# Quantum phase transition and Resurgence: Lessons from 3d N=4 SQED


Takuya Yoda

Department of Physics, Kyoto University



[Prog. Theor. Exp. Phys. (2021) 103B04. (arXiv: 2103.13654)]



Collaborators: Toshiaki Fujimori[A,D], Masazumi Honda[B], Syo Kamata[C], Takahiro Misumi[D,E], Norisuke Sakai[D]

Hiyoshi Phys. Keio U.[A], YITP[B], NCBJ[C], RECNS Keio[D], Akita->Kindai U.[E]



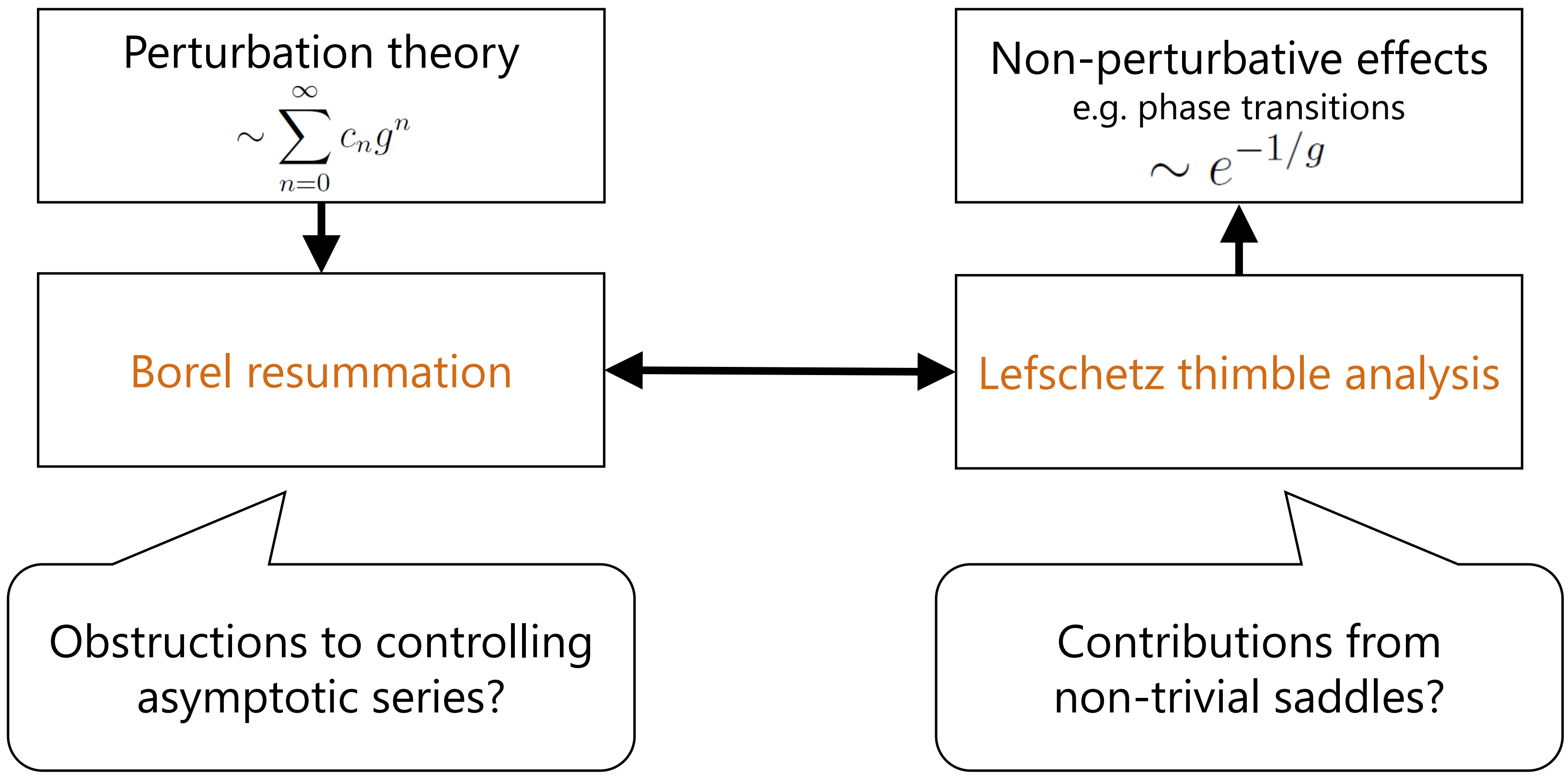
Oct./25/2021 YITP workshop QCD phase diagram and lattice QCD

# Resurgence theory

[J. Ecalle, 81]

Lectures and reviews, e.g.
[M. Marino, 12]
[D. Dorigoni, 19]
[I. Aniceto, G. Basar, R. Schiappa, 19]

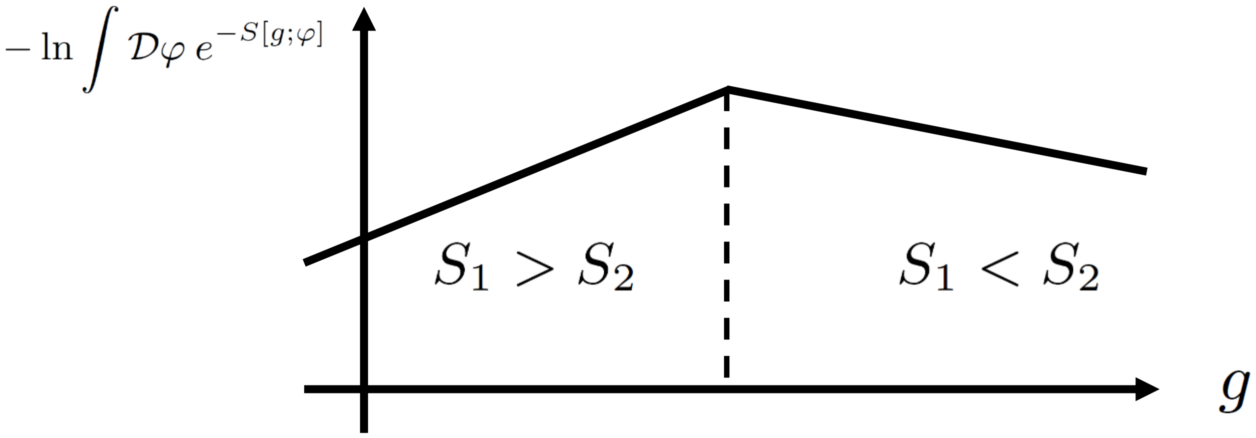
- One of the approaches to non-perturbative physics
- Decodes *non-perturbative information* from *perturbation theory*



# Phase transitions and resurgence

Common story:

1st order phase transition = Change of dominant saddles




Recent works:

- 0-dim Gross-Neveu, Nambu-Jona-Lasinio like model [T. Kanazawa, Y. Tanizaki, 15]
- 2dim Yang-Mills on lattice (reduced to Gross-Witten-Wadia) [G. Dunne et al., 16, 17, 18]

etc.

→ Is resurgence theory applicable to 2nd order phase transitions or more realistic QFTs?

# Brief summary

## Model

[Russo, Tierz, 17]

- $3\mathrm{dim}\ \mathcal{N}=4\ U(1)$ SUSY gauge theory + $2N_f$ hypermultiplets with charge 1
- Fayet-Illiopoulos parameter $\eta$, flavor mass $m$

→ 2nd order quantum phase transition at the large-flavor limit

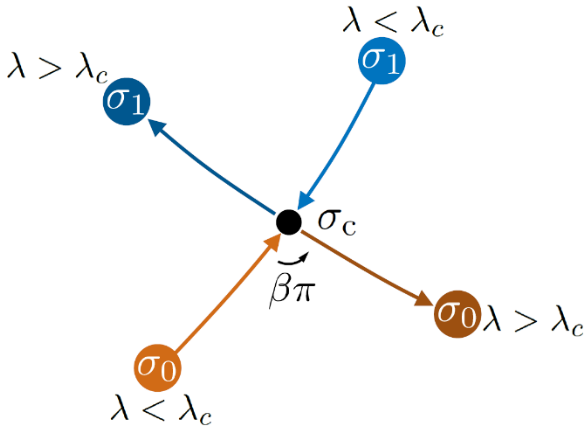
## Result: resurgence is applicable!

[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

2nd order phase transition = Simultaneous Stokes and Anti-Stokes phenomena

Change of the form of asymptotic expansion and change of dominant saddles

- The order of the phase transition is determined by the collision angle of saddles
- Such information is encoded in a perturbative series



# Contents


✓ Motivations and Brief summary (3)
- 2nd order phase transition in SQED3 (review) (4)
- Lefschetz thimble analysis (7)
- Borel resummation (4)
- Lessons from SQED3 (3)
- Conclusion and future works (1)

Total: 22

# SQED3

## Model:

$3\mathrm{dim}\ \mathcal{N}=4\ U(1)$ SUSY gauge theory with

- $2N_f$ hypermultiplets (charge 1)
- Fayet-Illiopoulos parameter $\eta$
- flavor masses $\pm m$

## Partition function:

Exactly computed on $S^3$ by SUSY localization technique

[Pestun, 12]
[A. Kapustin, B. Willett, I. Yaakov, 10]
[N. Hama, K. Hosomichi, S. Lee, 11]
[D. L. Jafferis, 12]

$$Z=\int_{-\infty}^{\infty} d\sigma \frac{e^{i\eta\sigma}}{\left[2\cosh\frac{\sigma+m}{2}\cdot 2\cosh\frac{\sigma-m}{2}\right]^{N_f}}$$

$\sigma$ : Coulomb branch parameter
i.e. constant configuration of the scalar belonging to the vector multiplet in $\mathcal{N}=2$ Language

# Saddles

't Hooft like limit:

[Russo, Tierz, 17]

$$N_f \to \infty, \quad \lambda \equiv \frac{\eta}{N_f} = \text{fixed}.$$

"Action":

$$S(\sigma) = N_f \left[ -i\lambda\sigma + \ln(\cosh\sigma + \cosh m) \right]$$

Saddles:

$$\sigma_n^{\pm} = \log\left( \frac{-\lambda \cosh m \pm i\Delta(\lambda, m)}{i + \lambda} \right) + 2\pi i n \quad (n \in \mathbb{Z}),$$

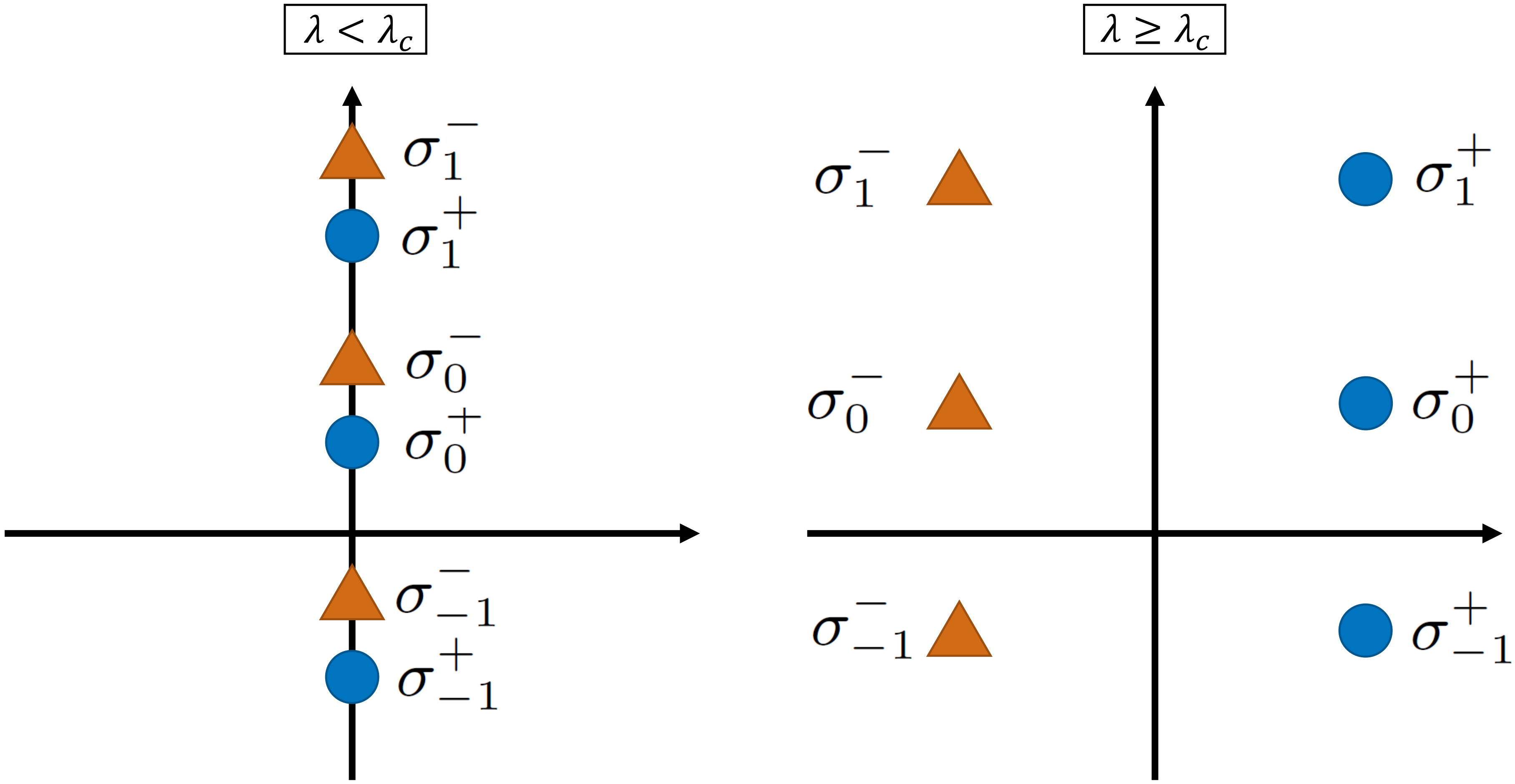
$$\Delta(\lambda, m) = \sqrt{1 - \lambda^2 \sinh^2 m}.$$

Something must happen at

$$\lambda_c \equiv \frac{1}{\sinh m}$$

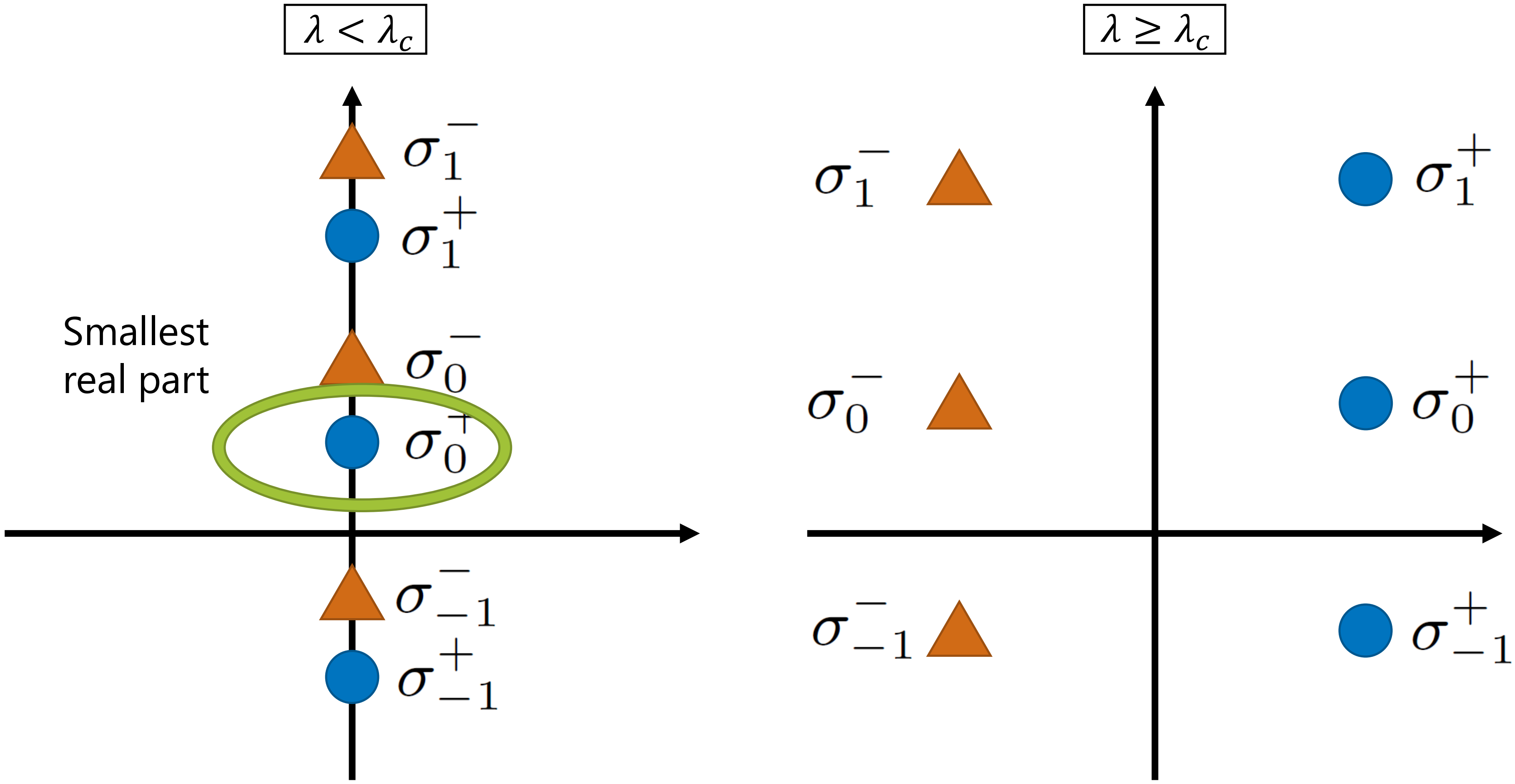
# Saddles

[Russo, Tierz, 17]



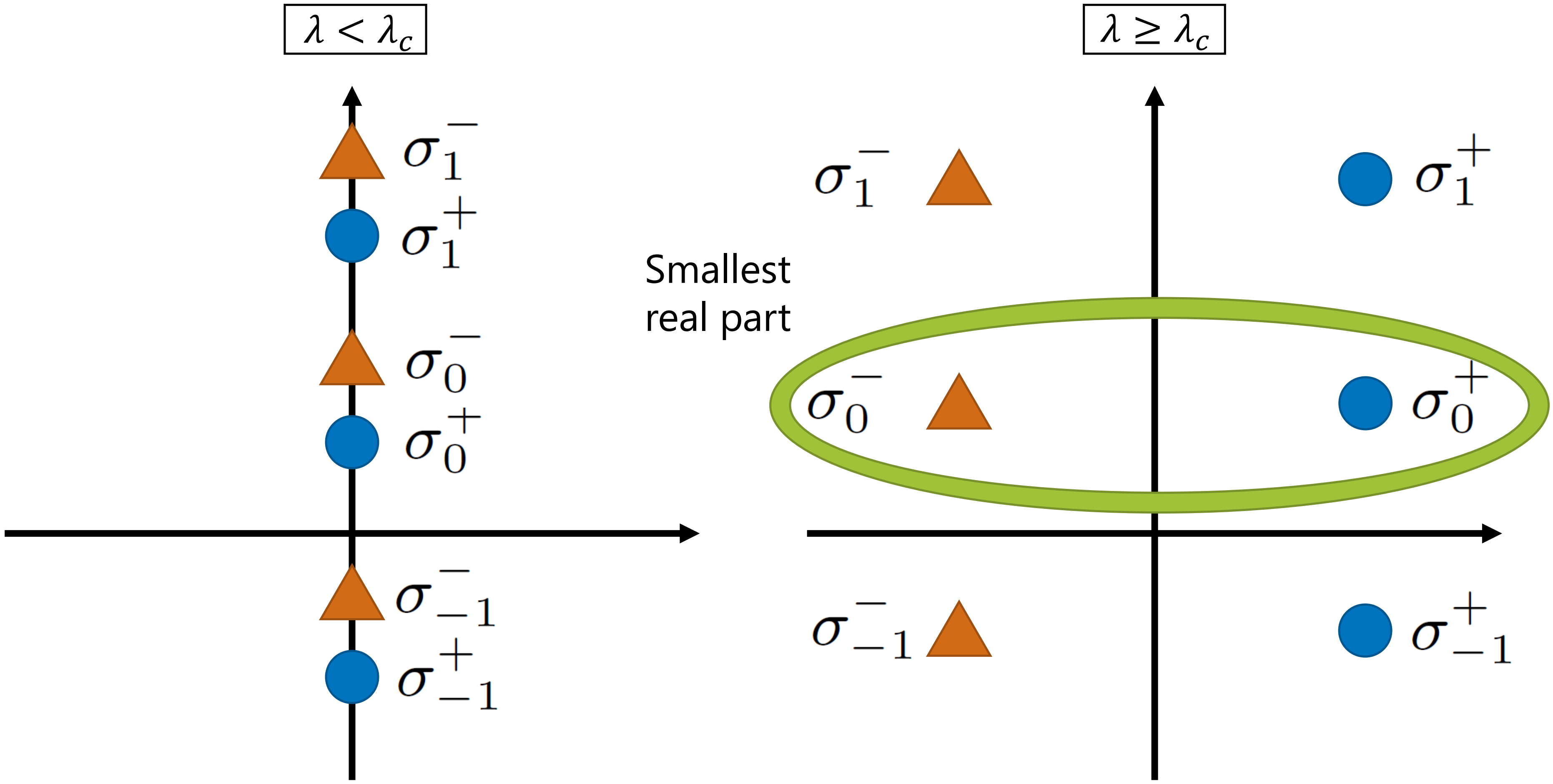
# Saddles

[Russo, Tierz, 17]



# Saddles

[Russo, Tierz, 17]



# $2^{nd}$ order phase transition

If the saddles of smallest real part contribute, [Russo, Tierz, 17]

$$\frac{d^2 F}{d\lambda^2} = \begin{cases} \frac{N_f}{1+\lambda^2}\left(1 + \frac{\cosh m}{\sqrt{1-\lambda^2 \sinh^2 m}}\right) & \lambda < \lambda_c \\ \frac{N_f}{1+\lambda^2} & \lambda \geq \lambda_c \end{cases}$$

→ $2^{nd}$ order phase transition

Questions:

- Smallest real part does not necessarily mean that such saddles contribute to the path integral. Can we ***justify it in more precise way***?
- Can we interpret the $2^{nd}$ order phase transition ***from the viewpoint of resurgence***, and ***draw lessons*** for generic QFTs?

We will answer to these questions, Yes!

# Contents

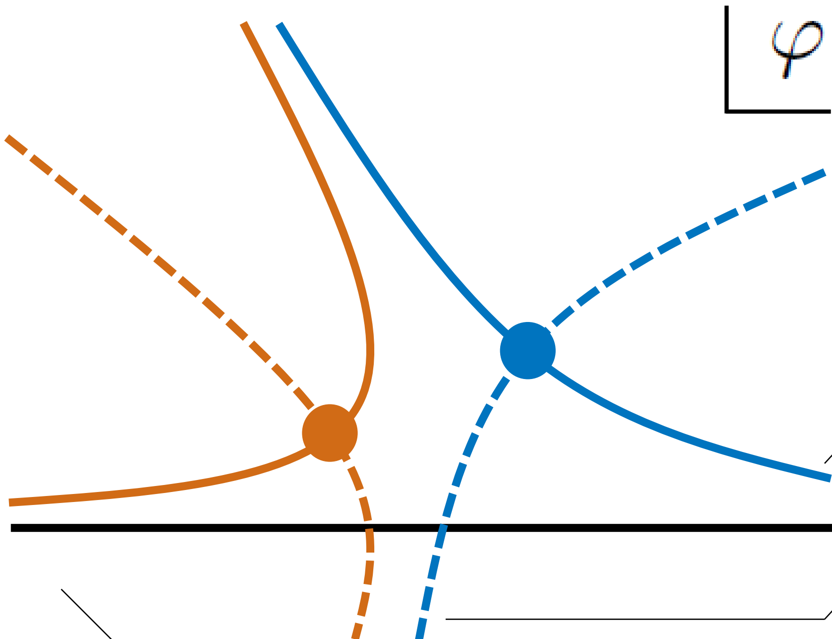

✓ Motivations and Brief summary (3)
✓ $2^{nd}$ order phase transition in SQED3 (review) (4)
- Lefschetz thimble analysis (7)
- Borel resummation (4)
- Lessons from SQED3 (3)
- Conclusion and future works (1)

Total: 22

# Lefschetz thimbles and dual-thimbles

Lefschetz thimbles = "Steepest descents" in configuration space
Dual-thimbles = "Steepest ascents" in configuration space

[E. Witten, 11]
[M. Cristoforetti et al., 12,13,14]
[H. Fujii et al., 13]
[G. Aarts, 13]
[A. Alexandru et al, 16]




$$\mathcal{J}_i : \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = +\overline{\frac{\delta S[\varphi]}{\delta\varphi}}, \quad \varphi(-\infty) = \varphi_i$$

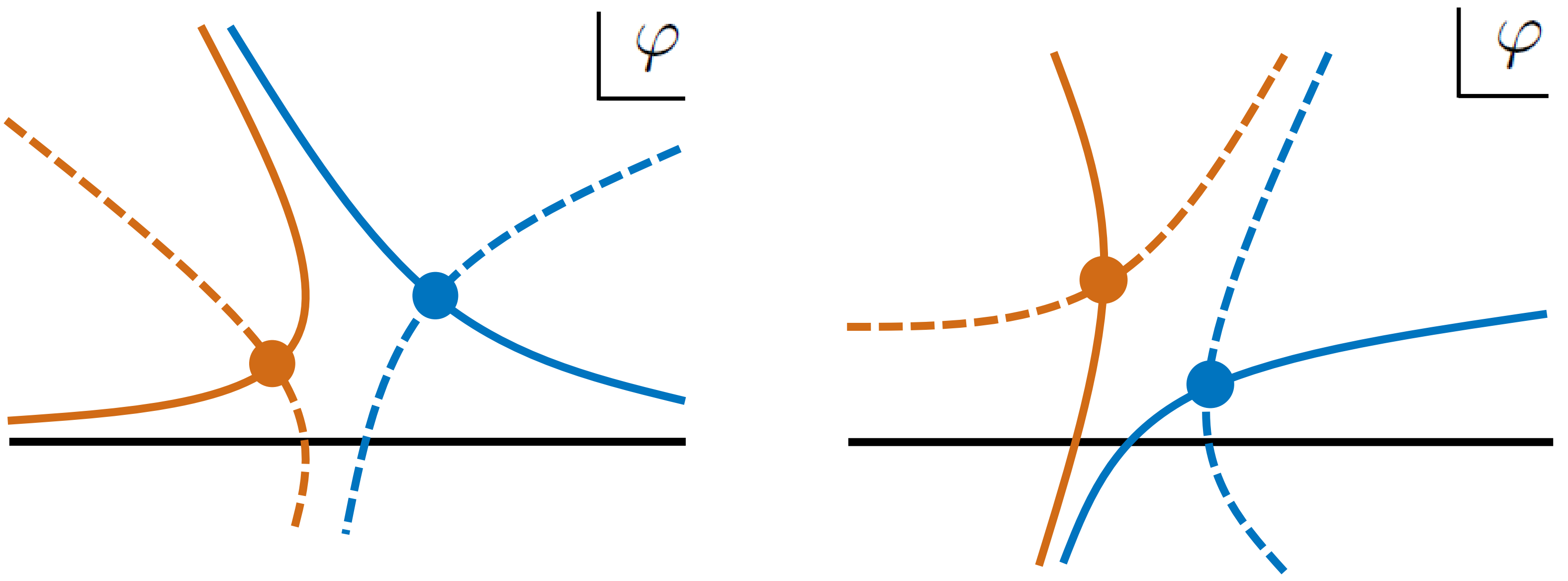
$$\mathcal{K}_i : \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = -\overline{\frac{\delta S[\varphi]}{\delta\varphi}}, \quad \varphi(-\infty) = \varphi_i$$

$$\left.\frac{\delta S[\varphi]}{\delta\varphi}\right|_{\varphi=\varphi_i} = 0$$

$$\mathcal{I} = \sum_i n_i \mathcal{J}_i$$ (thimble decomposition)

intersection number

# Stokes and anti-Stokes phenomena




Stokes phenomenon : Change of an intersection number, which occurs at

$$\mathrm{Im}[S[\varphi_i]] = \mathrm{Im}[S[\varphi_j]]$$

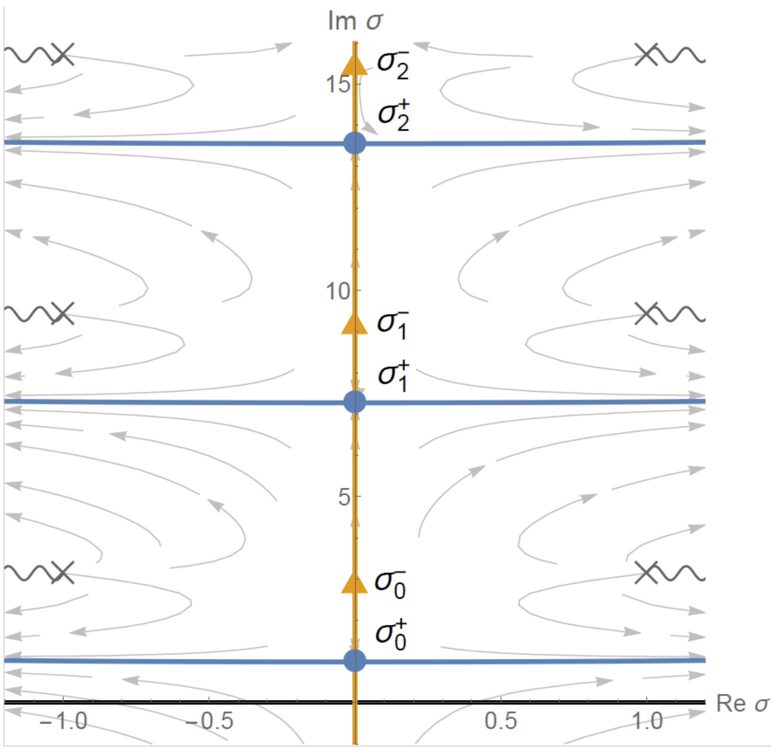
Anti-Stokes phenomenon: Change of dominant saddles, which occurs at

$$\mathrm{Re}[S[\varphi_i]] = \mathrm{Re}[S[\varphi_j]]$$

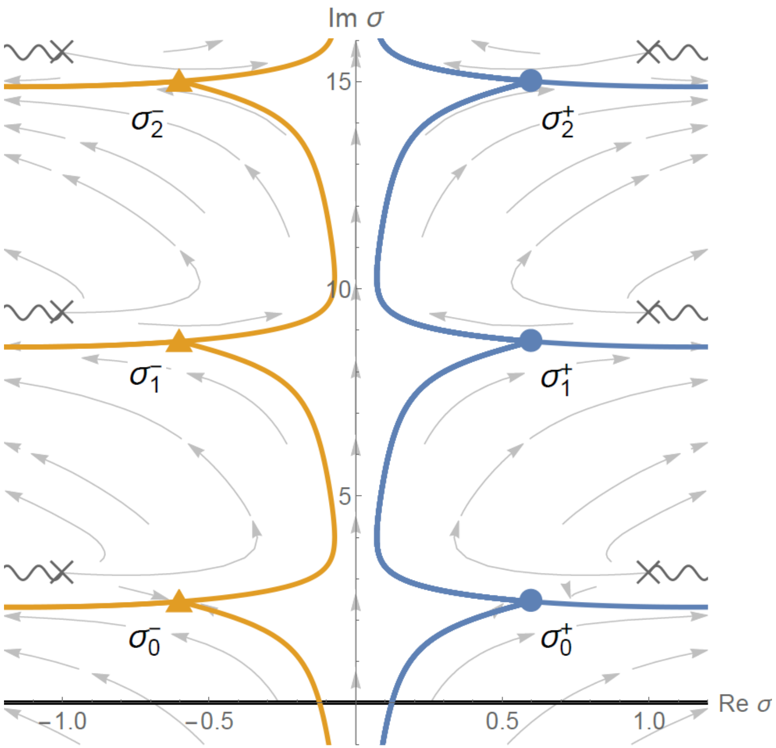
⟷ 1st order phase transition

# Stokes and anti-Stokes pheno. at the critical pt.

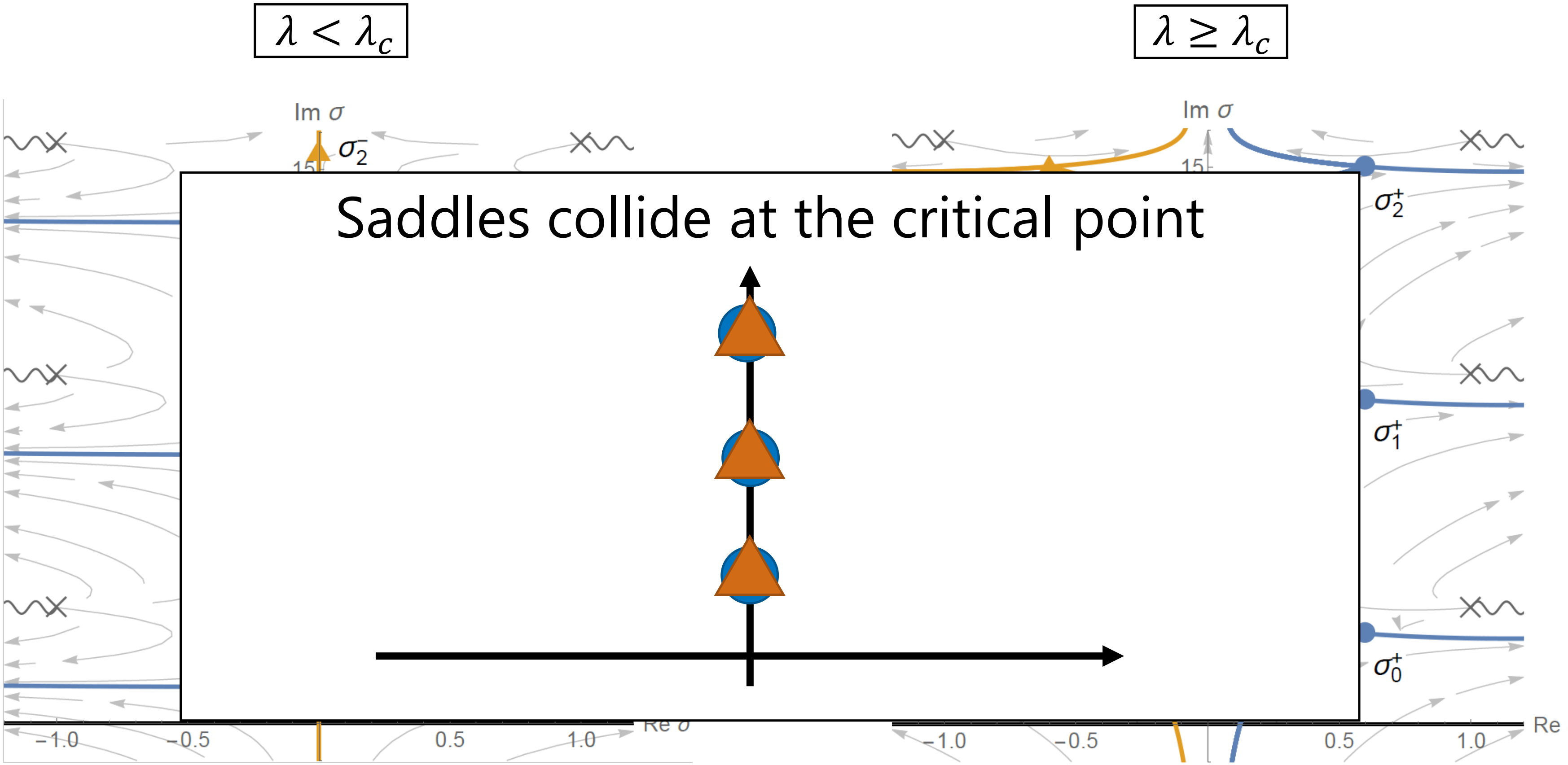
$\lambda < \lambda_c$




$\lambda \geq \lambda_c$



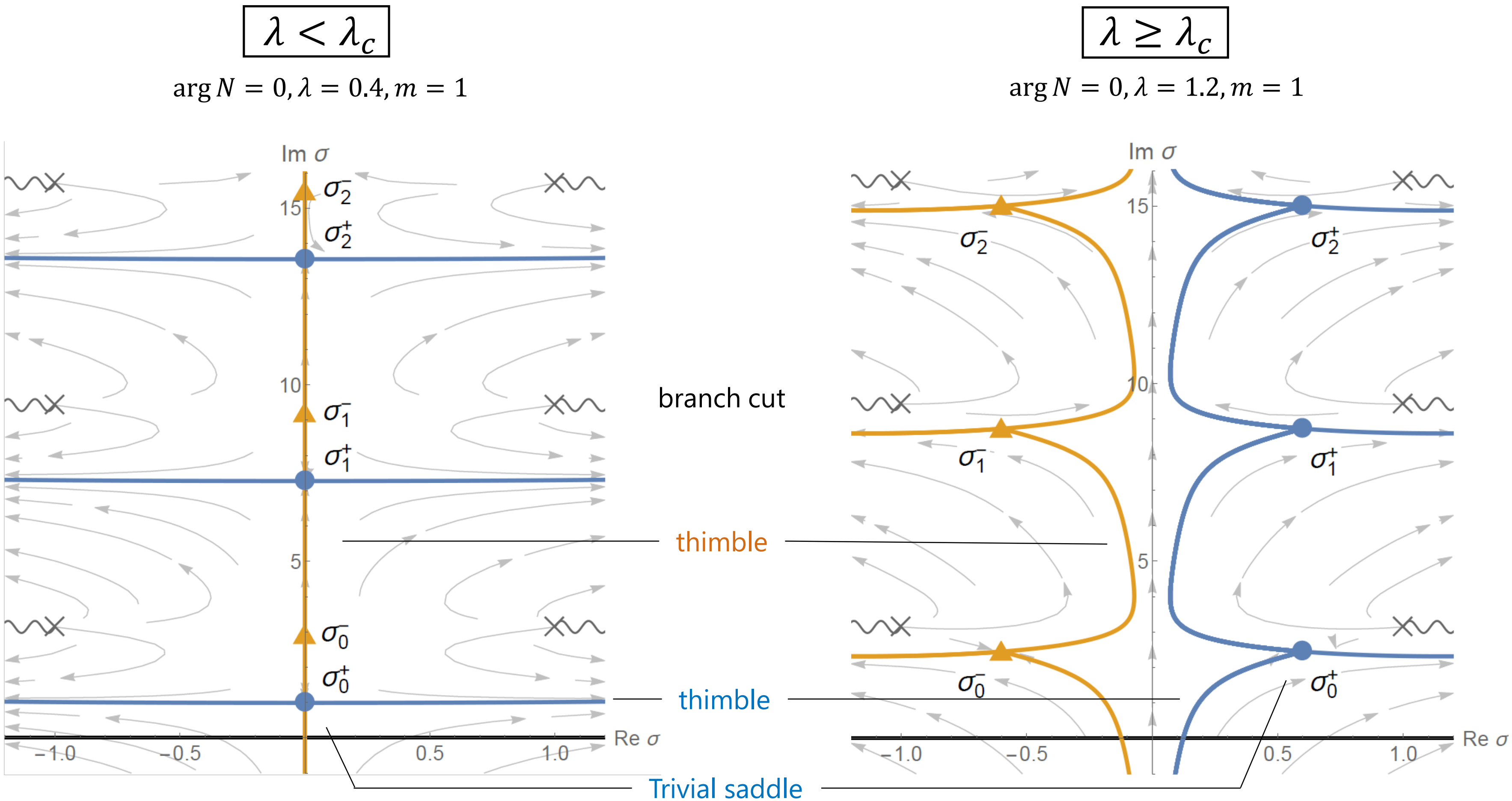
# Stokes and anti-Stokes pheno. at the critical pt.




At the critical point, a Stokes and an anti-Stokes phenomenon co-occur.

# Lefschetz thimble structure

[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

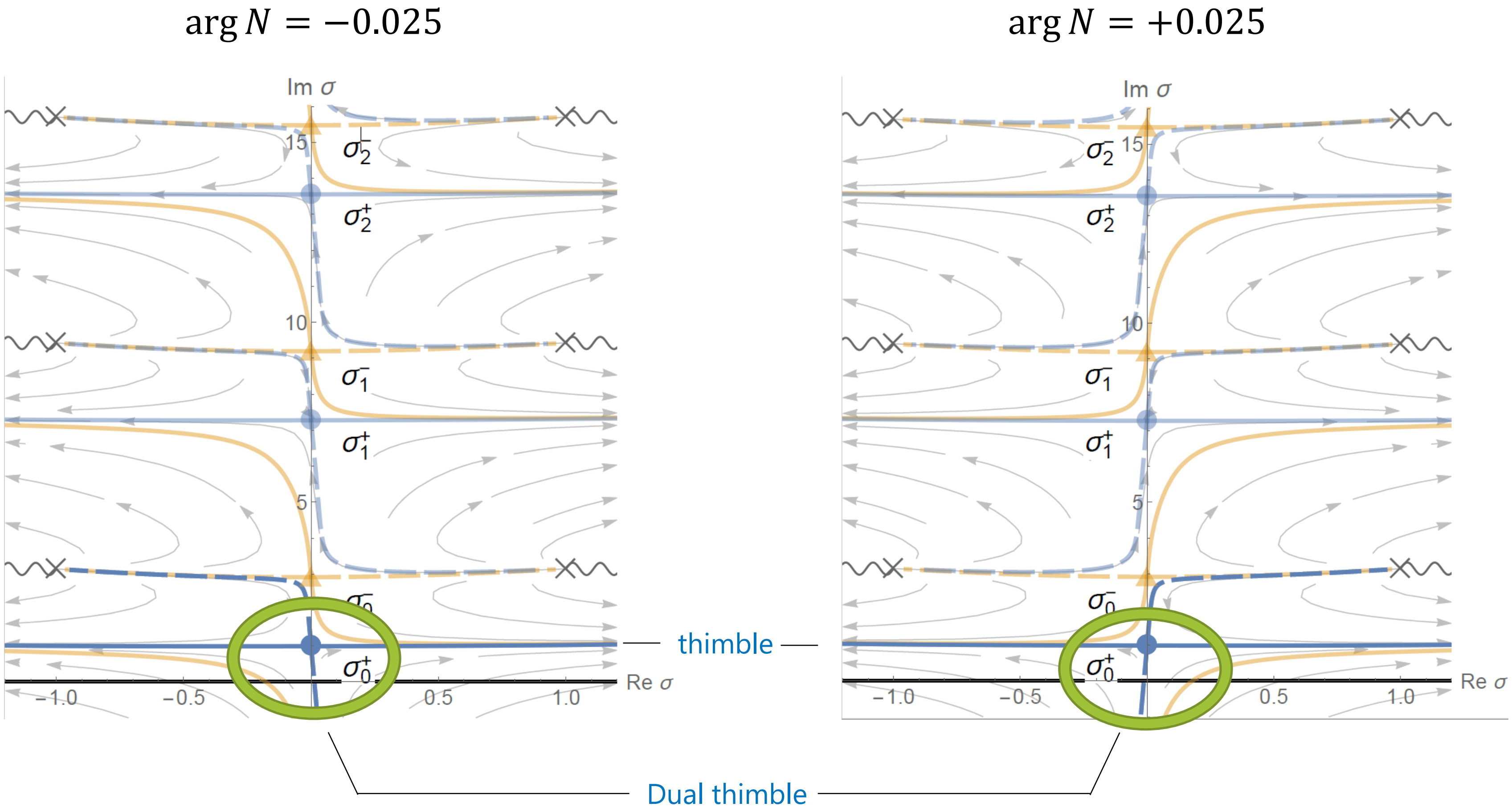



Thimble decomposition $\mathcal{I} = \sum_i n_i \mathcal{J}_i$ is ambiguous

→ Vary the phase of $N_f$

# Lefschetz thimble structure (subcritical)

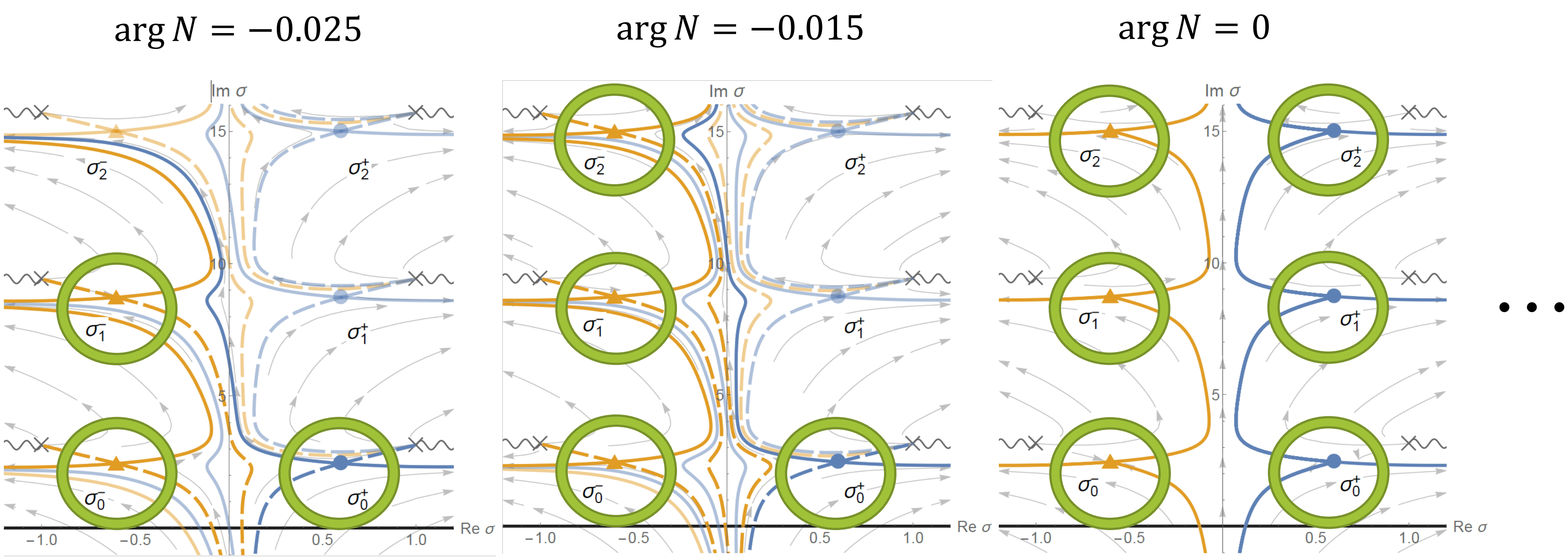
$\lambda < \lambda_c$




- No Stokes phenomenon
- Only the trivial saddle $\sigma_0^+$ contributes to the path integral

# Lefschetz thimble structure (supercritical)

$\boxed{\lambda \geq \lambda_c}$

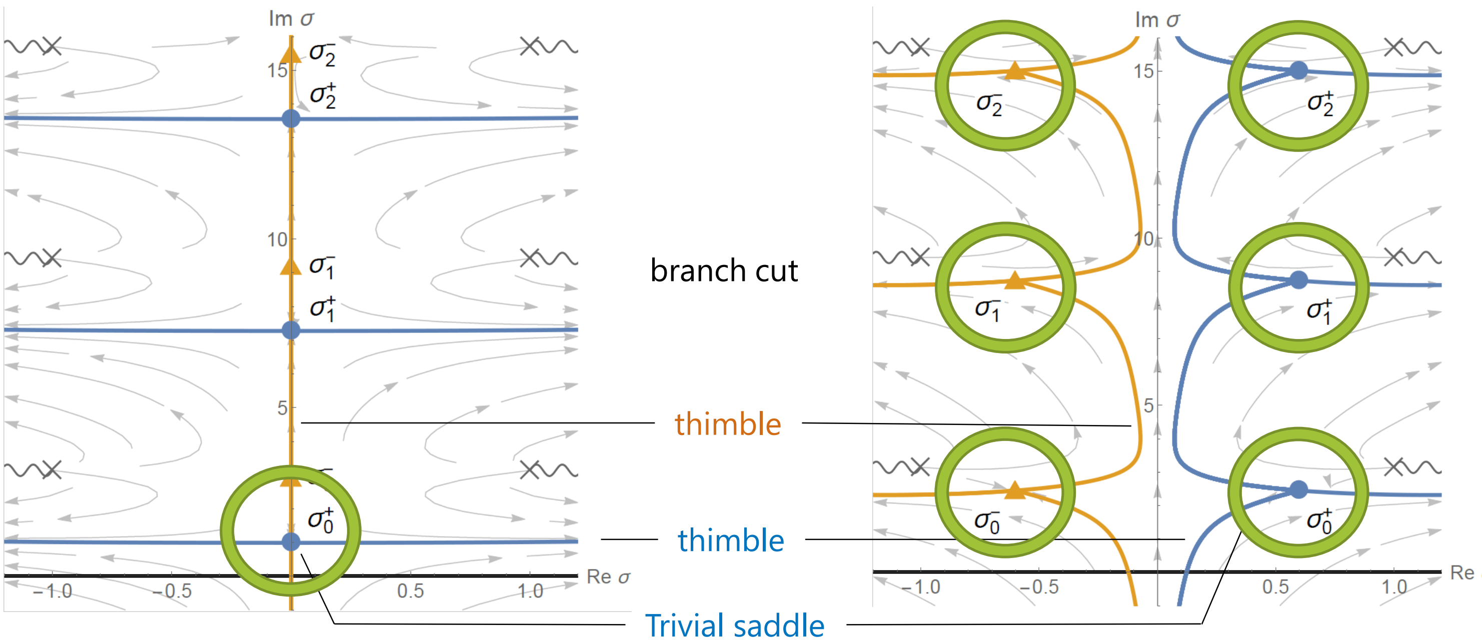



- Infinite number of Stokes phenomena occur around $\arg N_f = 0$
- Infinite number of saddles $\sigma_n^{\pm}$ contribute to the path integral

# Phase transition and Lefschetz thimble structure

Summary

| $\lambda < \lambda_c$ | $\lambda \geq \lambda_c$ |
|---|---|
| • No Stokes phenomenon<br>• Only the trivial saddle $\sigma_0^+$ contributes | • Infinite number of Stokes phenomena<br>• Infinite number of saddles $\sigma_n^\pm$ contribute<br>(Two of which $\sigma_0^+, \sigma_0^-$ survive the large-flavor limit) |




- Stokes and anti-Stokes phenomena occur at the same time

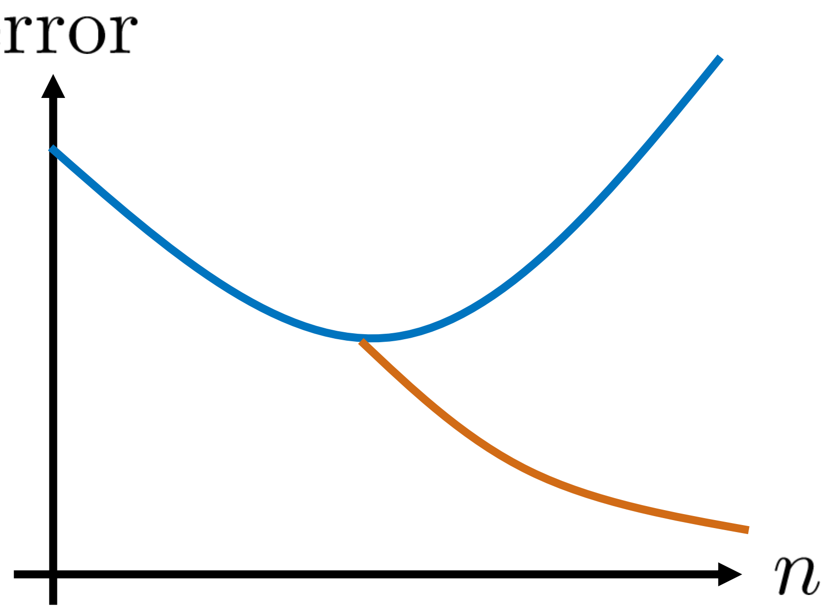
⟶ The phase transition is reinterpreted from the view point of thimbles

# Contents


- ✓ Motivations and Brief summary (3)
- ✓ $2^{nd}$ order phase transition in SQED3 (review) (4)
- ✓ Lefschetz thimble analysis (7)
- Borel resummation (4)
- Lessons from SQED3 (3)
- Conclusion and future works (1)

Total: 22

# Asymptotic series and resurgence structure




$$Z(g) = \sum_{n=0}^{\infty} c_n g^{n+r}, \quad c_n \sim n!$$

Resurgence structure

$$Z(g) = \sum_{n=0}^{\infty} c_n^{(0)} g^n + \sum_{n=0}^{\infty} c_n^{(1)} g^n e^{-S_1/g} + \cdots$$

[J. Ecalle, 81]
[M. Marino, 12]
[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]
[D. Dorigoni, 19]
[I. Aniceto, G. Basar, R. Schiappa, 19]

Resurgence theory:

the perturbative part knows the non-perturbative parts

# Borel resummation

Resuming an asymptotic series

[J. Ecalle, 81]

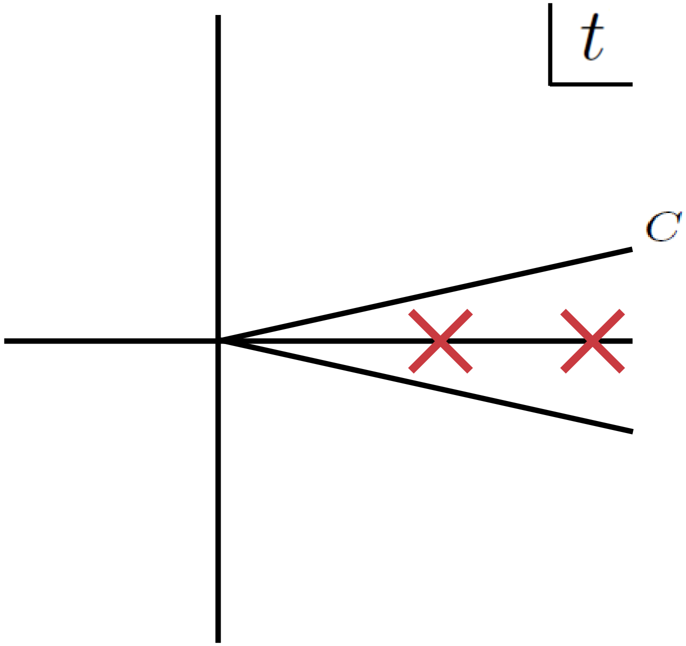
Lectures and reviews, e.g.
[M. Marino, 12]
[D. Dorigoni, 19]
[I. Aniceto, G. Basar, R. Schiappa, 19]

$$Z(g) = \sum_{n=0}^{\infty} c_n g^{n+r}, \quad c_n \sim n!$$ Asymptotic series

$$\mathcal{S}Z(g) = \int_C \mathrm{d}t \; e^{-t/g} \mathcal{B}Z(t)$$ Borel resummation

$$\mathcal{B}Z(t) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n+r)} t^{n+r-1}$$ Borel transformation

Borel transformation may have Borel singularities




Borel singularities imply
non-perturbative corrections!

# Non-trivial saddle and Borel singularities

[Lipatov, 77]

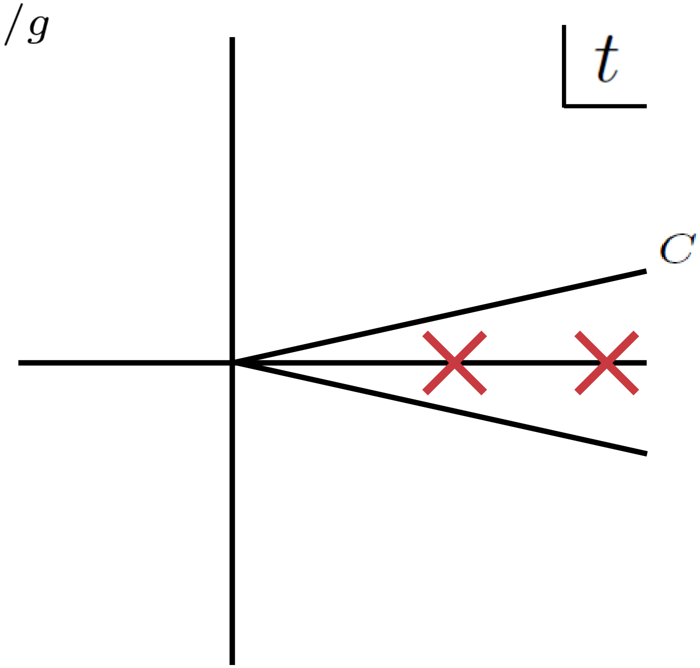
$$Z(g) = \int \mathcal{D}\varphi \, e^{-S[\varphi]/g} \sim \sum_{n=0}^{\infty} c_n g^n$$

$$c_n = \frac{1}{2\pi i} \oint \frac{\mathrm{d}g}{g^{n+1}} \int \mathcal{D}\varphi \, e^{-S[\varphi]/g}$$

$$\sim e^{-S[\varphi_*]/g_* - (n+1)\ln g_*}$$

$$\sim \frac{n!}{(S[\varphi_*])^{n+1}}$$

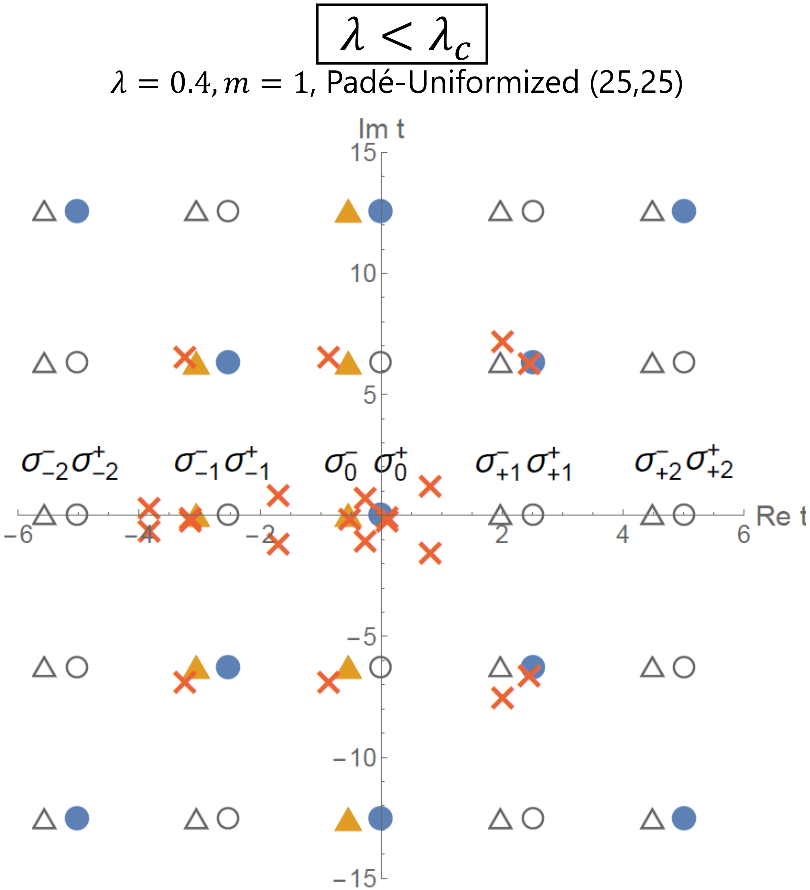
$$\mathcal{B}Z(t) \sim \sum_{n=0}^{\infty} \left(\frac{t}{S[\varphi_i]}\right)^n = \frac{1}{1 - \frac{t}{S[\varphi_i]}}$$

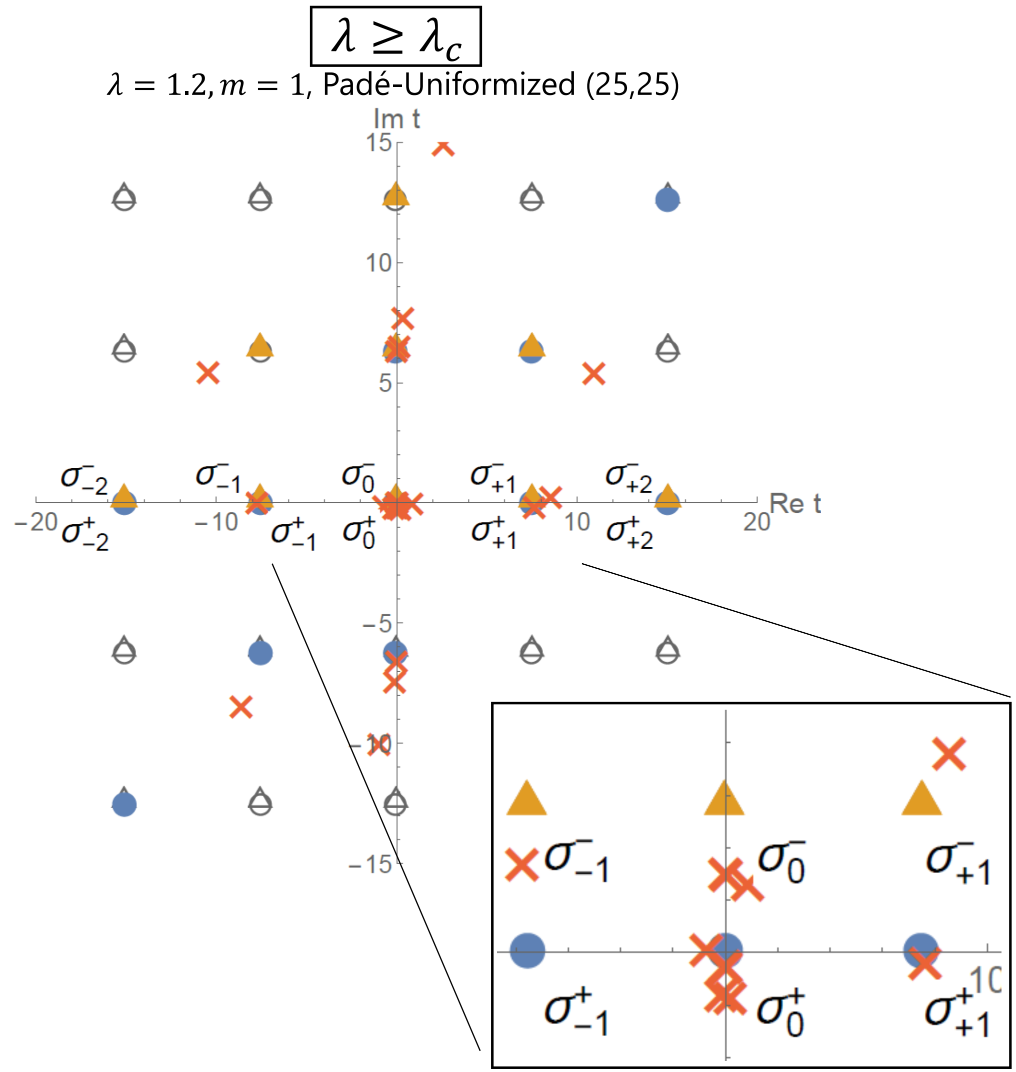



Non-trivial saddles are encoded in an asymptotic series

# Borel plane structure (improved)

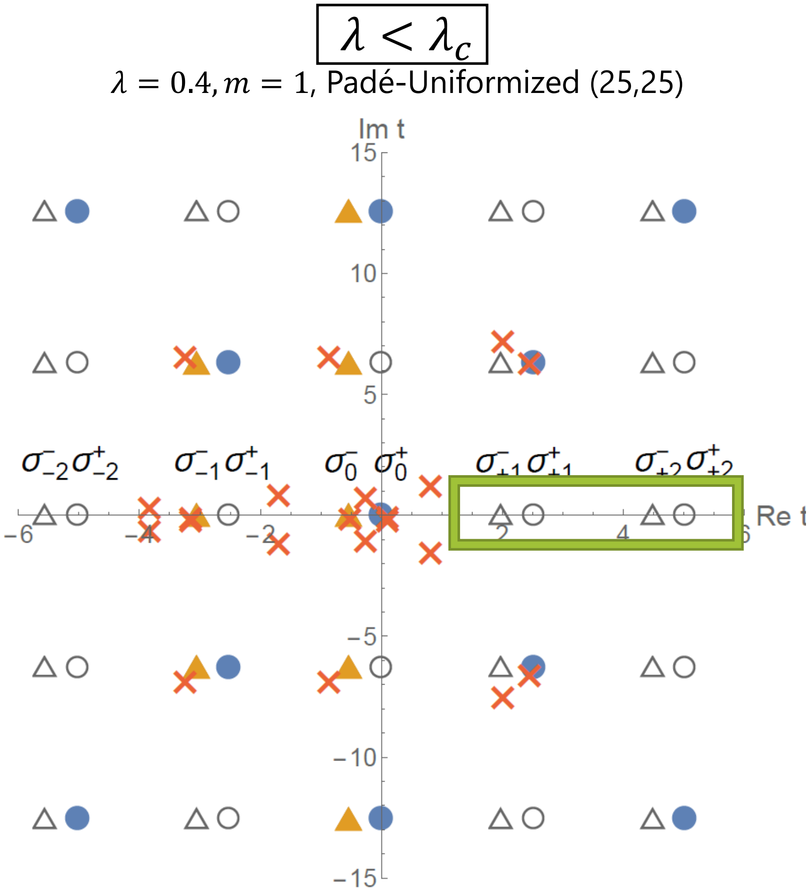
[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

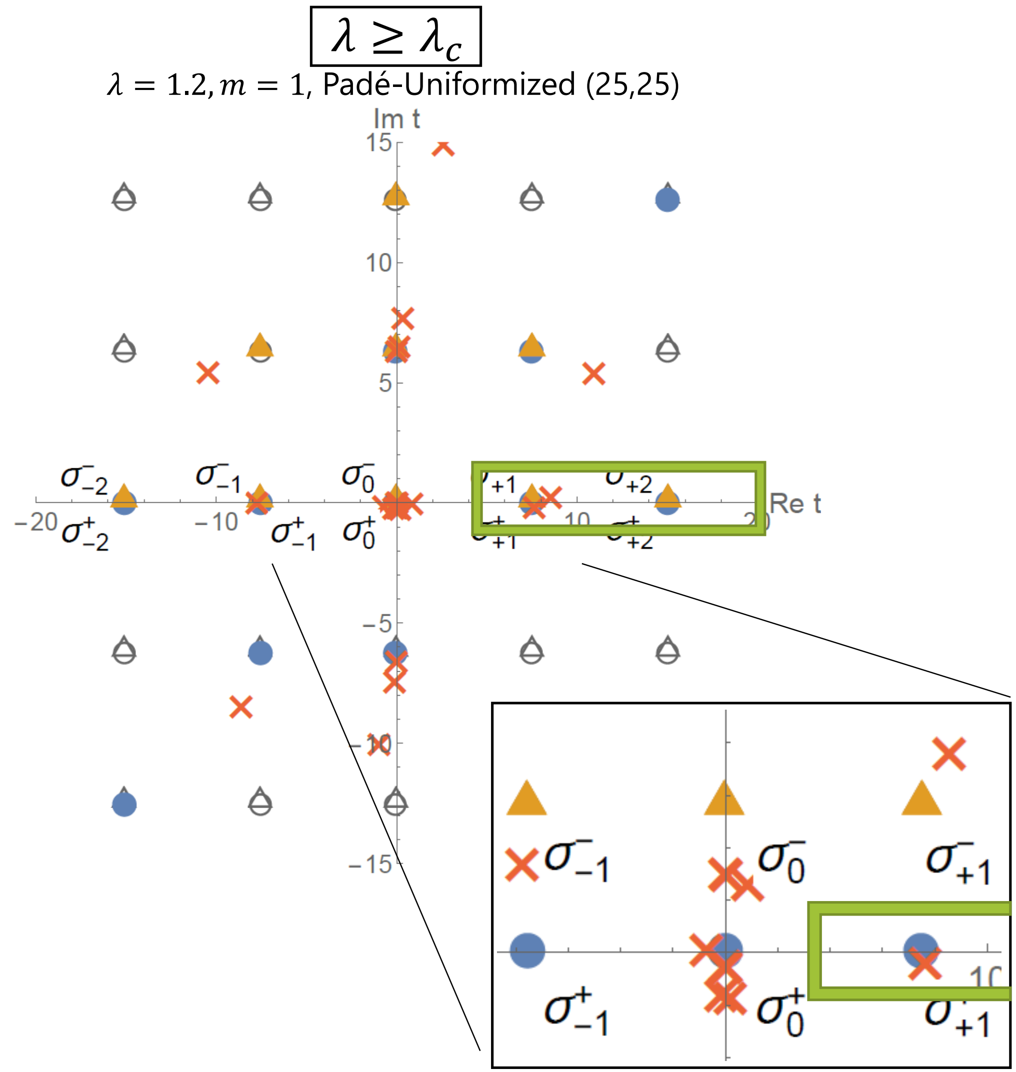


# Borel plane structure (improved)

[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]



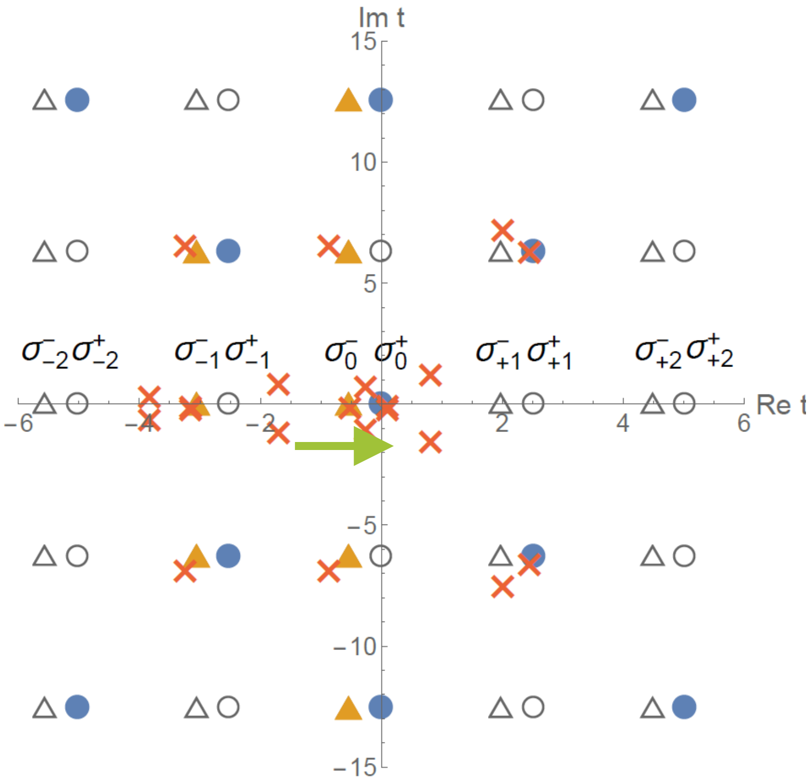

- The Stokes phenomena are encoded as Borel non-summability

# Borel plane structure (improved)

[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

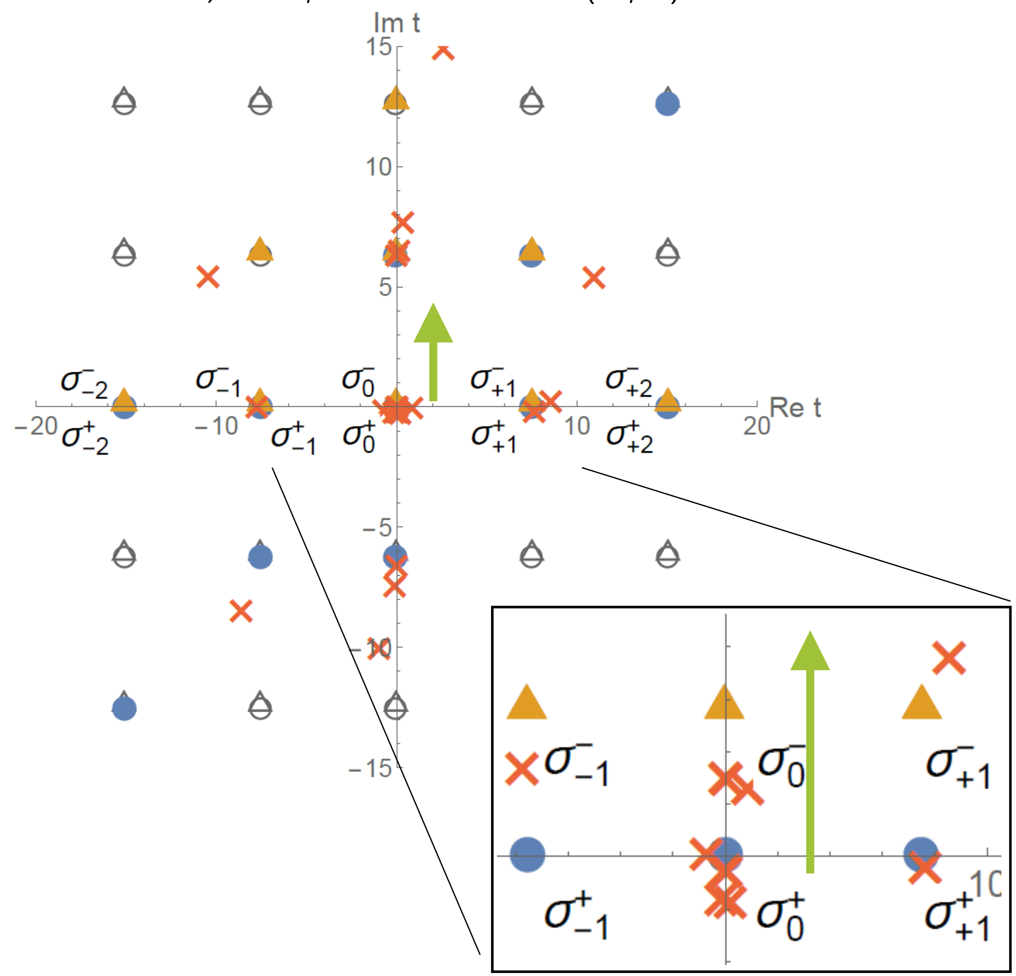
$\lambda < \lambda_c$

$\lambda = 0.4, m = 1$, Padé-Uniformized (25,25)




$\lambda \geq \lambda_c$

$\lambda = 1.2, m = 1$, Padé-Uniformized (25,25)




- The collision of saddles are encoded as collision of Borel singularities
- The anti-Stokes phenomenon is encoded as Borel singularities along the vertical axis

# Contents


- ✓ Motivations and Brief summary (3)
- ✓ $2^{nd}$ order phase transition in SQED3 (review) (4)
- ✓ Lefschetz thimble analysis (7)
- ✓ Borel resummation (4)
- Lessons from SQED3 (3)
- Conclusion and future works (1)

Total: 22

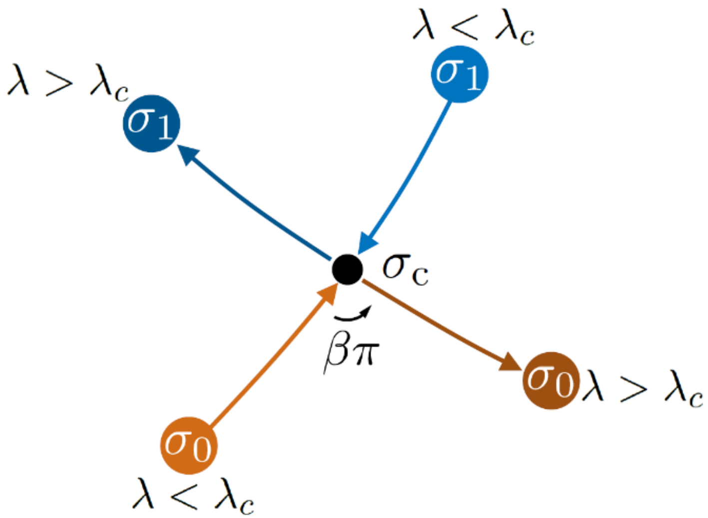
# Collision of saddles

[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

Consider

$$e^{-NF(\lambda)} = \int \mathrm{d}\sigma \; e^{-N\tilde{S}(\lambda;\sigma)}$$

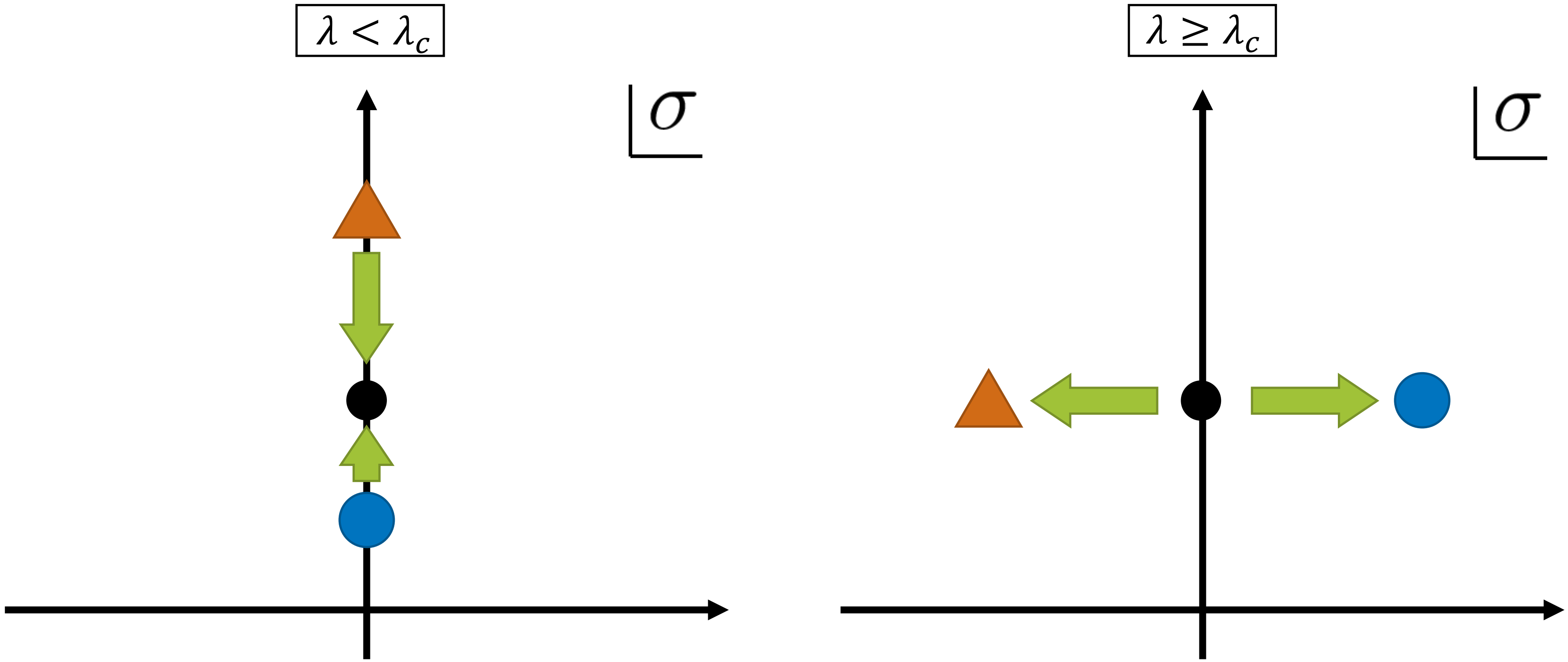
If the "action" is holomorphic, and $n$ saddles collide as




Then, the "action" value at $m$-th saddle is $\tilde{S}_m \simeq c_0 + T_m(\delta\lambda)^{(n+1)\beta}$

⟶ Phase transition is of order $\lceil (n+1)\beta \rceil$

# Revisiting the SQED3




$$n = 2, \beta\pi = \pi/2$$

→ Phase transition is of order $\lceil (2+1)/2 \rceil = 2$

# Revisiting the SQED3

The 2nd order phase transition corresponds to followings

Stokes and
anti-Stokes phenomena
at the same time

Lefschetz thimble analysis:

i. Contributing saddles jump as $\sigma_0^+ \to \sigma_0^+, \sigma_0^-$

ii. Two saddles collide with an angle $\pi/2$

iii. Infinite number of Stokes phenomena associated with $\sigma_{n}^{\pm}$ … ccur

Borel resummation:

I. Two Borel singularities collide and line up along the vertical axis

II. Two Borel singularities collide with an angle $\pi/2$

III. Large-flavor expansion becomes Borel non-summable

# Revisiting the SQED3

The 2nd order phase transition corresponds to followings

Lefschetz thimble analysis:

i. Contributing saddles jump as $\sigma_0^+ \to \sigma_0^+, \sigma_0^-$

ii. Two saddles collide with an angle $\pi/2$

iii. Infinite number of Stokes phenomena associa

Borel resummation:

I. Two Borel singularities collide and line up along the vertic l axis

II. Two Borel singularities collide with an angle $\pi/2$

III. Large-flavor expansion becomes Borel non-summable

The order of phase transition is decoded from “scattering angle”

# Revisiting the SQED3

The 2nd order phase transition corresponds to followings

Lefschetz thimble analysis:

i. Contributing saddles jump as $\sigma_0^+ \to \sigma_0^+, \sigma_0^-$

ii. Two saddles collide with an angle $\pi/2$

iii. Infinite number of Stokes phenomena associated with $\sigma_{n>0}^{\pm}$ occur

Borel resummation:

I. Two Borel singularities collide and line up alon

II. Two Borel singularities collide with an angle $\pi/$

III. Large-flavor expansion becomes Borel non-summable

Due to SUSY

$$Z = \int_{-\infty}^{\infty} d\sigma \frac{e^{i\eta\sigma}}{\left[2\cosh\frac{\sigma+m}{2} \cdot 2\cosh\frac{\sigma-m}{2}\right]^{N_f}}$$

# Revisiting the SQED3

The 2nd order phase transition corresponds to followings

Lefschetz thimble analysis:

i. Contributing saddles jump as $\sigma_0^+ \to \sigma_0^+, \sigma_0^-$

ii. Two saddles collide with an angle $\pi/2$

iii. Infinite number of Stokes phenomena associated with $\sigma_{n>0}^{\pm}$ occur

Borel resummation:

I. Two Borel singularities collide and line up along the vertical axis

II. Two Borel singularities collide with an angle $\pi/2$

III. Large-flavor expansion becomes Borel non-summable

They can be generalized as long as $e^{-NF(\lambda)} = \int \mathrm{d}\sigma \; e^{-N\tilde{S}(\lambda;\sigma)}$

# Contents


✓ Motivations and Brief summary (3)
✓ $2^{nd}$ order phase transition in SQED3 (review) (4)
✓ Lefschetz thimble analysis (7)
✓ Borel resummation (4)
✓ Lessons from SQED3 (3)
• Conclusion and future works (1)

Total: 22

# Conclusion and future works

## Question:

Is resurgence applicable to 2nd order phase transitions or more realistic QFTs?

## Answer: resurgence is applicable!

2nd order phase transition = simultaneous Stokes and anti-Stokes phenomenon

- The order of phase transition is determined by a collision of saddles
- It is decoded from a perturbative series

→ Generalized to other systems

## Future works:

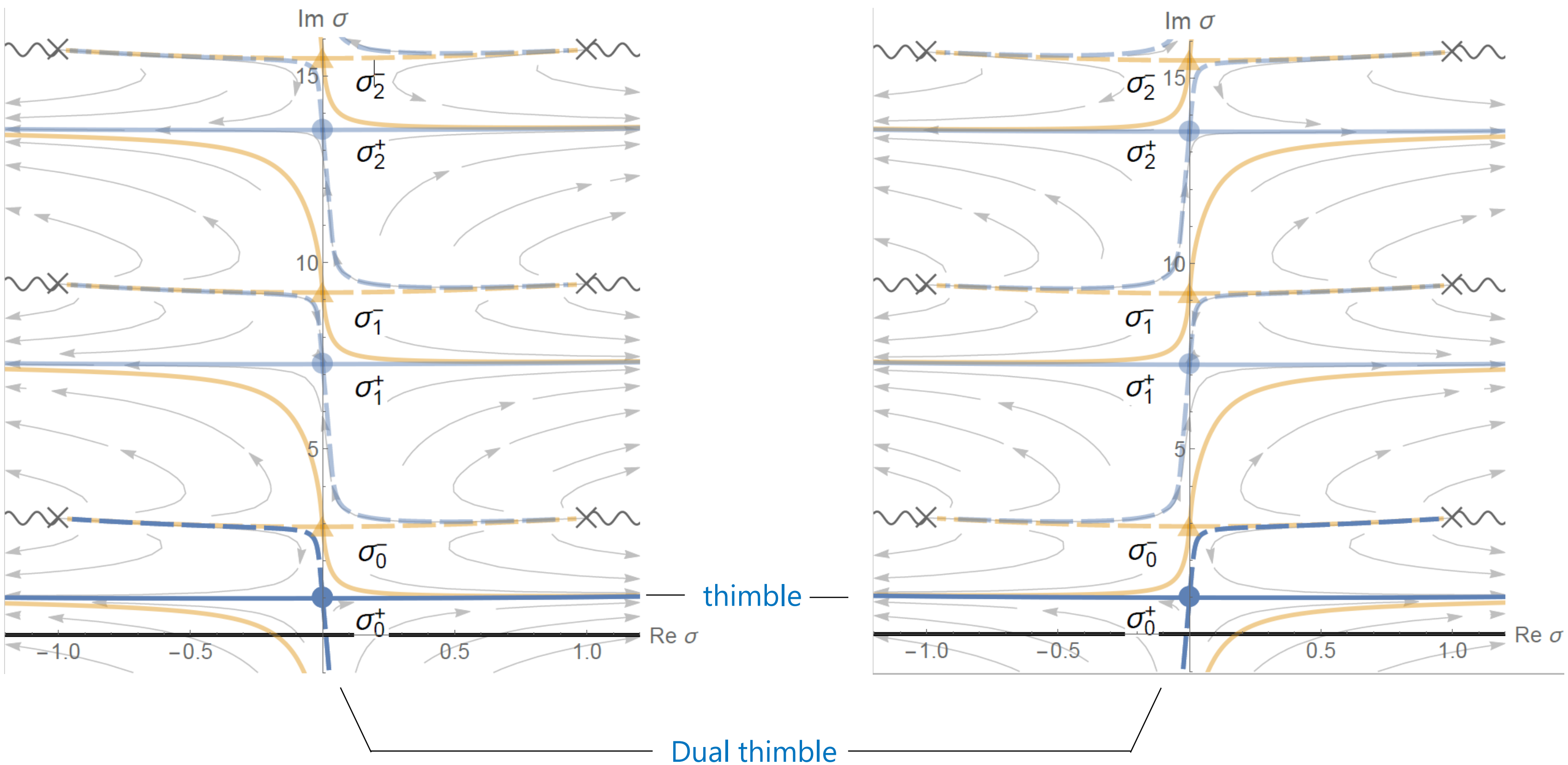
- Relation to Lee-Yang zeros ?
- Expansion with respect to other parameters ?
- Physical meaning of the phase transition ?

# Backups

# Lefschetz thimble structure (subcritical)

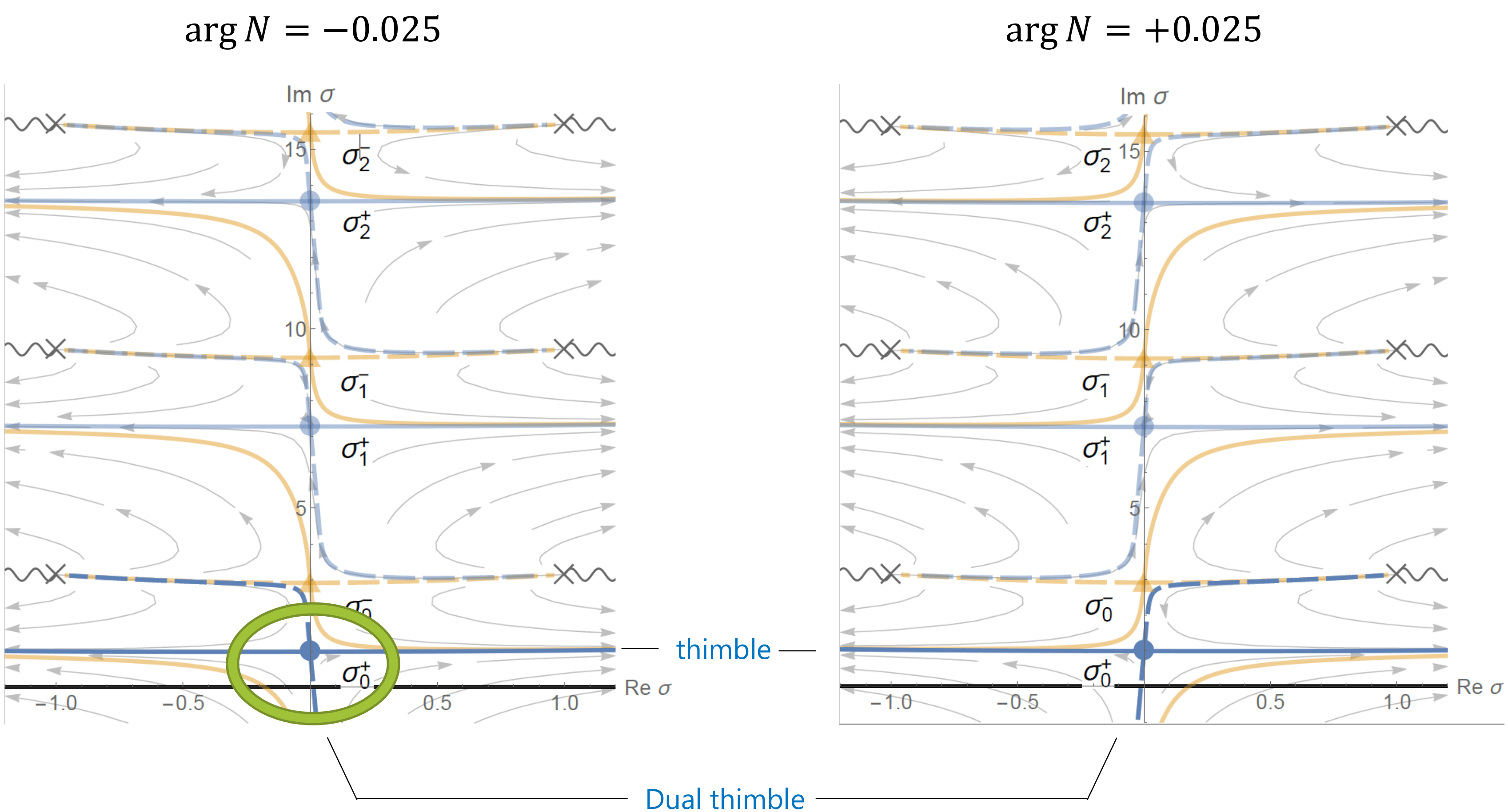
$\lambda < \lambda_c$

$\arg N = -0.025$ $\qquad\qquad$ $\arg N = +0.025$



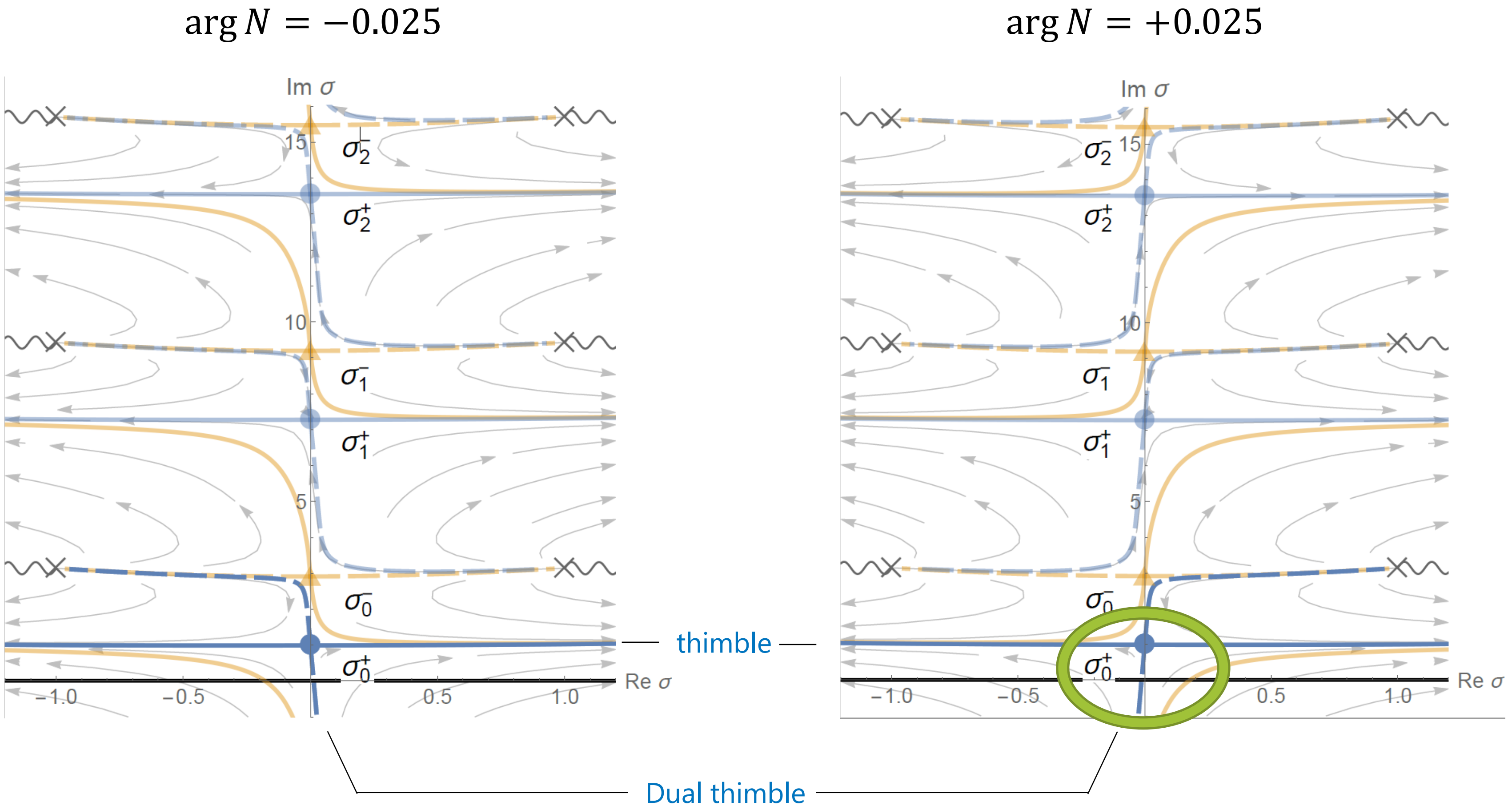
# Lefschetz thimble structure (subcritical)

$\lambda < \lambda_c$



# Lefschetz thimble structure (subcritical)

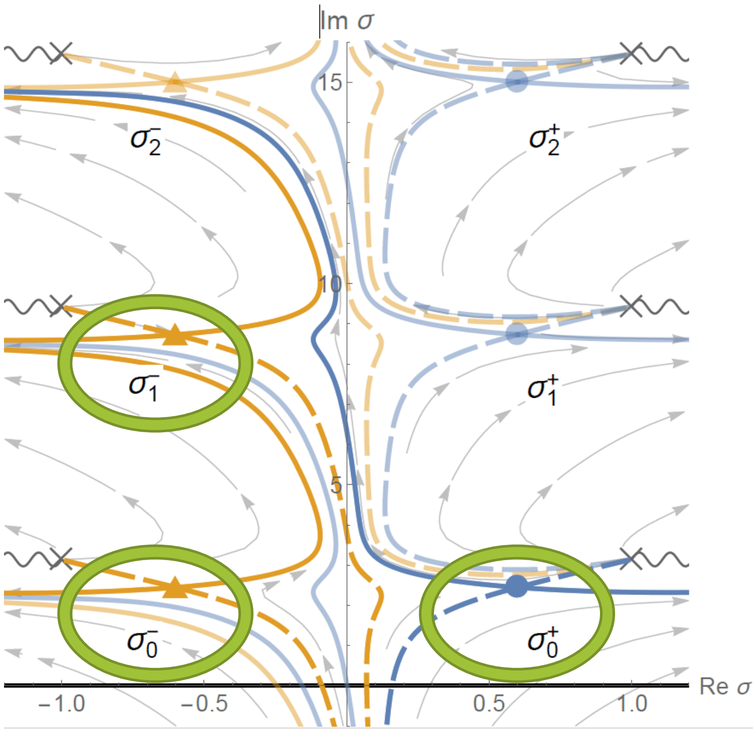
$\lambda < \lambda_c$



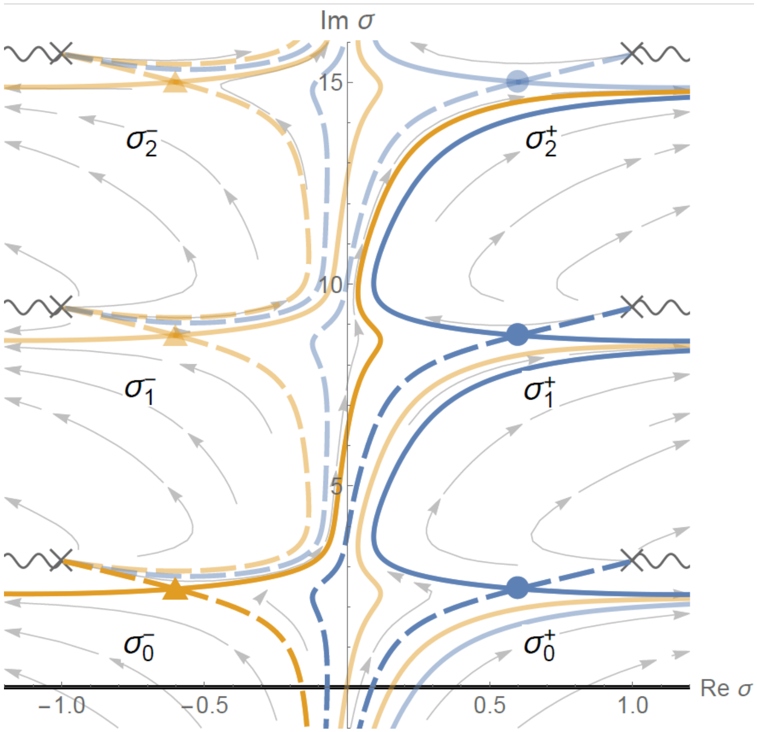
# Lefschetz thimble structure (supercritical)

$\boxed{\lambda \geq \lambda_c}$

$\arg N = -0.025$



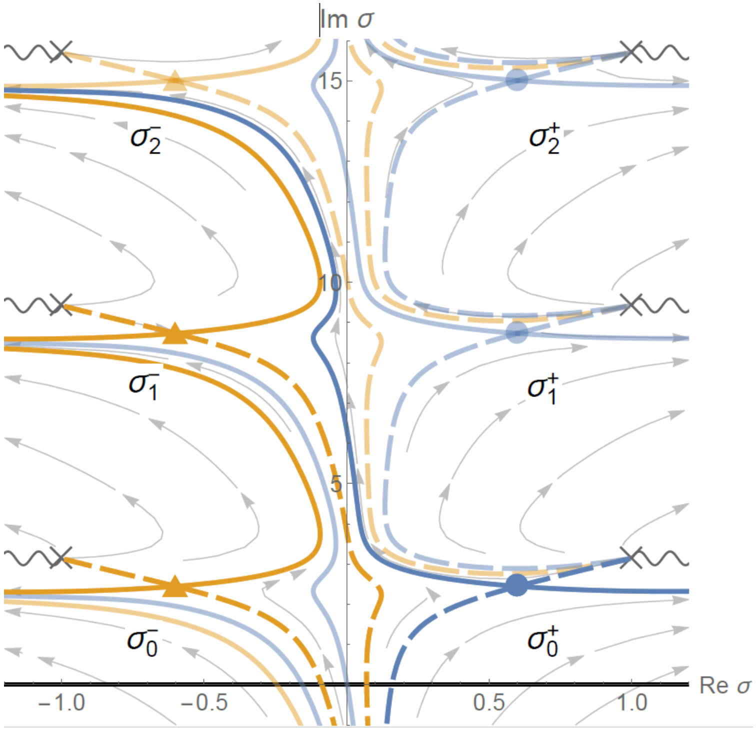

$\arg N = +0.025$



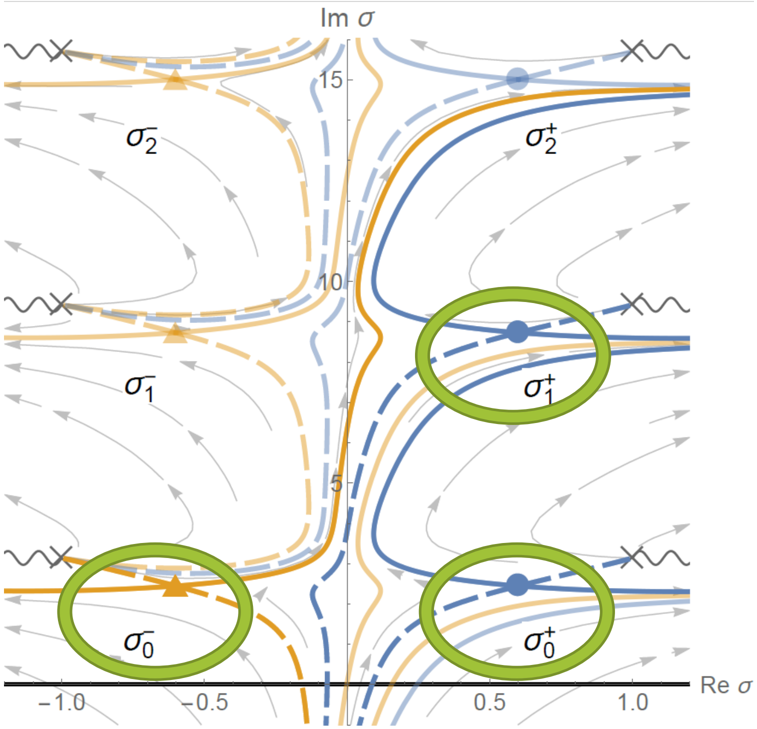
# Lefschetz thimble structure (supercritical)

$\boxed{\lambda \geq \lambda_c}$

$\arg N = -0.025$




$\arg N = +0.025$



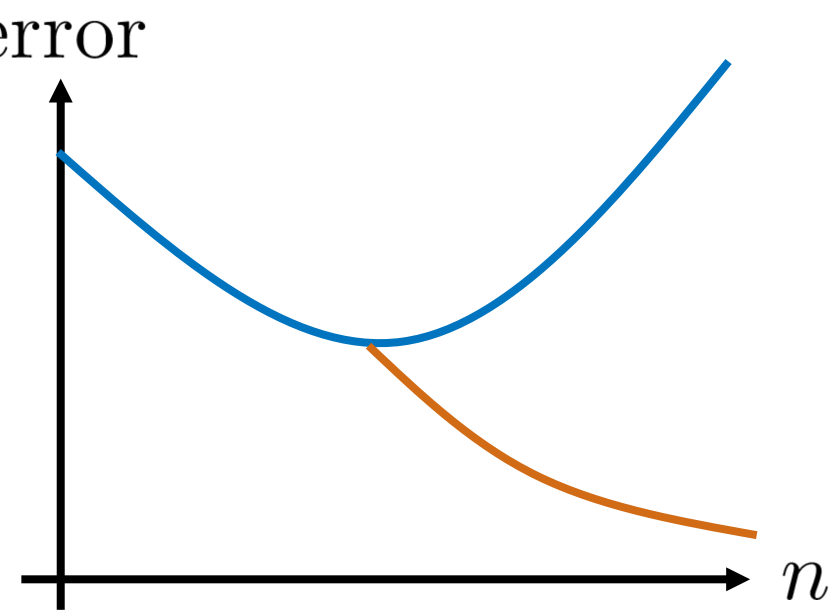
# Contents

✓ Motivations and Brief summary
- $2^{nd}$ order phase transition in SQ
- Lefschetz thimble analysis
- Borel resummation (11)
- Lessons from SQED3 (3)
- Conclusion and future works (1)

Total: 31

Before these, let me provide
*lightning introduction to resurgence*

# Asymptotic series and resurgence structure




$$Z(g) = \sum_{n=0}^{\infty} c_n g^{n+r}, \quad c_n \sim n!$$

Resurgence structure

$$Z(g) = \sum_{n=0}^{\infty} c_n^{(0)} g^n + \sum_{n=0}^{\infty} c_n^{(1)} g^n e^{-S_1/g} + \cdots$$

[J. Ecalle, 81]
[M. Marino, 12]
[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]
[D. Dorigoni, 19]
[I. Aniceto, G. Basar, R. Schiappa, 19]

Resurgence theory:

the perturbative part knows the non-perturbative parts

# Example: 0dim Sine-Gordon model

Partition function:

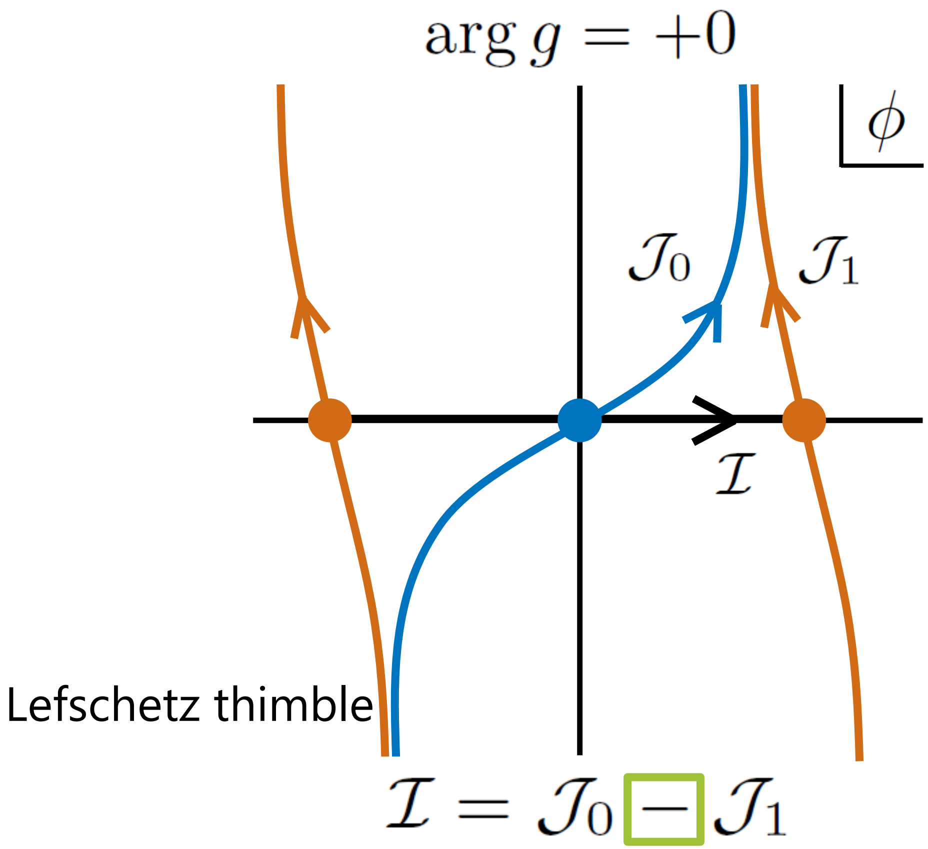
$$Z(g) := \sqrt{\frac{\pi}{2g}} e^{-1/4g} I_0(1/4g) = \frac{1}{\sqrt{2\pi g}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\varphi \, e^{-\frac{1}{2g}\sin^2\varphi} \quad \left(-\frac{\pi}{2} < \pm \arg(1/4g) < \frac{3\pi}{2}\right)$$

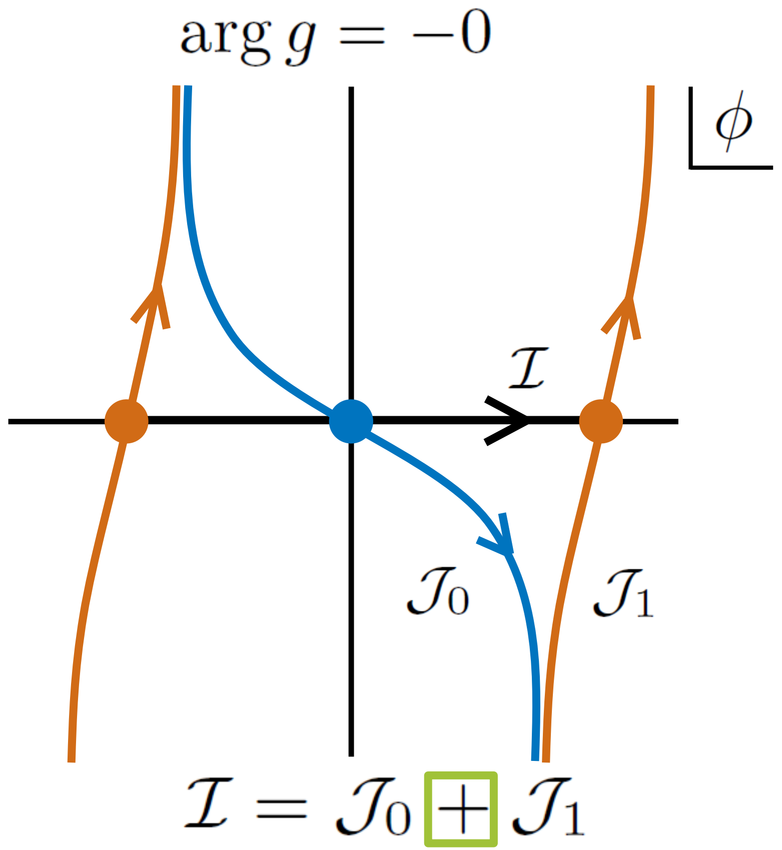
$$= \sum_{n=0}^{\infty} \frac{(+2)^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)} g^{n+1} \pm i e^{-1/2g} \sum_{n=0}^{\infty} \frac{(-2)^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)} g^{n+1}$$

Perturbative part — Non-perturbative part

# Observation 1

$$Z(g) = \frac{1}{\sqrt{2\pi g}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\varphi \, e^{-\frac{1}{2g}\sin^2\varphi} = \sum_{n=0}^{\infty} c_n^{(0)} g^{n+1} \pm i e^{-1/2g} \sum_{n=0}^{\infty} c_n^{(1)} g^{n+1}$$



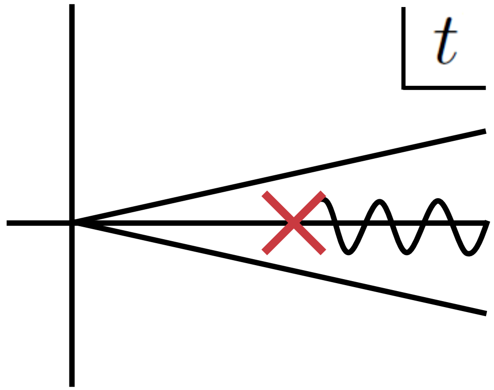



# Observation 2

$$\sum_{n=0}^{\infty} c_n^{(0)} g^{n+1} \to \int_0^{\infty} \mathrm{d}t\, e^{-t/g} \sum_{n=0}^{\infty} \frac{c_n^{(0)}}{\Gamma(n+1)} t^n$$

Borel resummation

$$= \int_0^{\infty} \mathrm{d}t\, e^{-t/g} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; 2t\right)$$




"Ambiguity" $\sim ie^{-1/2g}$

Recall

$$\sum_{n=0}^{\infty} \frac{(+2)^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)} g^{n+1} \pm ie^{-1/2g} \sum_{n=0}^{\infty} \frac{(-2)^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)} g^{n+1}$$

# Observation 3

$$\underline{c_n^{(0)}} = \frac{(+2)^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)}$$

$$\sim \frac{2^n \Gamma(n)}{\Gamma(1/2)^2} \left[ \boxed{1} + \frac{\boxed{-1/4}}{n-1} + \frac{\boxed{9/32}}{(n-1)(n-2)} + \frac{\boxed{-75/128}}{(n-1)(n-2)(n-3)} + \cdots \right]$$

$$\underline{c_n^{(1)}} = \frac{(-2)^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)}$$

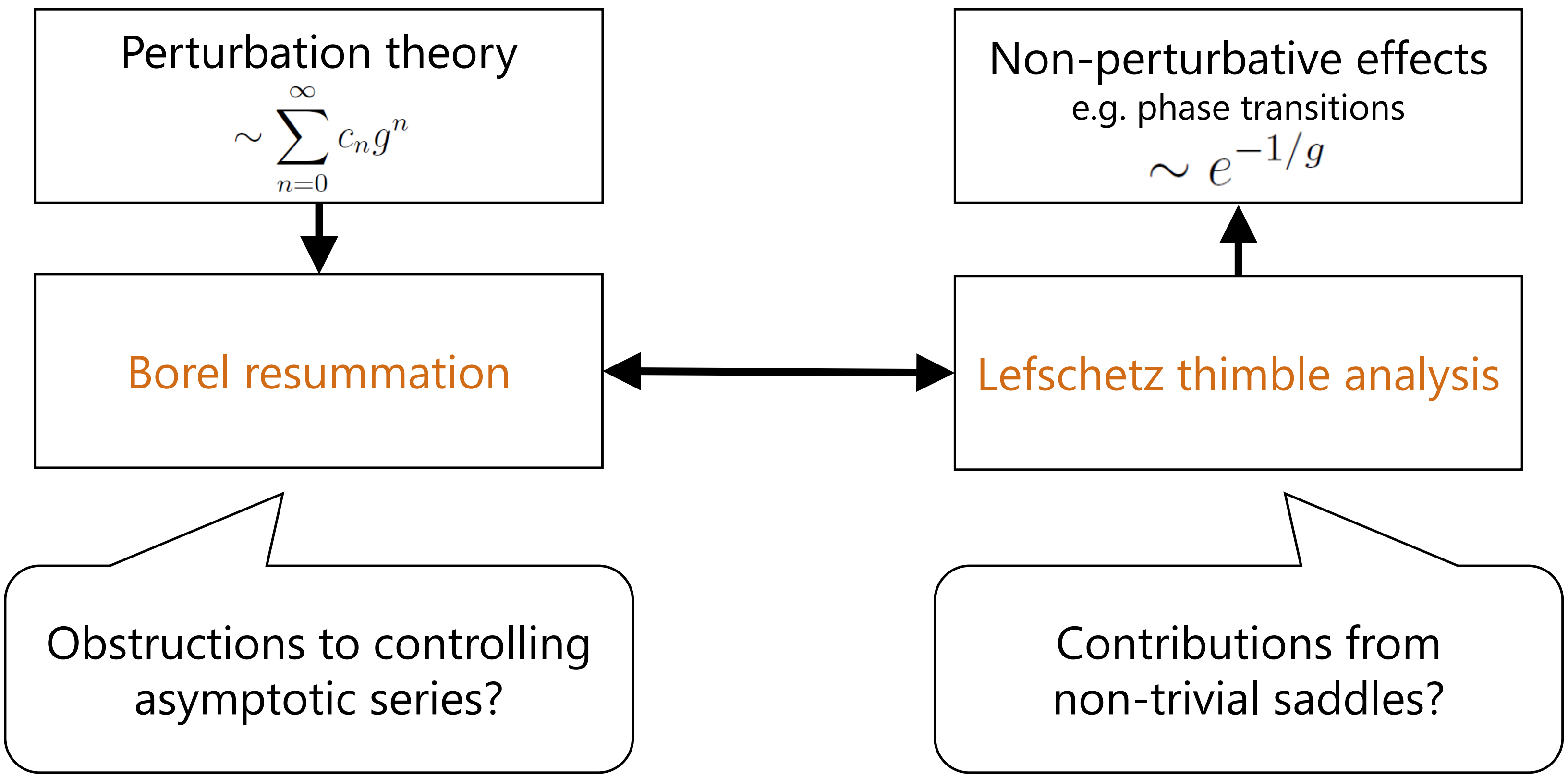
$$c_0^{(1)} = \boxed{1},\ c_1^{(1)} = 2^1 \left( \boxed{\frac{-1}{4}} \right),\ c_2^{(1)} = 2^2 \left( \boxed{\frac{9}{32}} \right),\ c_3^{(1)} = 2^3 \left( \boxed{\frac{-75}{128}} \right)$$

# Resurgence theory

[J. Ecalle, 81]

Lectures and reviews, e.g.
[M. Marino, 12]
[D. Dorigoni, 19]
[I. Aniceto, G. Basar, R. Schiappa, 19]

- One of the approaches to non-perturbative physics
- Decodes *non-perturbative information* from *perturbation theory*



# Application to the "path integral"

[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

Partition function:

$$Z = \int_{-\infty}^{\infty} \mathrm{d}\sigma \; e^{-S(\sigma)} \qquad S(\sigma) = N_f \left[ -i\lambda\sigma + \ln(\cosh\sigma + \cosh m) \right]$$

Thimble/dual-thimble equations:

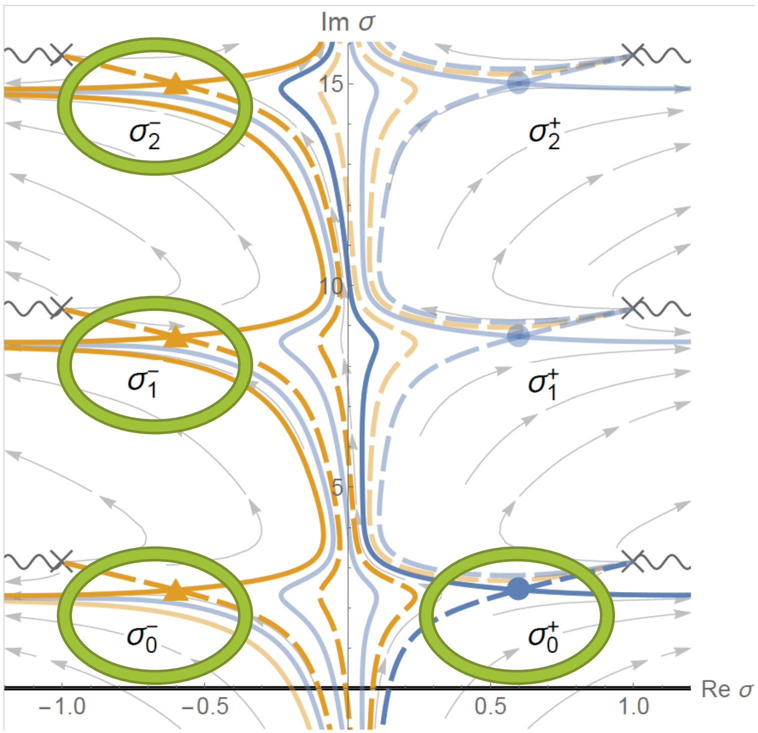
$$\mathcal{J}_n^{\pm} : \ \frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = +\overline{\frac{\mathrm{d}S(\sigma)}{\mathrm{d}\sigma}}, \quad \sigma(-\infty) = \sigma_n^{\pm}$$

$$\mathcal{K}_n^{\pm} : \ \frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = -\overline{\frac{\mathrm{d}S(\sigma)}{\mathrm{d}\sigma}}, \quad \sigma(-\infty) = \sigma_n^{\pm}$$

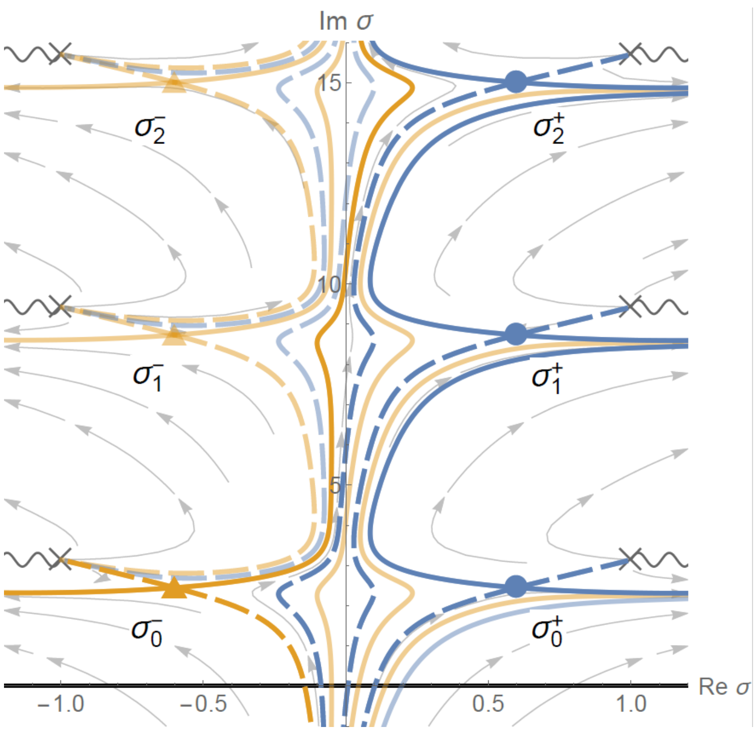
# Lefschetz thimble structure (supercritical)

$\boxed{\lambda \geq \lambda_c}$

$\arg N = -0.015$



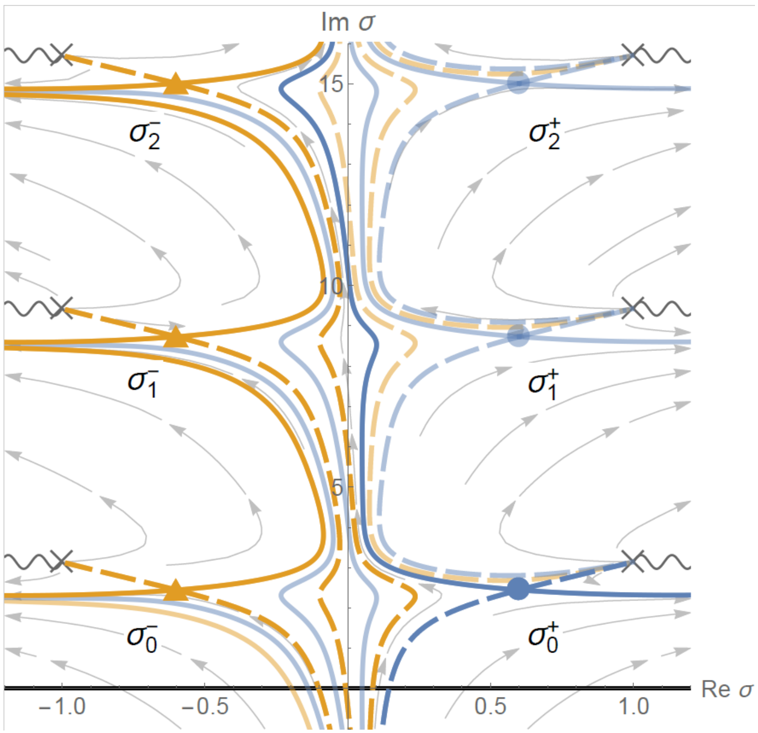

$\arg N = +0.015$



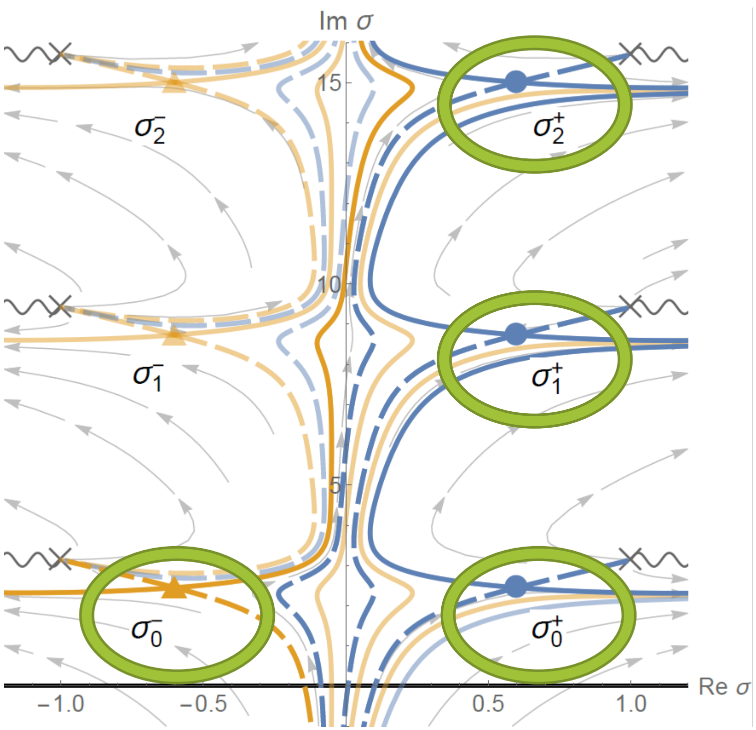
# Lefschetz thimble structure (supercritical)

$\boxed{\lambda \geq \lambda_c}$

$\arg N = -0.015$



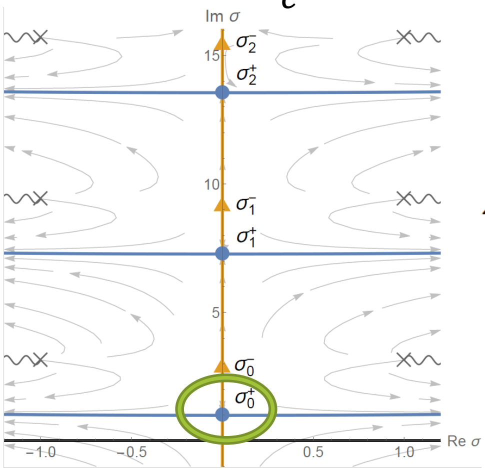

$\arg N = +0.015$



# What we do

## Large-flavor expansion around the trivial saddle:

$\lambda < \lambda_c$




asymptotic series!

$$Z(\lambda; N) = \frac{1}{2^N}\int \mathrm{d}\sigma\ e^{-NS(\lambda;\sigma)}, \quad S(\lambda;\sigma) = -i\lambda\sigma + \ln(\cos \quad + \cosh m)$$

$$\overset{\text{around } \sigma_0^+}{=} \frac{1}{2^N}\sqrt{\frac{2\pi}{NS''(\lambda;\sigma_0^+)}}\, e^{-NS(\lambda;\sigma_0^+)} \sum_{l=0}^{\infty} \frac{a_l(\lambda)}{N^l}$$

## Borel resummation:

$$\mathcal{S}Z(\lambda; N) = \frac{1}{2^N}\sqrt{\frac{2\pi}{NS''(\lambda;\sigma_0^+)}}\, e^{-NS(\lambda;\sigma_0^+)} \cdot N\int_C \mathrm{d}t\ e^{-Nt} \sum_{l=0}^{\infty} \frac{a_l(\lambda)}{\Gamma(l+1)} t^l$$

## Sub-questions in this part:

- Is the Borel plane structure ***consistent with the Lefschetz thimble structure***?
- Can we ***decode the phase transition*** from the perturbative series?

# But please wait (1/2): Borel-Padé approximation

Exact quantities:

$$F\left(\frac{1}{N_f}\right) = \sum_{\ell=0}^{\infty} \frac{a_l}{N_f^{\ell}}$$

asymptotic series

$$\mathcal{B}F(t) = \sum_{l=0}^{\infty} \frac{a_\ell}{\Gamma(\ell+1)} t^\ell$$

Borel transformation

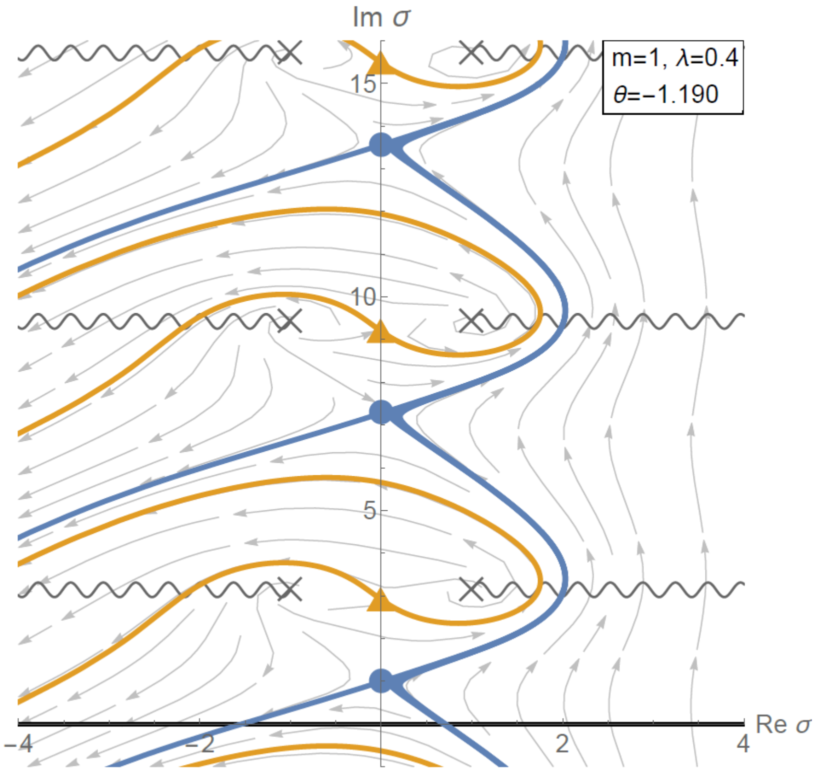
Approximate these from finite number of inputs

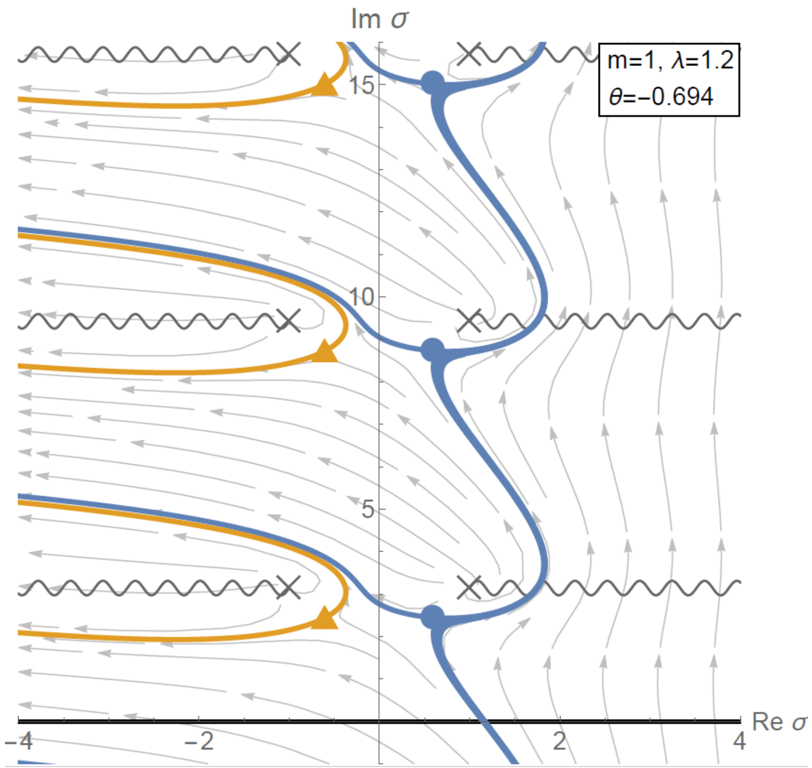
$$\mathcal{P}_{m,n}(t) = \frac{P_m(t)}{Q_n(t)}$$

Borel-Padé approximation

# But please wait (2/2): Larger $\theta = \arg N_f$

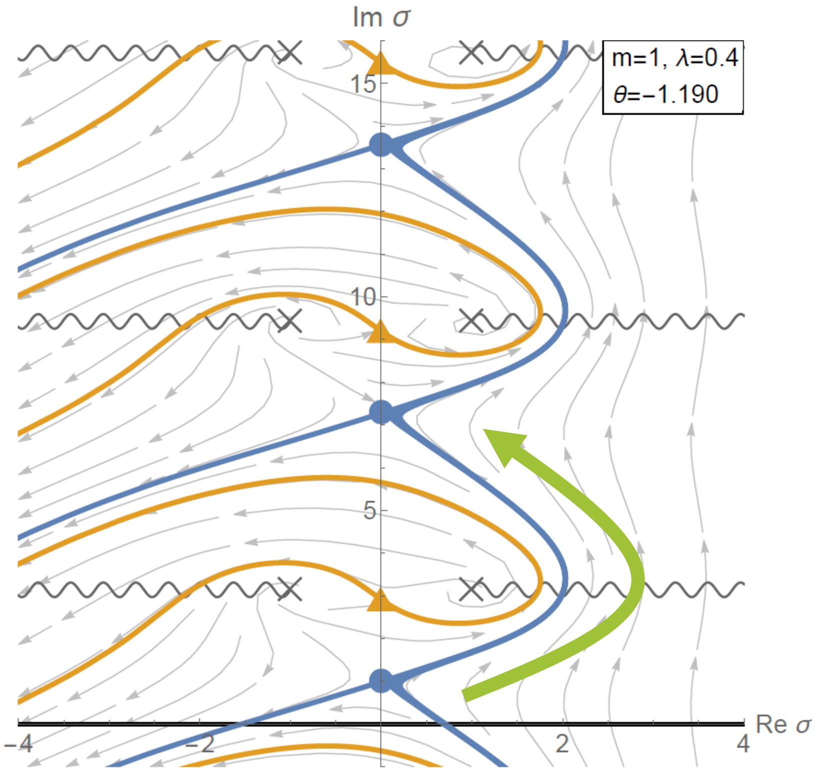
Stokes phenomena occur on different Riemann sheets

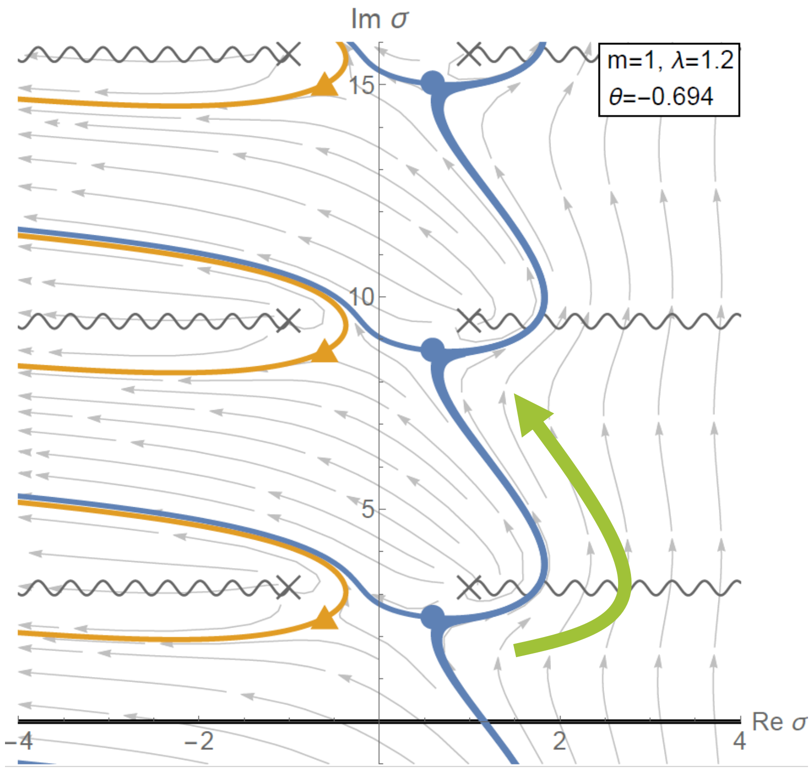


# But please wait (2/2): Larger $\theta = \arg N_f$

Stokes phenomena occur on different Riemann sheets



# Borel plane structure

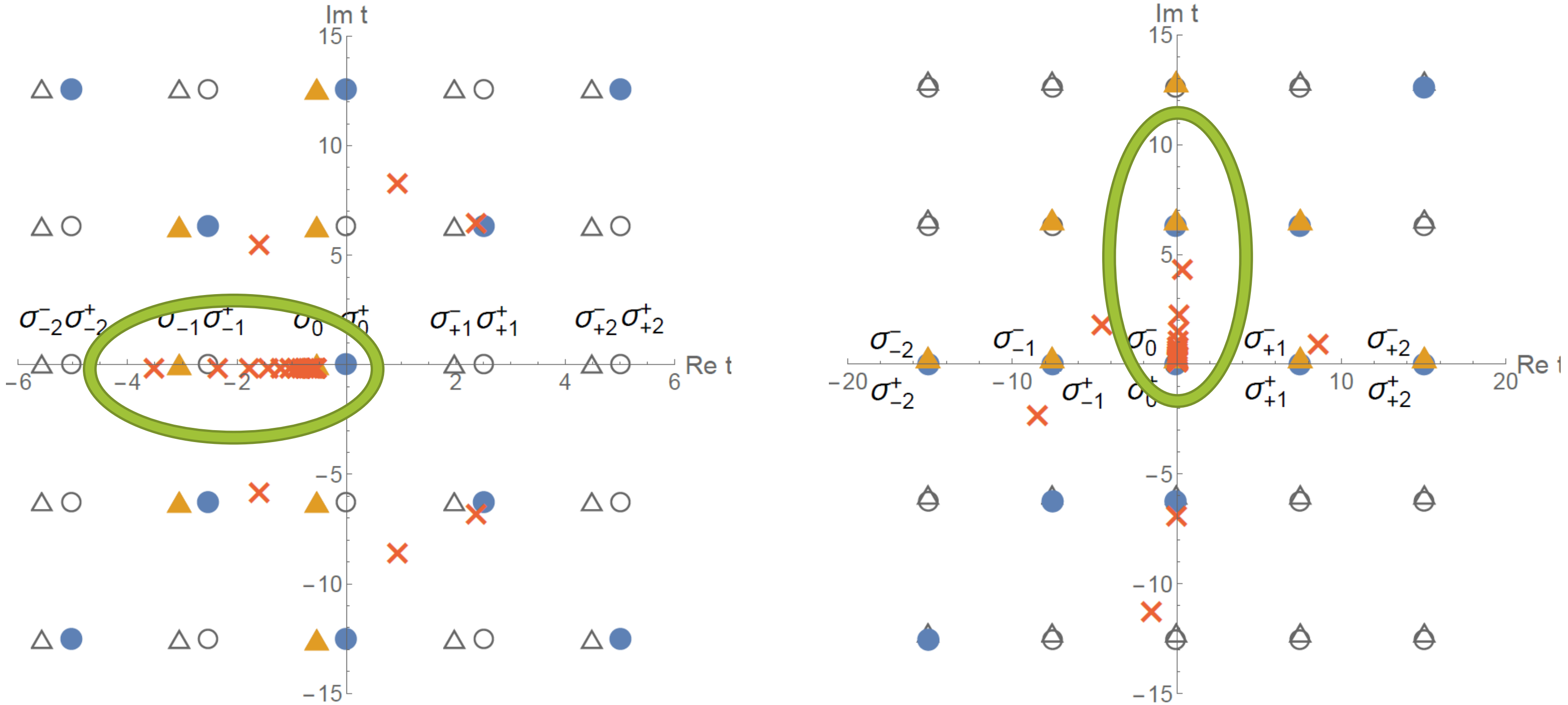
[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

$\lambda < \lambda_c$

$\lambda = 0.4, m = 1$, Padé (25,25)

$\lambda \geq \lambda_c$

$\lambda = 1.4, m = 1$, Padé (25,25)




Poles of the Padé approximant are consumed for branch cuts…

# Improvement: Padé-Uniformized approximation

[Costin, Dunne, 20]

Uniformize the Borel $t$-plane by a map:

$$t \mapsto u(t) = -\ln\left(1 - \frac{t}{s}\right)$$

Branch cut singularity at $t = s$ is eliminated

Perform the standard Padé approximation on the $u$-plane

$$\widetilde{\mathcal{B}F}(t) \simeq \mathcal{P}_{m,n}(u(t))$$

# Analytical study for large $\lambda$

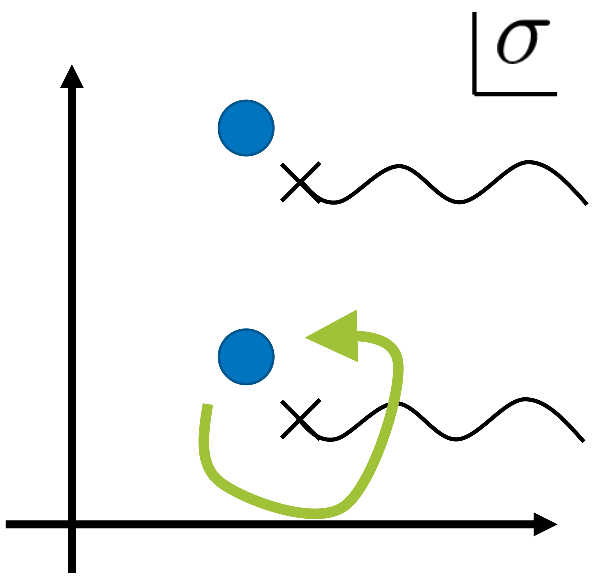
[T. Fujimori, M. Honda, S. Kamata, T. Misumi, N. Sakai, TY, 21]

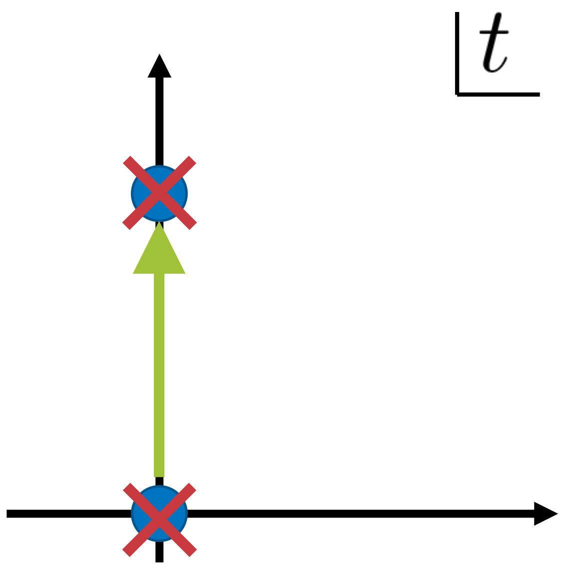
Lading contribution for large $\lambda$

$$F(N_f;\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\delta\sigma \; e^{-N_f(i\lambda\delta\sigma + \log(1 - i\lambda\delta\sigma))}$$

$$= \frac{1}{i\lambda}\int dt\; e^{-N_f t}\frac{W(-e^{t-1})}{1+W(-e^{t-1})}$$

$\sigma$-plane and Borel $t$-plane are directly related via $\delta\sigma = \frac{1}{i\lambda}\left(1 + W(-e^{t-1})\right)$



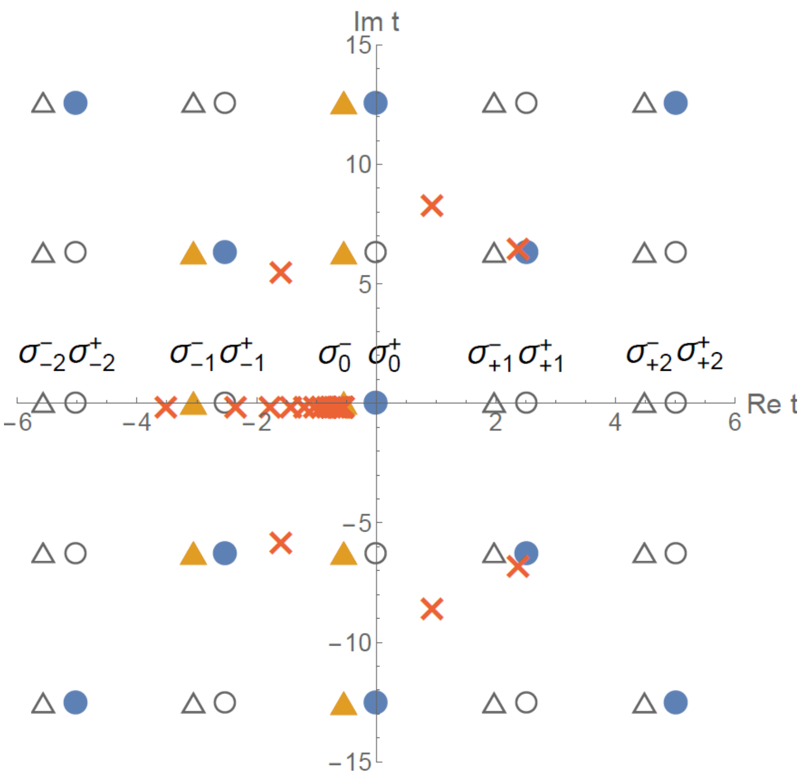



# Borel singularities

- The Borel plane structure is consistent with the Lefschetz thimble structure
- Still there are artifacts, and missing singularities far from the origin

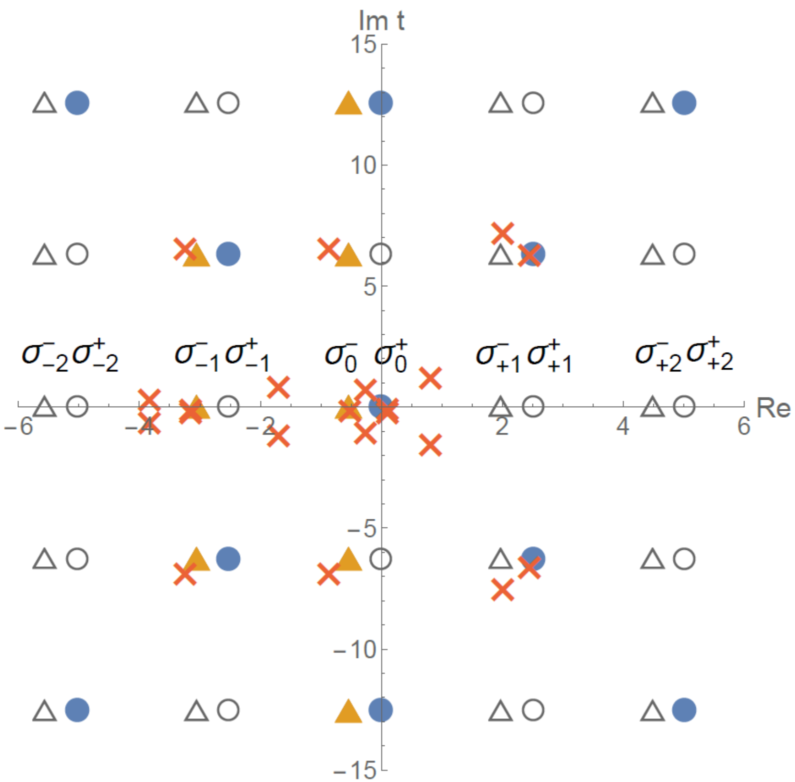
$\lambda < \lambda_c$

$\lambda = 0.4, m = 1$, Padé (25,25)




$\lambda < \lambda_c$

$\lambda = 0.4, m = 1$, Padé-Uniformized (25,25)

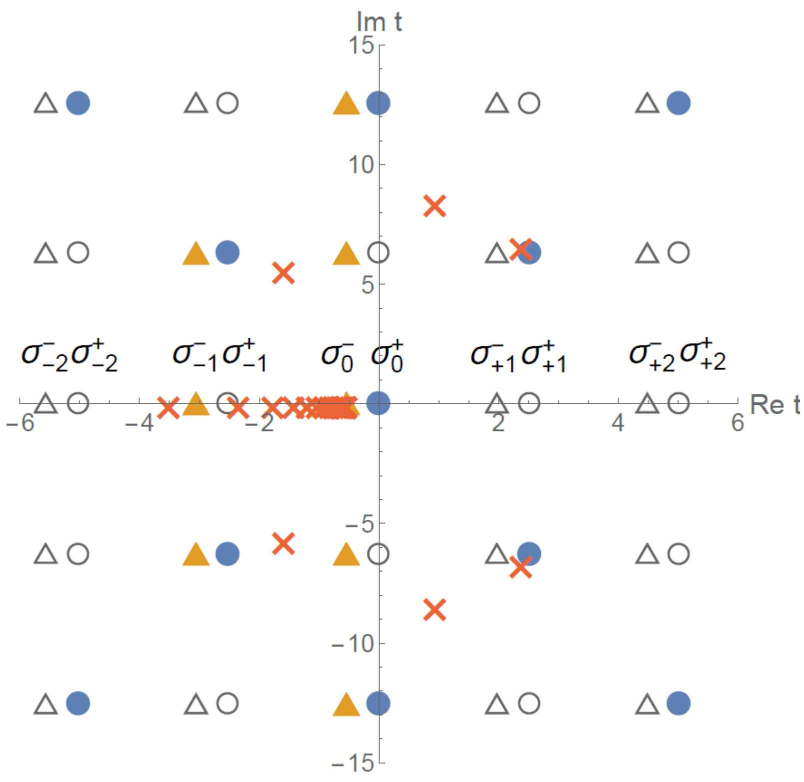


# Borel singularities

- The Borel plane structure is consistent with the Lefschetz thimble structure
- Still there are artifacts, and missing singularities far from the origin

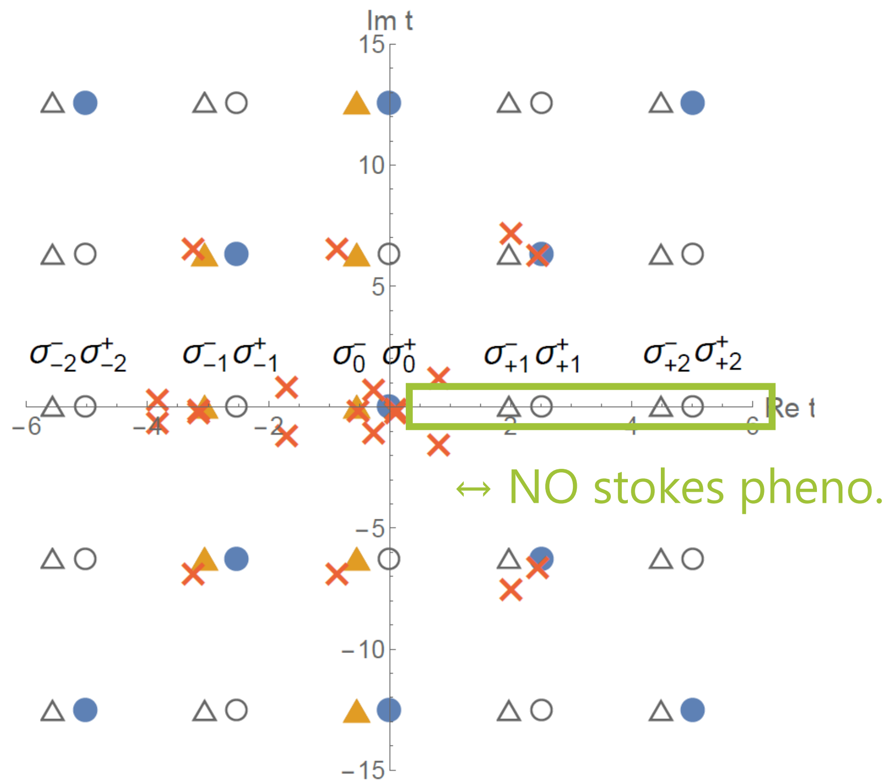
$\lambda < \lambda_c$

$\lambda = 0.4, m = 1$, Padé (25,25)




$\lambda < \lambda_c$

$\lambda = 0.4, m = 1$, Padé-Uniformized (25,25)

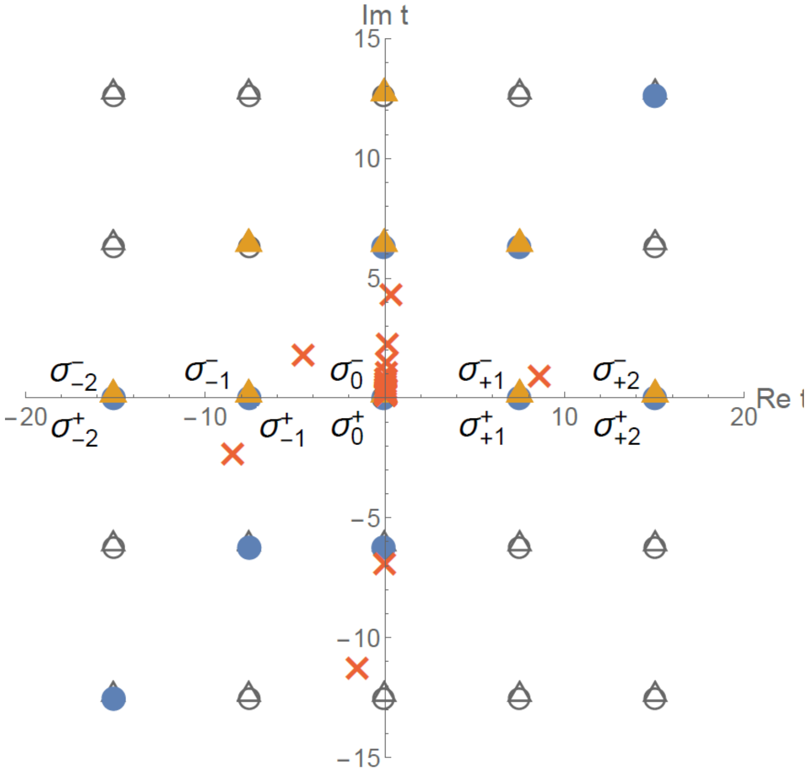


# Borel singularities

- The Borel plane structure is consistent with the Lefschetz thimble structure
- Still there are artifacts, and missing singularities far from the origin

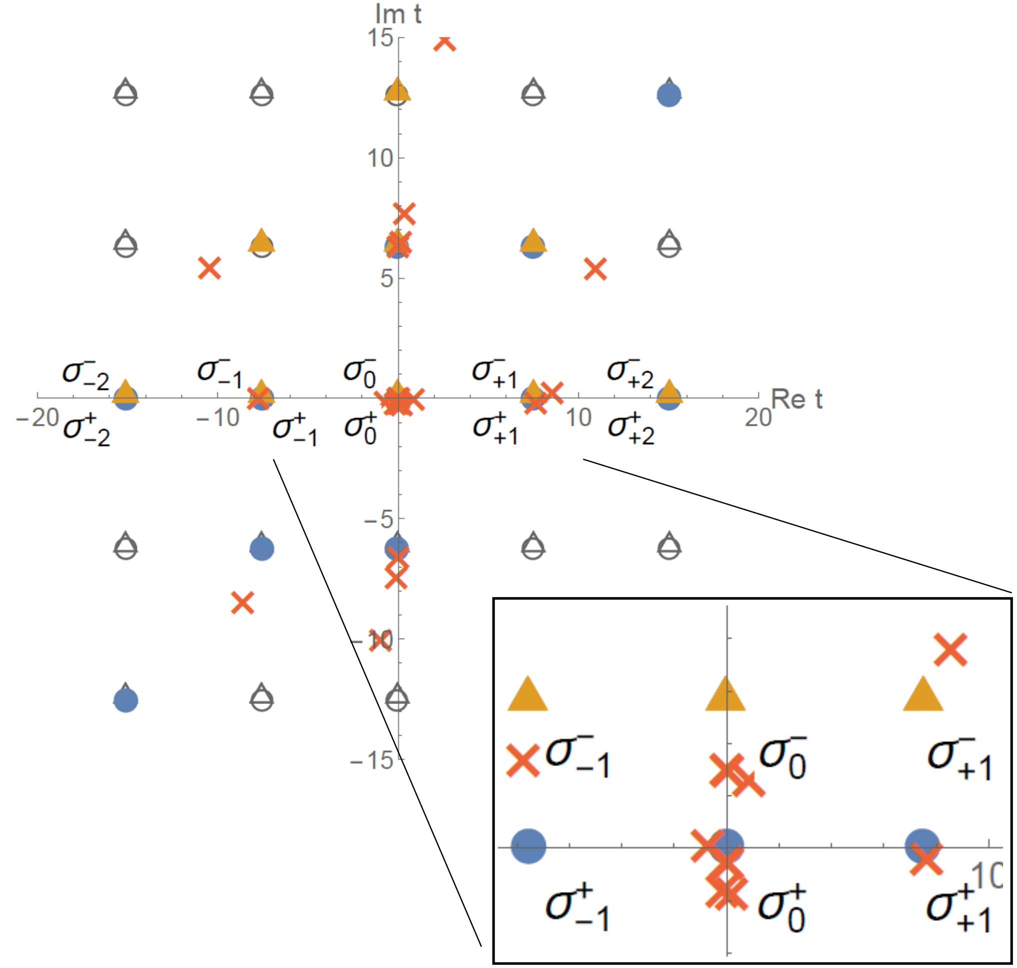
$\lambda \geq \lambda_c$

$\lambda = 1.4, m = 1$, Padé (25,25)




$\lambda \geq \lambda_c$

$\lambda = 1.2, m = 1$, Padé-Uniformized (25,25)

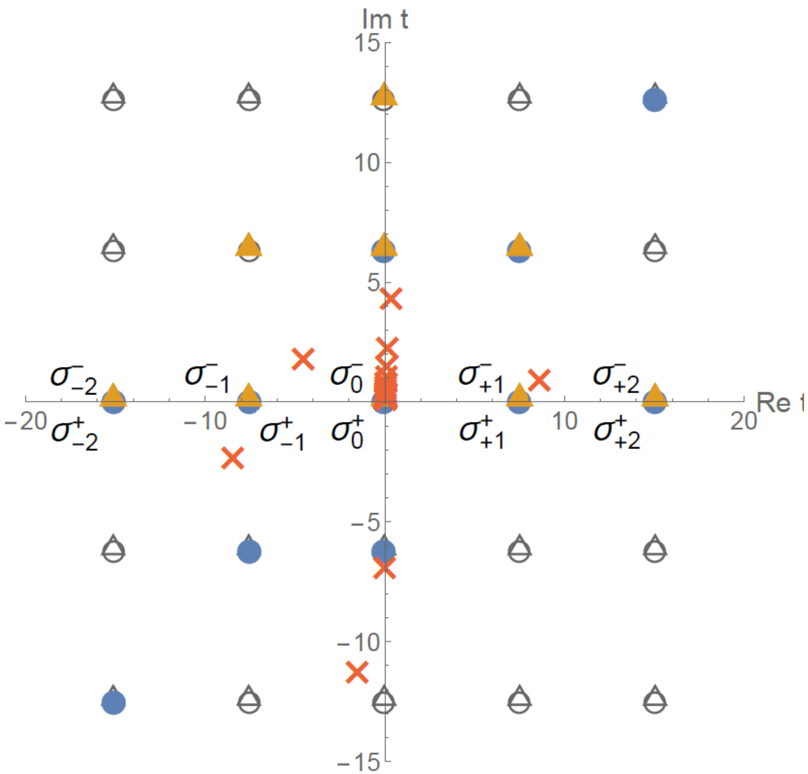


# Borel singularities

- The Borel plane structure is consistent with the Lefschetz thimble structure
- Still there are artifacts, and missing singularities far from the origin

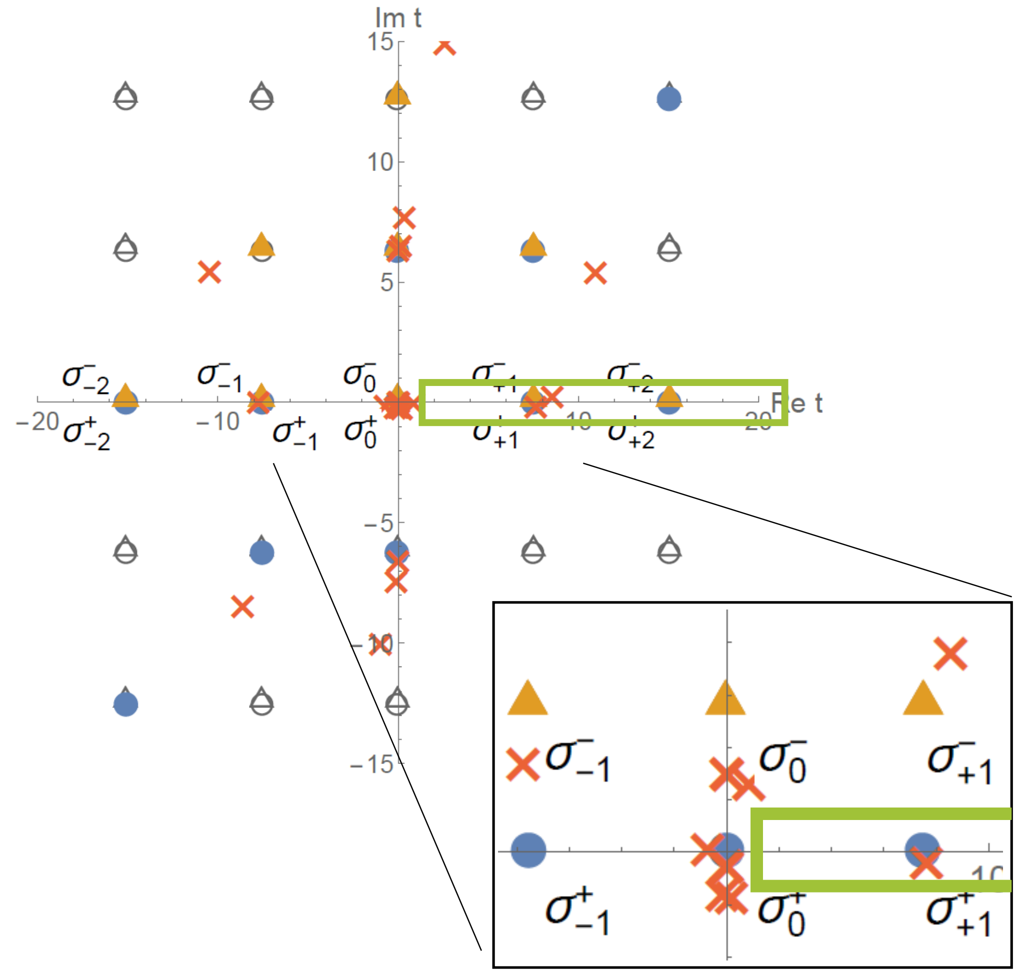
$\lambda \geq \lambda_c$

$\lambda = 1.4, m = 1,$ Padé (25,25)




$\lambda \geq \lambda_c$

$\lambda = 1.2, m = 1,$ Padé-Uniformized (25,25)



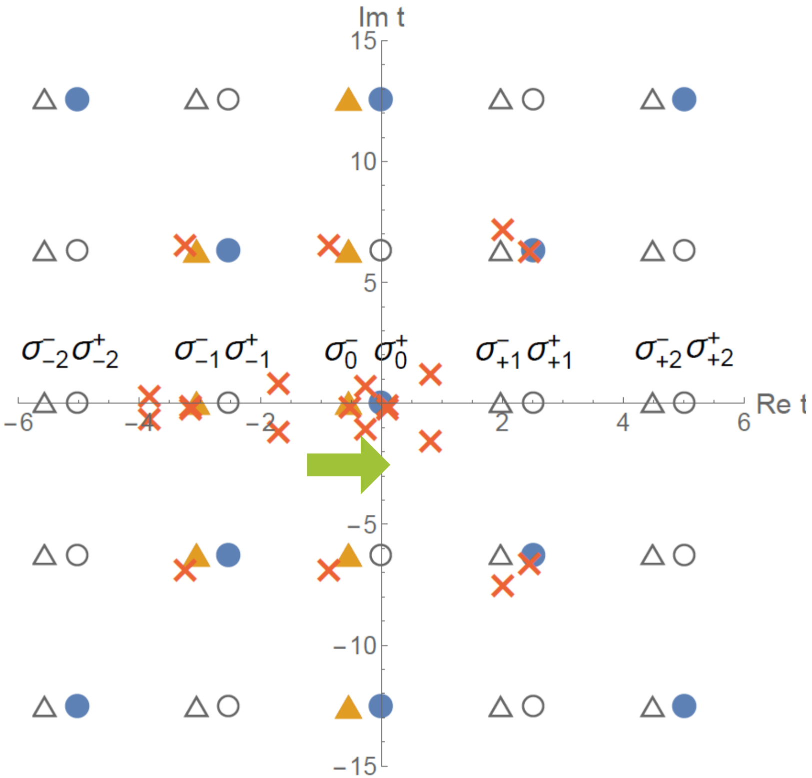
# The order of the phase transition

2nd order phase transition corresponds to

collision of two saddles with the reflection angle $\pi/2$

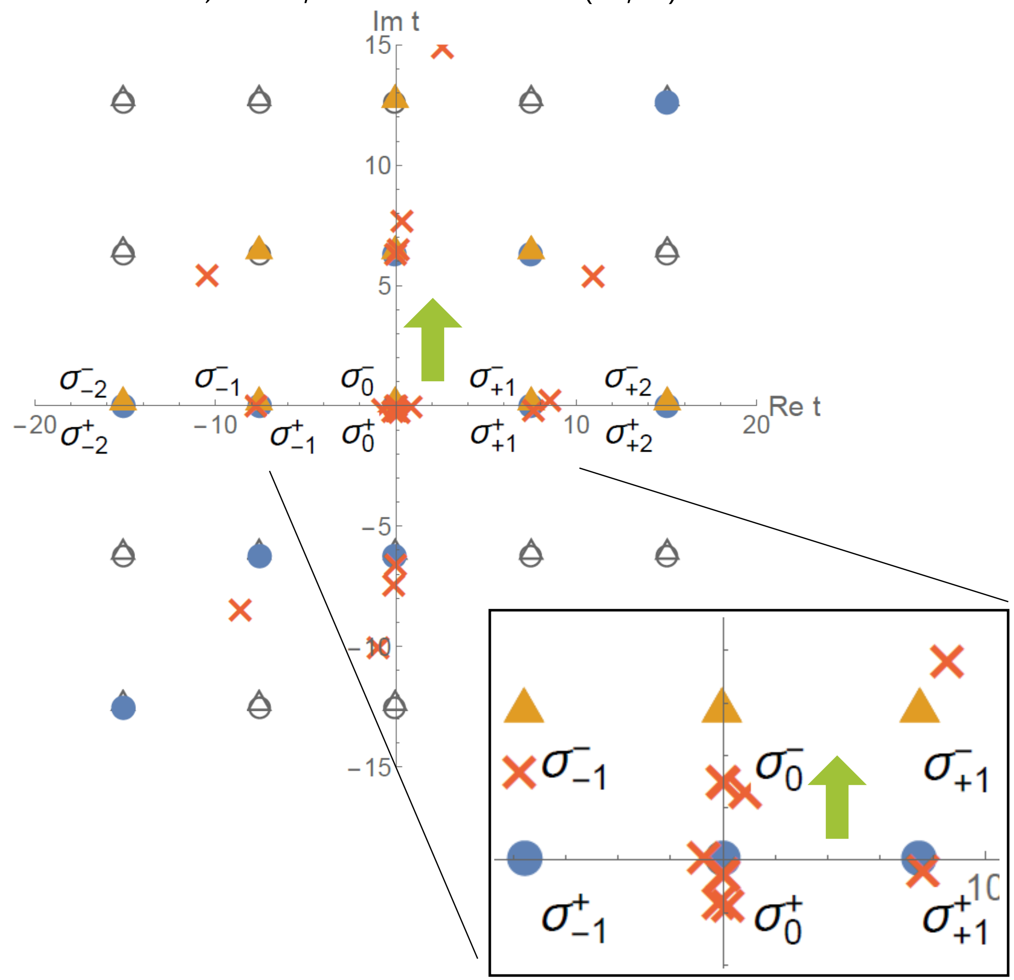
$\lambda < \lambda_c$

$\lambda = 0.4, m = 1$, Padé-Uniformized (25,25)




$\lambda \geq \lambda_c$

$\lambda = 1.2, m = 1$, Padé-Uniformized (25,25)



# Lefschetz thimble analysis

## 0dim Sine-Gordon model

[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]

$$Z(g) = \frac{1}{(2\pi g)^{1/2}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi \; e^{-S(\phi)/g}, \quad S(\phi) = \frac{1}{2}\sin^2\phi$$

## Saddles and Lefschetz thimbles

$$0 = \frac{\mathrm{d}S(\phi)}{\mathrm{d}\phi} \Rightarrow \phi = 0, \pm\frac{\pi}{2}$$

Trivial saddle and non-trivial saddles

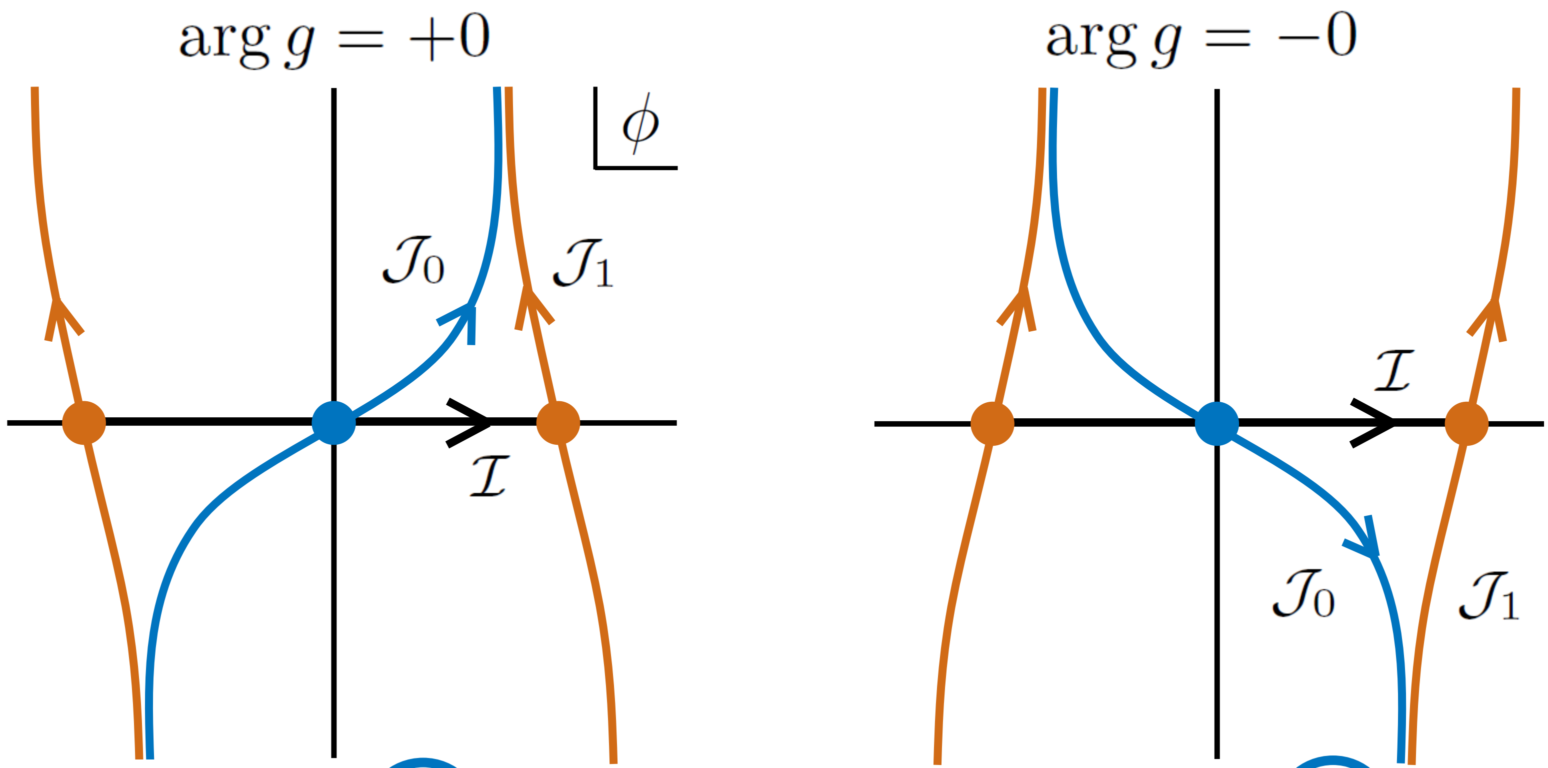
$$\mathcal{J}_i : \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = \overline{\frac{\mathrm{d}S}{\mathrm{d}\phi}}, \quad \phi(-\infty) = \phi_i \qquad \mathrm{Im}\, S(\phi(t)) = \mathrm{const.}, \quad \mathrm{Re}\, S(\phi_i) \leq \mathrm{Re}\, S(\phi(t))$$

$$\mathcal{K}_i : \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = -\overline{\frac{\mathrm{d}S}{\mathrm{d}\phi}}, \quad \phi(-\infty) = \phi_i \qquad \mathrm{Im}\, S(\phi(t)) = \mathrm{const.}, \quad \mathrm{Re}\, S(\phi_i) \geq \mathrm{Re}\, S(\phi(t))$$

# Lefschetz thimble analysis

Around $\arg g = 0$, [Cherman, Dorigoni, Unsal, 14] [Cherman, Koroteev, Unsal, 14]

Stokes phenomenon associated with the trivial saddle




$$\mathcal{I} = \mathcal{J}_0 - \mathcal{J}_1 \qquad \mathcal{I} = \mathcal{J}_0 + \mathcal{J}_1$$

NO Stokes phenomenon associated with the non-trivial saddles

# Borel resummation

[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]

Perturbation theory around the trivial saddle diverges

$$Z(g) = \frac{1}{(2\pi g)^{1/2}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi \; e^{-\frac{1}{2g}\sin^2\phi}$$

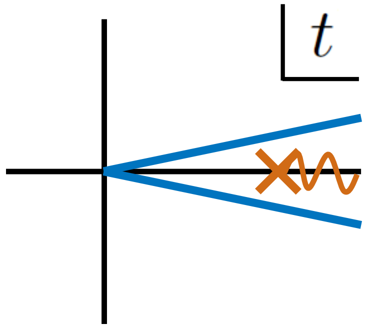
$$\overset{\text{around } \phi=0}{=} e^{-S(0)/g} \cdot \frac{1}{g} \sum_{n=0}^{\infty} \frac{(+2)^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)} g^{n+1}$$

There is a Borel singularity (and a branch cut) around $\arg g = 0$

$$\mathcal{S}Z(g) = \int_C \mathrm{d}t \; e^{-t/g} \mathcal{B}Z(t)$$

$$= e^{-S(0)/g} \cdot \frac{1}{g} \int_C \mathrm{d}t \; e^{-t/g} \sum_{n=0} \frac{2^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)^2} (+t)^n$$

$$= e^{-S(0)/g} \cdot \frac{1}{g} \int_C \mathrm{d}t \; e^{-t/g} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; +2t\right)$$



# Borel resummation

[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]

Perturbation theory around a non-trivial saddle diverges

$$Z(g) = \frac{1}{(2\pi g)^{1/2}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi \; e^{-\frac{1}{2g}\sin^2\phi}$$

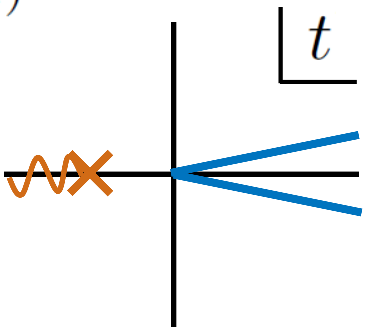
$$\overset{\text{around } \phi=\pi/2}{=} i e^{-S(\pi/2)/g} \cdot \frac{1}{g} \sum_{n=0}^{\infty} \frac{(-2)^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)} g^{n+1}$$

There is NO Borel singularity (nor branch cut) around $\arg g = 0$

$$\mathcal{S}Z(g) = \int_C \mathrm{d}t \; e^{-t/g} \mathcal{B}Z(t)$$

$$= i e^{-S(\pi/2)/g} \cdot \frac{1}{g} \int_C \mathrm{d}t \; e^{-t/g} \sum_{n=0} \frac{2^n \Gamma(n+1/2)^2}{\Gamma(1/2)^2 \Gamma(n+1)^2} (-t)^n$$

$$= i e^{-S(\pi/2)/g} \cdot \frac{1}{g} \int_C \mathrm{d}t \; e^{-t/g} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; -2t\right)$$



# Resurgence structure

[Cherman, Dorigoni, Unsal, 14]
[Cherman, Koroteev, Unsal, 14]

The two types of ambiguities cancel
and the location of the Borel singularity agrees with $S\left(\frac{\pi}{2}\right) = 1/2$

$$\mathcal{S}Z(g) = \mathcal{S}_{\pm}Z(g)|_{\text{around } \phi=0} \mp \mathcal{S}Z(g)|_{\text{around } \phi=\pi/2}$$

$$= e^{-S(0)/g} \cdot \frac{1}{g}\int_{C^{\pm}} \mathrm{d}t\ e^{-t/g}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;+2t\right)$$

$$\mp ie^{-S(\pi/2)/g} \cdot \frac{1}{g}\int_{C} \mathrm{d}t\ e^{-t/g}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right)$$

$$= \mathrm{Re}\ \mathcal{S}_{\pm}Z(g)|_{\text{around } \phi=0}$$

Information of non-trivial saddles is
encoded in perturbation theory around the trivial saddle

# Large-flavor expansion

[Fujimori, Honda, Kamata, Misumi, Sakai, TY, to appear]

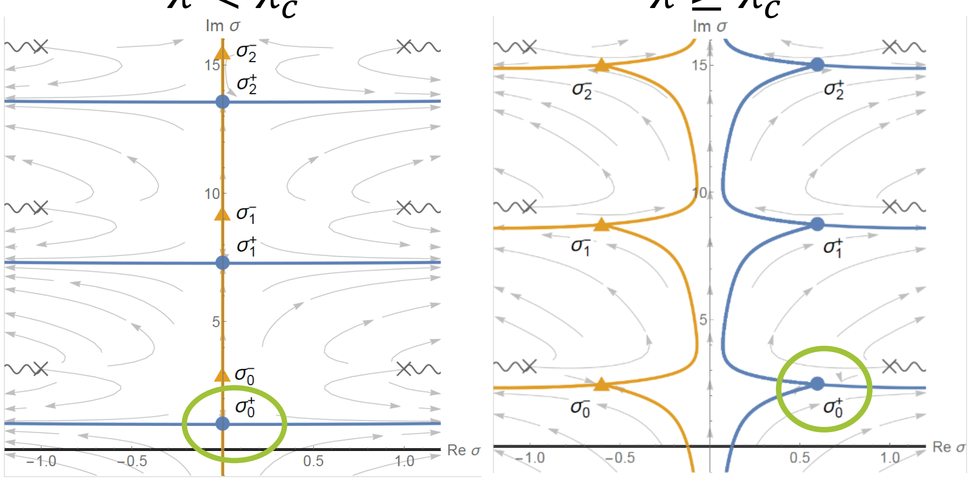
Consider the Borel resummation of $1/N$ expansion to see how thimbles' structure is encoded

$$Z(\lambda;N)=\frac{1}{2^N}\int \mathrm{d}\sigma\ e^{-NS(\lambda;\sigma)},\qquad S(\lambda;\sigma)=-i\lambda\sigma+\ln(\cosh\sigma+\cosh m)$$

$$\overset{\text{around }\sigma_0^+}{=}\frac{1}{2^N}\sqrt{\frac{2\pi}{NS''(\lambda;\sigma_0^+)}}e^{-NS(\lambda;\sigma_0^+)}\sum_{l=0}^{\infty}\frac{a_l(\lambda)}{N^l}$$

$$\mathcal{S}Z(\lambda;N)=\frac{1}{2^N}\sqrt{\frac{2\pi}{NS''(\lambda;\sigma_0^+)}}e^{-NS(\lambda;\sigma_0^+)}\cdot N\int_C \mathrm{d}t\ e^{-Nt}\underline{\sum_{l=0}^{\infty}\frac{a_l(\lambda)}{\Gamma(l+1)}t^l}$$

$\lambda<\lambda_c$ $\qquad\qquad$ $\lambda\geq\lambda_c$




Does perturbation theory around the trivial saddle know non-trivial saddles and the phase transition?