Role and subtlety of lattice chiral symmetry in finite temperature QCD

prepared for “YITP workshop: QCD phase diagram and lattice QCD”

Based on the works by the JLQCD collaboration:
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Topological susceptibility

$$\chi_t \equiv \frac{\langle Q^2 \rangle}{V}$$

Interesting, yet difficult for many reasons:

- **Non-integer on the lattice:** need some smoothing, not perfect though.
- **Topology freezing:** gets worse as the continuum limit approached.
- **Index theorem:** related to fermion zero-modes.
- **Suppressed by sea quarks:** near-zero modes are most relevant.
- **Measure of quark condensate:** again near-zero modes.

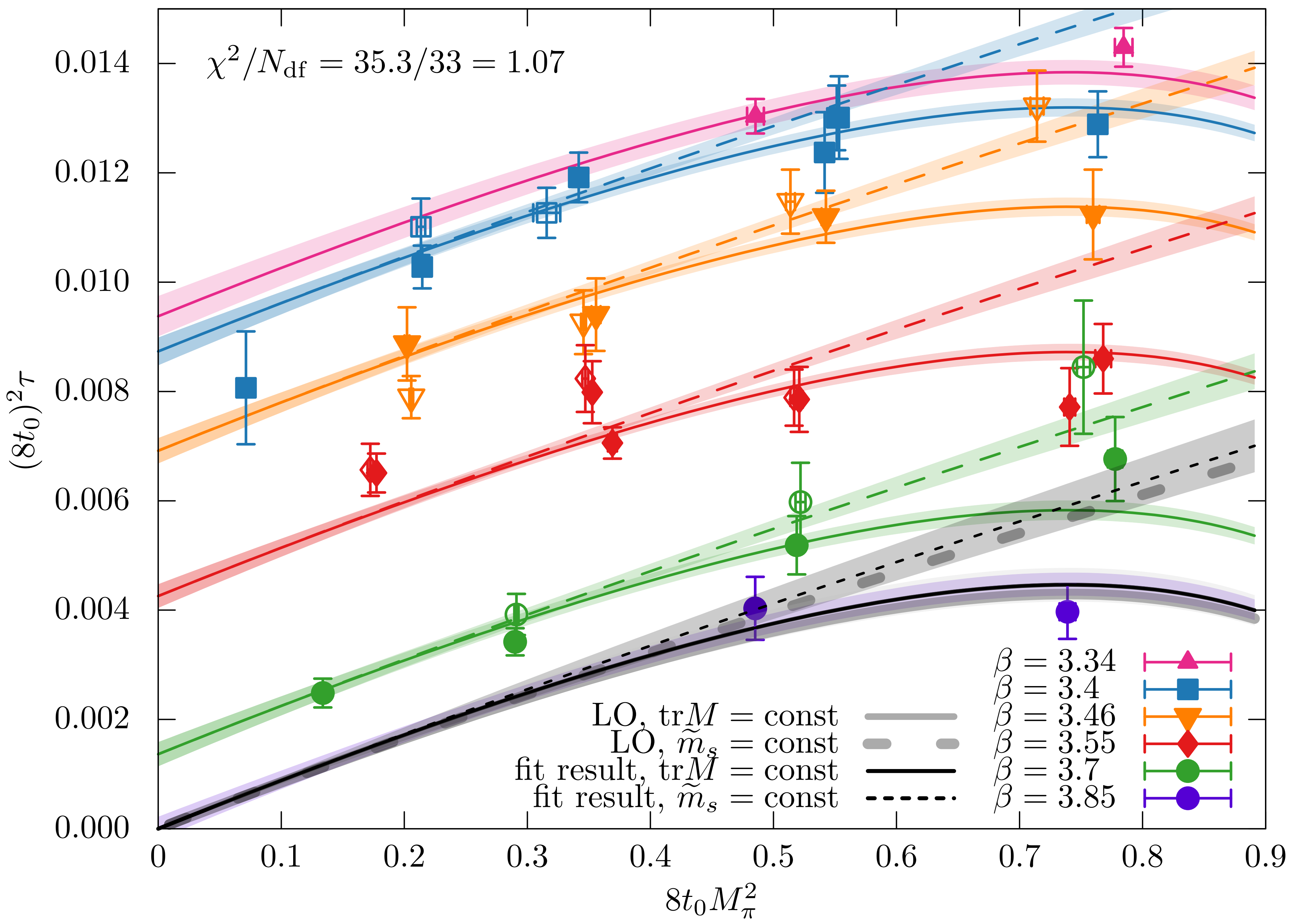
$$\left[\prod_k (m^2 + \lambda_k^2)\right]^{N_f}$$

$$\chi_t = \frac{m_q \Sigma}{N_f}$$

Very sensitive to the discretization effect for (near-)zero modes.

Large discretization effect for topological susceptibility?

RQCD (2021)



O(a)-improved Wilson fermion

Large error, > O(100%), near 0.1 fm

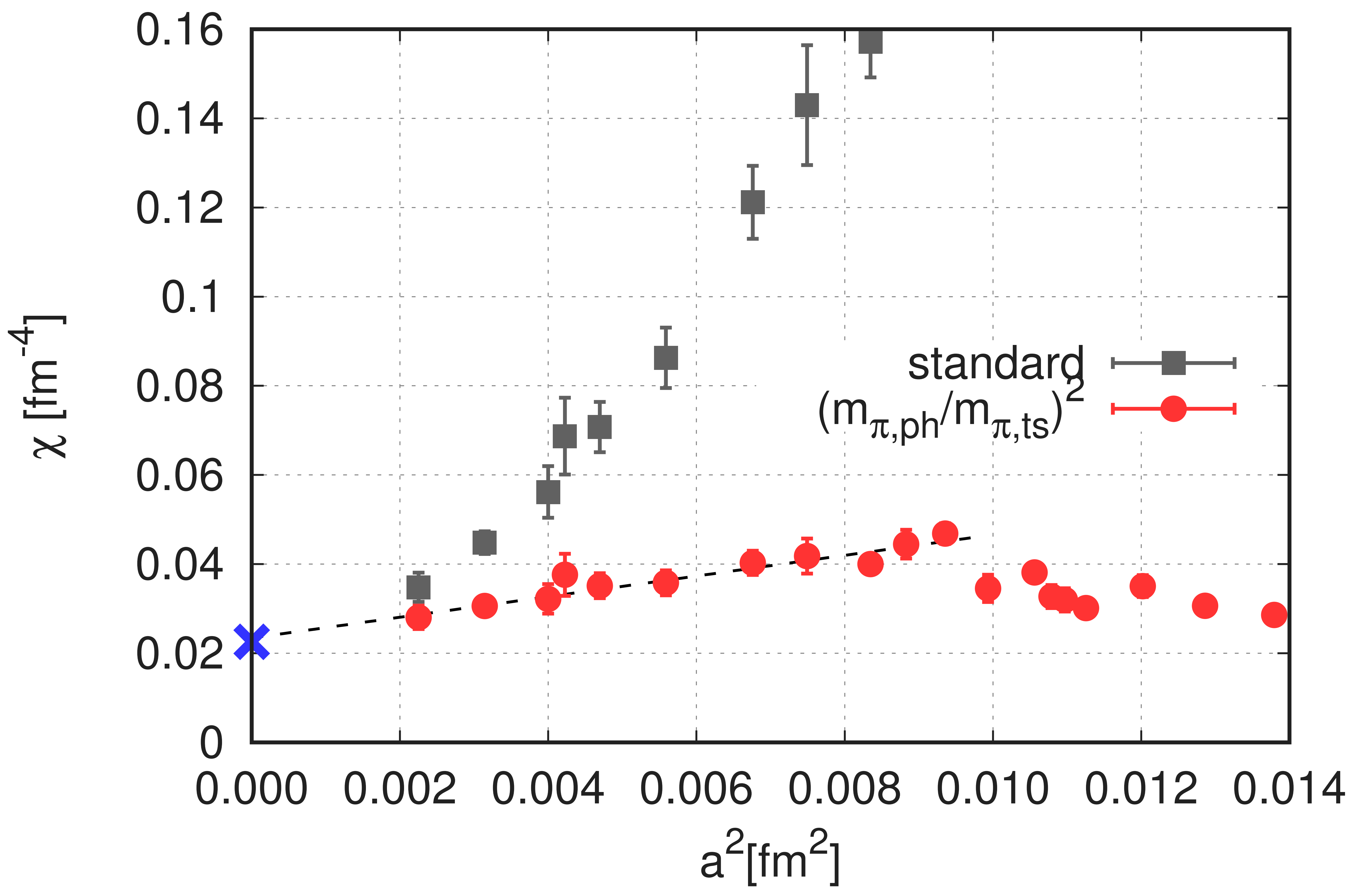
Suppression by fermion determinant

$$\left[\prod_k (m^2 + \lambda_k^2)\right]^{N_f}$$

doesn't work properly at finite a.

Large discretization effect for topological susceptibility?

BMW, Nature (2016)



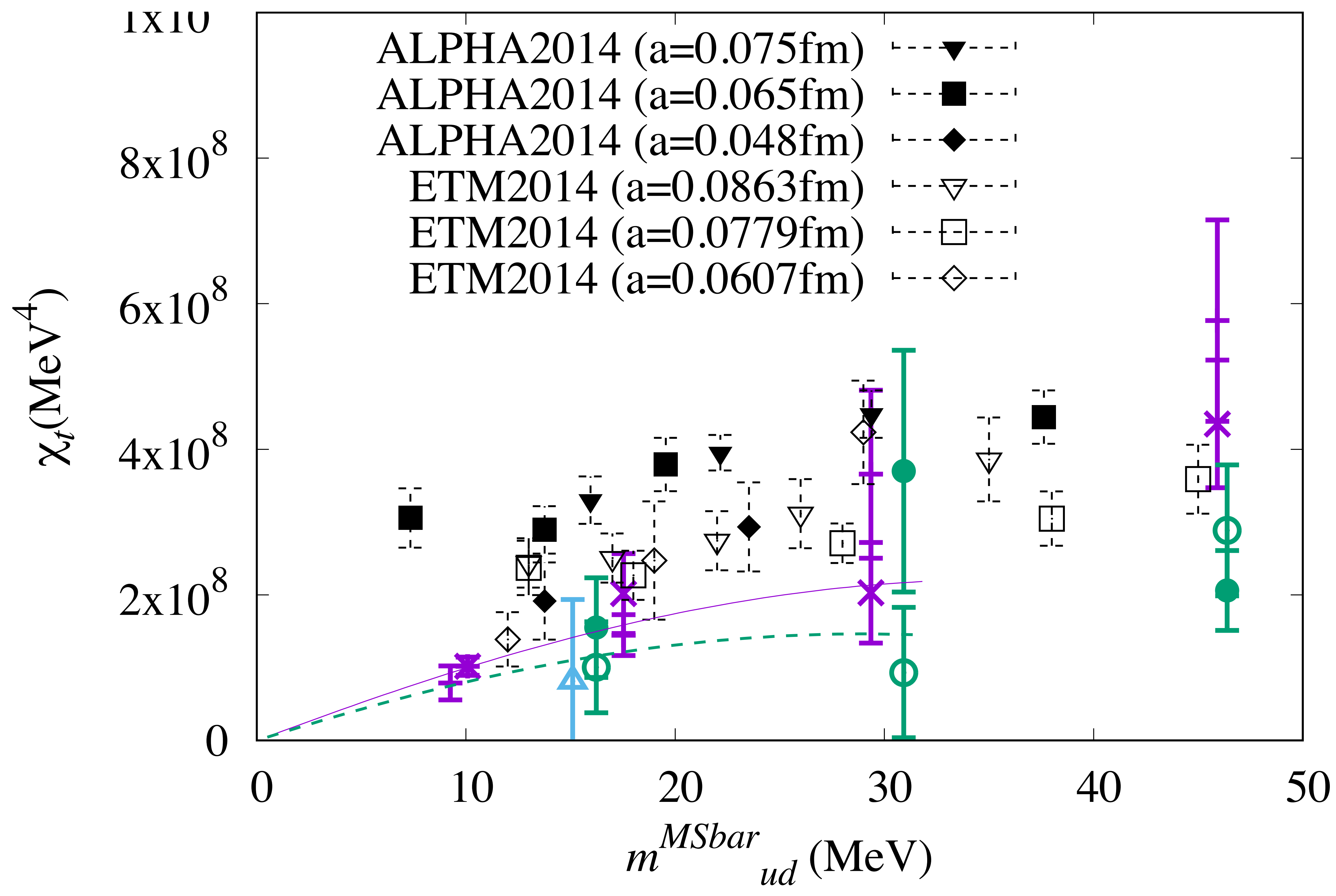
Stout-staggered (not improved)

Huge error, >> O(100%), at a = 0.06 fm.

Large error, O(100%), even after correcting the taste-breaking. (Topological charge couples to taste-singlet fermionic determinant.)

Large discretization effect for topological susceptibility?

JLQCD (2017)



Is $\chi_t = \frac{m_q \Sigma}{N_f}$ reproduced?

- ALPHA: O(a)-improved Wilson fermion
- ETM: twisted mass fermion, χ_t from spectral sum
- JLQCD: domain-wall fermion; 2 lattice spacings

Huge, large, or modest discretization effects depending on the fermion formulation, and how to define Q.

Near-zero modes?

Common belief: the discretization effect appears as $O(a^2)$
How does it affect the near-zero modes?

$$\lambda \left[1 + O(a^2\Lambda^2)\right]$$

✘ Chiral symmetry violated;
zero mode not protected.

$$\lambda + O(a^2\Lambda^3)$$

Probably, too large.

To my knowledge, there is no argument to apply
Symanzik effective theory for Dirac eigenmodes.

$U_A(1)$ susceptibility

$U_A(1)$ susceptibility

To probe the $U_A(1)$ violation in the vacuum, e.g.

$$\Delta_{\pi-\delta} = \int d^4x \, [\langle \pi^a(x)\pi^a(0)\rangle - \langle \delta^a(x)\delta^a(x)\rangle]$$

- Disc. error would be of $O(a^2\Lambda^4)$.

Eigenvalue decomposition:

$$\Delta_{\pi-\delta} = \int_0^\infty d\lambda \, \rho(\lambda) \frac{2m^2}{(m^2+\lambda^2)^2}$$

- Log divergent in UV

- More sensitive to low-lying eigenvalue spectrum compared to

$$\Sigma = \int_0^\infty d\lambda \, \rho(\lambda) \frac{2m}{m^2+\lambda^2}$$

Actually, probes lowest-lying modes almost exclusively.

$U_A(1)$ susceptibility

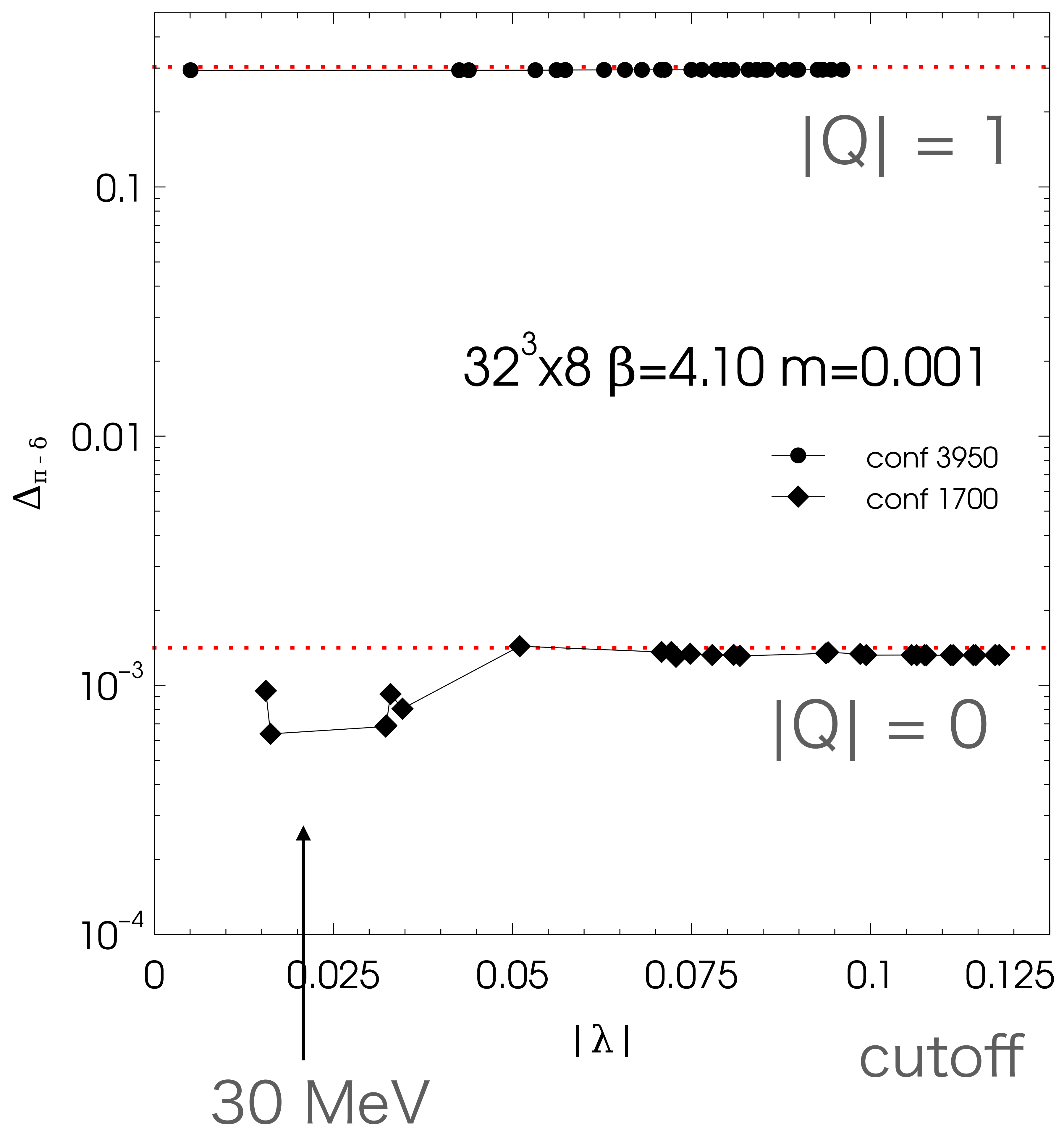
$$\Delta_{\pi-\delta} = \int_0^\infty d\lambda\, \rho(\lambda) \frac{2m^2}{(m^2+\lambda^2)^2}$$

Contributions from low-modes:

- Zero mode is actually dominant (x100).
- It is a volume-dependent statement, though. Zero-mode contribution is suppressed eventually as $1/V^{1/2}$. (Better to subtract from the beginning.)

- Can we identify the zero-mode unambiguously? Chiral symmetry is crucial.

JLQCD (2016)



Domain-wall fermion is not good enough

Ginsparg-Wilson relation

- Exact chiral symmetry is realized on the lattice, if the Dirac operator satisfies

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

- Domain-wall fermion is one implementation; overlap fermion is another.
- Exact chirality is achieved when lattice size in 5th dim, $L_s \to \infty$, otherwise, chiral symmetry is inexact.

- Violation can be studied using an operator

$$\Delta_{GW} \equiv \hat{\gamma}_5 H + H\hat{\gamma}_5,$$
$$\hat{\gamma}_5 \equiv \gamma_5 - H$$

with $H = \gamma_5 D$.

Residual mass

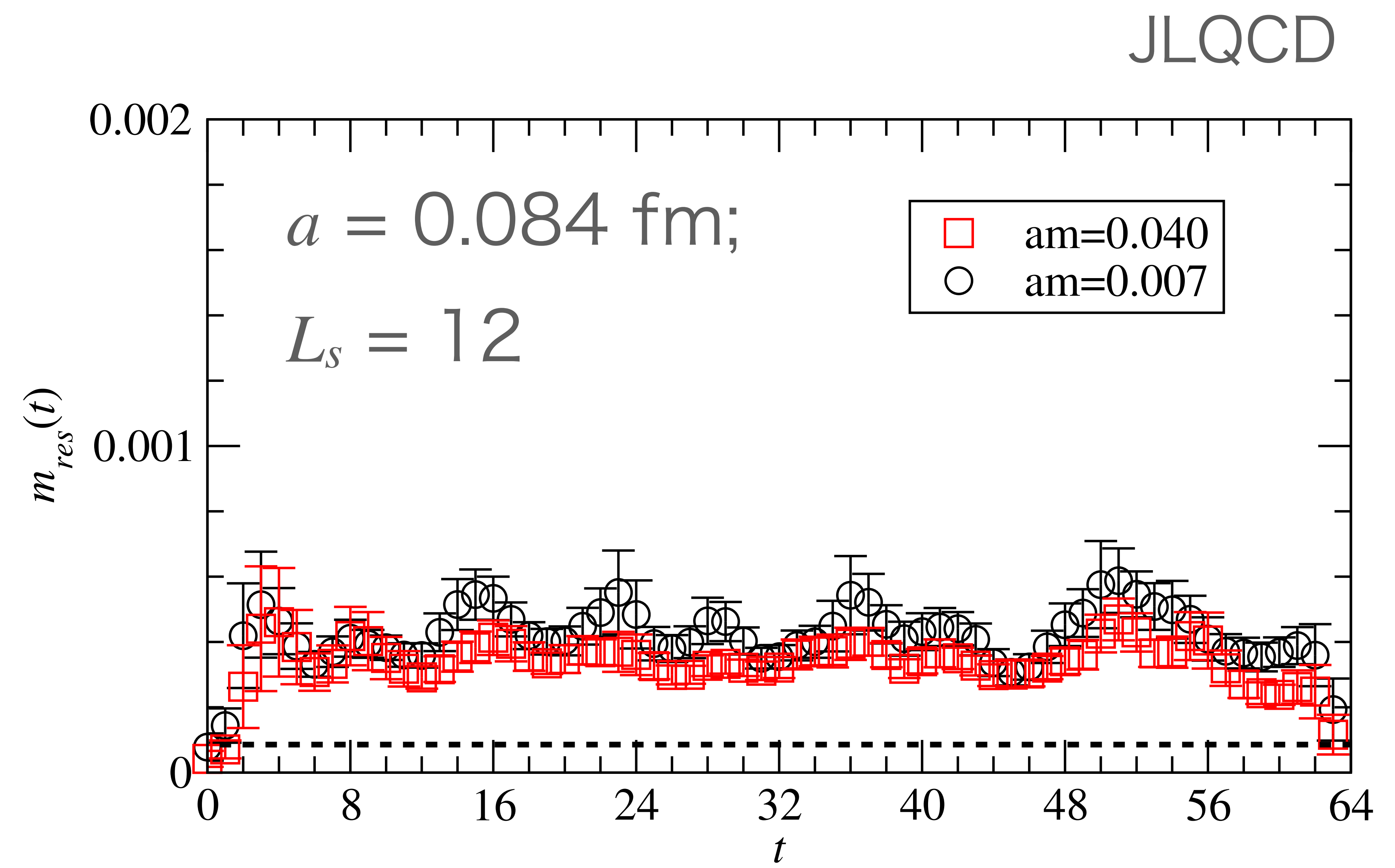
$m_{\rm res}$: parametrizes the effect of violation. Then, use Symanzik effective theory to estimate potential errors, as $O(am_{res})$. But, how do you define $m_{\rm res}$? Try

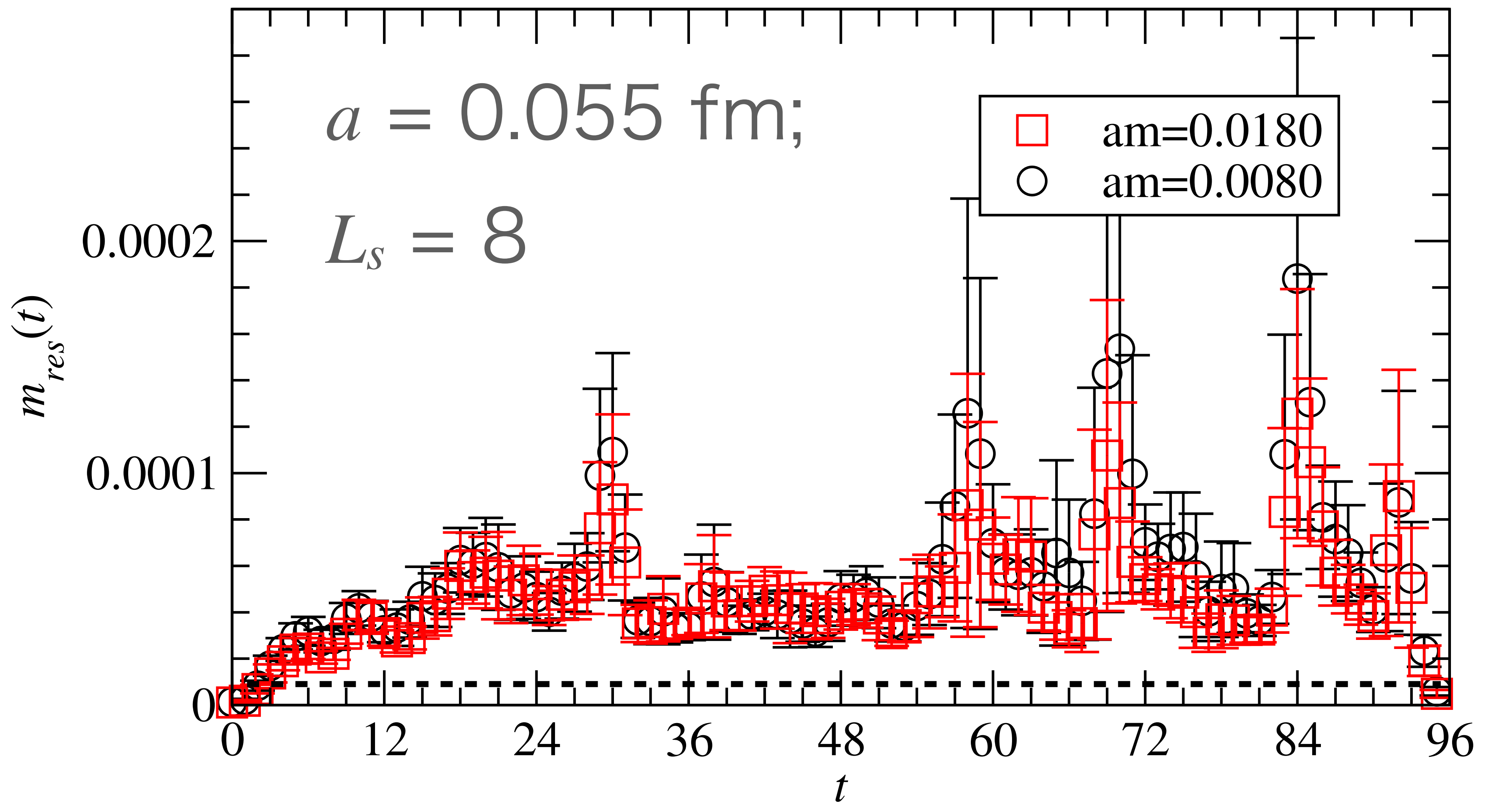
$$am_{res}(t) = \frac{\sum_{\boldsymbol{x},\boldsymbol{y}} \langle \mathrm{Tr}[G^\dagger(\boldsymbol{x},t;\boldsymbol{y},0)\Delta_{GW} G(\boldsymbol{x},t;\boldsymbol{y},0)]\rangle}{\sum_{\boldsymbol{x},\boldsymbol{y}} \langle \mathrm{Tr}[G^\dagger(\boldsymbol{x},t;\boldsymbol{y},0) G(\boldsymbol{x},t;\boldsymbol{y},0)]\rangle}$$

or

$$am_{res}(t) = \frac{\sum_{t,\boldsymbol{x},\boldsymbol{y}} \langle \mathrm{Tr}[G^\dagger(\boldsymbol{x},t;\boldsymbol{y},0)\Delta_{GW} G(\boldsymbol{x},t;\boldsymbol{y},0)]\rangle}{\sum_{t,\boldsymbol{x},\boldsymbol{y}} \langle \mathrm{Tr}[G^\dagger(\boldsymbol{x},t;\boldsymbol{y},0) G(\boldsymbol{x},t;\boldsymbol{y},0)]\rangle}$$

Not unique. Much smaller at short distances.





Even defined for eigenmodes

- Violation may and does depend on the states. We can study more details through

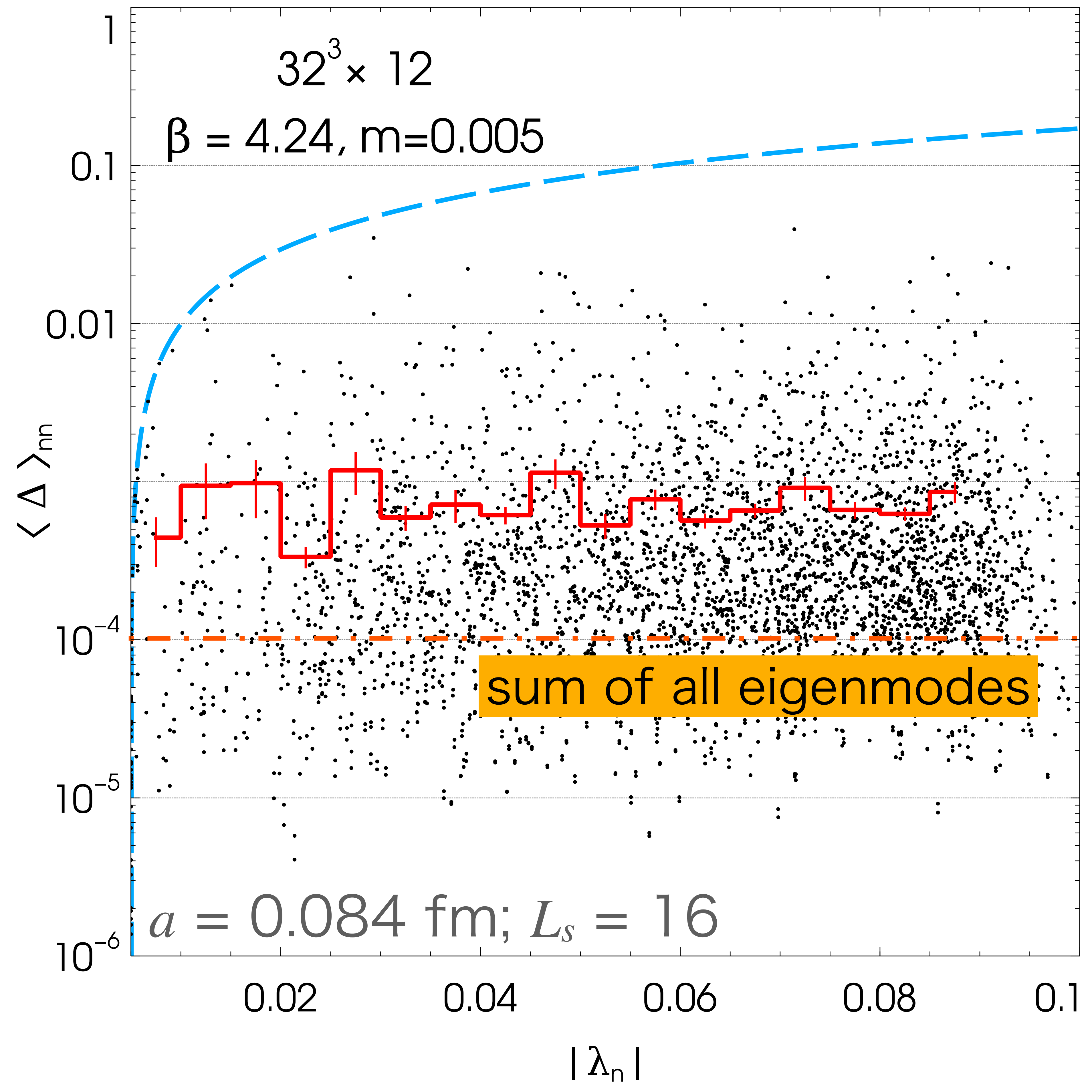
$$\langle \Delta_{GW} \rangle_{nn} \equiv \langle \psi_n | \Delta_{GW} | \psi_n \rangle$$

for each eigenstate $|\psi_n\rangle$ of H.

- Violation is typically enhanced for low-lying modes; the effect for individual eigenmode is very different (x100).

What is its effect on (near-)zero modes?

JLQCD (2016)



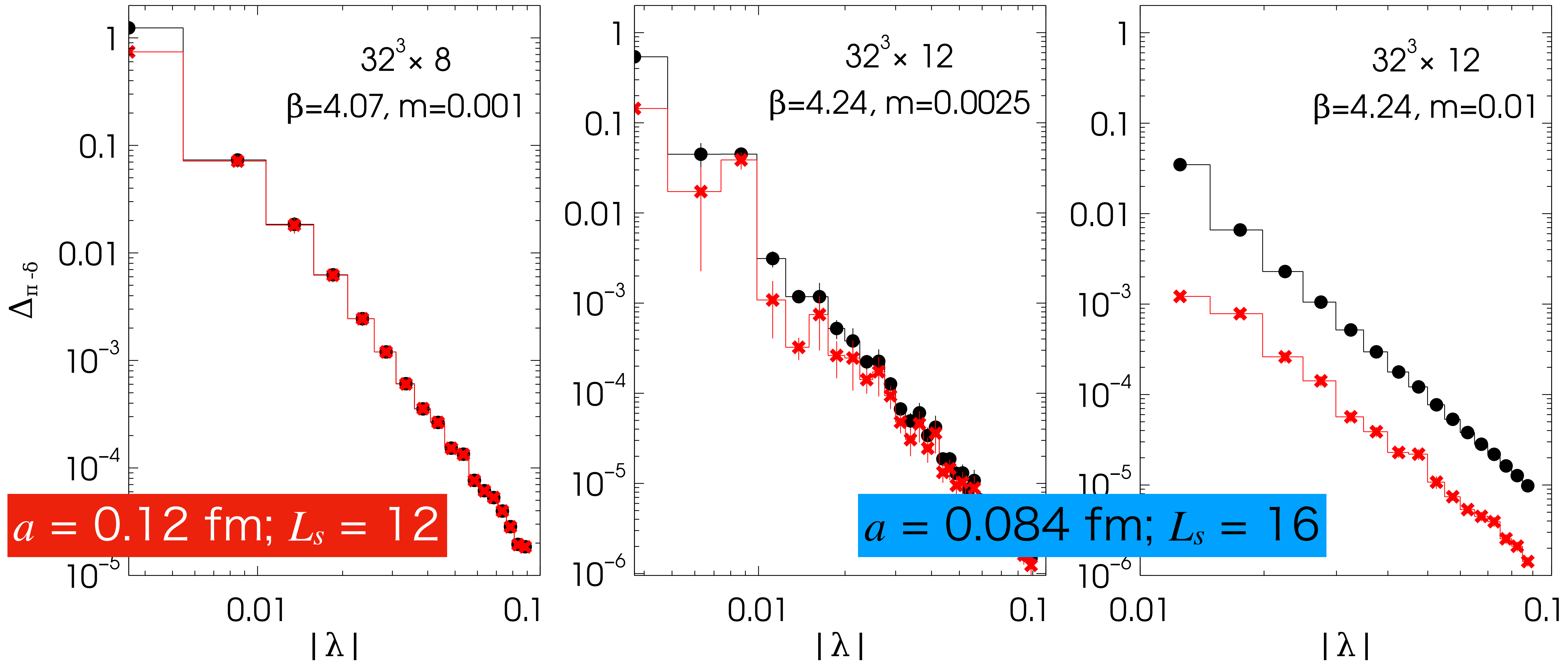
Eigenvalue decomposition

Identify the effect of the GW violation through eigenmodes.

$$\Delta_{\pi-\delta} = \frac{1}{V(1-m^2)^2}\sum_n \frac{2m^2(1-\lambda_n^2)^2}{\lambda_n^4} + \frac{1}{V(1-m)^2}\sum_n \left[\frac{h_{nn}}{\lambda_n} - \frac{4g_{nn}}{\lambda_n}\right]$$

red: GW violating

JLQCD (2016)



Signal is dominated by the GW violating effect, especially for coarse lattice and/or light quarks.

Subtlety of $U_A(1)$ susceptibility

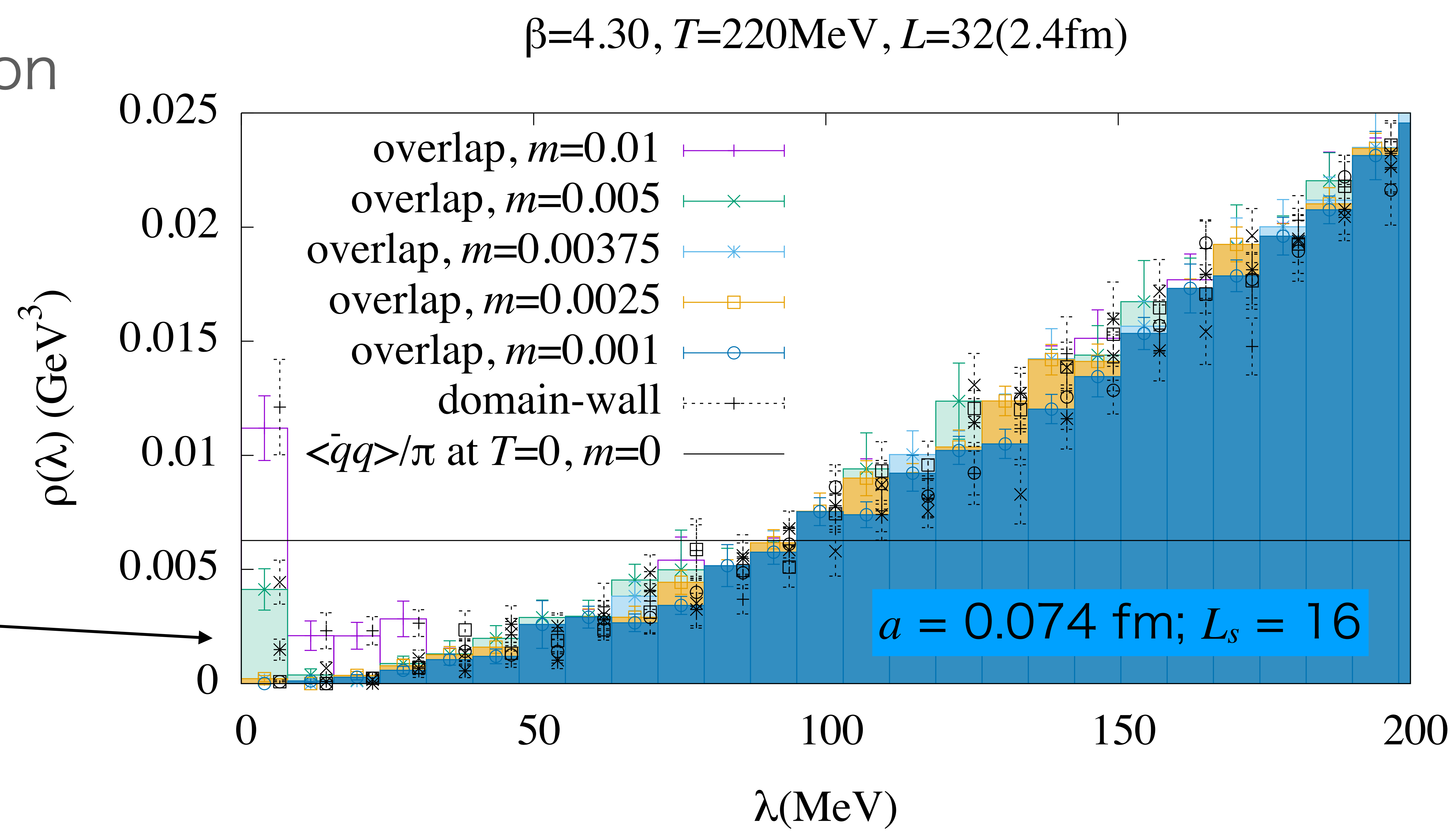
1. Dominated by (near-zero) modes.
2. Near-zero modes vulnerable to the GW violation.

Want to be careful:

- Reweighting to the exact GW fermion (that we call overlap)
- Not very significant for fine lattice and heavier quarks.

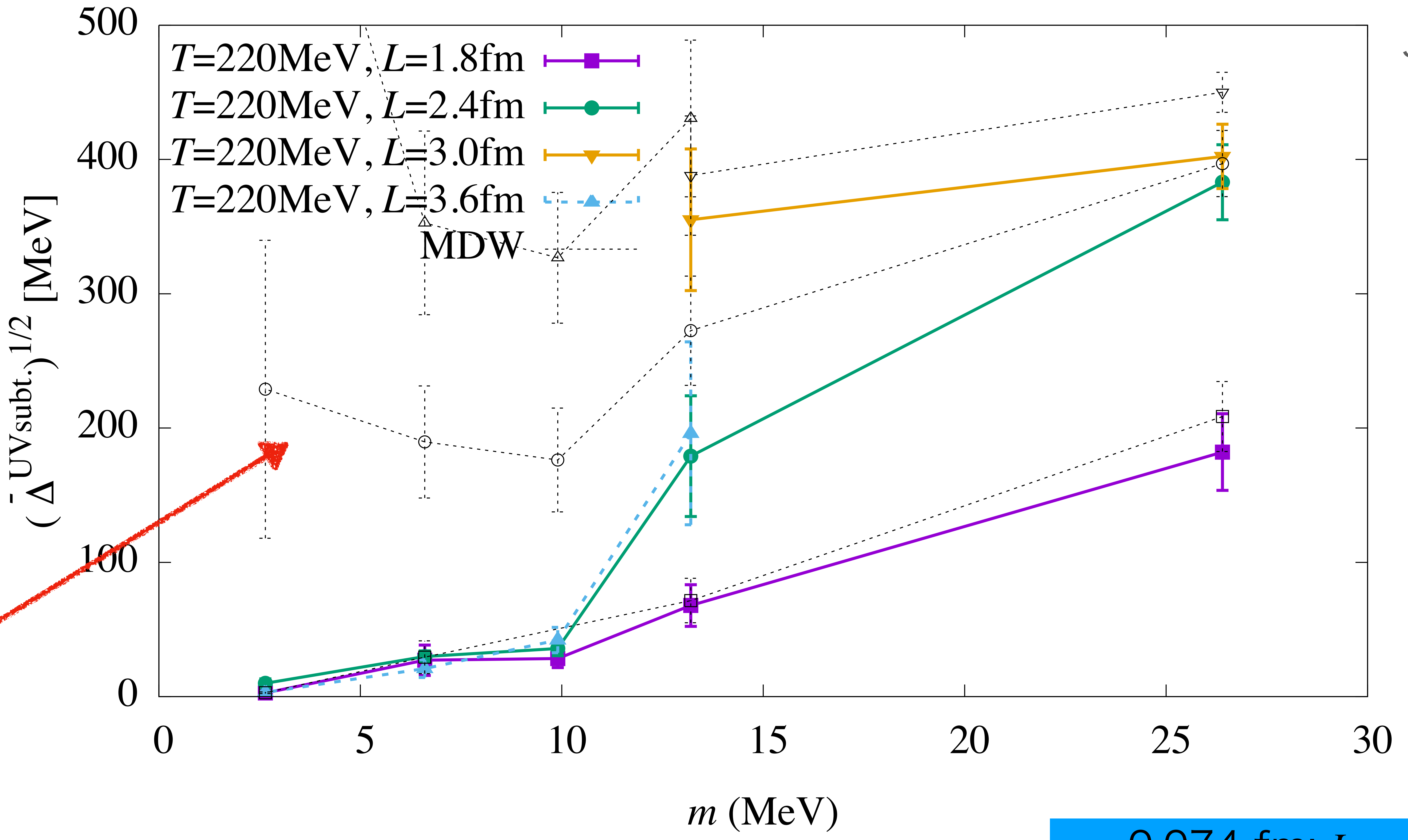
- But, if you focus on the most important region …

JLQCD (2020)



$U_A(1)$ susceptibility after (exact) zero-mode subtraction

JLQCD (2020)



Danger of the (tiny) GW violation.

m_{res} = 0.14(6) MeV

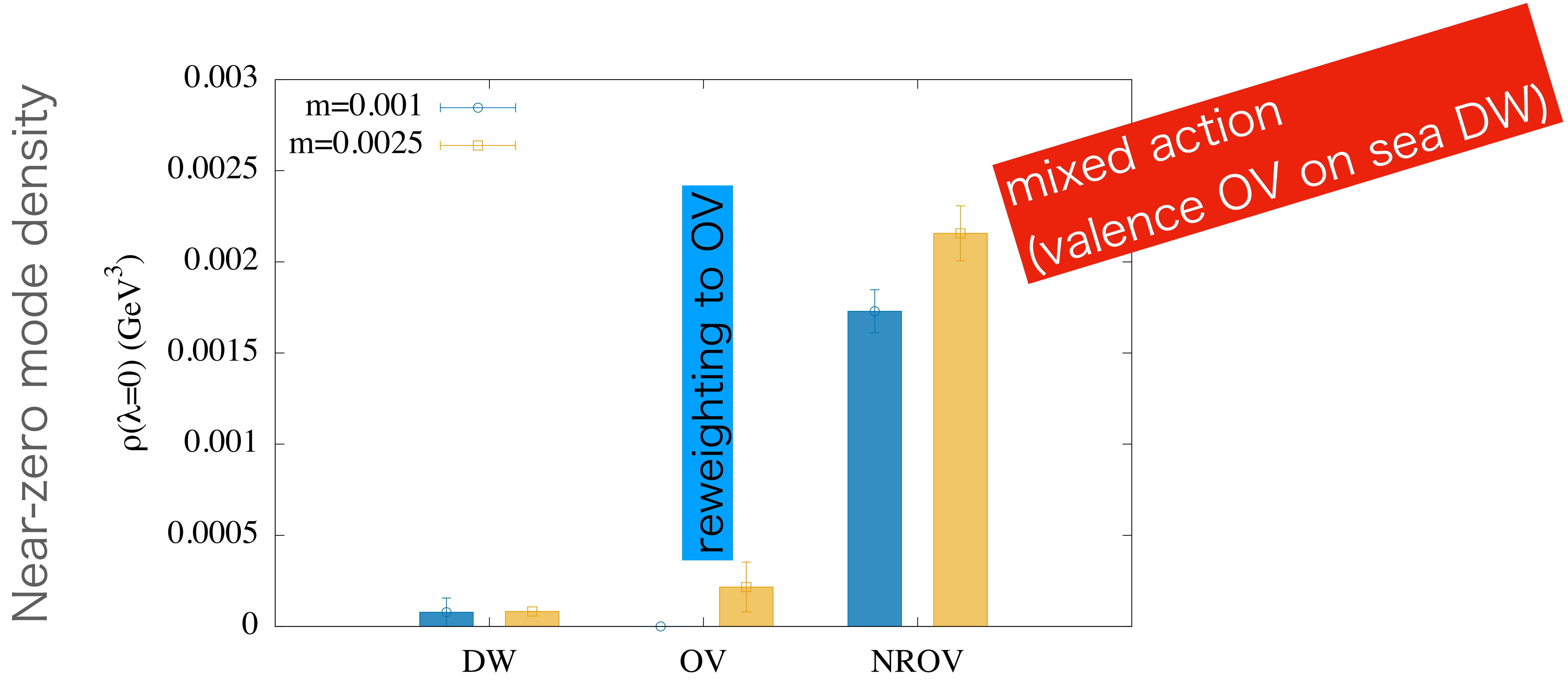
a = 0.074 fm; L_s = 16

Even with Mobius domain-wall fermion at a fine lattice ⋯

More subtleties

Mixed action?

Do not try to improve your valence quark only.



Discretization error can become huge (x10).
Continuum extrapolation would hardly work.

Staggered fermion

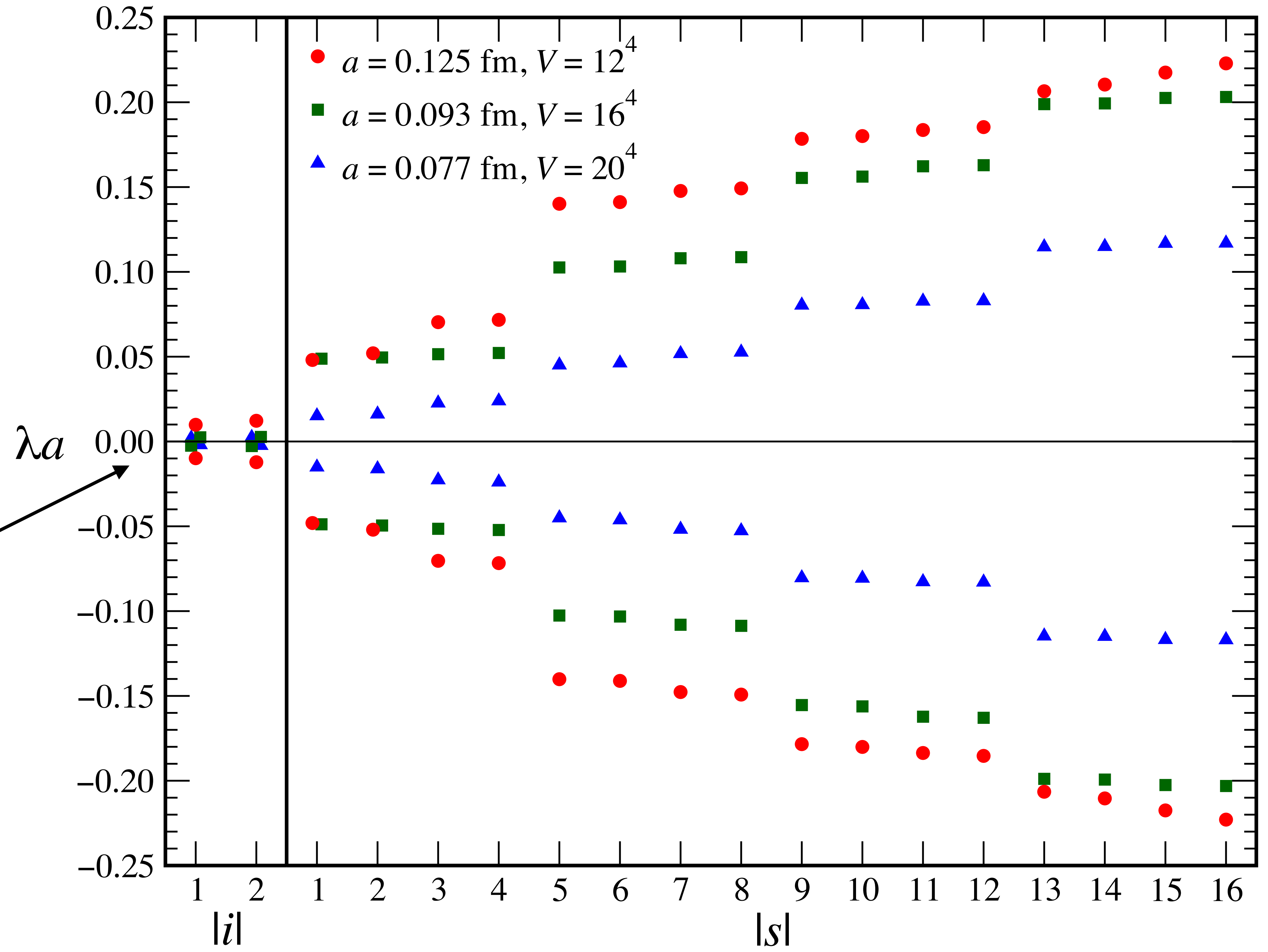
What is the effect of **taste-breaking** to

$$\Delta_{\pi-\delta} = \int_0^\infty d\lambda\, \rho(\lambda) \frac{2m^2}{(m^2+\lambda^2)^2}$$

- 4 Eigenvalues for each continuum-like mode. Non-degeneracy is a sign of chiral violation.
- Typical size with HISQ ~ 10 MeV.

It would be in a dangerous region for the $U_A(1)$ susceptibility. When m = 5 MeV, it may over-/under-estimate it by a factor of O(10).

HPQCD and Fermilab (2011)



On a |Q| = 1 config.

Summary

- Physics of near-zero mode in finite temperature QCD is interesting: topological susceptibility, $U_A(1)$ susceptibility, etc.
- In the lattice calculation, chiral symmetry is extremely important. Unless it is satisfied super-precisely, your calculation will end up with large (or huge) discretization effect. The continuum limit with a^2 may be an illusion.
- Relevance to physics is extraordinary:
  - poster by H. Fukaya (Wed), “What is chiral susceptibility probing?”
  - talk by K. Suzuki (Thu), “Axial U(1) anomaly at high temperature with chiral fermions”
  - See also, poster by I. Kanamori (Wed), “2+1 Flavor Fine Lattice Simulations for Finite Temperature with Domain Wall Fermions” and Y. Nakamura, “Finite temperature phase transition for three flavor QCD with Mobius-domain wall fermions”