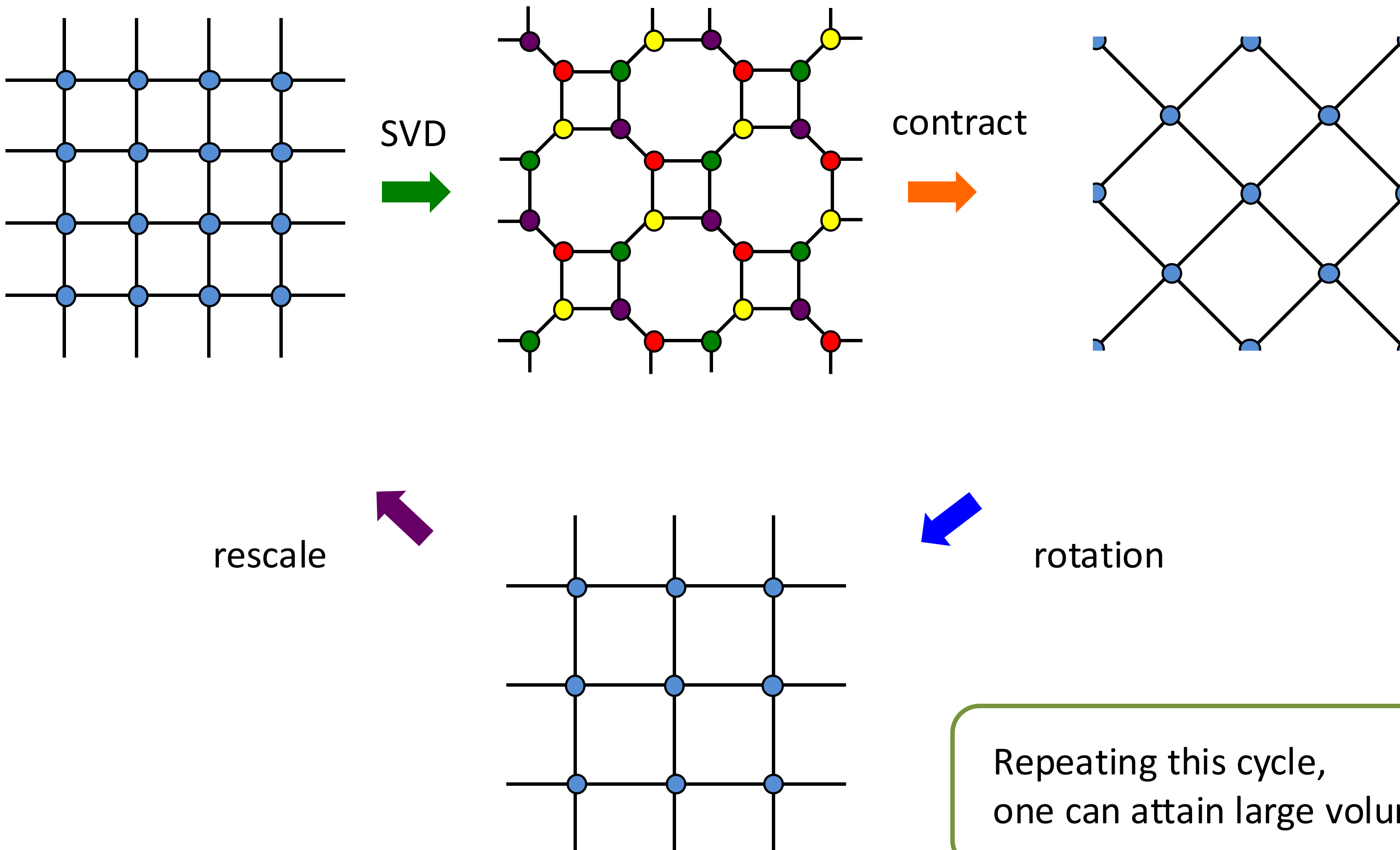
contraction



renormalization!

Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)



Repeating this cycle,
one can attain large volume

Road to lattice QCD

- 4-dimensional space-time
  - Efficient coarse-graining scheme even for higher dimensional system
- Color
  - “Armillary sphere” Yosprakob 2023, Yosprakob+Okunishi 2024
- Flavor
  - Yosprakob+ 2023, Akiyama 2023

Tensor network rep. of Z

depends on property of field and interaction

- Scalar field (non-compact)
  - Orthonormal basis expansion

    Shimizu mod.phys.lett. A27,1250035(2012), Lay & Rundnick PRL88,057203(2002)
  - Gauss Hermite quadrature Sakai et al., JHEP03(2018)141

- Gauge field (compact : SU(N), CP(N) etc.)
  - Character expansion : maintain symmetry, better convergence

    Meurice et al., PRD88,056005(2013)

- Fermion field (Dirac/Majorana)

  Shimizu & Kuramashi PRD90,014508(2014), ST & Yoshimura PTEP(2015)043B01
  - Grassmann number $\theta^2=0$ -> finite sum

$$e^{\phi\theta} = 1 + \phi\theta = \sum_{n=0}^{1} (\phi\theta)^n$$

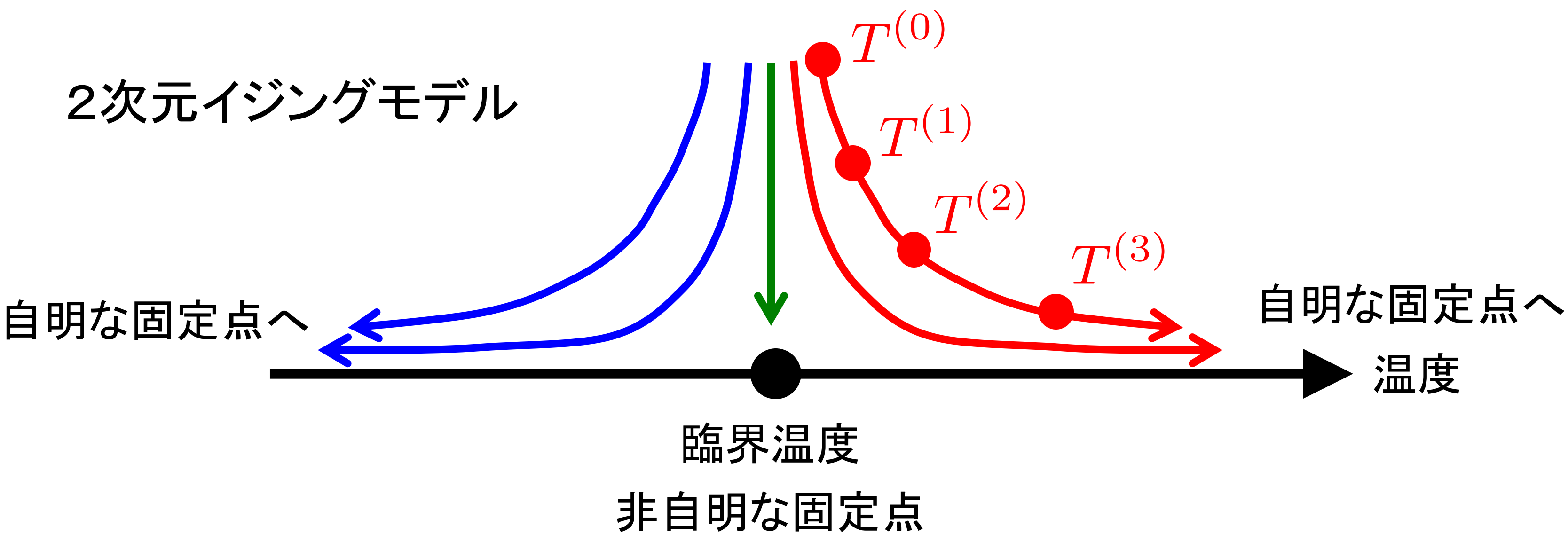
In principle, we can treat any fields

テンソルのフロー

ある温度での
初期テンソル

$$T^{(0)} \xrightarrow{\mathcal{R}} T^{(1)} \xrightarrow{\mathcal{R}} T^{(2)} \xrightarrow{\mathcal{R}} T^{(3)} \xrightarrow{\mathcal{R}} \cdots$$

２次元イジングモデル



高温相の自明な固定点テンソル

初期テンソル

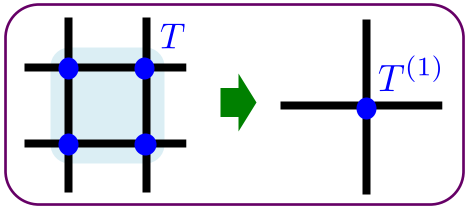
高温極限 β → 0

$$T = \begin{bmatrix} 1 & 0 & 0 & \tanh\beta \\ 0 & \tanh\beta & \tanh\beta & 0 \\ 0 & \tanh\beta & \tanh\beta & 0 \\ \tanh\beta & 0 & 0 & (\tanh\beta)^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

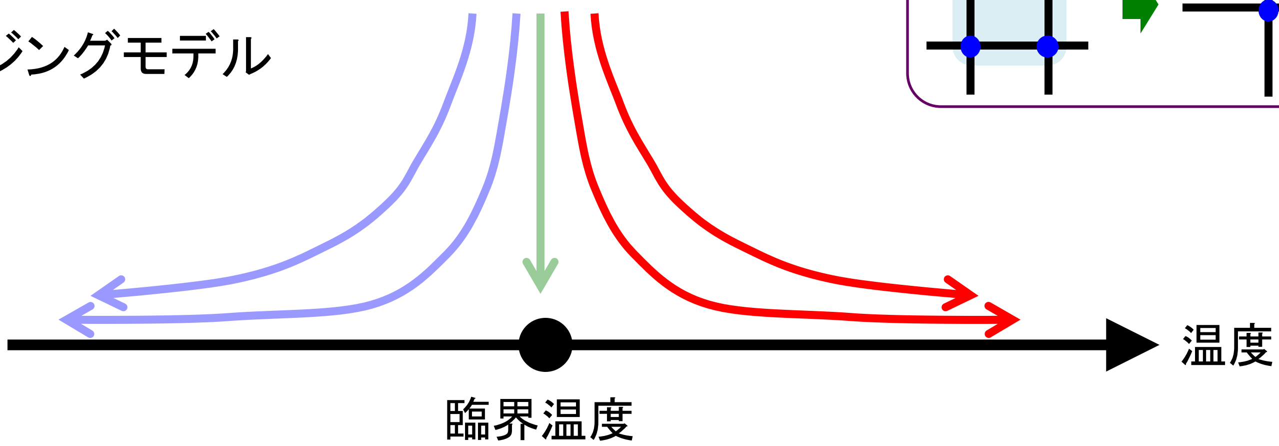
1成分

$$S_{\mathrm{ent}} = 0$$

高温相の自明な固定点テンソル



２次元イジングモデル



低温相の自明な固定点テンソル

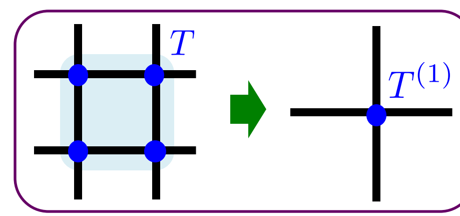
低温極限 $\beta \to \infty$

初期テンソルに適当
な変換を施してから

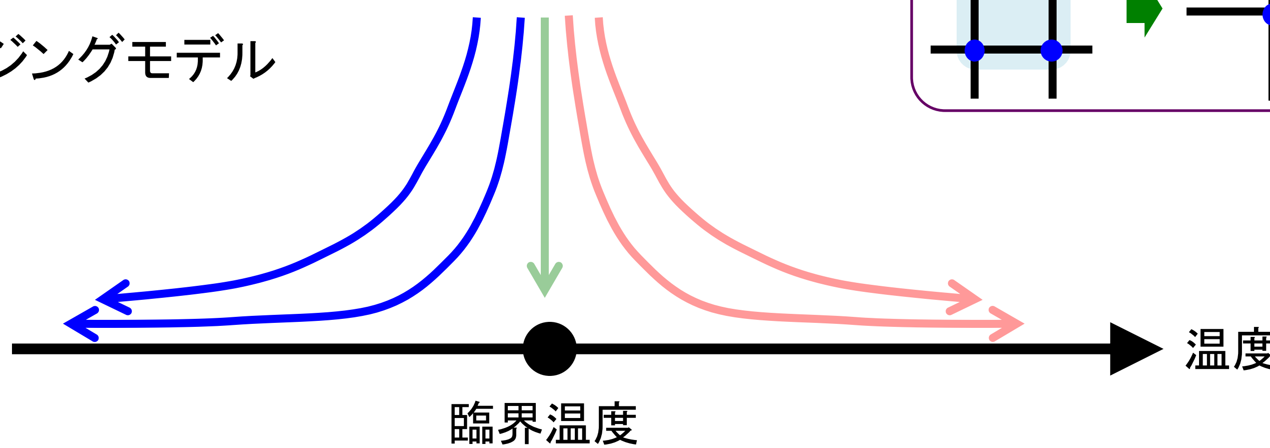
$$T \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

２成分 $\longrightarrow$ $S_{\mathrm{ent}} = \log 2$

低温相の自明な固定点テンソル



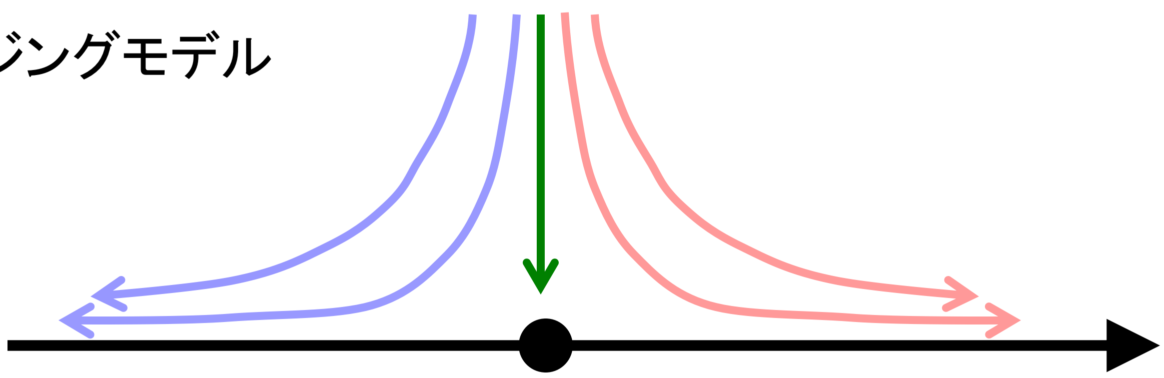
２次元イジングモデル



臨界温度の非自明な固定点テンソルは？

その場合、解析的にテンソルを調べることはできない。
非自明な固定点テンソルは無限次元？
数値的にしか解析できない（有限次元で）

２次元イジングモデル



MC法とTN法(TRG)の比較

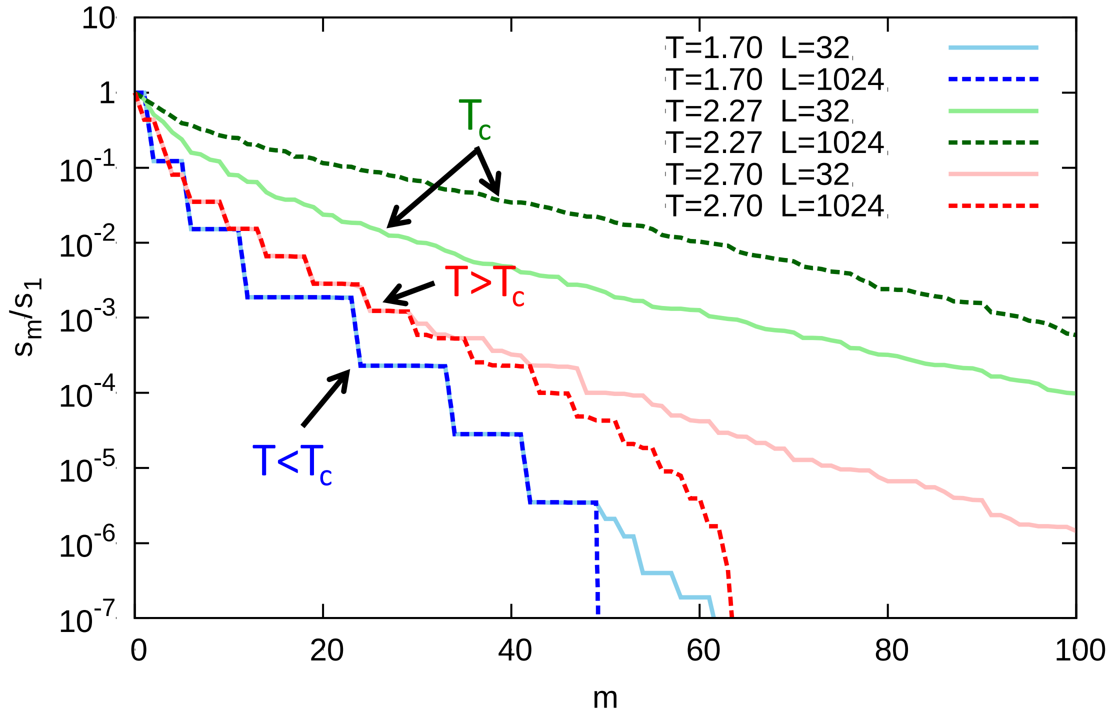
	MC法	TN法(TRG)
分配関数の扱い方	ボルツマン因子を確率として扱う	分配関数をテンソルネットワーク表示する
効率性を確保するための技法	重点サンプリング	特異値分解による情報圧縮
誤差	統計誤差	系統誤差 D_{cut}
符号問題(複素作用問題)	あり	なし(∵確率解釈がない)
その他の問題点	臨界減速	臨界点近傍で情報圧縮の効率が低下

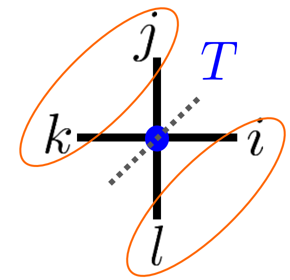
クラスターアルゴリズム等で解決

２次元であれば、克服可能
Evenbly & Vidal 2014, Gu et al., 2015

Hierarchy of singular value

2D Ising model D_{cut}=32





$$T_{(kj)(il)} = \sum_m u_{(kj)m}\sigma_m v_{m(il)}$$

$$L = 2^{\#\mathrm{RG\ step}}$$

$$\begin{aligned} T_c &= 2/[\ln(1+\sqrt{2})] \\ &= 2.269\ldots \end{aligned}$$

- Off criticality: good hierarchy (small S)
- Near criticality: hierarchy gets worse (large S)

like critical slowing down in MC

Tensor network renormalization (TNR) Evenbly&Vidal 2014 can help the situation

Renormalization group

$$H(K;\{s\}) = \sum_i K_i \mathcal{O}_i(\{s\})$$

s : spins

$$\mathcal{Z} = \sum_{\{s\}} e^{-\beta H}$$

Block spin transf., Migdal-Kadanoff RG

$$H = \sum_i K_i \mathcal{O}_i \xrightarrow{\mathcal{R}} H' = \sum_i K'_i \mathcal{O}'_i$$

$$K' = \mathcal{R}_K(K)$$

$K^* = \mathcal{R}_K(K^*)$: fixed point

target : critical exponent etc

TRG

$$T_{ijkl}(K)$$

i, j, k, l : indexes

$$\mathcal{Z} = \sum_{i,j,k,l,\ldots} \prod T_{ijkl} \cdots$$

SVD + contraction

$$\sum_{i,j,k,l,\ldots} \prod T_{ijkl} \cdots \xrightarrow{\mathcal{R}} \sum_{i,j,k,l,\ldots} \prod T'_{ijkl} \cdots$$

$$T' = \mathcal{R}_T(T)$$

$T^* = \mathcal{R}_T(T^*)$: fixed point tensor

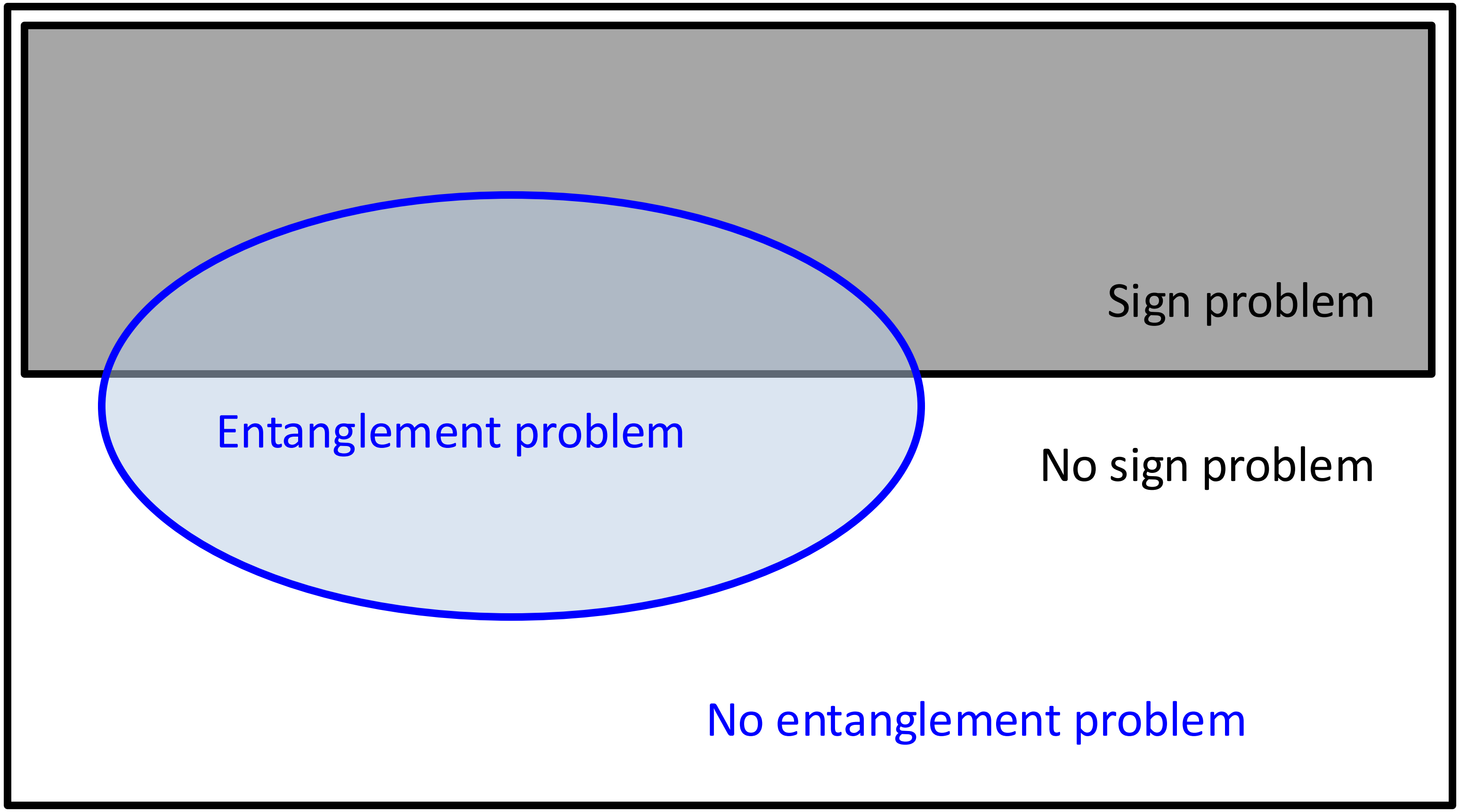
target : partition function etc

Kadanoff et al, Rev.Mod.Phys.86,647(2014)

From point of view of MC

From point of view of TN

Space of theory



Is there a region where any method cannot access ?

Is entanglement problem NP-hard ?

SVD

For simplicity

rank-k $m \times n$ real matrix A ($m \geq n \geq k$) is given by

$$A = U \Lambda V^T$$ (full SVD)

$$\begin{cases} U : m \times m \text{ orthonormal matrix} : U=(u_1, u_2, \ldots, u_m), \quad U^T U = U U^T = I_m \\ V : n \times n \text{ orthonormal matrix} : V=(v_1, v_2, \ldots, v_n), \quad V^T V = V V^T = I_n \\ \Lambda : m \times n \text{ diagonal matrix} : \Lambda = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \end{cases}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > \sigma_{k+1} = \ldots = \sigma_n = 0$)

$$A = \sum_{l=1}^{k} \sigma_l u_l v_l^T$$ Decomposition using rank-1 tensor

Low-rank approximation

- ランク r (< k) 以下のすべての行列の中で、ターゲットの行列 A に「最も近い」行列 X は？

$$\min_{X \in \mathbb{R}^{m \times n}, \mathrm{rank}(X) \leq r} ||A - X||$$

- 「最も近い」＝フロベニウスノルム $||A\text{-}X||_{\mathrm{F}}$ が最小

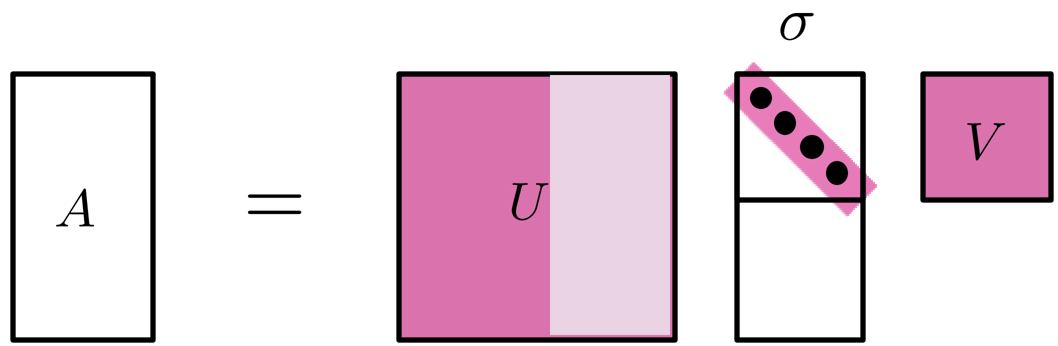
$$||Y||_{\mathrm{F}}^2 \equiv \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^2 = \mathrm{tr}\left[YY^T\right]$$

以後はこのノルムを使う

- 具体的な「最も近い」行列は？

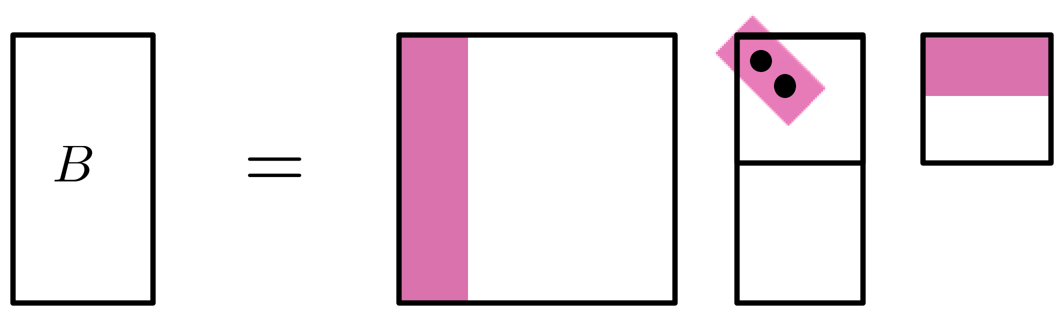
Best approximation of A ?

$$A = \sum_{l=1}^{k} \sigma_l u_l v_l^T$$



truncated at r

$$B = \sum_{l=1}^{r} \sigma_l u_l v_l^T$$



Proof of Eckart & Young theorem

ランクが $r\ (< k)$ の行列 X を考え、$||A\text{-}X||_{\mathrm{F}}$ を最小化すればよい

$$
\begin{aligned}
||A - X||_{\mathrm{F}}^2 &= \mathrm{tr}\left[(A - X)(A - X)^T\right] \\
&= \mathrm{tr}\left[UU^T(A - X)VV^T(A - X)^T\right] \\
&= \mathrm{tr}\left[U^T(A - X)V(U^T(A - X)V)^T\right] \\
&= \mathrm{tr}\left[(\Lambda - G)(\Lambda - G)^T\right] \\
&= \sum_{i=1}^{k}(\sigma_i - g_{ii})^2 + \sum_{i=k+1}^{n} g_{ii}^2 + \sum_{i \neq j} g_{ij}^2
\end{aligned}
$$

$$
\boxed{\begin{aligned} UU^T &= I_m \\ VV^T &= I_n \end{aligned}}
$$

$$
\boxed{\begin{aligned} \Lambda &= U^T AV \\ G &= U^T XV \end{aligned}}
$$

G と X の特異値は同じ

Proof of Eckart & Young theorem

ランクが $r\ (< k)$ の行列 X を考え、$||A\text{-}X||_\mathrm{F}$ を最小化すればよい

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&= \mathrm{tr}\left[UU^T(A-X)VV^T(A-X)^T\right] \\
&= \mathrm{tr}\left[U^T(A-X)V(U^T(A-X)V)^T\right] \\
&= \mathrm{tr}\left[(\Lambda-G)(\Lambda-G)^T\right] \\
&= \sum_{i=1}^{k}(\sigma_i-g_{ii})^2+\sum_{i=k+1}^{n}g_{ii}^2+\sum_{i\neq j}g_{ij}^2
\end{aligned}
$$

$$
\boxed{\begin{aligned} UU^T &= I_m \\ VV^T &= I_n \end{aligned}}
$$

$$
\boxed{\begin{aligned} \Lambda &= U^TAV \\ G &= U^TXV \end{aligned}}
$$

G と X の特異値は同じ

ランク r で、$||A\text{-}X||_\mathrm{F}$ を最小化する G は、$g_{ii}=\sigma_i\ (i=1,\ldots,r)$ 以外の要素はすべてゼロ

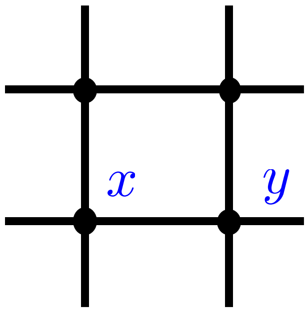
$$G=\mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_r,0,\ldots,0) \Rightarrow X=UGV^T=\sum_{l=1}^{r}\sigma_l u_l v_l^T$$

Tensor network rep. for 2D Ising

e.g. 2D Ising model

$$\mathcal{Z} = \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y\right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y)$$

Nearest Neighbor



Tensor network rep. for 2D Ising

e.g. 2D Ising model

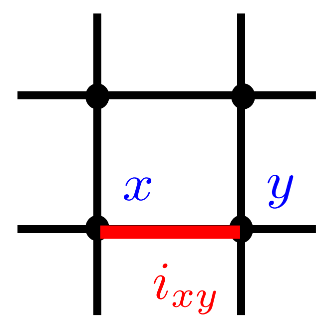
$$
\begin{aligned}
\mathcal{Z} &= \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y \right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y) \\
&= (\cosh\beta)^{2V} \sum_{\{s\}} \prod_{<x,y>} \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}}
\end{aligned}
$$

V = # of lattice sites

$$
\begin{aligned}
\exp(\beta s_x s_y) &= \cosh(\beta s_x s_y) + \sinh(\beta s_x s_y) \\
&= \cosh\beta + s_x s_y \sinh\beta \\
&= \cosh\beta (1 + s_x s_y \tanh\beta) \\
&= \cosh\beta \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}}
\end{aligned}
$$

$s_x = \pm 1$

new d.o.f.



bond variable

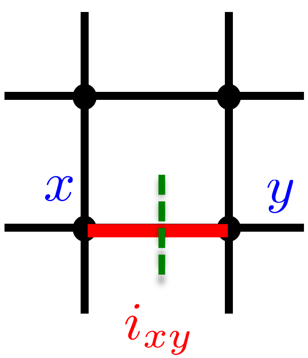
Tensor network rep. for 2D Ising

e.g. 2D Ising model

$$\mathcal{Z} = \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y\right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y)$$

$$= (\cosh\beta)^{2V} \sum_{\{s\}} \prod_{<x,y>} \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}}$$

$$= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{<x,y>} (s_x \sqrt{\tanh\beta} \cdot s_y \sqrt{\tanh\beta})^{i_{xy}}$$



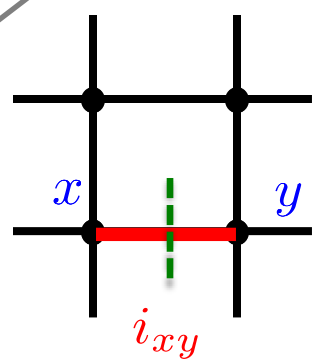
Tensor network rep. for 2D Ising

e.g. 2D Ising model

$$
\begin{aligned}
\mathcal{Z} &= \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y\right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y) \\
&= (\cosh\beta)^{2V} \sum_{\{s\}} \prod_{<x,y>} \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{<x,y>} (s_x \sqrt{\tanh\beta} \cdot s_y \sqrt{\tanh\beta})^{i_{xy}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{x} (s_x \sqrt{\tanh\beta})^{i_{xy}} (s_x \sqrt{\tanh\beta})^{i_{xz}} (s_x \sqrt{\tanh\beta})^{i_{xw}} (s_x \sqrt{\tanh\beta})^{i_{xv}}
\end{aligned}
$$

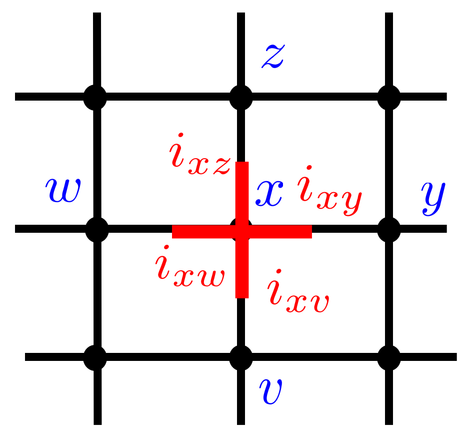
$$\prod_{<x,y>}$$

bond



$$\prod_{x}$$

lattice point



Tensor network rep. for 2D Ising

e.g. 2D Ising model

$$
\begin{aligned}
\mathcal{Z} &= \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y\right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y) \\
&= (\cosh\beta)^{2V} \sum_{\{s\}} \prod_{<x,y>} \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{<x,y>} (s_x\sqrt{\tanh\beta} \cdot s_y\sqrt{\tanh\beta})^{i_{xy}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{x} (s_x\sqrt{\tanh\beta})^{i_{xy}} (s_x\sqrt{\tanh\beta})^{i_{xz}} (s_x\sqrt{\tanh\beta})^{i_{xw}} (s_x\sqrt{\tanh\beta})^{i_{xv}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{x} (\sqrt{\tanh\beta})^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} s_x^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \prod_{x} (\sqrt{\tanh\beta})^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} \sum_{s_x=\pm 1} s_x^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \prod_{x} \boxed{(\sqrt{\tanh\beta})^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} 2\delta(\mathrm{mod}(i_{xy}+i_{xz}+i_{xw}+i_{xv}, 2))}
\end{aligned}
$$

summation is done

$$= T_{i_{xy} i_{xz} i_{xw} i_{xv}}$$

new d.o.f. : index of tensor

Tensor network rep. for 2D Ising

e.g. 2D Ising model

$$\mathcal{Z} = 2^V (\cosh\beta)^{2V} \sum_{..,i,j,k,l,m,n,o,..} \cdots T_{ijkl} T_{mnio} \cdots$$

$$T_{ijkl} = (\sqrt{\tanh\beta})^{i+j+k+l} \delta(\mathrm{mod}(i+j+k+l), 2)$$

$$\begin{bmatrix} T_{0000} & T_{0001} & T_{0010} & T_{0011} \\ T_{0100} & T_{0101} & T_{0110} & T_{0111} \\ T_{1000} & T_{1001} & T_{1010} & T_{1011} \\ T_{1100} & T_{1101} & T_{1110} & T_{1111} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \tanh\beta \\ 0 & \tanh\beta & \tanh\beta & 0 \\ 0 & \tanh\beta & \tanh\beta & 0 \\ \tanh\beta & 0 & 0 & (\tanh\beta)^2 \end{bmatrix}$$

size and elements of tensor depend on a model

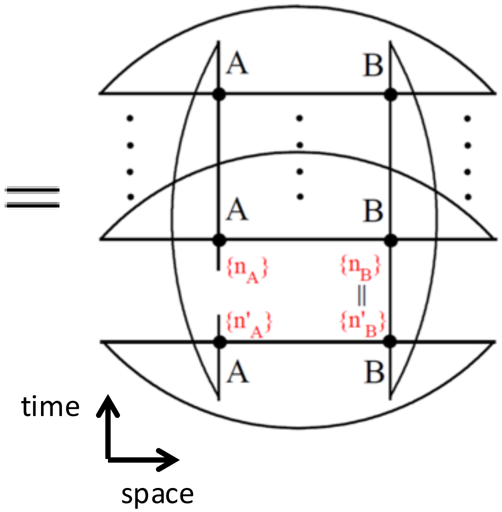
(previous) Tensor network study

EE can be directly computed without relying on replica method

- TN representation of ρ_A

Bazavov+ 2016

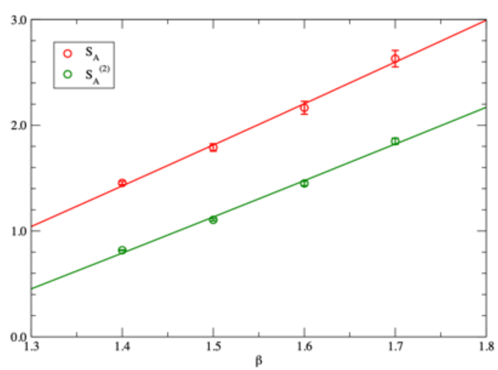
$$(\rho_A)_{\{n_A\},\{n'_A\}} =$$



$$\Rightarrow \quad S_A = -\mathrm{Tr}\rho_A \log \rho_A = -\sum_i \lambda_i \log \lambda_i$$

Luo+Kuramashi 2023

- 1+1dim O(3)



for off-critical

central charge

$$S_A = \frac{c}{3} \log \xi \qquad \xi^{-1} \propto \beta e^{-2\pi\beta}$$

correlation length

$\Rightarrow$ c=1.97(9)

Kuramashi+Luo 2023

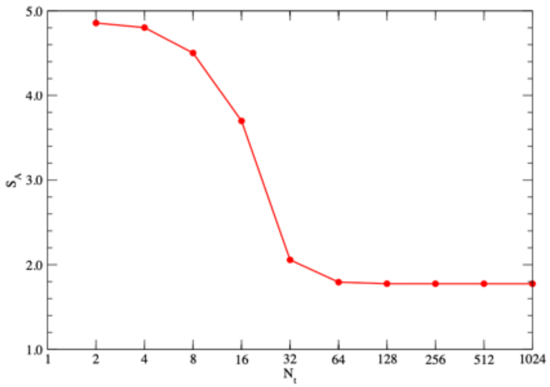


Figure 3: N_t dependence of entanglement entropy at $\beta = 1.5$. The bond dimension is $D_{\mathrm{cut}} = 130$.

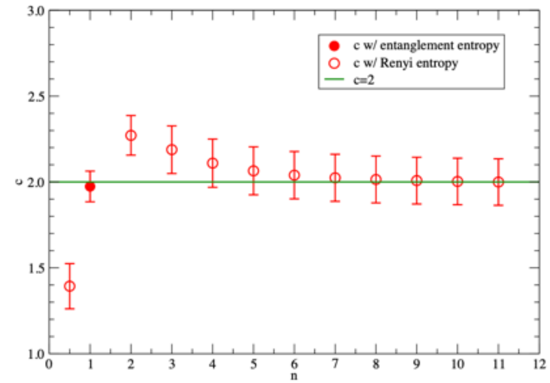


Figure 11: n dependence of central charge c obtained from nth-ordr Rényi (open) and entanglement (closed) entropies. Solid line denotes $c = 2$ to guide your eyes.