Peeking into the θ vacuum

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Refs. Kitano, Matsudo, NY, Yamazaki, PLB822, 136657 (2021)
Kitano, NY, Yamazaki, JHEP02 (2021) 073

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Introduction

$$\theta\text{-term:}\quad \mathscr{L}_\theta = -\, i\theta \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

- None of symmetries of the SM constrains the value of θ (usually called $\bar{\theta}$).

  Why is $\theta \ll 1$? (strong CP problem)

- **Effects of the θ term on various observables** have been actively studied experimentally, theoretically and on the lattice.

Focus on **the effect of θ on the vacuum of 4d SU(N) YM.**

How to explore the effect of θ on the vacuum

Look at vacuum energy density, $f(\theta)$

$$e^{-Vf(\theta)} = \frac{Z(\theta)}{Z(0)}$$

where

$$Z(\theta) = \int \mathscr{D}U\, e^{-S_{\rm YM} + i\theta Q}$$

$$Q = \int d^4x\, q(x)$$

- For $SU(N)$ YM theory,
  $Q \in \mathbb{Z} \Rightarrow f(\theta) = f(\theta + 2\pi)$
  $S_{\rm YM}$ is CP even $\Rightarrow f(\theta) = f(-\theta)$

$$\Downarrow$$

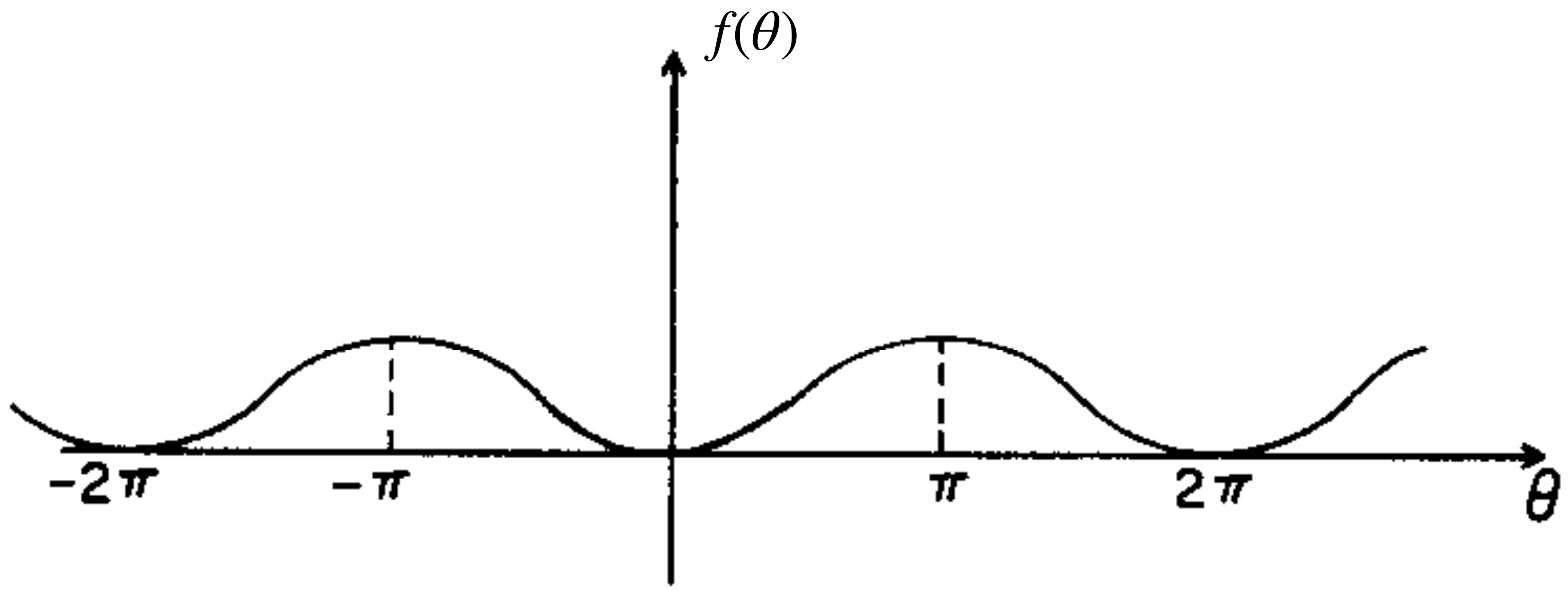
$$f(\pi - \theta') = f(\pi + \theta')$$

Action is CP symmetric at $\theta = 0$ and π.

Expected θ dependence: instanton vs large-N_c

Dilute instanton gas approximation (DIGA)

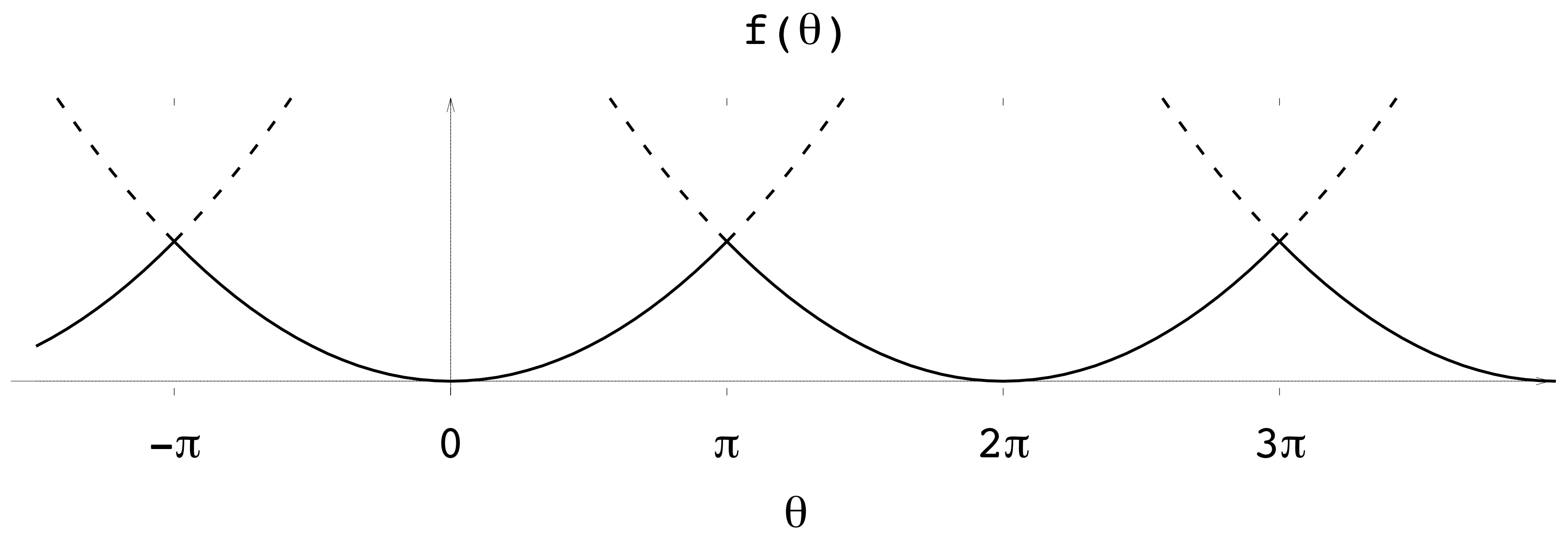
$\Rightarrow f(\theta) = \chi(1 - \cos\theta)$



- a single branch
- smooth everywhere

Large N_c argument [Witten (1980, 1998)]

$\Rightarrow f(\theta) = \chi/2 \min_{k\in\mathbb{Z}}(\theta + 2\pi k)^2 + O(1/N_c^2)$



- many branches
- spontaneous CPV (1st order PT) at $\theta = \pm(2n+1)\pi$
- order parameter $df(\theta)/d\theta|_{\theta=\pi} = -i\langle q(x)\rangle_{\theta=\pi}$

the behavior of $f(\theta)$ around $\theta = \pi$

Lesson from 2d CP^{N-1} model

$$\mathscr{L} = \frac{N}{2g}\overline{D_\mu z}\, D_\mu z - i\theta q$$

z : N-component complex scalar field with $\bar{z}z = 1$

$$D_\mu = \partial_\mu + iA_\mu \ , \quad A_\mu = i\bar{z}\partial_\mu z$$

$$q(x) = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu = \frac{i}{2\pi}\epsilon_{\mu\nu}\overline{D_\mu z}\, D_\mu z$$

- Good testing ground for 4d $SU(N)$ YM because of many similarities

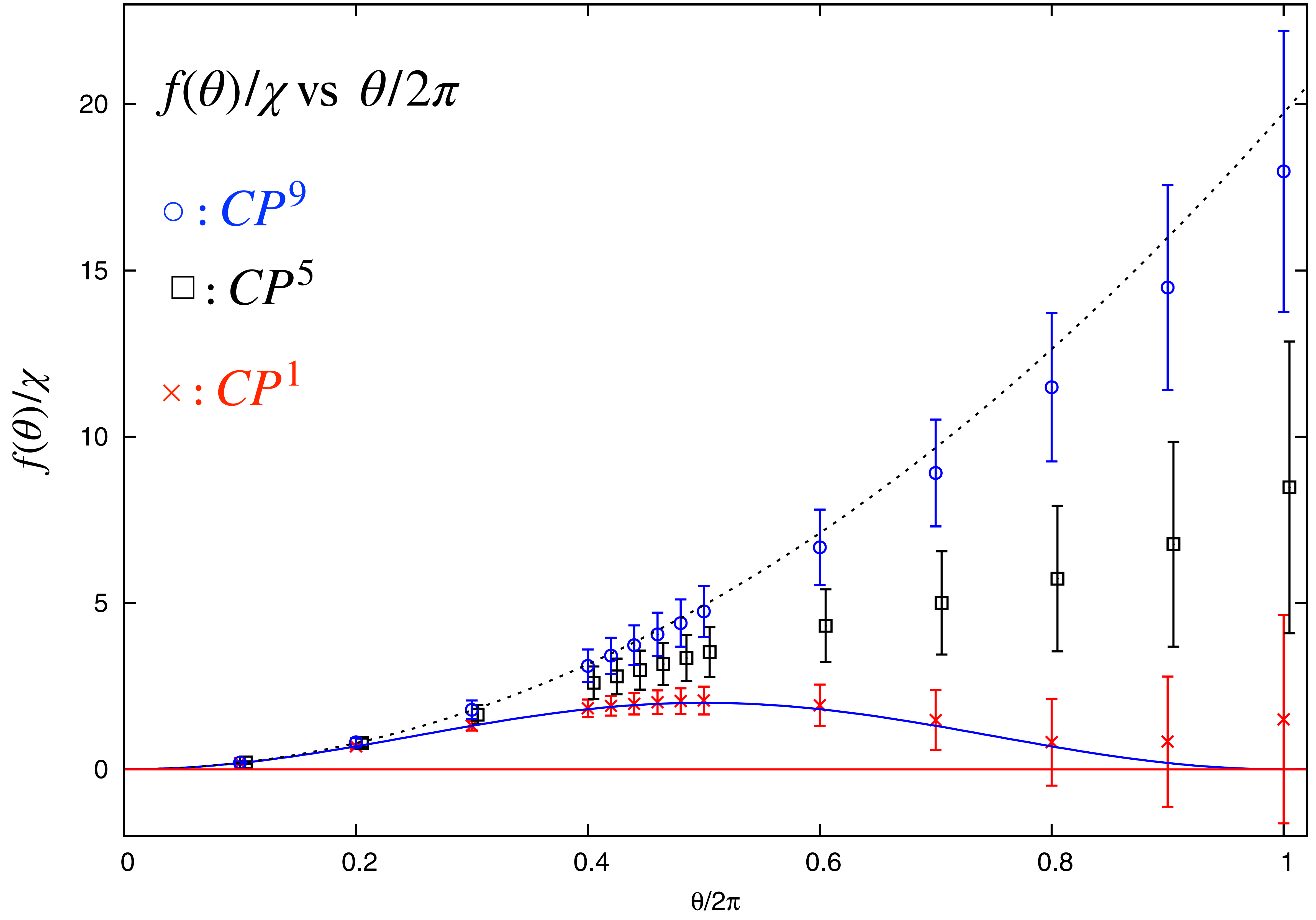
  [asymptotically free, dynamical mass gap, instanton, $1/N$ expandable, ...]

- At $\theta = \pi$, gapped and CP broken for $N \geq 3$
- But CP^1 (*i.e.* $N = 2$) **is exceptional !**

  $\Rightarrow$ gapless and no CPV at $\theta = \pi$ ($\Longleftrightarrow$Haldane conjecture)

$f(\theta)$ in 2d CP^{N-1} model (lattice)

[Keith-Hynes and Thacker (2008)]



$N = 10,\ 6,\ 2$ were studied.

- CP^9 consistent with large N
- $f(\theta)/\chi$ decreases as N decreases.
- CP^9 and CP^5 shows CPV at $\theta = \pi$.
- CP^1 is consistent with the DIGA

$$f(\theta) = \chi(1 - \cos\theta)$$

Lattice calculations of $f(\theta)$ in 4d $SU(N)$

$$\mathcal{L}_\theta = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - i\theta q$$

- "i" makes direct lattice calculation difficult/impossible (sign problem).

- Taylor expansion around $\theta = 0$

$$f(\theta) = \frac{\chi}{2}\theta^2(1 + b_2\theta^2 + b_4\theta^4 + \cdots)$$

and determines each coefficient on the lattice by

$$\chi = \frac{\langle Q^2 \rangle_{\theta=0}}{V}$$

$$b_2 = -\frac{\langle Q^4 \rangle_{\theta=0} - 3\langle Q^2 \rangle^2_{\theta=0}}{12\langle Q^2 \rangle_{\theta=0}}$$

$$b_4 = \frac{\langle Q^6 \rangle_{\theta=0} - 15\langle Q^2 \rangle_{\theta=0}\langle Q^4 \rangle_{\theta=0} + 30\langle Q^2 \rangle^3_{\theta=0}}{360\langle Q^2 \rangle_{\theta=0}}$$

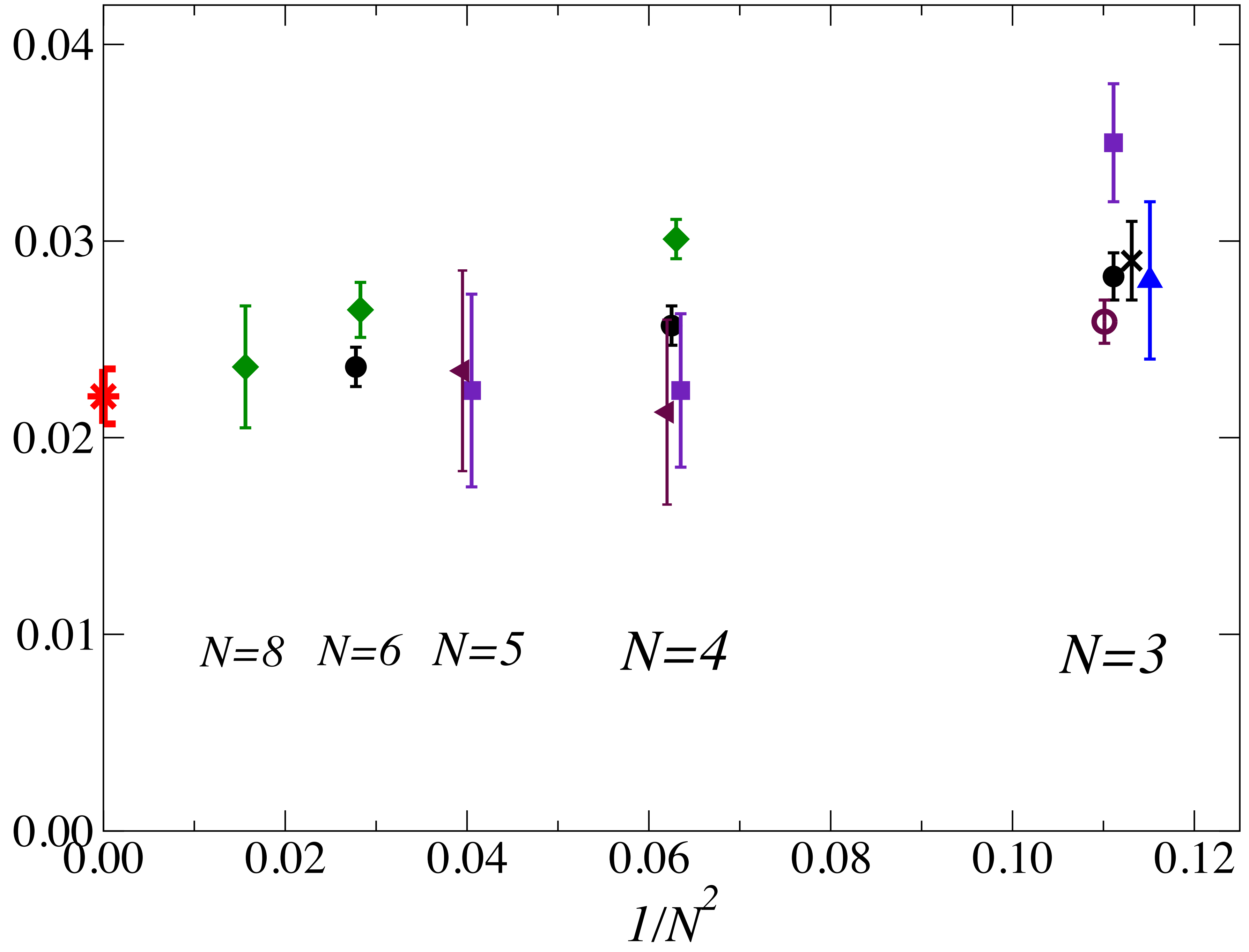
$$\vdots$$

c.f. $f(\theta) = \chi(1 - \cos\theta) \;\Rightarrow b_2 = -\frac{1}{12}, \;\cdots$

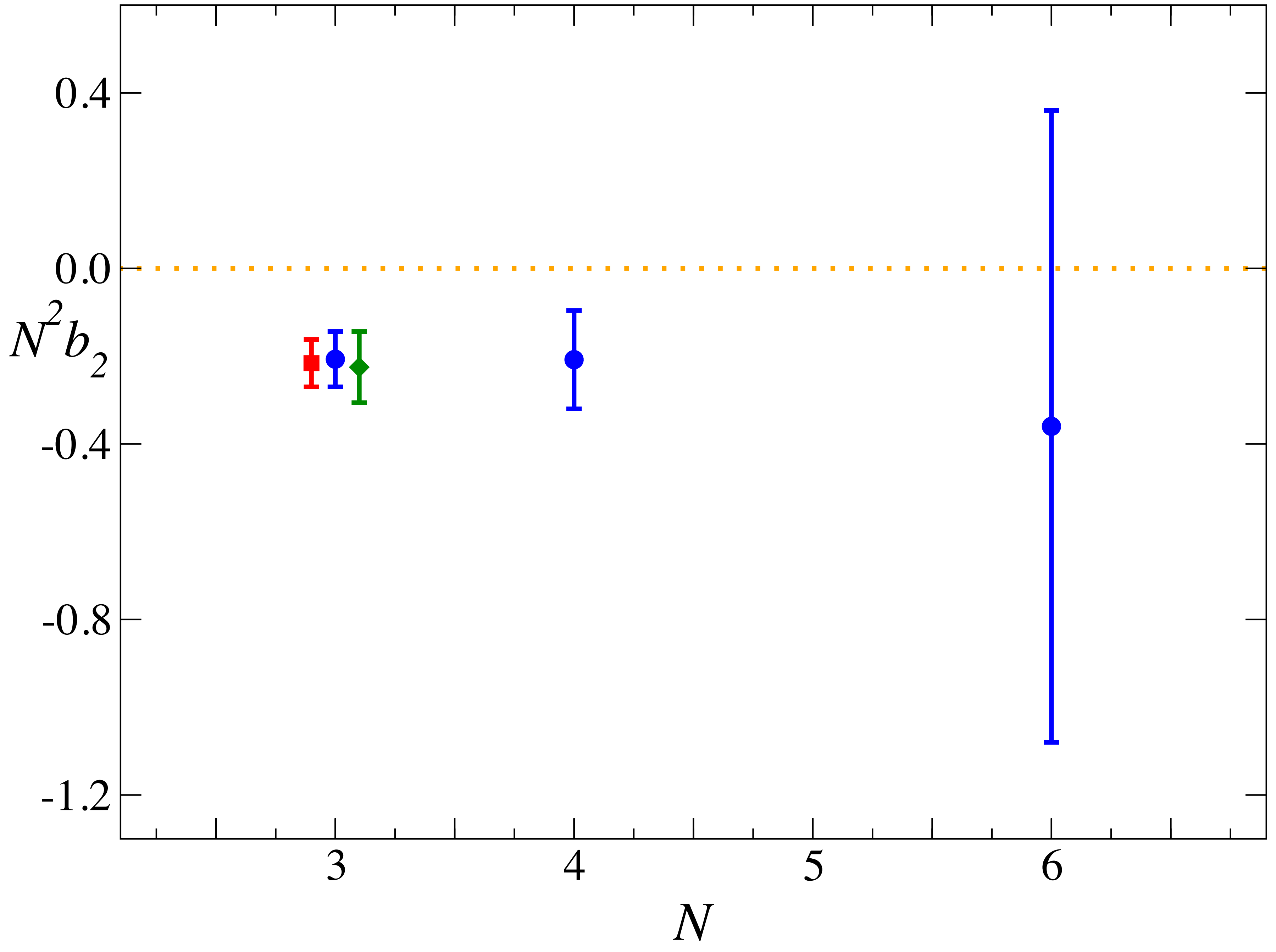
First two coefficients

[Review by Vicari and Panagopoulos (2018)]

$$\chi/\sigma^2 = C_\infty + \frac{c_2}{N_c^2} + O(1/N_c^4)$$



$$b_2 = \frac{\bar{b}_2}{N_c^2} + O(1/N_c^4)$$

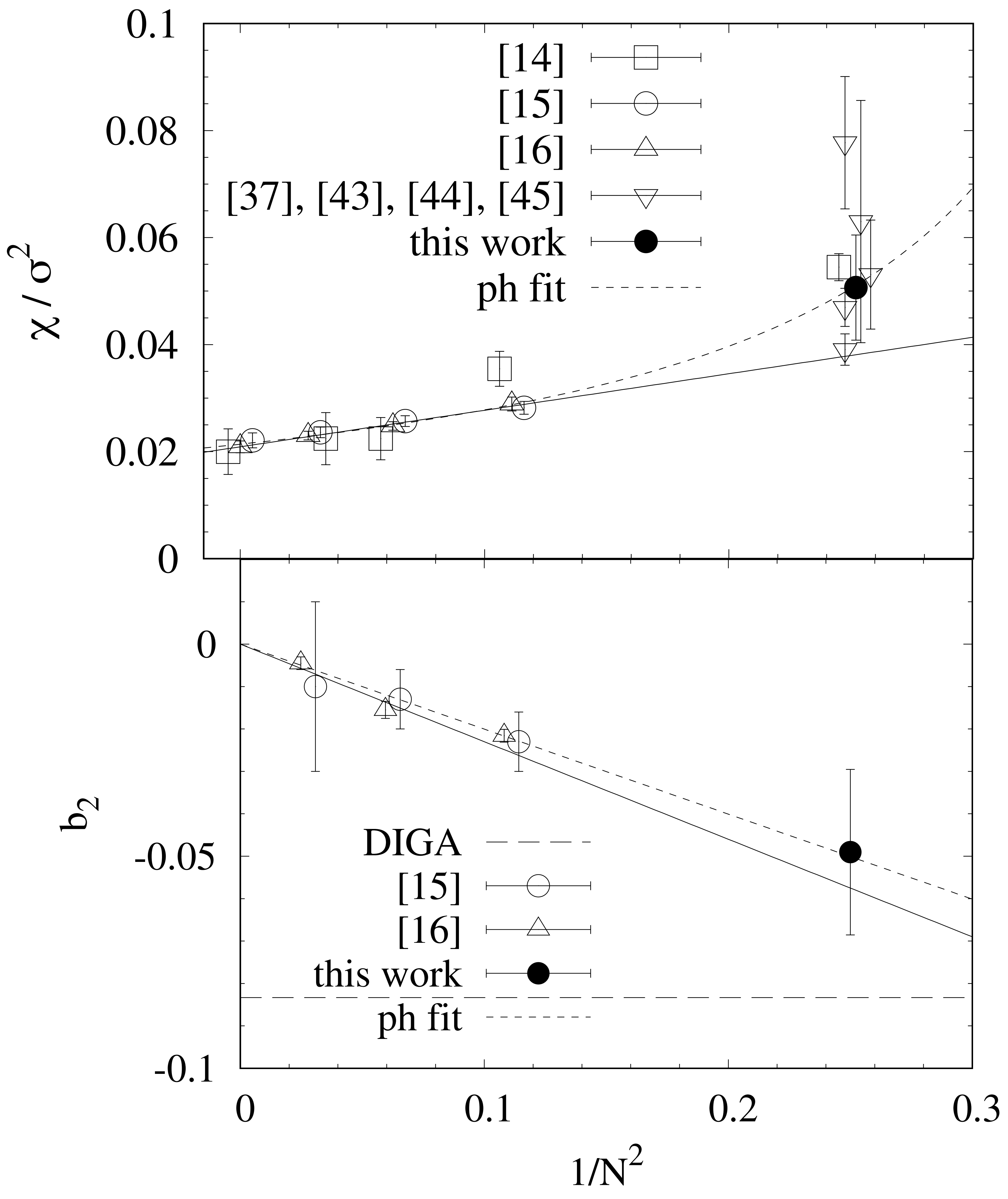


Correction to the large N_c limit looks small ⇒ nothing special happens down to $N_c = 3$

Is N_c =2 exceptional like 2d CP^1?

χ and b_2 in $SU(2)$ at $T = 0$

[Kitano, NY, Yamazaki (2021)]



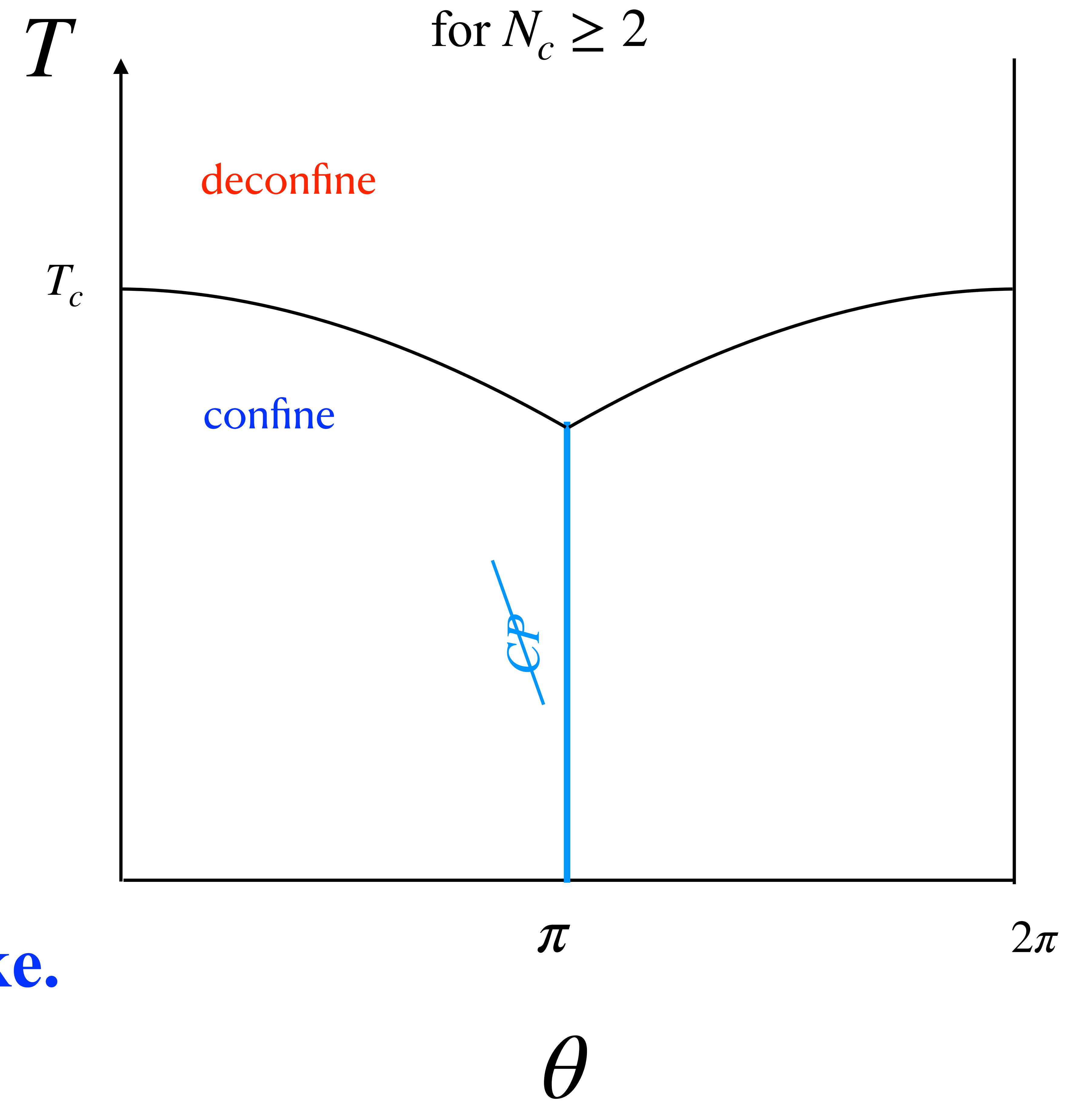
- smoothly connected to large N_c limit
  $\Rightarrow N_c \in \mathbb{Z}$ can be analytically continued to $\mathbb{R}$
  $\Rightarrow f(\theta)$ is a smooth function of N_c
- $b_2 \neq -\dfrac{1}{12}$ (*i.e.* not instanton-like)

Speculated that SU(2) belongs to large N_c class and *CPV* takes places at $\theta = \pi$.

Conjectured θ-T phase diagram

- Large N_c argument $\Rightarrow$ CPV in the vacuum at $\theta = \pi$
- At high T, instanton calc. $\Rightarrow$ no CPV at $\theta = \pi$ at high T
- "For general N_c, CP has to be broken at $\theta = \pi$ if the vacuum is in the confining phase."
  [Gaiotto, et al.(2017)], [Kitano, Suyama, NY(2017)]
- Numerical evidences and our speculation $\Rightarrow$ *CPV* for $N_c \geq 2$
- $T_c(\theta)$ has been known for SU(3) YM around $\theta = 0$.
  [D'Elia, Negro(2012, 2013)].

4d $SU(N_c)$ YM



We want to check **the conjecture that $SU(2)$ is large N_c-like.**

$\Rightarrow$ Naive lattice method fails. How to check?

A method without any expansion

[Keith-Hynes and Thacker (2008)]

[Kitano,Matsudo,NY,Yamazaki(2021)]

Replace Q with $Q_{\text{sub}} = \sum_{x \in V_{\text{sub}}} q(x) \;\; \notin \mathbb{Z}$

where $V_{\text{sub}} = l^4$ is a sub-volume.

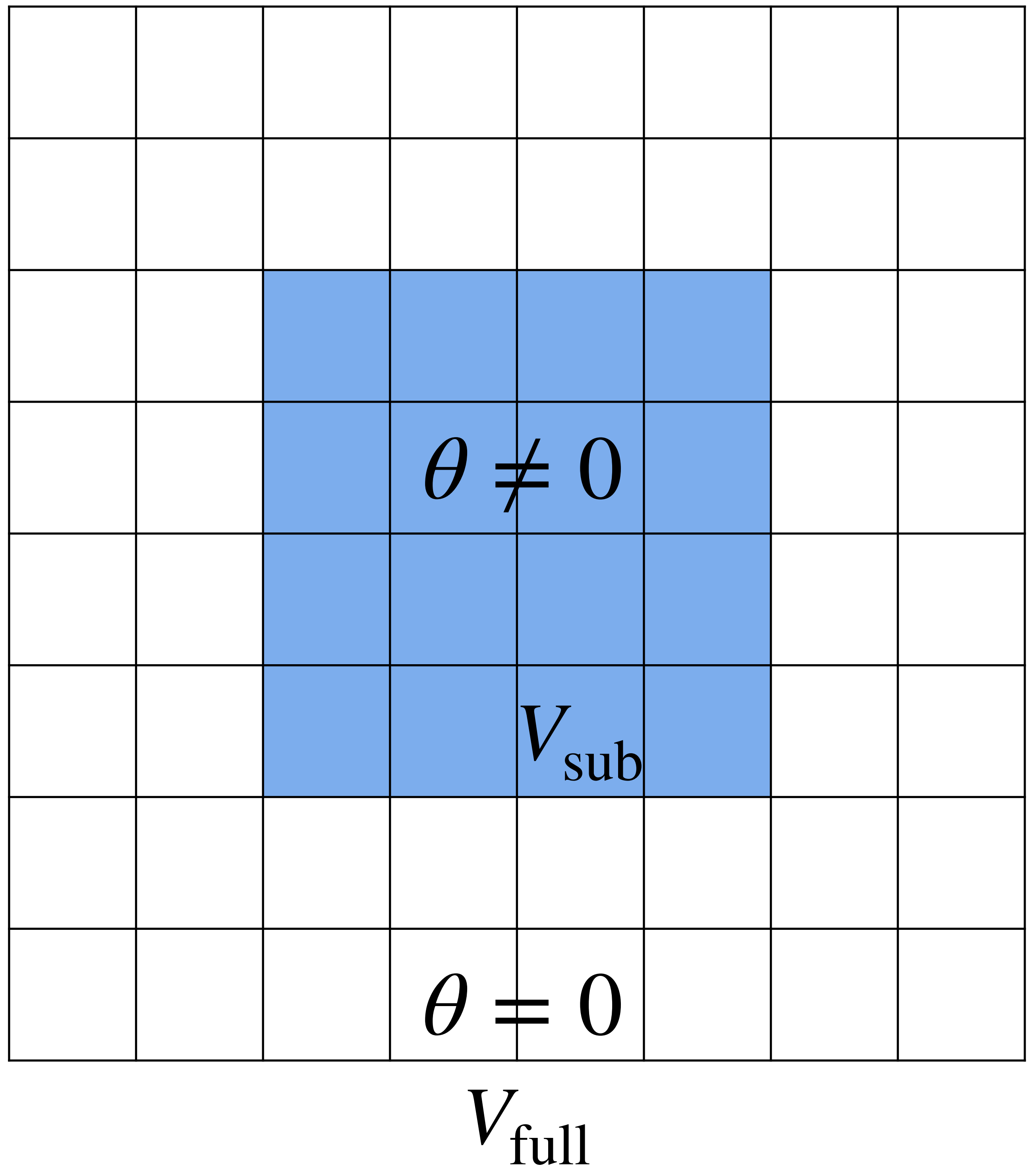
$$e^{-V_{\text{sub}} f_{\text{sub}}(\theta)} = \frac{Z_{\text{sub}}(\theta)}{Z(0)} = \frac{1}{Z(0)} \int \mathcal{D}U \; e^{-S_g + i\theta Q_{\text{sub}}} = \langle e^{i\theta Q_{\text{sub}}} \rangle$$

$$f_{\text{sub}}(\theta) = -\frac{1}{V_{\text{sub}}} \ln\langle \cos(\theta Q_{\text{sub}}) \rangle$$

$$f(\theta) = \lim_{V_{\text{sub}} \to \infty} f_{\text{sub}}(\theta) = \lim_{l \to \infty} \left\{ f(\theta) + \frac{s(\theta)}{l} + O(1/l^2) \right\}$$

with $s(\theta)$ surface tension

sub-volume method



Some remarks on the sub-volume method

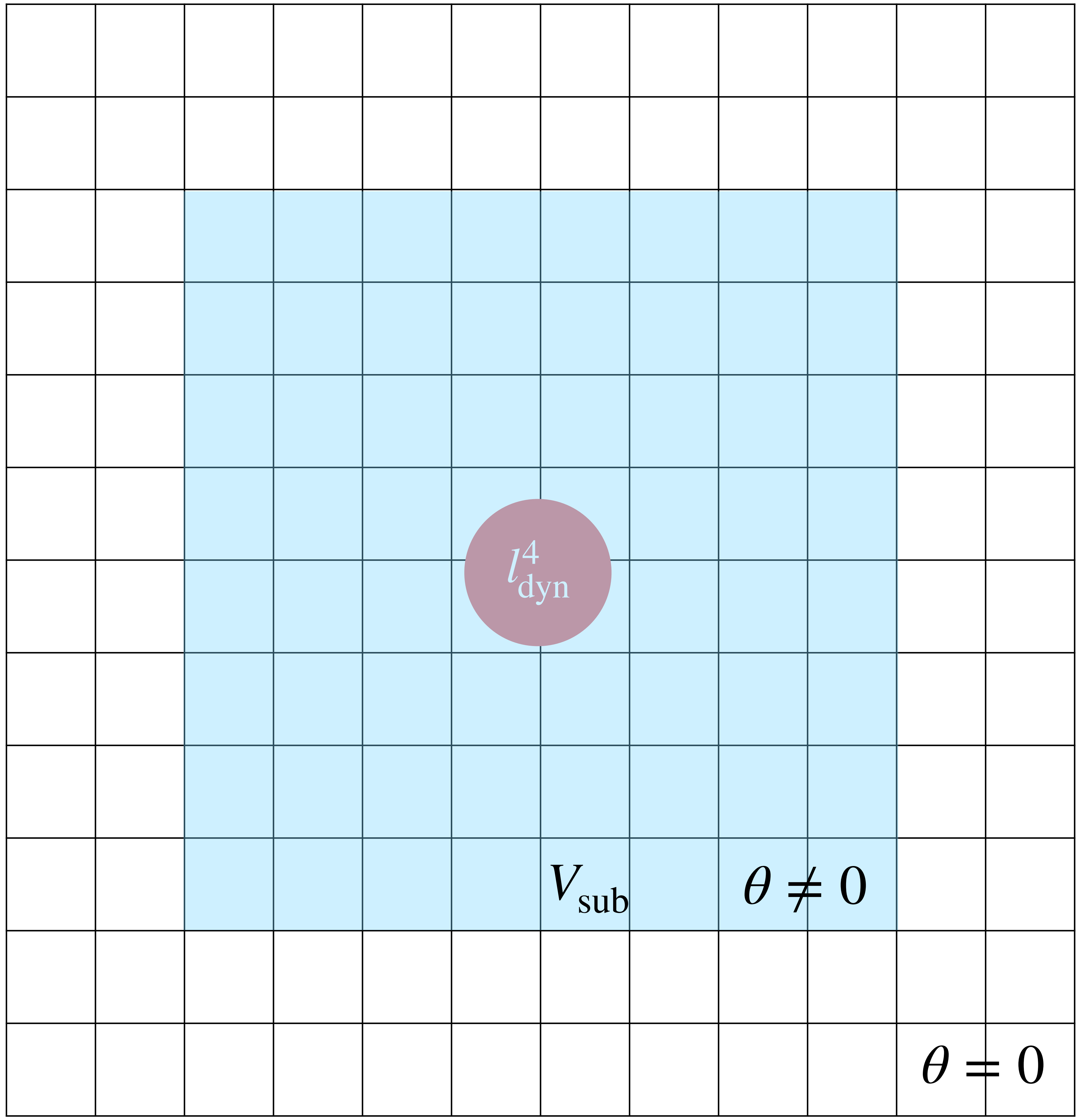
What is the suitable range for $V_{\rm sub}$?

- $V_{\rm sub} \gg l_{\rm dyn}^4$ ($l_{\rm dyn}$: dynamical length scale)

- As long as $V_{\rm sub} \gg l_{\rm dyn}^4$, $f_{\rm sub}(\theta)$ is expected to show a scaling behavior, $f_{\rm sub}(\theta) = f(\theta) + \frac{s(\theta)}{l} + O(1/l^2)$. (*c.f.* static potential)

- As $V_{\rm sub} \to V_{\rm full}$, finite size effects may appear. $\Rightarrow V_{\rm sub} \ll V_{\rm full}$.

$$l_{\rm dyn}^4 \ll V_{\rm sub} \ll V_{\rm full}$$



Some remarks on the sub-volume method (cont'd)

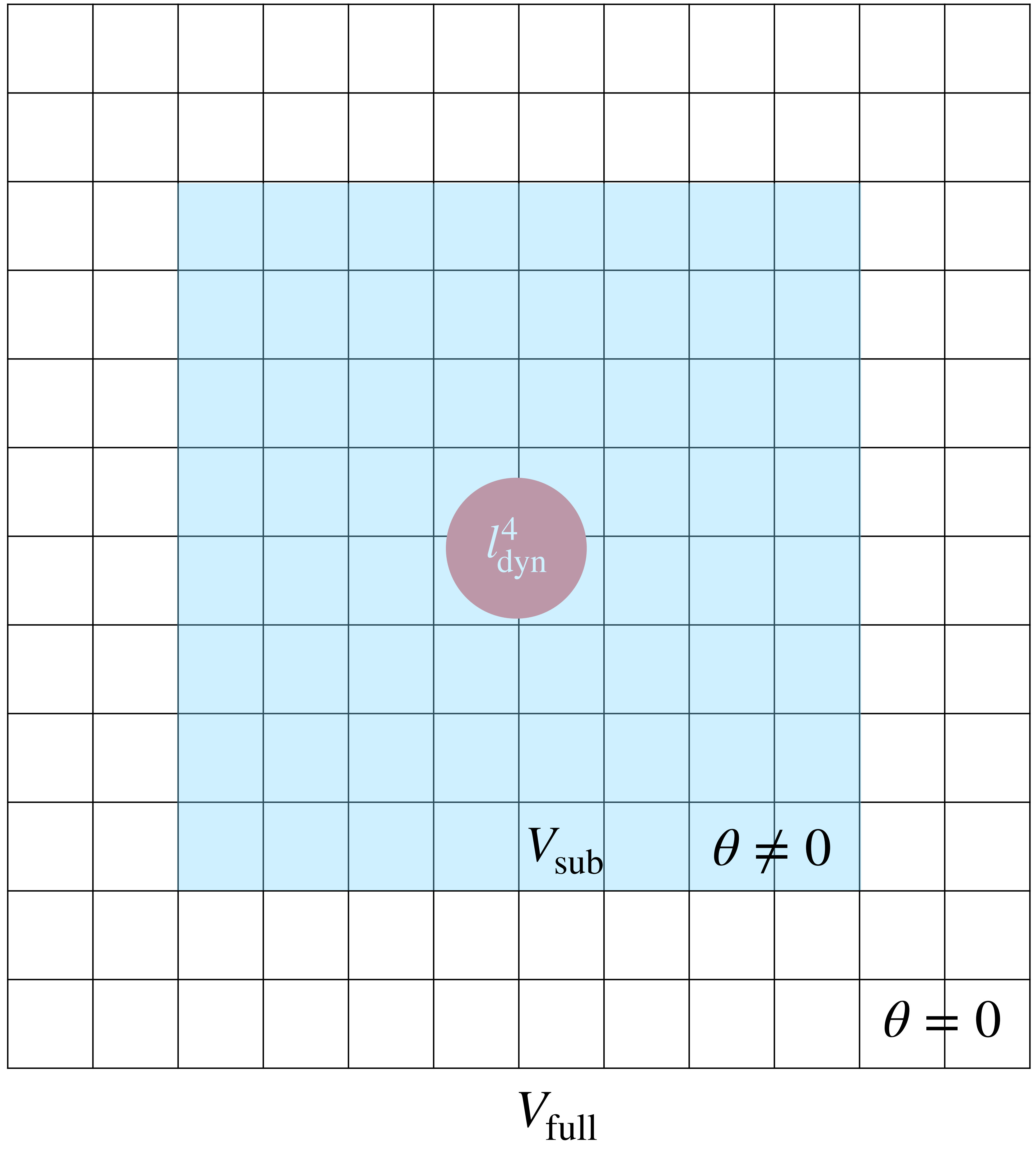
$$f_{\rm sub}(\theta) = -\frac{1}{V_{\rm sub}} \ln\langle \cos(\theta Q_{\rm sub}) \rangle$$

The method fails when $\langle \cos(\theta Q_{\rm sub}) \rangle \leq 0$.

Since $Q_{\rm sub}$ increases with $V_{\rm sub}$, $V_{\rm sub}$ may have an upper bound, $V_{\rm sub}^{\rm max}(\theta)$.

Requirement: $l_{\rm dyn}^4 \ll V_{\rm sub}^{\rm max}(\theta)$

If the window is open, possible to peek into the θ -vacuum.

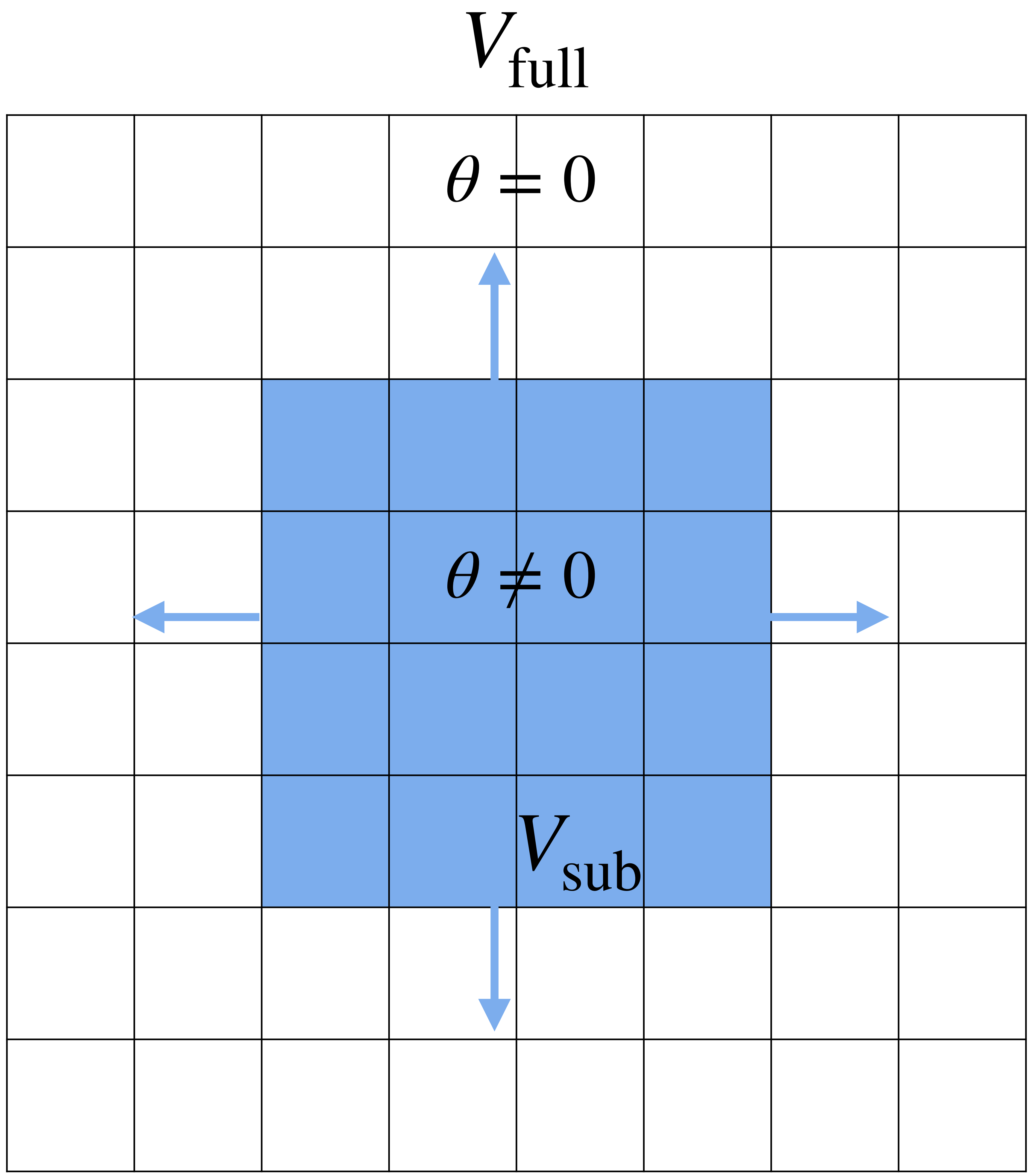


Lattice parameters and observables

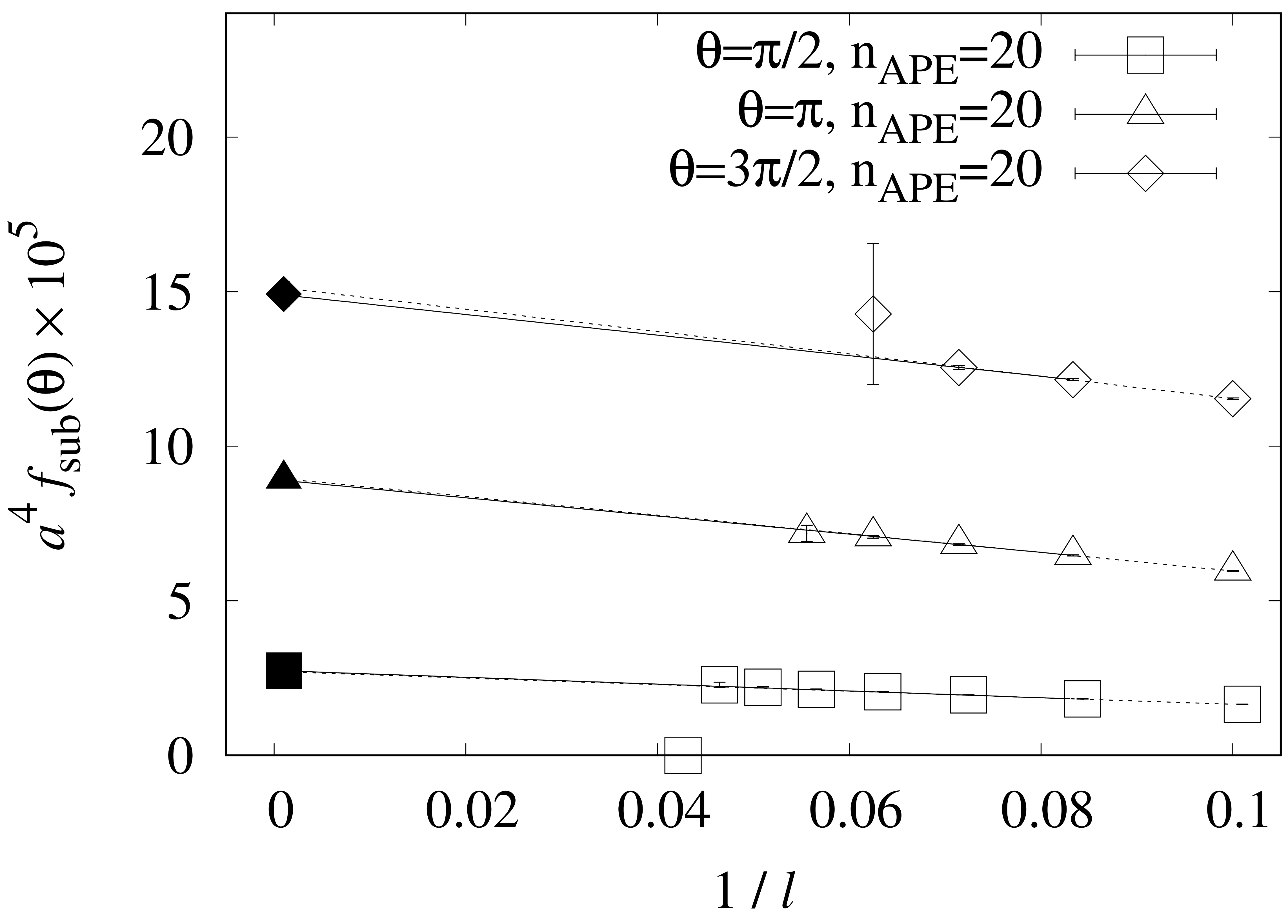
- $SU(2)$ YM theory by Symanzik improved gauge action
- $\beta = \dfrac{4}{g^2} = 1.975$ [cf. $1/(aT_c) = 9.50$]
- $V_{\rm full} = 24^3 \times \{48,\ 8,\ 6\}$ $(T = 0,\ 1.2T_c,\ 1.6T_c)$
- Periodic boundary condition in all directions
- # of configs = { 68000 , 5000 , 5000 }
- $V_{\rm sub} = l^4$ for T=0 and $V_{\rm sub} = l^3 \times N_T$ for finite T
- $Q_{\rm sub} = \sum_{x \in V_{\rm sub}} q(x)$ is calculated after APE smearing and substituted in

  ✓ $f(\theta) = -\lim_{V_{\rm sub}\to\infty} \dfrac{1}{V_{\rm sub}} \ln\langle \cos(\theta Q_{\rm sub}) \rangle$

  ✓ $\dfrac{df(\theta)}{d\theta} = \lim_{V_{\rm sub}\to\infty} \dfrac{1}{V_{\rm sub}} \dfrac{\langle Q_{\rm sub} \sin(\theta Q_{\rm sub}) \rangle}{\langle \cos(\theta Q_{\rm sub}) \rangle}$



$l \to \infty$ limit at $T = 0$

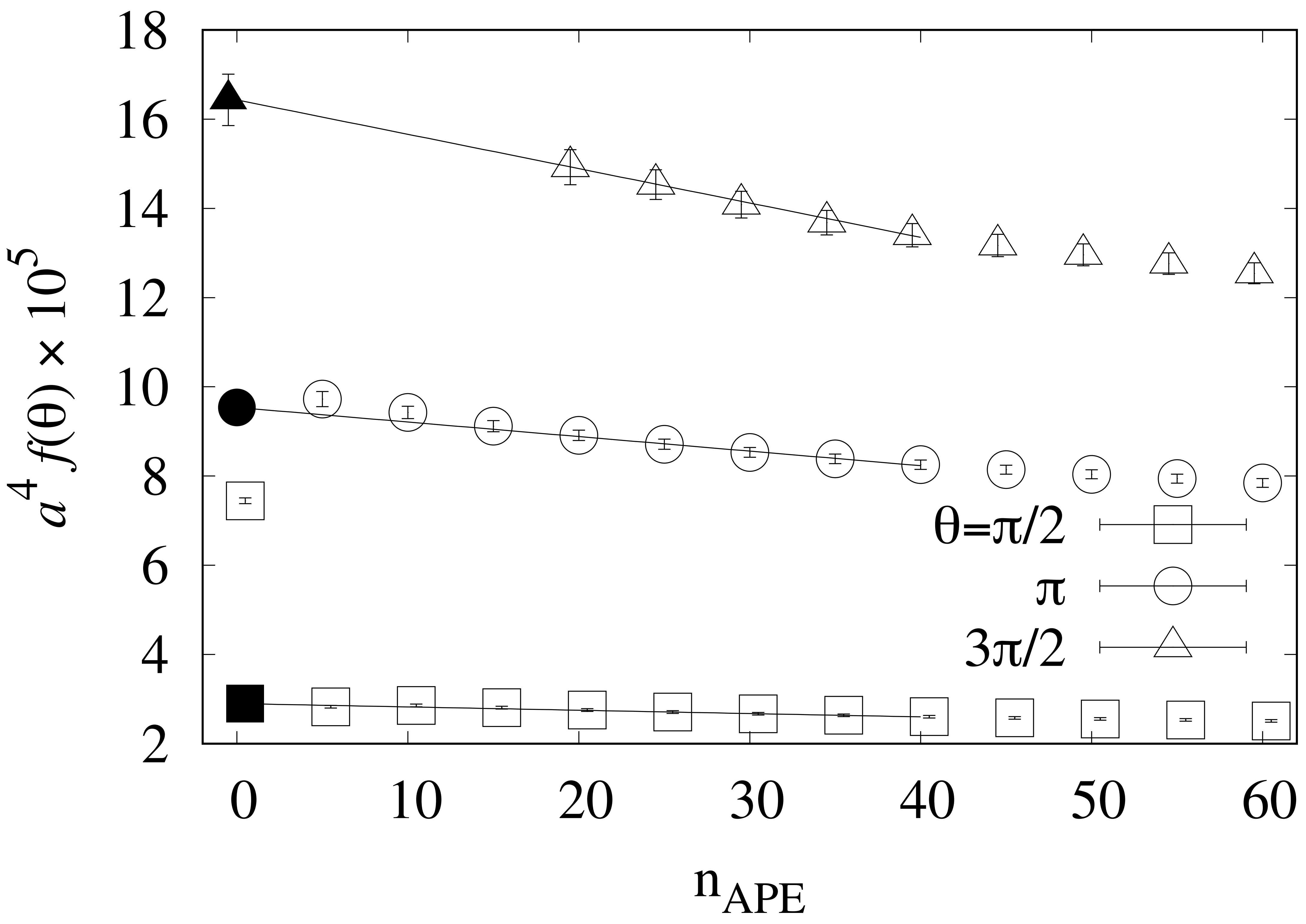


- $V_{\mathrm{sub}} = l^4$ with $l \in \{10,\ 12,\ \cdots,\ 20\}$
- Data in the range of $l_{\mathrm{dyn}}^4 \ll V_{\mathrm{sub}} \ll V_{\mathrm{full}}$ are fitted to

$$f_{\mathrm{sub}}(\theta) = f(\theta) + \frac{as(\theta)}{l}$$

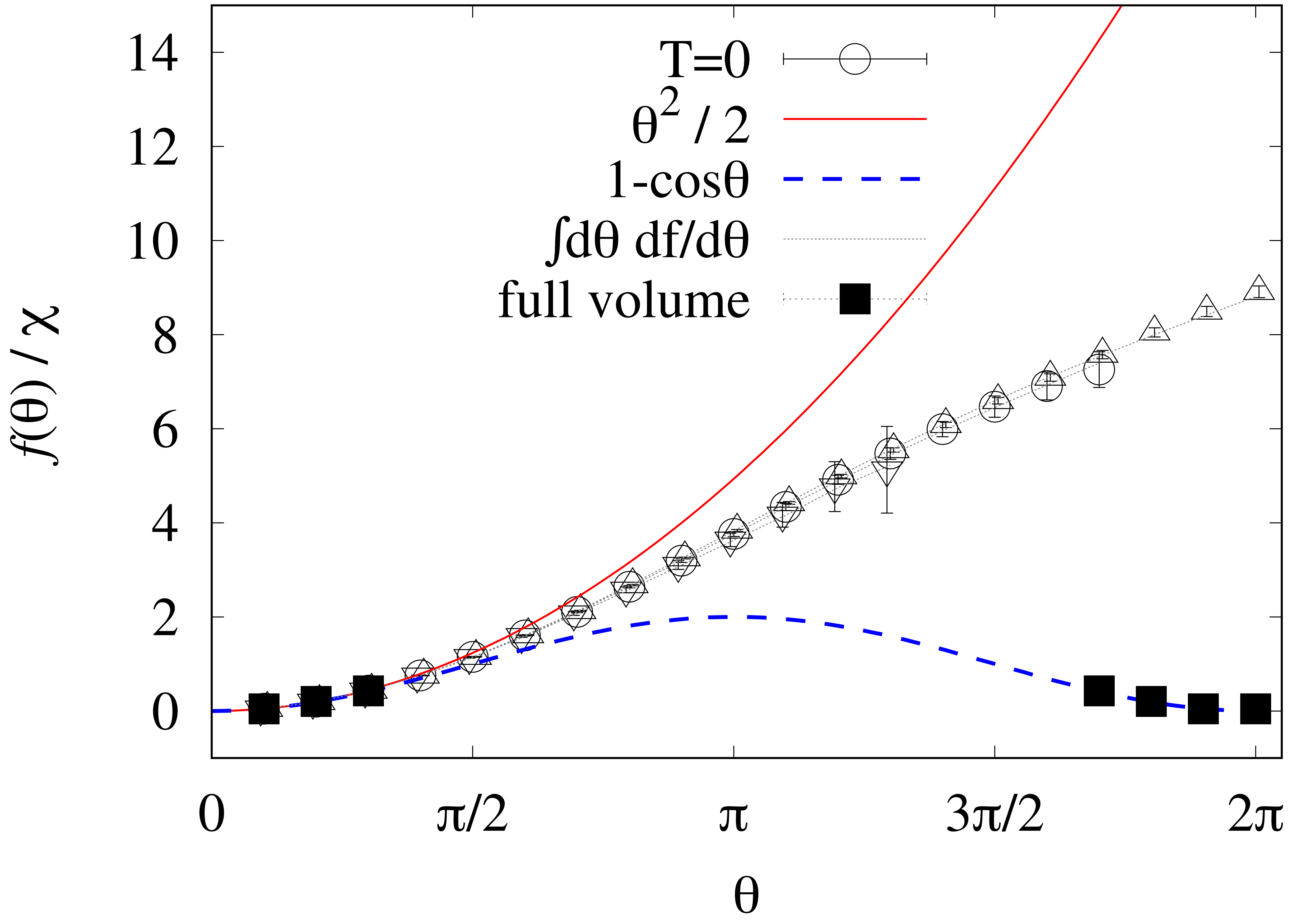
- Linear extrapolation works well.

$n_{\rm APE} \to 0$ limit at $T = 0$



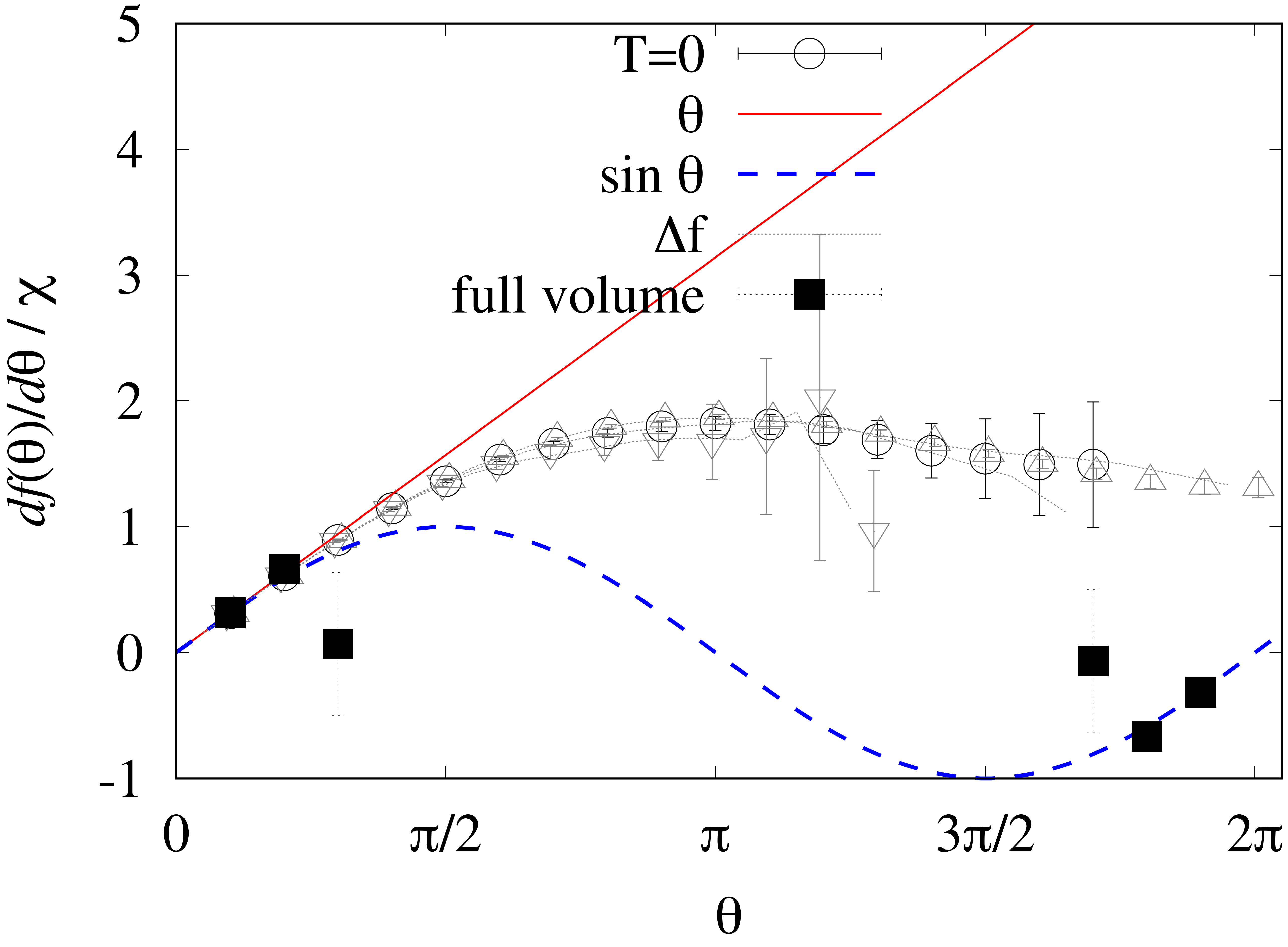
- Fit range $n_{\rm APE} = [20, 40]$ determined in [Kitano, NY, Yamazaki (2021)].
- Linear fit works well.
- Monotonic function $f(\pi) < f(3\pi/2)$

θ dependence of $f(\theta)$ at $T = 0$



- Succeed to calculate up to $\theta \sim 3\pi/2$
- Monotonically increasing function
- Inconsistent with DIGA
- Re-weighting (=full volume) method works only around $\theta = 0$.
- Numerical consistency with $\int d\theta \frac{df}{d\theta}$

$df(\theta)/d\theta$ at $T = 0$

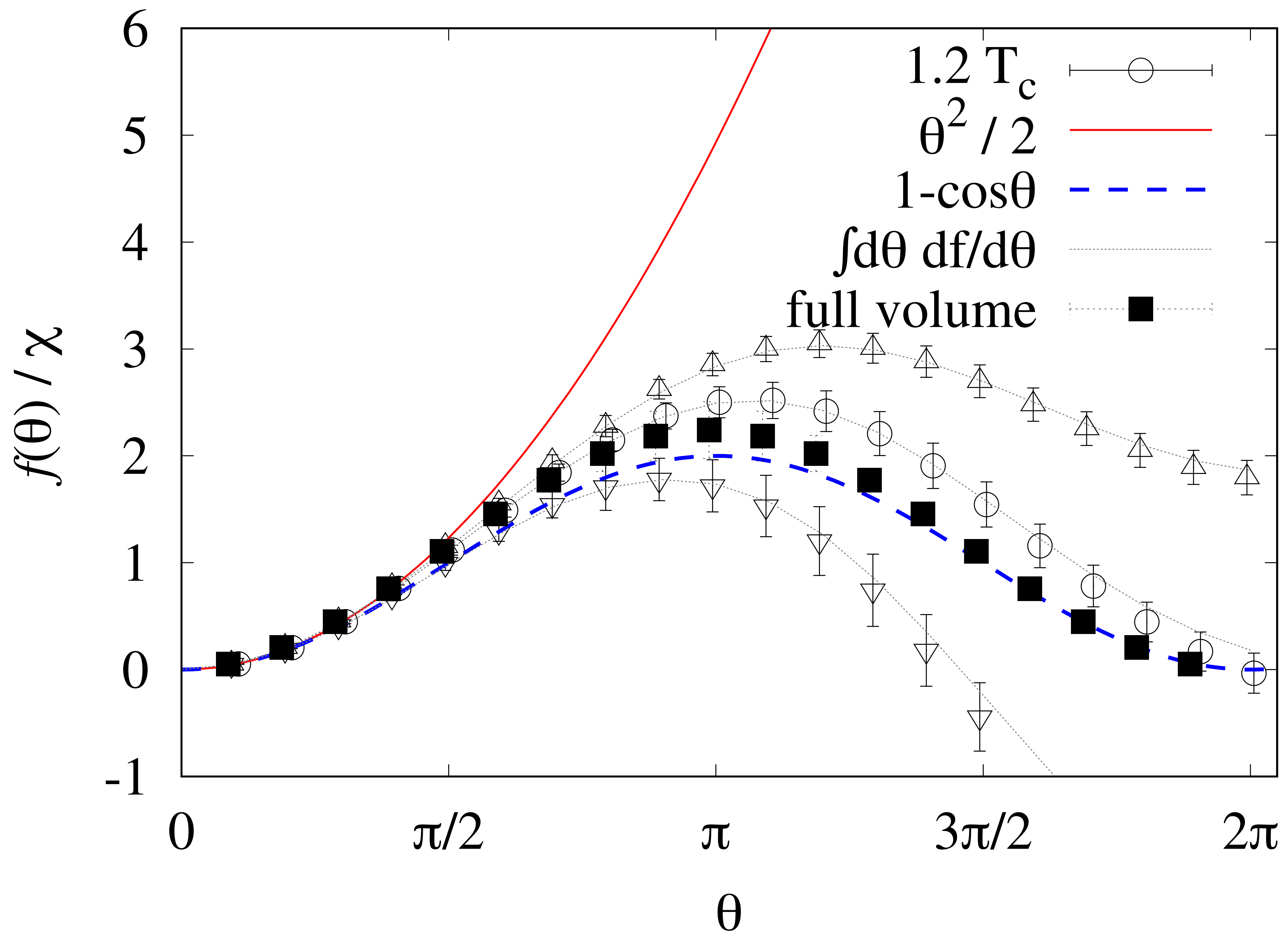


- Order parameter is non-zero

$$\left. df(\theta)/d\theta \right|_{\theta=\pi} = -i\langle\, q(x)\,\rangle_{\theta=\pi} \neq 0$$

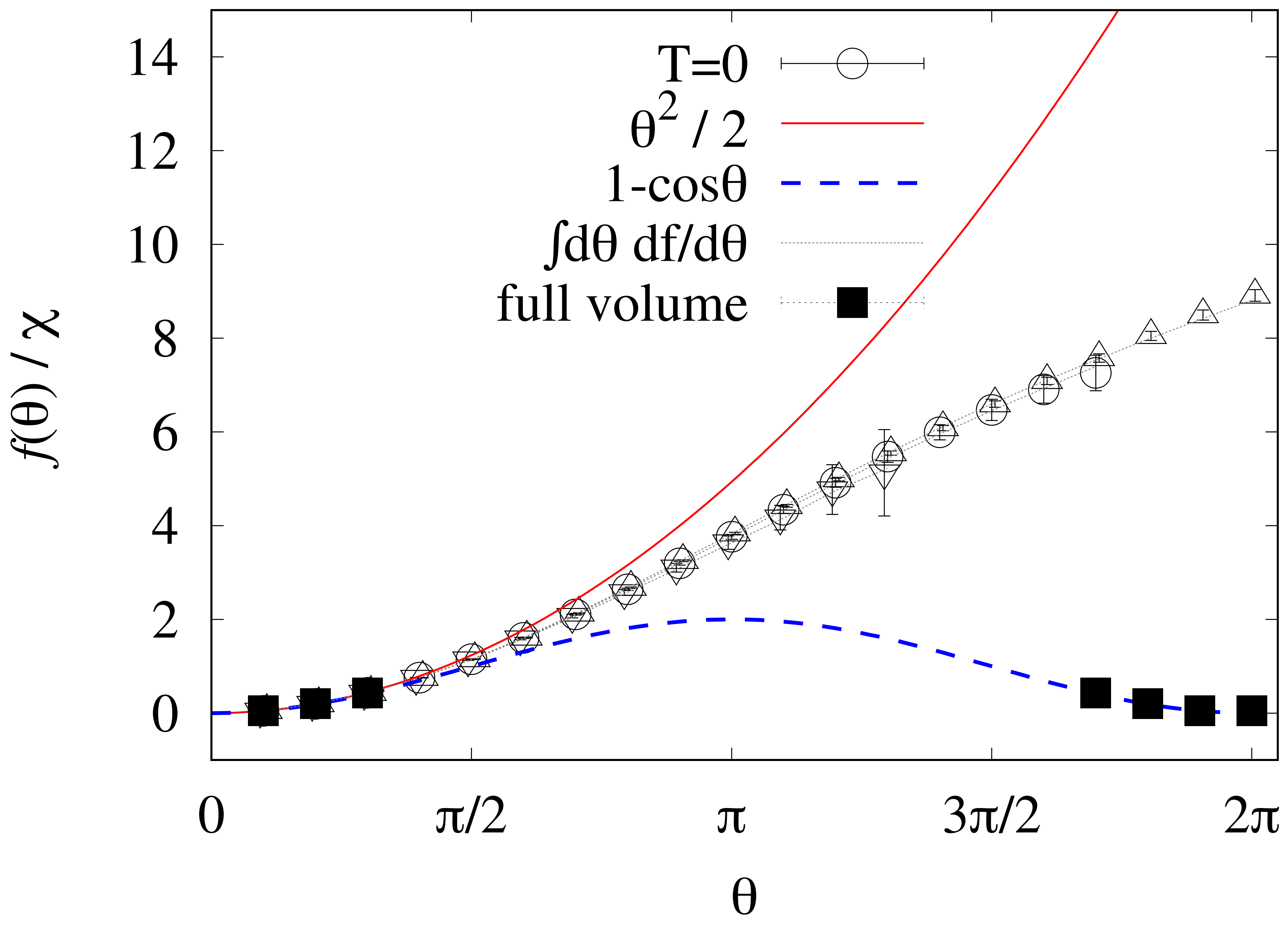
⇒ **spontaneous CPV at $\theta = \pi$**

θ dependence of $f(\theta)$ at $T = 1.2T_c$

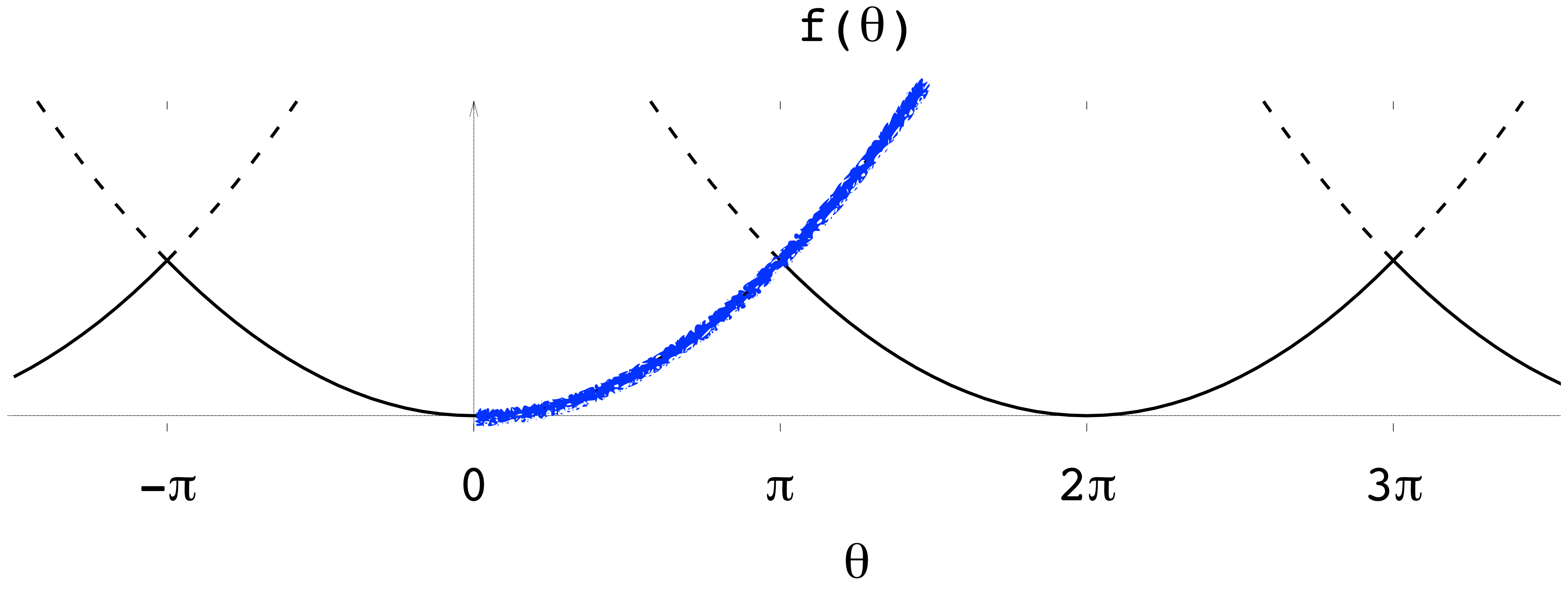


- Large systematic uncertainty due to ambiguity of the scaling region
- Within large uncertainty, consistent with the DIGA, and $df(\theta)/d\theta\Big|_{\theta=\pi} \approx 0 \Rightarrow$ no *CPV* for $T \geq 1.2\, T_c$
- Similar results at $T = 1.6\, T_c$

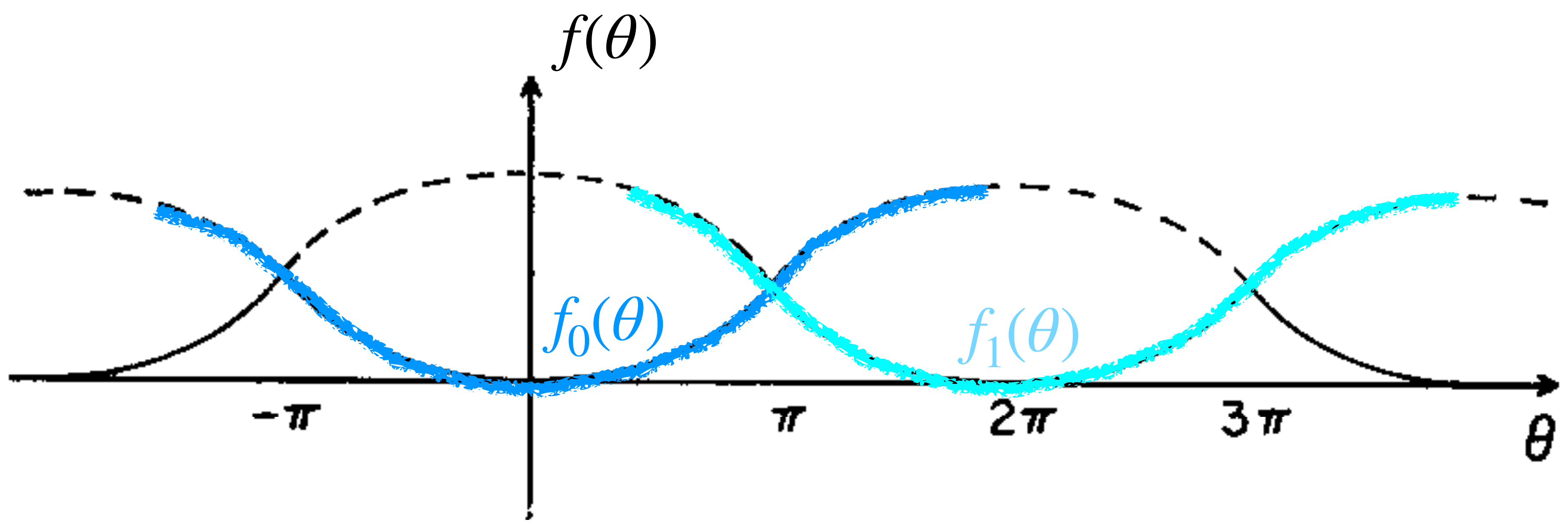
$f(\pi - \theta) = f(\pi + \theta)\,?$



We are looking at



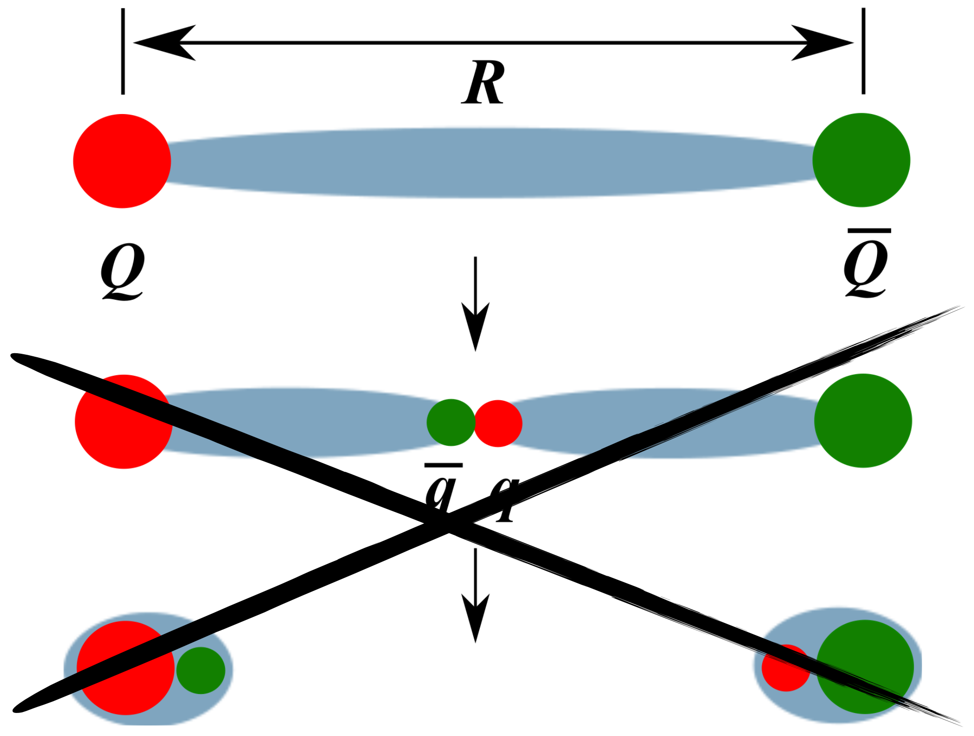
$f(\pi - \theta) = f(\pi + \theta)$? (Cont'd)



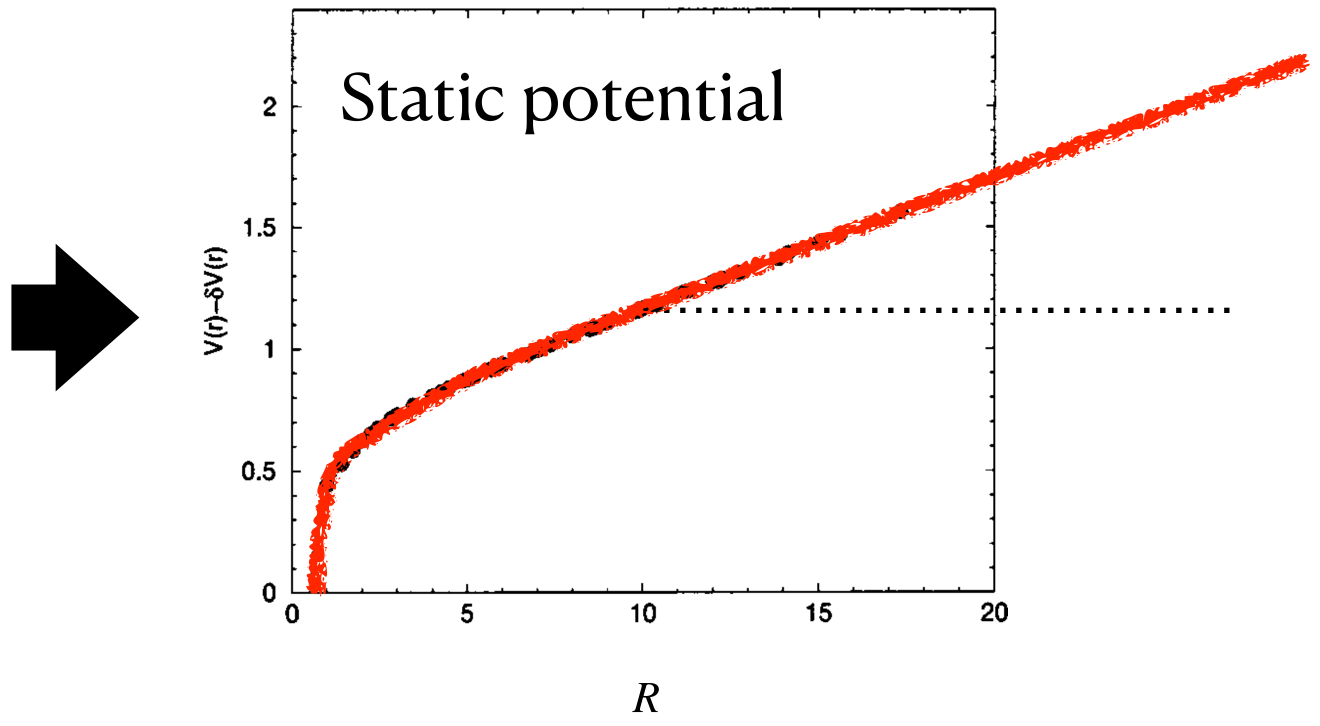
- Sub-volume method stick to trace the original branch even after passing the crossing points.

- Similar experience in the the static potential on the dynamical configs, where"string breaking" is expected to occur at large separation but …

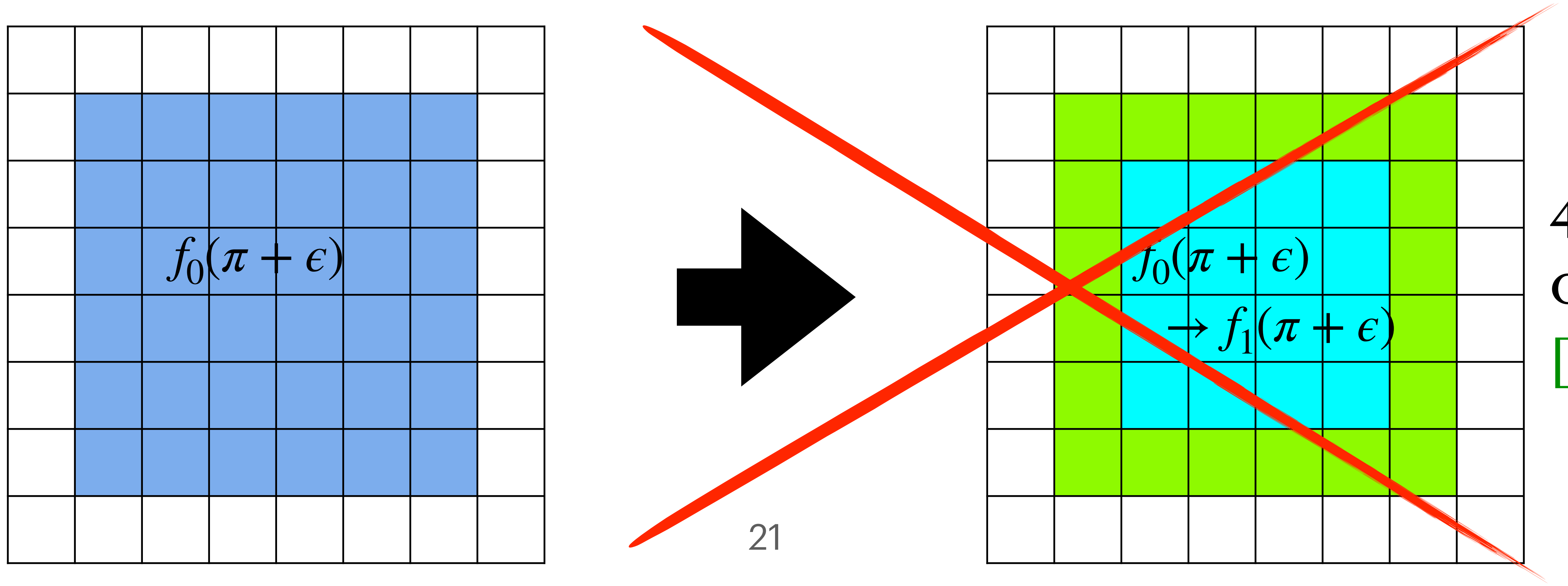
  ⇒ Nothing but overlap problem



Chernodub (2010)



- In the present case, the transition from domain to a bag-like object is expected to occur but …



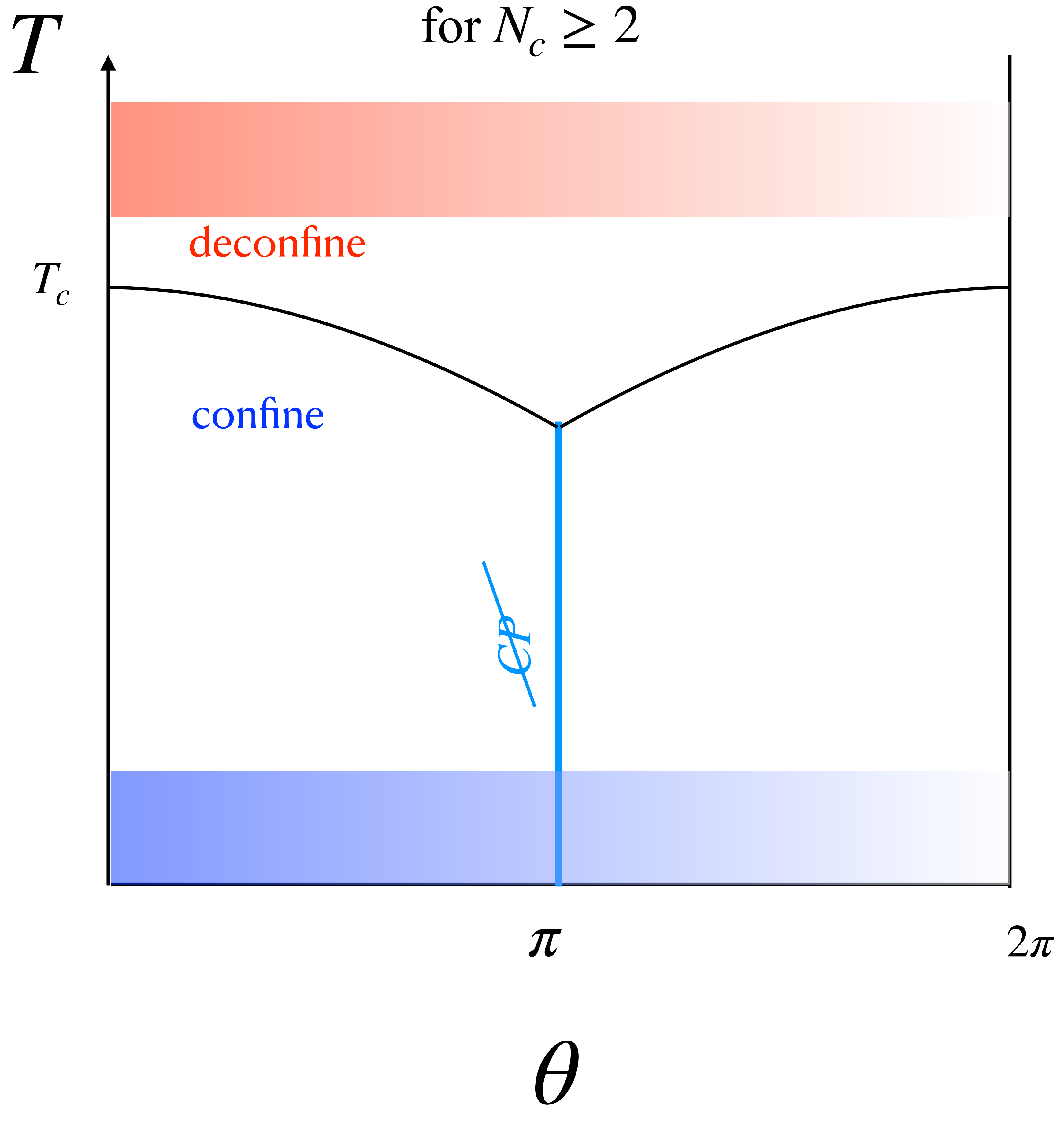
4d SU(N) YM has an topological object called "glue bag". [Luscher (1978)].

Summary and conclusion

- We have developed a sub-volume method, which enables us to calculate $f(\theta)$ up to $\theta \sim 3\pi/2$ in SU(2) Yang-Mills theory.
- Combining with the theory requirement $f(\pi - \theta) = f(\pi + \theta)$, our result provides with the evidence for spontaneous *CPV* at $\theta = \pi$ for the vacuum.
  $\Rightarrow N_c$=2 belongs to large N class (not like CP^1 model).
- The same method reproduces the result consistent with the DIGA, $f(\theta) \sim \chi(1 - \cos\theta)$, above $1.2\,T_c$.

4d $SU(N_c)$ YM

for $N_c \geq 2$



Future studies

- In order for everything to be clear, we want to look at the θ vacuum rather than peek into.
- Exploring the location of $T_c(\theta)$
  [D'Elia, Negro(2012, 2013)]
- Also interesting to apply the sub-volume method to the finite density system.
- Other application ?

4d $SU(N_c)$ YM