# The nature of the high temperature phase of Yang-Mills theory:

***Night thoughts on the Polyakov loop***

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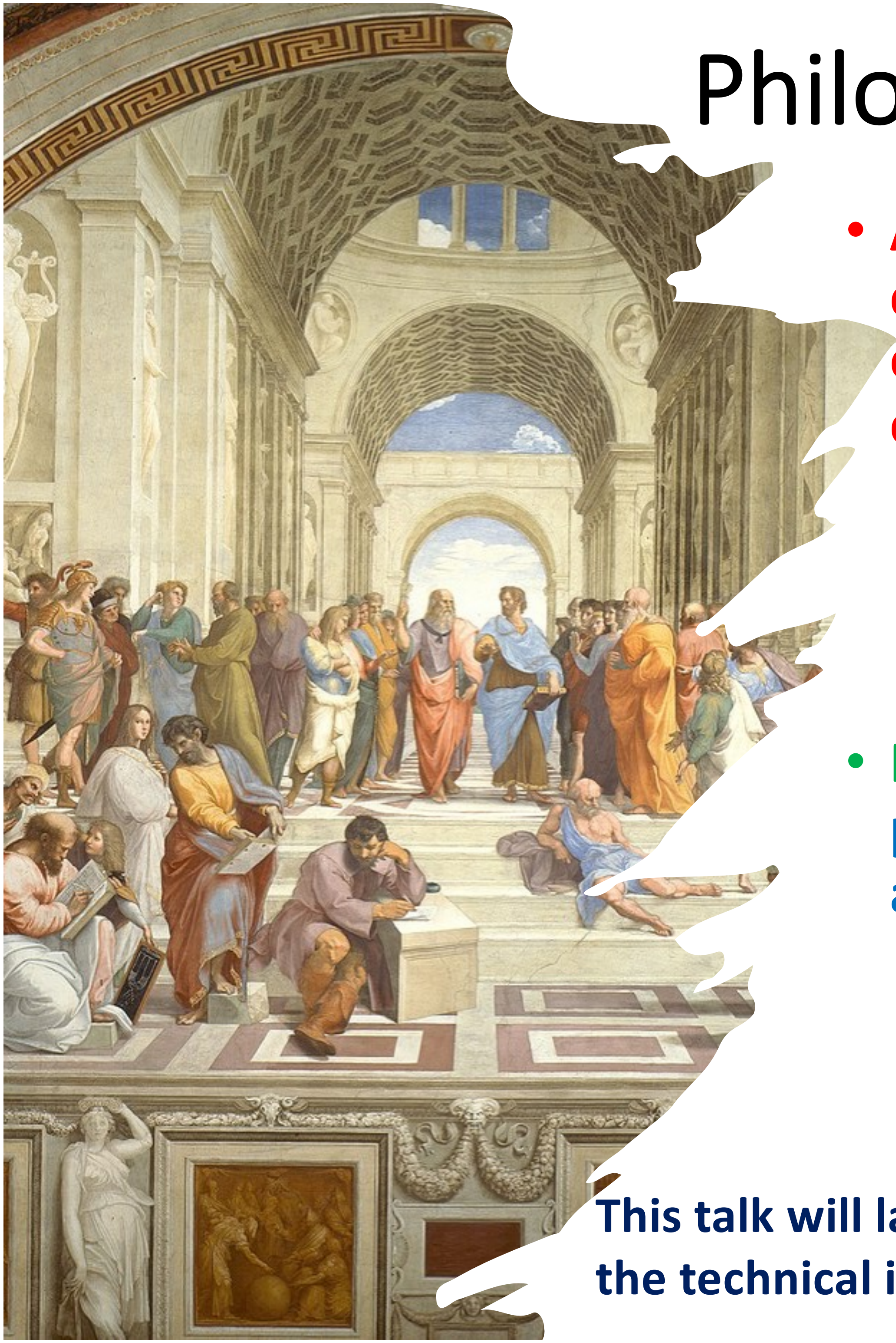
*Paper in preparation*

U.S. DEPARTMENT OF ENERGY | Office of Science

- **Yang-Mills theory has first-order phase transition. Conventional wisdom: low-temperature phase confining and the high temperature phase deconfined.**
  - **Principal reason for belief that high temperature phase is deconfined: analysis based on the Polyakov loop.**

- **Paper explores possibility that Polyakov loop has been misinterpreted and that the high temperature phase is also confining.**

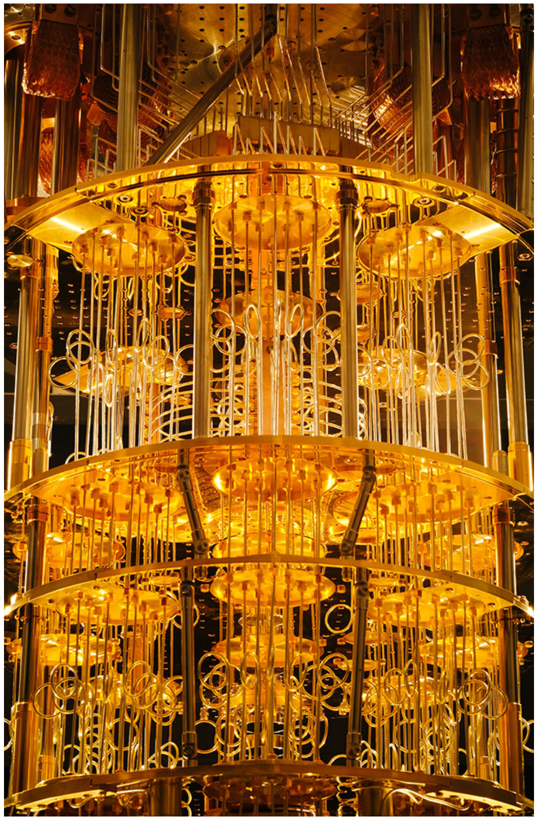


# Philosophical Issues

- **At best,  Polyakov loop and its correlators show that quarks (or other d.o.f carrying fundamental color charges) are deconfined.**
  - **Yang-Mills  has no d.o.f. carrying fundamental color charges.  The real issue of confinement involves the gluons!**
- **Related philosophical issue: order parameters based on PL do not apply to QCD.**
  - **Lack of free quarks in nature—described by QCD—led to postulation of confinement in the first place**

**This talk will largely ignore these issues and focus on the technical issues associated with the Polyakov loop.**

Technical issue are serious

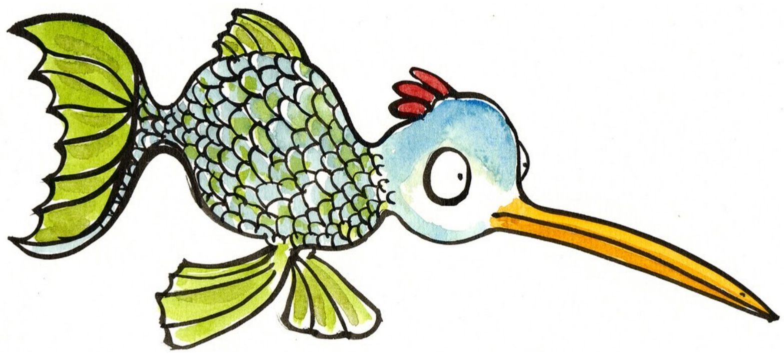


Recent work on quantum computing has acted refocus attention on

- Hamiltonian formulations of gauge theories.
- Real time dynamics

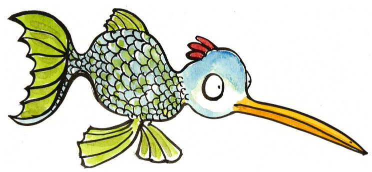
This forces one to confront technical issues with Polyakov loop observables

- **Polyakov loop is strange beast–neither fish nor fowl.**

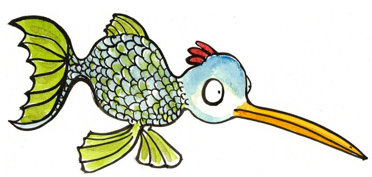


- **Only defined in gauge invariant way in context of Euclidean space functional (with periodic boundary conditions imposing finite temperatures).**
  - **Meaning in Minkowski space—space where physical events occur is—at best—obscure.**
  - **Does not correspond to a gauge-invariant quantum mechanical operator.**
    - **Meaning in Hamiltonian treatments with a Hilbert space is—at best—obscure.**
    - **Meaning for systems for systems that are out of thermal equilibrium is—at best—obscure.**

- The expectation value of the Polyakov loop and the string tension obtained from its correlator are ill-defined as gauge-invariant objects



  - except in the context of thermally equilibrated systems computed in a particular way (via functional integrals in Euclidean space with periodic boundary conditions).



# The Polyakov loop

$$L(\vec{x};\beta) \equiv \mathrm{Tr}\left[\overleftrightarrow{L}(\vec{x};\beta)\right]$$

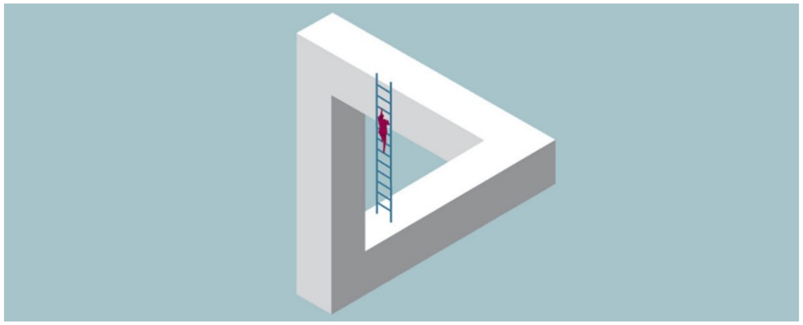
$$\overleftrightarrow{L}(\vec{x};\beta) \equiv \left[T\left(\exp\left(i\int_{0\tau_0}^{\tau_0+\beta} d\tau g A_0(\vec{x},\tau)\right)\right.\right.$$

**Treat as an operator with $\beta$ a parameter (not necessarily connected to temp). NOT gauge invariant under general gauge transformations.**

**Invariant under special gauge transformations satisfying $\Omega(\vec{x},\tau+\beta) = \Omega(\vec{x},\tau+\beta)$ which preserves the periodic boundary condition on the gauge fields $A_\mu(\vec{x},\tau+\beta) = A_\mu(\vec{x},\tau)$ (and ensures correct thermal expectation values for gauge-invariant operators at *$T=1/\beta$*).**

**PL—although not a gauge-invariant operator—*might* be regarded as a gauge-invariant "object"—but only in the context of functional integrals for thermal systems implemented in the conventional way with these boundary conditions.**

# A paradox



- **Many quantum computing applications use a Hamiltonian formalism in the *$A_0$=0* gauge.**

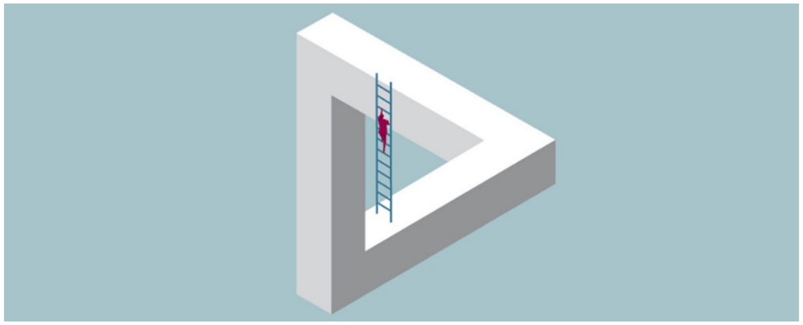
  **Such treatments are supposed to be capable of computing all physical observables—at least in principle.**

  **In any treatment with *$A_0$=0*, *L= 1* by construction, regardless of temperature.**

- **But functional integral treatment—which are also supposed to be capable of computing all physical observables—gives *<L>=0* in low-T phase.**

- **Usual technical explanation: In functional integral, the Polyakov loop defined on topologically nontrivial space–a generalized cylinder where time wraps around; prevents *$A_0$ = 0* from be implementable over all Euclidean times by any continuous gauge transformation.**

# A paradox



- **But paradox remains: Functional integrals and the Hamiltonian treatment in the temporal gauge are supposed to be capable of describing any physical observable.**
- **Another way to see the paradox: From perspective of Hamiltonian approach, the topology associated with the Polyakov loop does not appear to exist: How can it emerge when the physics is re-expressed as Euclidean space functional integral?**
- **Yet another way to see the paradox: From perspective of Hamiltonian approach, center transformations have no meaning. How can can center symmetry emerge when the physics is re-expressed as Euclidean space functional integral?**

**Talk *explores* three heretical views about the Polyakov loop, and what it tells us about the high temperature phase of Yang-Mills Theory**

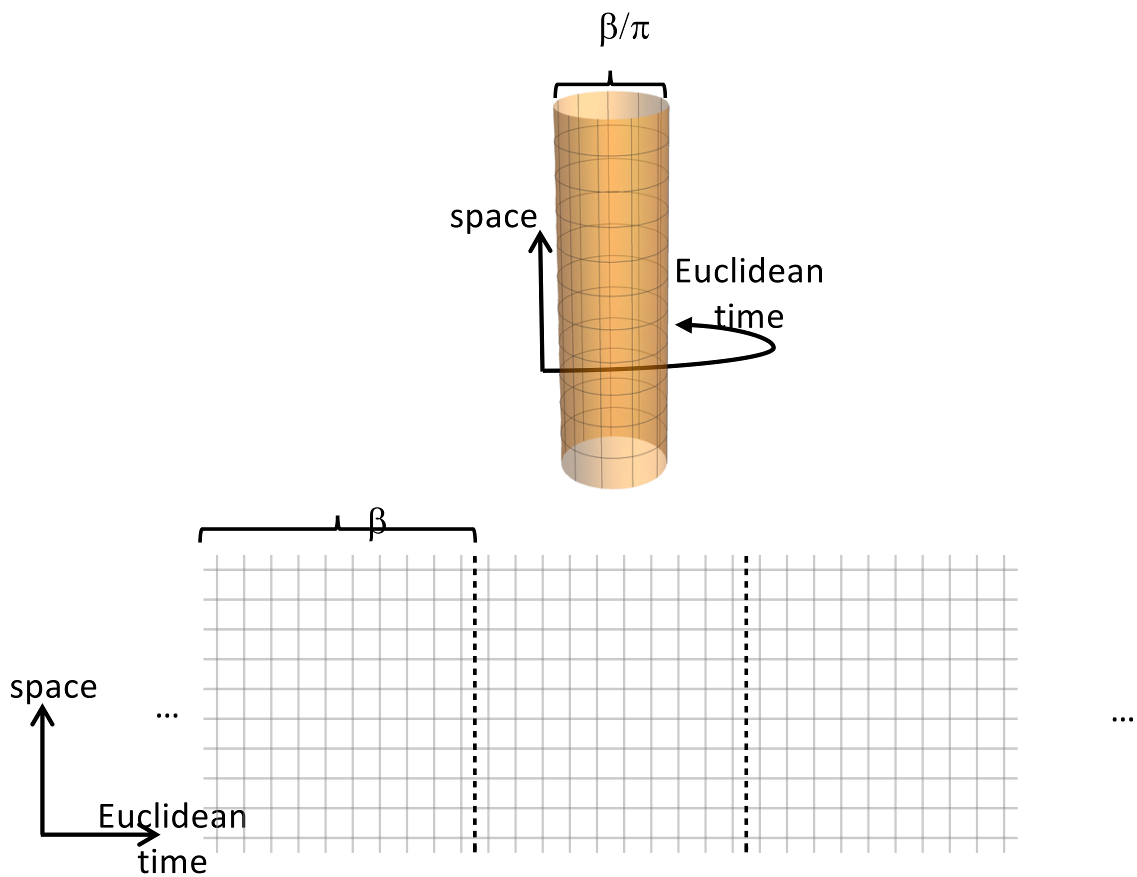

- **Polyakov loop is not gauge invariant in functional integrals using most general boundary conditions consistent with finite temperatures.**
  - **With most general b.c., no topology and no Z(Nc) center symmetry breaking.**
- **No connection between Polyakov loop observables and the free energy of systems with static color charges.**
- **No reliable evidence that the high-temperature phase of Yang-Mills deconfines fundamental color charges—high T phase could be confining.**

# FUNCTIONAL INTEGRALS AND GAUGE INVARIANCE AT NON-ZERO TEMPERATURE

- **Functional integrals for bosonic field theories at non-zero temperatures impose periodic b.c.s in on the fields with the periodicity fixed by *$\beta$ = 1/T:***

$$\phi(\vec{x}, \tau + \beta) = \phi(\vec{x}, \tau)$$

- **Two equivalent viewpoints:**
  - **Space-time is unbounded but field values repeat with periodicity $\beta$.**
  - **Space-time has topology of cylinder with circumference of $\beta$ in the temporal direction.**
  - **The topology of the cylinder plays no role.**




**Same result if you calculate on cylinder or on one period of unbounded space with periodic fields: topology of cylinder irrelevant.**

- More subtle with a gauge theory. One can impose periodic boundary condition on gluon fields: $A_\mu(\vec{x}, \tau + \beta) = A_\mu(\vec{x}, \tau)$

Non-gauge theory can be viewed as either periodic in full space-time or on a cylinder.

- If you view a gauge theory as being on a cylinder and require gauge transformations to be nonsingular on cylinder then $\Omega(\vec{x}, \tau + \beta) = \Omega(\vec{x}, \tau)$

- Makes *L* gauge invariant.
  - Topology of the cylinder is needed for this to make sense.

- Means center transformations not allowable gauge transformations.
  - The spontaneous breaking of center symmetry is only possible if one takes this view.

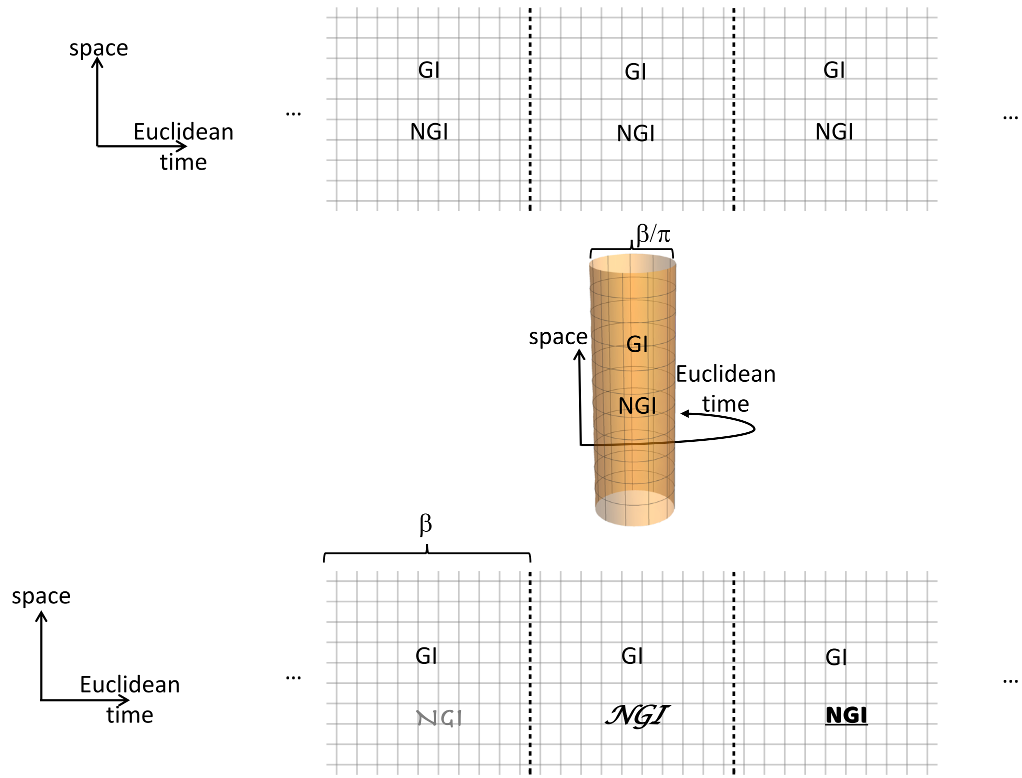
- **Subtlety: the gluon field contains both physical information and unphysical information (associated with gauge redundancy).**
  - **The physical information associated with the gauge field—all gauge-invariant combinations of the fields—clearly needs to be periodic.**
  - ***A priori*, no need for periodicity of unphysical d.o.f. !**
  - **Most general b.c. consistent with thermal physics *only* need ensure gauge invariant combos of gauge fields are periodic.**

- **Start with a periodic gauge field defined on an infinite space-time, make an arbitrary aperiodic gauge transformation $\Omega$ on the full space-time on the fields: only gauge-variant quantities would be affected, these become aperiodic.**

**Functional integral evaluated over one period with bc.**

$$A_\mu(\vec{x},\tau) = \Omega(\vec{x},\tau+\beta)A_\mu(\vec{x},\tau+\beta)\Omega^\dagger(\vec{x},\tau+\beta) - \frac{i}{g}\left(\partial_\mu\Omega(\vec{x},\tau+\beta)\right)\Omega^\dagger(\vec{x},\tau+\beta)$$

**has same expectation value for all gauge-invariant operators as with periodic b.c.**




- **Upper cartoon: typical gauge configuration consistent with periodic boundary conditions. Fully equivalent to the gauge fields on cylinder.**
- **Lower cartoon: typical gauge configuration consistent with the more general b.c. The GI part is same as top, NGI differs and differs on each segment.**

- Functional integrals with more general b.c. has same results for thermal expectation values of any gauge-invariant operators as with simple periodic b.c.

But with more general b.c....

- Space-time does not have the topology of cylinder.
- The Polyakov loop (and its correlators) are not gauge-invariant.
  - Expectation values of these will yield zero—as with any purely gauge-variant quantity.
- Center transformations become ordinary gauge transformations and thus center symmetry cannot break.

# Two ways to impose boundary conditions.

**Equivalent for all expectation values of gauge-invariant operators but different for Polyakov loop and related beasts.**

**Which b.c. is appropriate for these?**

$$A_\mu(\vec{x}, \tau + \beta) = A_\mu(\vec{x}, \tau)$$

**Standard periodic b.c.**

$$A_\mu(\vec{x}, \tau) = \Omega(\vec{x}, \tau + \beta) A_\mu(\vec{x}, \tau + \beta) \Omega^\dagger(\vec{x}, \tau + \beta)$$

$$- \frac{i}{g} \left( \partial_\mu \Omega(\vec{x}, \tau + \beta) \right) \Omega^\dagger(\vec{x}, \tau + \beta)$$

**Most general b.c. consistent with correct thermal expectation values**

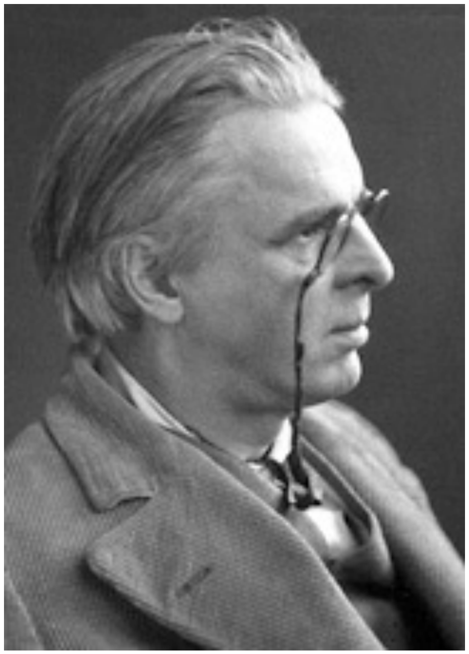

**Heretical view:**

**The more general b.c. is the appropriate one!!!**

- **The Polyakov loops behaves like any other mathematical object that does not correspond to a gauge invariant operator—it vanishes in the functional integral.  The PL is NOT physical!**

- The paradox associated with the difference between Polyakov observables in Hamiltonian treatments in temporal gauge and the functional integral treatments is resolved.
  - Explains the lack of topology in Hamiltonian treatment—it is not there in functional treatment either.
  - Explains lack of center symmetry in Hamiltonian treatment —it is not there in functional treatment either.
- Explains why Polyakov loop has no meaning in Minkowski space—it also has no meaning in Euclidean.

# Center Symmetry and Confinement



**The Irish poet, William Butler Yeats might have seemed to have been prescient about confinement. In 1920 he described the Yang Mills phase transition in his apocalyptic poem *Second Coming*:**

**"Things fall apart,**
**the centre cannot hold"**

***i.e.* deconfinement in Yang Mills**

**Yeats won the Nobel Prize in 1923 for his work but, inexplicably, it was in literature rather than physics.**

**But perhaps the Nobel committee was correct, and the committee realized that Yeats was using the wrong boundary conditions!**

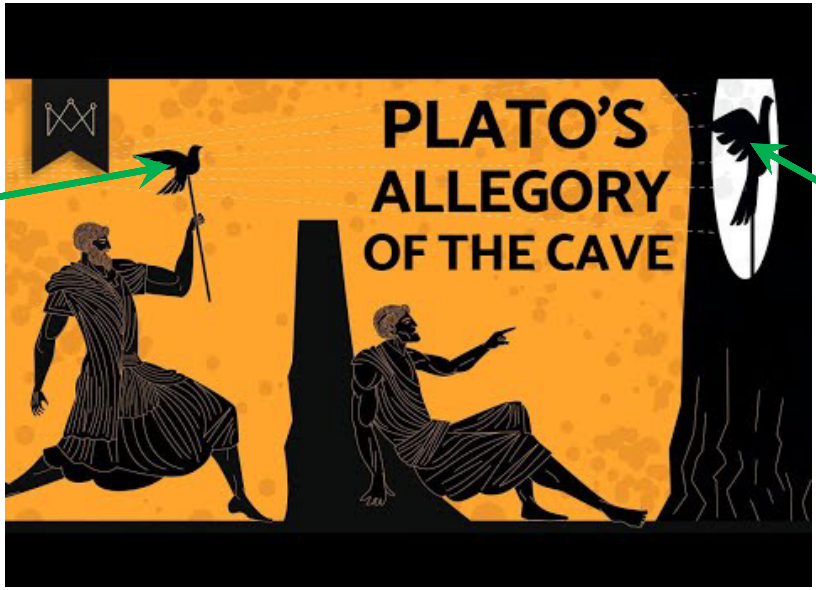
- **If the correlator of the Polyakov loop is unphysical so is the string tension obtained from it and one can conclude nothing about confinement from vanishing string tension.**

- **A puzzle: In context of functional integrals with periodic b.c., string tension acts as order parameter that cleanly distinguish between the low temperature phase (non-vanishing string tension) and high temperature phase (vanishing string tension). How is this possible if the Polyakov loop is unphysical?**

**Resolution:** All physical observables (which are based on gauge invariant operators) can be computed with either b.c. ; same phase transition.

If one use the conventional periodic ones, the order parameters based on the Polyakov loop are *correlated* with the phase, even though the order parameters have no direct physical meaning themselves.

They act like shadows on the wall of Plato's cave. They tell us something real about the phases without being real themselves.




Physically being in the high temperature phase

Signature of high temperature phase from Polyakov loop

- There is a potential flaw with this explanation.
  - **Standard argument: $C_{L^*L}$ (correlator of PL) related to free energies of a system with color charges in the fundamental and anti-fundamental (infinitely massive quarks) added to system and separated by some distance.**
  - **This is physical quantity and expressed via gauge invariant operators.**
  - **Allows for the computation of a string tension which vanishes in the high temp phase indicating deconfinement.**

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) \equiv \langle L^*(\vec{x_j}, \beta) L(\vec{x_i}, \beta) \rangle_{T=\frac{1}{\beta}}$$
$$= \frac{\int \mathcal{D}A_\mu(\vec{x}, t) L^*(\vec{x_j}, \beta) L(\vec{x_i}, \beta) \exp\left(-\int_0^\beta dt \int d^3x \mathcal{L}(A)\right)}{\int \mathcal{D}A_\mu(\vec{x}, t) \exp\left(-\int_0^\beta dt \int d^3x \mathcal{L}(A)\right)}$$

**Correlator of a Polyakov loop and its conjugate**

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \exp(-\beta F_{ij})$$

**Standard interpretation**

**Free energy is physical**

$$\sigma(T)_{\mathrm{PL}} = \lim_{|R| \to \infty} \frac{\log\left(C_{L^*L}(\vec{x} + \vec{R}, \vec{x}; T)\right)}{|R|}$$

**String tension computed from PL appears to be physical**

- However, standard argument relating $C_{L^*L}$ to free energies *appears* to suffer from subtle—but fatal—bugs.

**Allows heretical View:**

**There is no connection between Polyakov loop observables and the free energy of systems with static color charges.**

# Derivation of Standard Connection of $C_{L*L}$ and Free Energy

**Introduce fundamental static color charges at discrete points $\vec{x}_i$ (think arbitrarily heavy quark)**

$S^\dagger_{i;c}$ **Is a fermionic operator that places color charge *c* at** $\vec{x}_i$

$$\overline{S}^\dagger_{i;c} \equiv \frac{\epsilon_{abc} S^\dagger_{i;a} S^\dagger_{i;b}}{2}$$ **Places antifundamental color charge at** $\vec{x}_i$

**These propagate in time but not space.**

**Using standard functional integral methods:**

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \left\langle \overline{S}_{j;c'}(\beta) S_{i;c}(\beta) S^\dagger_{i;c}(0) \overline{S}^\dagger_{jc'}(0) \mathcal{P}_0 \right\rangle_T$$

$$= \mathrm{Tr}_{\mathrm{phys}} \left[ \exp(-\beta H) \overline{S}_{j;c'}(\beta) S_{i;c}(\beta) S^\dagger_{i;c}(0) \overline{S}^\dagger_{j;c'}(0) \mathcal{P}_0 \right]$$

$$= \sum_{k \in \mathrm{phys}} \langle \psi_k | \exp(-\beta H) \overline{S}_{j;c'}(\beta) S_{i;c}(\beta) S^\dagger_{i;c}(0) \overline{S}^\dagger_{j;c'}(0) \mathcal{P}_0 | \psi_k \rangle$$

$\text{Tr}_{\text{phys}}$ **indicates trace over the physical states only—that is states satisfying the color Gauss law.**

$$\mathcal{P}_0 \equiv \prod_{n=1}^{\infty} \left( n - \sum_{j,c} S^{\dagger}_{j,c} S_{j,c} \right)$$

**Projector onto states with no fundamental color charges, *i.e.* pure Y.M. states**

**Insert standard Euclidean time evolution for operators**

$$\mathcal{O}(\tau + \beta) = \exp(\beta H)\mathcal{O}(\tau)\exp(-\beta H)$$

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \text{Tr}_{\text{phys}} \left[ \overline{S}_{j;c'}(0) S_{i;c}(0) \exp(-\beta H) S^{\dagger}_{i;c}(0) \overline{S}^{\dagger}_{j;c'}(0) \mathcal{P}_0 \right]$$

**Use cyclic property of trace to rewrite as**

$$\begin{aligned} C_{L^*L}(\vec{x}_i, \vec{x}_j; T) &= \text{Tr}_{\text{phys}} \left[ \overline{S}_{j;c'}(0) S_{i;c}(0) \exp(-\beta H) S^{\dagger}_{i;c}(0) \overline{S}^{\dagger}_{j;c'}(0) \mathcal{P}_0 \right] \\ &= \text{Tr}_{\text{phys}} \left[ \exp(-\beta H) \mathcal{P}_{ij} \right] \end{aligned}$$

$$\mathcal{P}_{ij} = S^{\dagger}_{i;c}(0) \overline{S}^{\dagger}_{j;c'}(0) \mathcal{P}_0 \overline{S}_{j;c'}(0) S_{i;c}(0)$$

**gauge invariant**

$\mathcal{P}_{ij}$ **projects on to states with a fundamental color charge at $\vec{x}_i$ and an anti-fundamental charge at $\vec{x}_j$**

# Derivation of Standard Connection of $C_{L*L}$ and Free Energy

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \mathrm{Tr}_{\mathrm{phys}} \left[ \exp(-\beta H) \mathcal{P}_{ij} \right]$$

**In general,** $\mathrm{Tr}_{\mathrm{phys}}[\, e^{-\beta H} \, \mathcal{P}_C]$ **where** $\mathcal{P}_C$ **is a a gauge invariant projector under to states satisfying some physical constraint is the partition function subject to that constraint:**

$$\mathrm{Tr}_{\mathrm{phys}}[\, e^{-\beta H} \, \mathcal{P}_C] = Z_C(\beta) = \exp(-\beta F_C)$$

**As advertised**

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \exp(-\beta F_{ij})$$

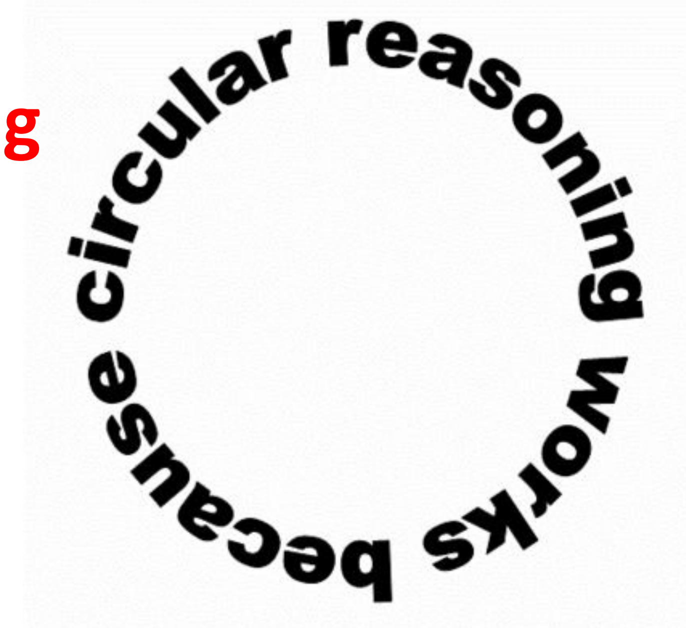
**This is physical and nonzero and apparently justifies the use of the conventional periodic b.c. and not the more general aperiodic ones.**

# But…

- **The preceding analysis is flawed in two ways:**

1. **The analysis uses circular reasoning. The first step implicitly starts with the standard periodic b.c. rather than the more general one to conclude that only the standard periodic b.c. is consistent.**
2. **The analysis starting from the standard periodic b.c. makes use of an illegitimate mathematical step.**

# Argument depends on circular reasoning




Starting point of the analysis:

Calculate correlator of PL and operators making and annihilating charge at $\tau=0$ and $\beta$ using standard functional integral methods.

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \left\langle \overline{S}_{j;c'}(\beta) S_{i;c}(\beta) S^\dagger_{i;c}(0) \overline{S}^\dagger_{jc'}(0) \mathcal{P}_0 \right\rangle_T$$

Valid only if the correlator of the Polyakov loop and the correlator of the creation and annihilation operator of the color charges are gauge invariant.

Neither are gauge invariant operators. Might be regarded as gauge invariant “objects”, but only if the periodic b.c. imposed at the outset

# Argument depends on circular reasoning


Only by assuming implicitly periodic b.c., does one conclude that the free energy is physical (gauge invariant), which implies $C_{L*L}$ is too, requires the use the periodic b.c.

Had one imposed the more general b.c. at the beginning one would have seen that

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \left\langle \overline{S}_{j;c'}(\beta) S_{i;c}(\beta) S^\dagger_{i;c}(0) \overline{S}^\dagger_{jc'}(0) \mathcal{P}_0 \right\rangle_T$$

holds only in a trivial way yielding no new information since

$C_{L^*L}(\vec{x}_i, \vec{x}_j; T)$ =0

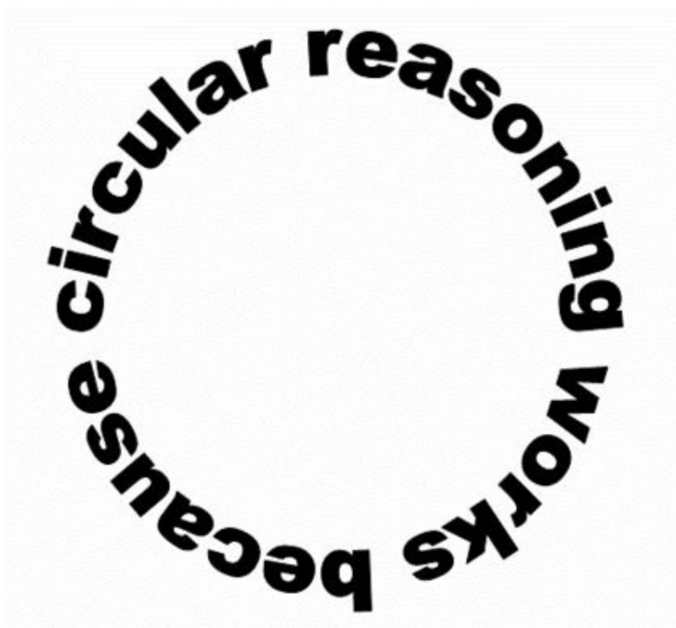
$\left\langle \overline{S}_{j;c'}(\beta) S_{i;c}(\beta) S^\dagger_{j\vec{x}_j;c}(0) \overline{S}^\dagger_{i;c'}(0) \mathcal{P}_0 \right\rangle_T$ =0

and 0=0 .

## Had one imposed the more general b.c. one would see no contradictions.

## Argument depends on circular reasoning

Ironic to use circular reason to deduce properties of a loop!!




# An illegitimate step

**Assume that conventional periodic b.c. holds.**

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \mathrm{Tr}_{\mathrm{phys}}\left[\overline{S}_{j;c'}(0) S_{i;c}(0) \exp(-\beta H) S^{\dagger}_{i;c}(0) \overline{S}^{\dagger}_{j;c'}(0) \mathcal{P}_0\right]$$

**Cannot deduce the free energy form directly from this.** $S^{\dagger}_{j;c} 0 \overline{S}^{\dagger}_{j;c'}(0)$

**And** $\overline{S}_{j;c'}(0) S_{i;c}(0)$ **are not gauge invariant and acting on physical states they may produce states outside the physical space.**

**To express things in a gauge invariant form, had to invoke cyclic property of the the trace.**

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \mathrm{Tr}_{\mathrm{phys}}\left[\overline{S}_{j;c'}(0) S_{i;c}(0) \exp(-\beta H) S^{\dagger}_{i;c}(0) \overline{S}^{\dagger}_{j;c'}(0) \mathcal{P}_0\right]$$

$$\Downarrow$$

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \mathrm{Tr}_{\mathrm{phys}}\left[\overline{S}_{j;c'}(0) S_{i;c}(0) \exp(-\beta H) S^{\dagger}_{i;c}(0) \overline{S}^{\dagger}_{j;c'}(0) \mathcal{P}_0\right]$$

$$= \mathrm{Tr}_{\mathrm{phys}}\left[\exp(-\beta H) \mathcal{P}_{ij}\right]$$

But…**Cyclic property of trace only valid for operators that map from a Hilbert space onto itself. The space here is the physical Hilbert space—states that satisfy the Gauss Law constraint.**

**The operator** $\overline{S}_{j;c'}(0)S_{i;c}(0)$ **could take one outside this space and invalidates step. Check this:**

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \mathrm{Tr}_{\mathrm{phys}}\left[\overline{S}_{j;c'}(0)S_{i;c}(0)\exp(-\beta H)S^\dagger_{i;c}(0)\overline{S}^\dagger_{j;c'}(0)\mathcal{P}_0\right]$$

$$= \sum_{k\in\mathrm{phys}} \langle\psi_k|\overline{S}_{j;c'}(0)S_{i;c}(0)\exp(-\beta H)S^\dagger_{i;c}(0)\overline{S}^\dagger_{j;c'}(0)\mathcal{P}_0|\phi_k\rangle$$

**The color charge operators add color charges without affecting color E-field acting on a state that satisfies color Gauss law, yields one that does not—*i.e.* the state is unphysical**

**Physical state; satisfies Gauss law**

**Cyclic property not valid & this invalidates free energy result!!**

## A simple calculation, taking into account physical space yields:

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \mathrm{Tr}_{\mathrm{unphys}} \left[ e^{-\beta H} S^{\dagger}_{i;c}(0) \overline{S}^{\dagger}_{j;c'}(0)\, \mathcal{P}_0\, \mathcal{P}_{\mathrm{phys}}\, \overline{S}_{j;c'}(0) S_{i;c}(0) \right]$$

$$\neq \mathrm{Tr}_{\mathrm{phys}} \left[ \exp(-\beta H) \mathcal{P}_{ij} \right]$$

**Where $\mathcal{P}_{phys}$ projects onto states satisfying Gauss law and $\mathrm{Tr}_{\mathrm{unphys}}$ sums over eigenstates of (1- $\mathcal{P}_{phys}$) with eigenvalue unity.**

**Result starting from conventional periodic b.c. has only unphysical states contributing and no connection to free energy of physical states with static color charges!!**


**Heretical View:**

**There is no connection between Polyakov loop observables and the free energy of systems with static color charges.**

Provides additional argument that the appropriate boundary conditions are the more general ones and not the usual periodic ones.

- Starting with the assumption that the standard period boundary conditions are valid
  - one obtains the result that all of the contributions to $C_{L^*L}$ only comes from unphysical states—ones that do not satisfy the color Gauss law when evaluated in a Hamiltonian picture.
- This suggests that the correlator of Polyakov loops is without direct physical content

The first two heretical propositions have been discussed: 1. Polyakov loop is not gauge invariant in functional integrals using most general thermal boundary conditions. 2. No connection between Polyakov loop observables and the free energy.

- **First two heretical ideas imply the third.**
- **The third heresy: No reliable evidence that the high-temperature phase of Yang-Mills deconfines fundamental color charges.**
  - **The evidence that fundamental color charges are deconfined at high temperatures come from studies of observables related to the Polyakov loop.**

**Raises the possibility that fundamental color charges are confined in both high temp and low temp phases.**

# Back up slides

- **Whether or not fundamental color charges are deconfined in the high temperature phase might be answered by lattice studies without recourse to the Polyakov loop.**
  - **One “simply” adds in explicit heavy quarks, takes the appropriate limits in the correct order and then calculates the observables that act as signatures of deconfinement.**
    - **However, apart from possible technical challenges, there are some subtle issues of principle as to what observables one should use and in what order various limits should be taken.  Theses issues are currently being explored.**

# Derivation of $C_{L^*L}$

**Cyclic property of trace only valid for operators that map from a Hilbert space onto itself. The space here is the physical Hilbert space—states that satisfy the Gauss Law constraint.**

**The operator $\overline{S}_{j;c'}(0)S_{i;c}(0)$ could take one outside this space (and in fact does) and this invalidates step.**

- **Check: Rewrite initial expression as trace over full Hilbert space, with projector onto the physical space.**
  **In full space, cyclic property valid**

$$
\begin{aligned}
C_{L^*L}(\vec{x}_i, \vec{x}_j; T) &= \mathrm{Tr}_{\mathrm{phys}}\left[\overline{S}_{j;c'}(0)S_{i;c}(0)\exp(-\beta H)S^{\dagger}_{j;c}(0)\overline{S}^{\dagger}_{j;c'}(0)\mathcal{P}_0\right] \\
&= \mathrm{Tr}_{\mathrm{full}}\left[\overline{S}_{j;c'}(0)S_{i;c}(0)\exp(-\beta H)S^{\dagger}_{j;c}0\overline{S}^{\dagger}_{j;c'}(0)\mathcal{P}_0\mathcal{P}_{\mathrm{phys}}\right] \\
&= \mathrm{Tr}_{\mathrm{full}}\left[\exp(-\beta H)S^{\dagger}_{j;c}(0)\overline{S}^{\dagger}_{j;c'}(0)\,\mathcal{P}_0\,\mathcal{P}_{\mathrm{phys}}\,\overline{S}_{j;c'}(0)S_{i;c}(0)\right]
\end{aligned}
$$

Contribution from physical states

$$C_{L^*L}(\vec{x}_i, \vec{x}_j; T) = \mathrm{Tr}_{\mathrm{full}}\left[\exp(-\beta H) S^\dagger_{j;c}(0) \overline{S}^\dagger_{j;c'}(0)\, \mathcal{P}_0\, \mathcal{P}_{\mathrm{phys}}\, \overline{S}_{j;c'}(0) S_{i;c}(0)\right]$$

$$= \mathrm{Tr}_{\mathrm{full}}\left[\exp(-\beta H) S^\dagger_{j;c}(0) \overline{S}^\dagger_{j;c'}(0)\, \mathcal{P}_0\, \mathcal{P}_{\mathrm{phys}}\, \overline{S}_{j;c'}(0) S_{i;c}(0) \mathcal{P}_{\mathrm{phys}}\right]$$

$$+ \mathrm{Tr}_{\mathrm{full}}\left[\exp(-\beta H) S^\dagger_{j;c}(0) \overline{S}^\dagger_{j;c'}(0)\, \mathcal{P}_0\, \mathcal{P}_{\mathrm{phys}}\, \overline{S}_{j;c'}(0) S_{i;c}(0) (1 - \mathcal{P}_{\mathrm{phys}})\right]$$

Contribution from unphysical states

**However, $\mathcal{P}_{\mathrm{phys}} \overline{S}_{j;c'}(0) S_{i;c}(0)\, \mathcal{P}_{\mathrm{phys}} = 0$ this can be shown formally but easy to understand intuitively:**

Projects on to states satisfying Gauss Law

$$\mathcal{P}_{\mathrm{phys}} \underbrace{\overline{S}_{j;c'}(0) S_{i;c}(0)}\, \mathcal{P}_{\mathrm{phys}}$$

Projects on to states satisfying Gauss Law

**Adds color charges without change color E-field. If acting on configuration satisfying Gauss law yields configuration which doesn't. If acting on a physical state yields an unphysical one.**

$$\mathcal{P}_{\rm phys}\overline{S}_{j;c'}(0)S_{i;c}(0)\,\mathcal{P}_{\rm phys} = 0$$

$$\begin{aligned} C_{L^*L}(\vec{x}_i, \vec{x}_j; T) &= \mathrm{Tr}_{\rm full}\left[\exp(-\beta H) S^\dagger_{j;c}(0)\overline{S}^\dagger_{j;c'}(0)\,\mathcal{P}_0\,\mathcal{P}_{\rm phys}\,\overline{S}_{j;c'}(0)S_{i;c}(0)\mathcal{P}_{\rm phys}\right] \\ &+ \mathrm{Tr}_{\rm full}\left[\exp(-\beta H) S^\dagger_{j;c}(0)\overline{S}^\dagger_{j;c'}(0)\,\mathcal{P}_0\,\mathcal{P}_{\rm phys}\,\overline{S}_{j;c'}(0)S_{i;c}(0)(1-\mathcal{P}_{\rm phys})\right] \end{aligned}$$

$$= \mathrm{Tr}_{\rm unphys}\left[e^{-\beta H} S^\dagger_{j;c}(0)\overline{S}^\dagger_{j;c'}(0)\,\mathcal{P}_0\,\mathcal{P}_{\rm phys}\,\overline{S}_{j;c'}(0)S_{i;c}(0)\right]$$