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Correlated Dirac Eigenvalues and Axial Anomaly in Chiral Symmetric QCD

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in collaboration with
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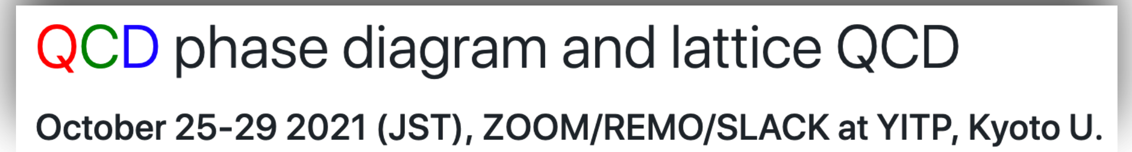
based on PRL 126 (2021) 082001, arXiv:2010.14836


QCD phase diagram and lattice QCD

October 25-29 2021 (JST), ZOOM/REMO/SLACK at YITP, Kyoto U.

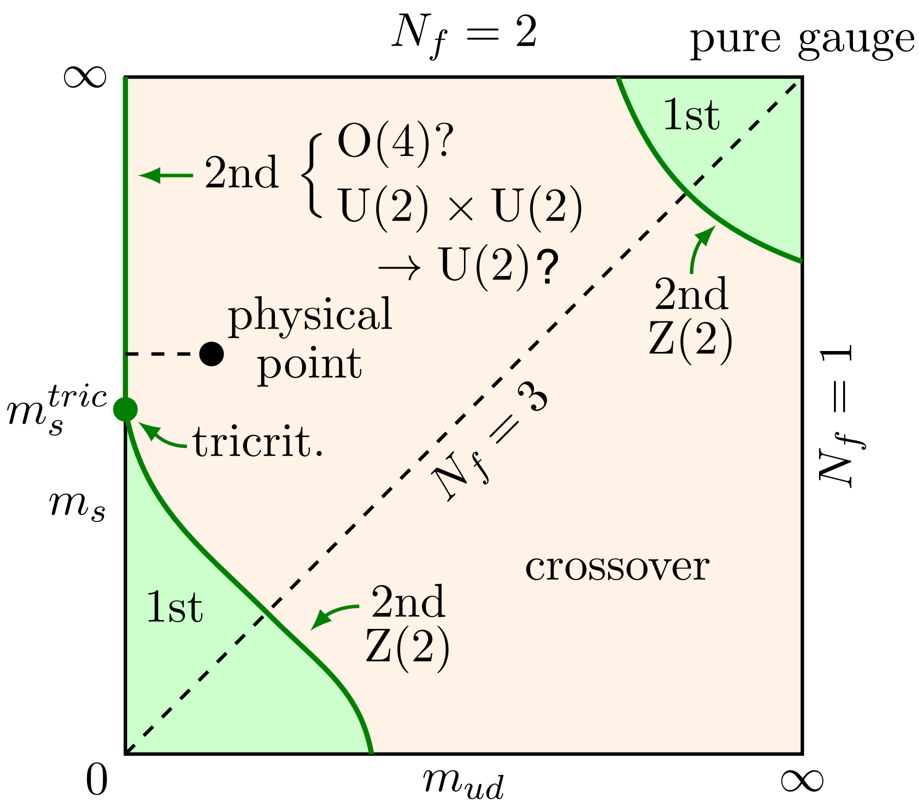
Outline

- Motivation
- $\partial^n \rho / \partial^n m_l$ & C_{n+1} and U$_A$(1) symmetry
- Lattice Setup
- Results
- Summary & Conclusions

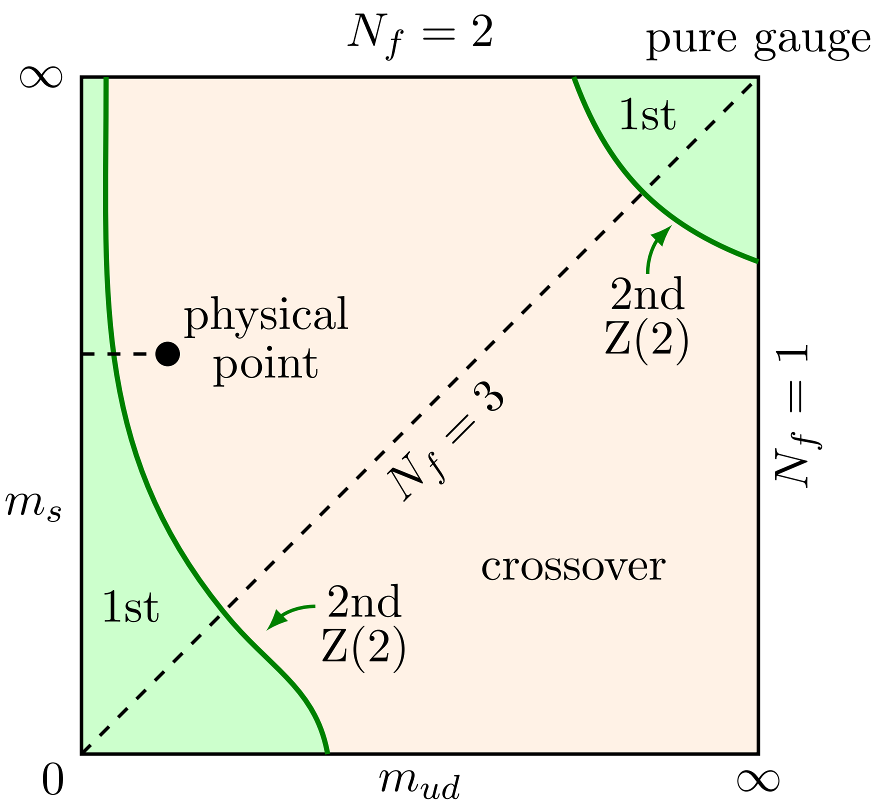
$U_A(1)$ symmetry & Chiral phase transition

The nature of chiral phase transition depends on how the axial anomaly manifests itself at T~T_c

scenario (1):



scenario (2):



Pisarski, Wilczek PRD 29 (1984) 338
Butti et. al., JHEP 08 (2003) 029
Pelissetto & Vicari, PRD 88 (2013) 105018
Grahl, PRD 90 (2014) 117904

Signatures of symmetry restorations

- Susceptibilities defined as integrated two point correlation functions of the local operators, e.g. $\chi_\pi = \int d^4x\langle \pi^i(x)\pi^i(0)\rangle$ HotQCD PRD 86 (2012) 094503

$$\begin{tikzcd}
\chi_{5,\mathrm{con}} & \pi : \bar{q}\gamma_5\frac{\tau}{2}q \arrow[r, leftrightarrow, "SU_L(2)\times SU_R(2)"] \arrow[d, leftrightarrow, "U_A(1)"'] & \sigma : \bar{q}q \arrow[d, leftrightarrow, "U_A(1)"] & \chi_{\mathrm{con}}+\chi_{\mathrm{disc}} \\
\chi_{\mathrm{con}} & \delta : \bar{q}\frac{\tau}{2}q \arrow[r, leftrightarrow, "SU_L(2)\times SU_R(2)"'] & \eta : \bar{q}\gamma_5 q & \chi_{5,\mathrm{con}}-\chi_{5,\mathrm{disc}}
\end{tikzcd}$$

$$SU_L(2)\times SU_R(2)$$

$$\chi_\pi - \chi_\delta = \chi_{\mathrm{disc}}$$

$$U_A(1)$$

$$\chi_\pi - \chi_\delta = \chi_{\mathrm{disc}} = 0$$

$$\langle\bar{\psi}\psi\rangle = \int_0^\infty \mathrm{d}\lambda \frac{4m_l\rho}{\lambda^2+m_l^2} \qquad \chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda \frac{8m_l^2\rho}{(\lambda^2+m_l^2)^2}$$

$$\chi_{\mathrm{disc}} = \int_0^\infty \mathrm{d}\lambda \frac{4m_l\partial\rho/\partial m_l}{\lambda^2+m_l^2}$$

Novel relation: quark mass derivatives of ρ & C₂

Eigenvalue spectrum for (2+1)-flavor QCD:

$$\rho(\lambda, m_l) = \frac{T}{VZ[U]}\int D[U]e^{-S_G[U]}\mathrm{det}[\not{D}[U]+m_s]\times(\mathrm{det}[\not{D}[U]+m_l])^2\rho_U(\lambda)$$

Eigenvalue spectrum for a given configuration: $\rho_U(\lambda) = \sum_j \delta(\lambda-\lambda_j)$

Partition function: $Z[U] = \int D[U]e^{-S_G[U]}\mathrm{det}[\not{D}[U]+m_s]\times(\mathrm{det}[\not{D}[U]+m_l])^2$

$$\mathrm{det}[\not{D}[U]+m_l] = \prod_j(+\mathrm{i}\lambda_j+m_l)(-\mathrm{i}\lambda_j+m_l) = \exp\left(\int_0^\infty \mathrm{d}\lambda\rho_U(\lambda)\mathrm{In}[\lambda^2+m_l^2]\right)$$

⇒

$$\boxed{\frac{V}{T}\frac{\partial\rho}{m_l} = \int_0^\infty \mathrm{d}\lambda_2\frac{4m_l\,C_2}{\lambda_2^2+m_l^2}}$$

$$C_2(\lambda,\lambda_2) = \langle\rho_U(\lambda)\rho_U(\lambda_2)\rangle - \langle\rho_U(\lambda)\rangle\langle\rho_U(\lambda_2)\rangle$$

Novel relation: $\partial^n \rho/\partial^n m_l$ & $\mathrm{C}_{\mathrm{n+1}}$

$$\frac{V}{T}\frac{\partial^2\rho}{m_l^2} = \int_0^\infty \mathrm{d}\lambda_2 \frac{4(\lambda_2^2 - m_l^2)\,C_2}{(\lambda_2^2 + m_l^2)^2} + \int_0^\infty \mathrm{d}\lambda_2 \mathrm{d}\lambda_3 \frac{(4m_l)^2\,C_3}{(\lambda_2^2 + m_l^2)(\lambda_3^2 + m_l^2)}$$

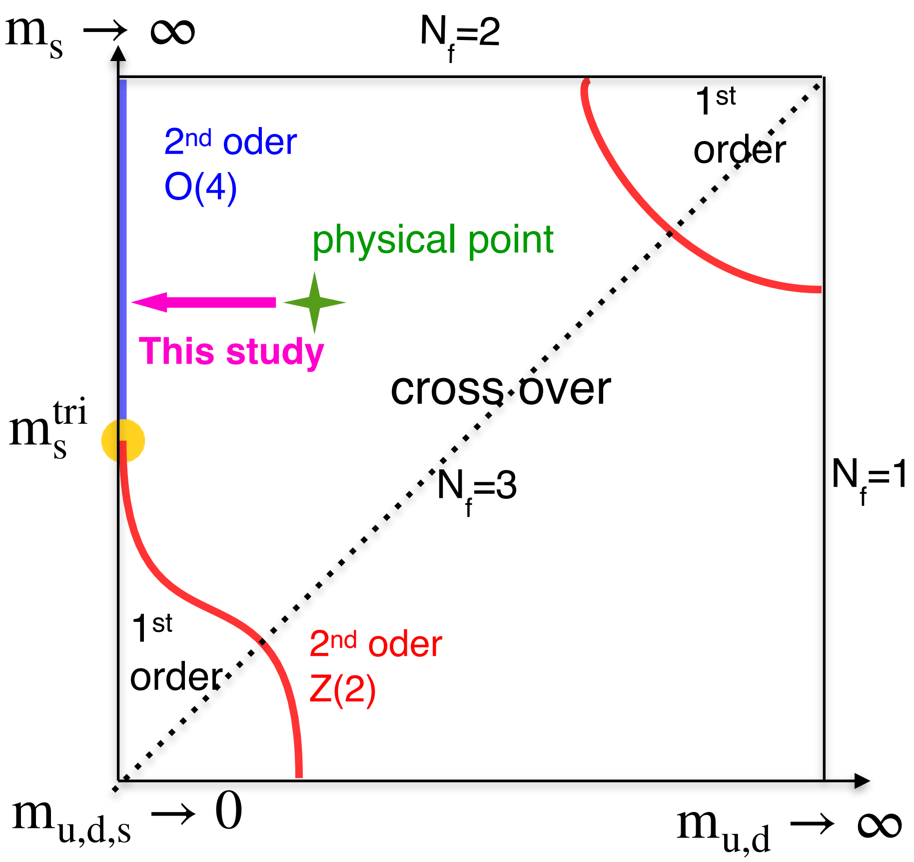
…… ……

…… ……

$$C_n(\lambda_1, \ldots, \lambda_n; m_l) = \left\langle \prod_{i=1}^{n} [\rho_U(\lambda_i) - \langle \rho_U(\lambda_i) \rangle] \right\rangle$$

Lattice Setup

- At T~205 MeV (1.6 T_c)
- HISQ/tree action
- N_f = 2+1:
  - ☑ N_τ = 8, 12, 16 (a = 0.12, 0.08, 0.06 fm)
  - ☑ m_s^{phy}/m_l = 20, 27, 40, 80, 160 (m_π = 160, 140, 110, 80, 55 MeV)
  - ☑ $4 \leq N_\sigma/N_\tau \leq 9$
- ρ is obtained from the Chebyshev filtering technique



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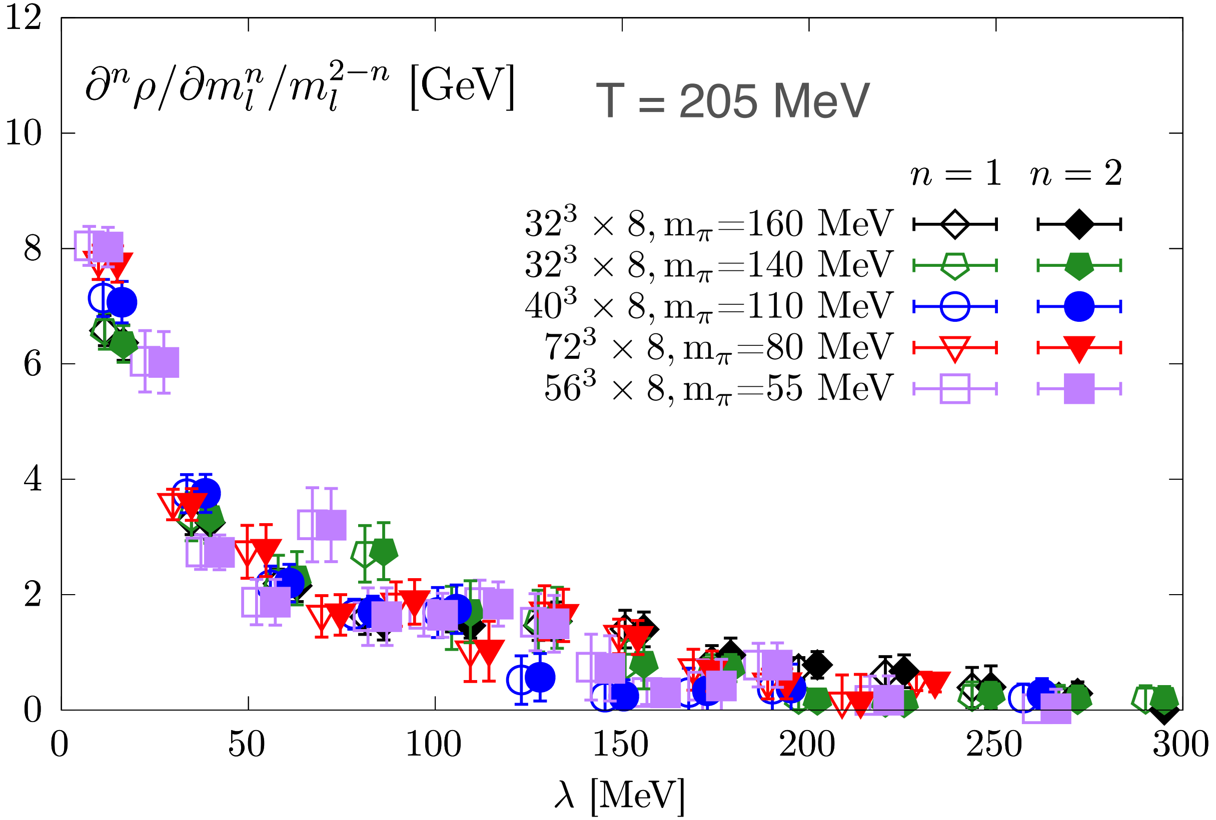
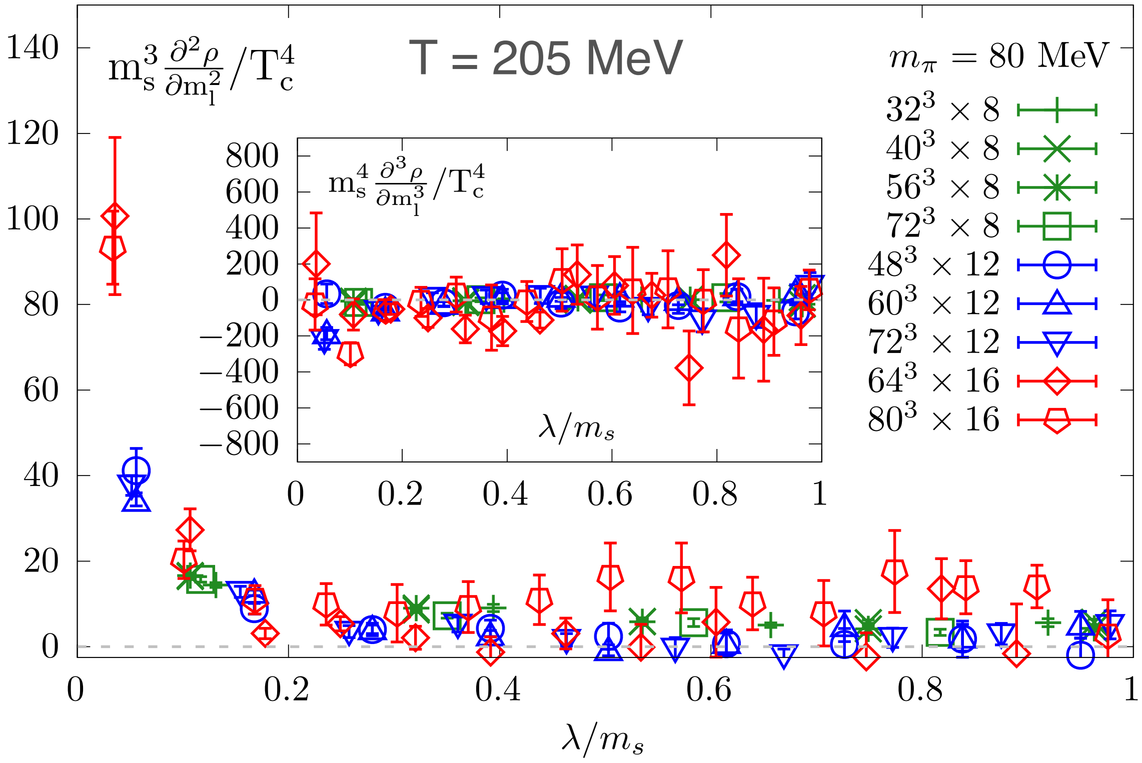
1^{st} and 2^{nd} mass derivatives of ρ on N_τ=8 lattices



- $m_l^{-1}(\partial\rho/\partial m_l) \approx \partial^2\rho/\partial m_l^2$
- Quark mass independent
- Peaked structure develops at $\lambda \to 0$ and drops rapidly towards zero for $\lambda/\mathrm{T} > 1$

2^{nd} and 3^{rd} mass derivatives of ρ: volume and *a* dependences



- Peaked structure becomes sharper towards continuum limit
- Mild volume dependence
-

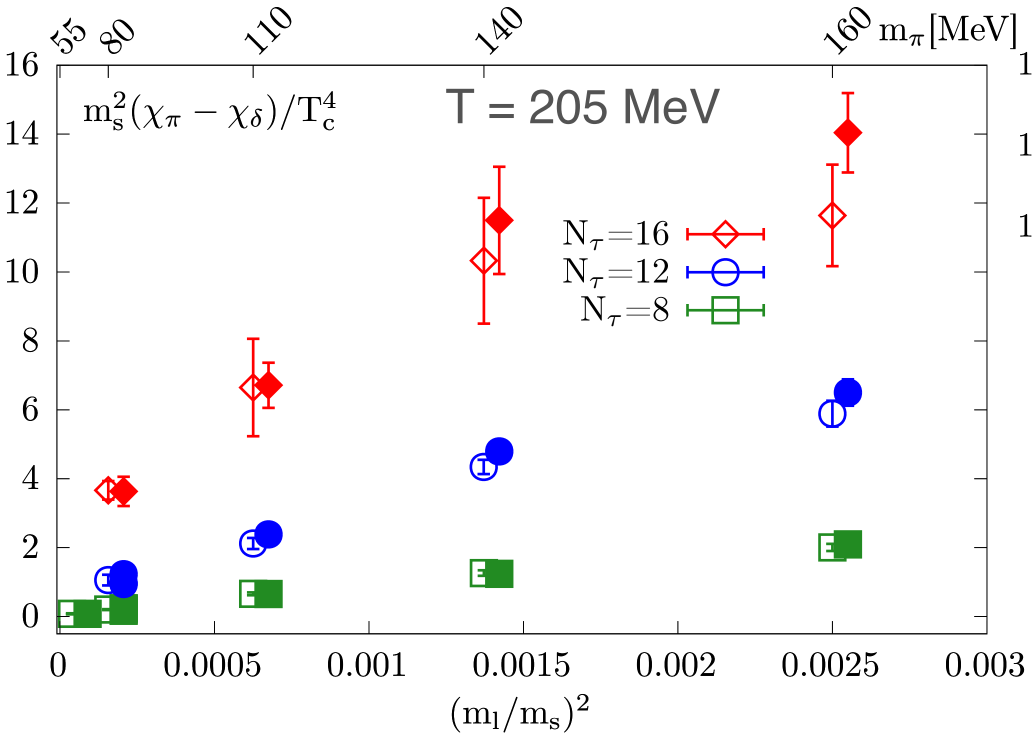
$$\partial^3\rho/\partial m_l^3 \approx 0$$

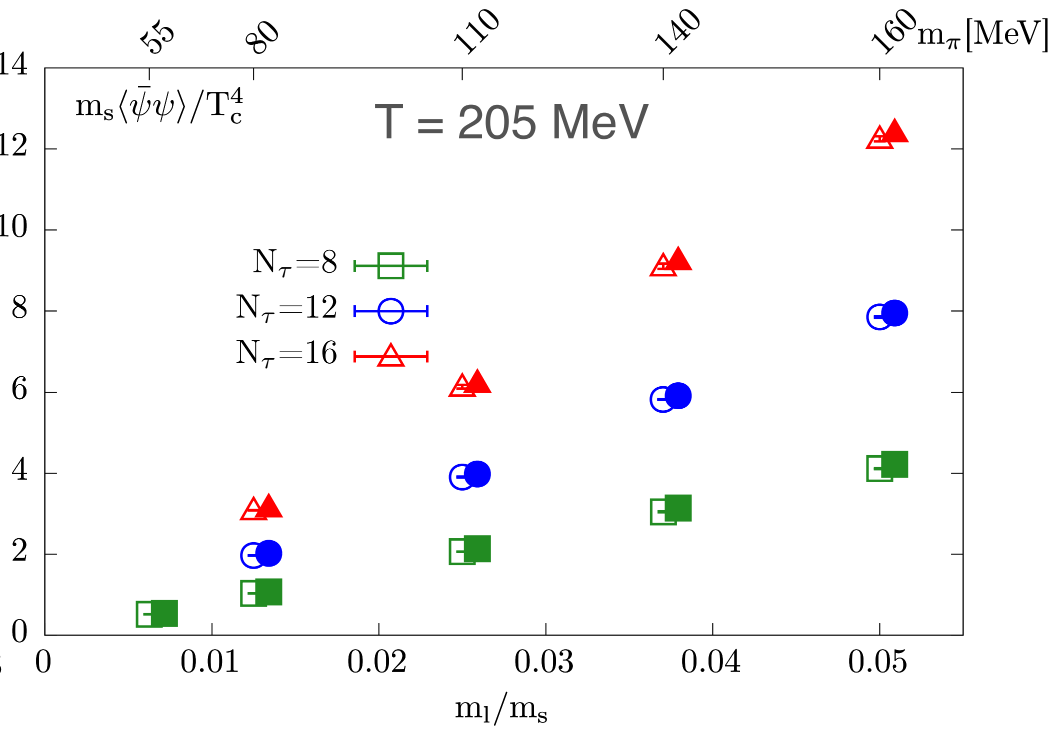
$$m_l^{-1}\left(\partial\rho/\partial m_l\right) \approx \partial^2\rho/\partial m_l^2$$

$$\Downarrow$$

$$\rho(\lambda \to 0, m_l \to 0) \propto m_l^2$$

Quantities related to ρ



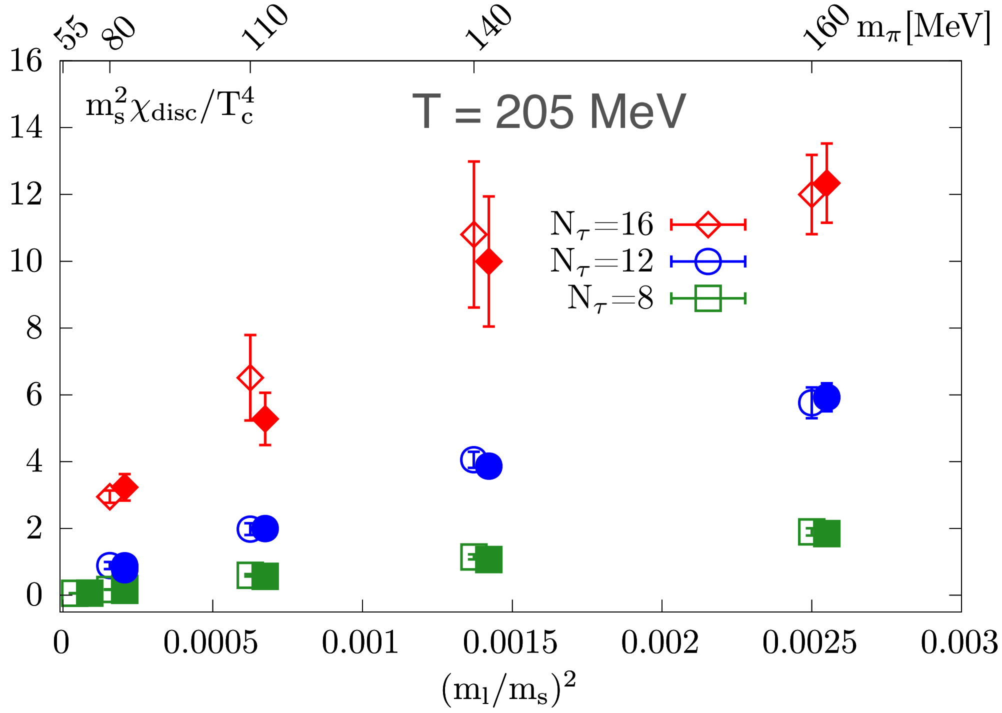


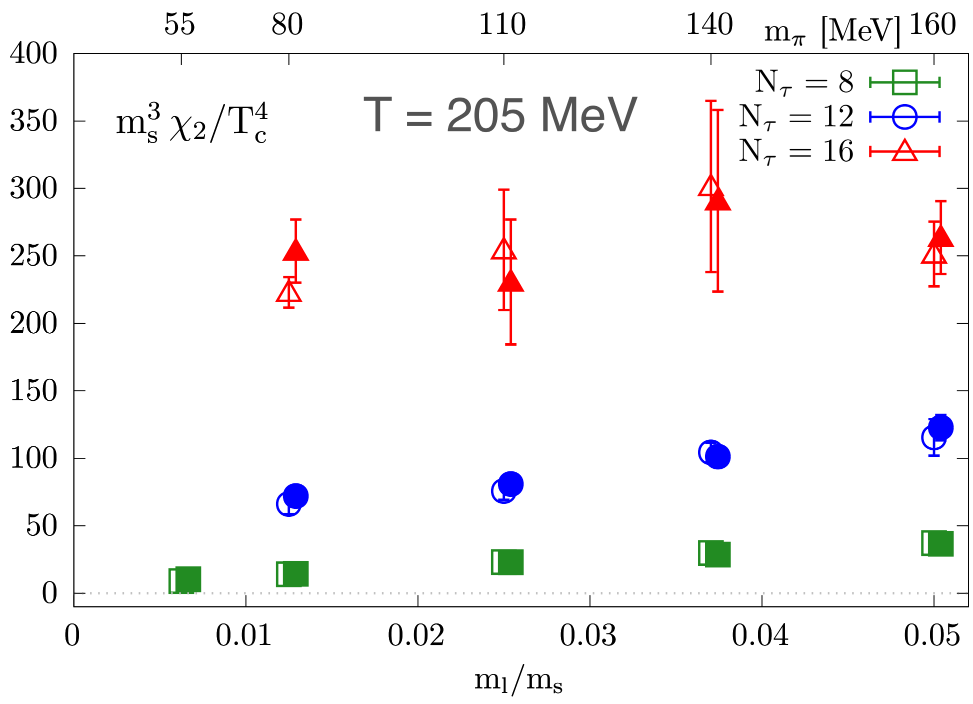
Open symbol: Results obtained from the fermion matrix inversion
Filled symbol: reconstructed from ρ

$$\chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda \frac{8m_l^2\rho}{(\lambda^2+m_l^2)^2} \qquad \langle\bar{\psi}\psi\rangle = \int_0^\infty \mathrm{d}\lambda \frac{4m_l\rho}{\lambda^2+m_l^2}$$

$\langle\bar{\psi}\psi\rangle$ is reproduced very well from ρ

Quantities related to 1st and 2nd mass derivatives of ρ



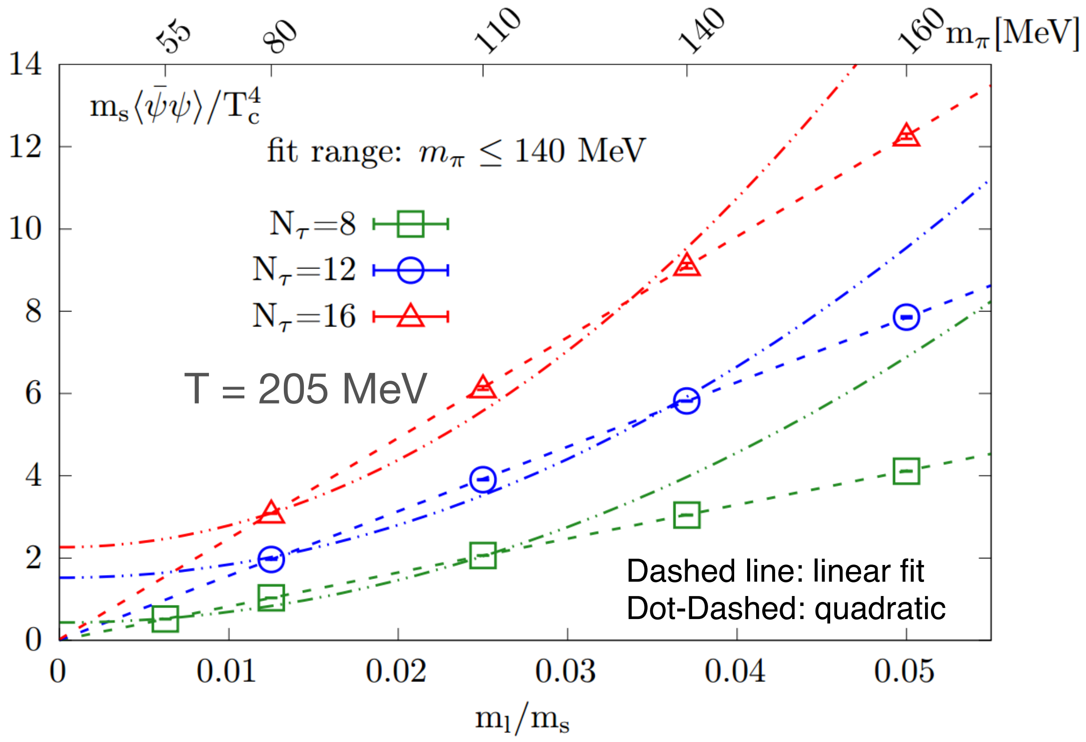


$$\chi_{\rm disc} = \int_0^\infty d\lambda \frac{4m_l \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

$$\chi_2 = \int_0^\infty d\lambda \frac{4m_l \partial^2\rho/\partial m_l^2}{\lambda^2 + m_l^2}$$

χ_{disc} and χ_2 are successfully reproduced from the 1st and 2nd mass derivatives of ρ

$SU_L(2)xSU_R(2)$ symmetry restoration



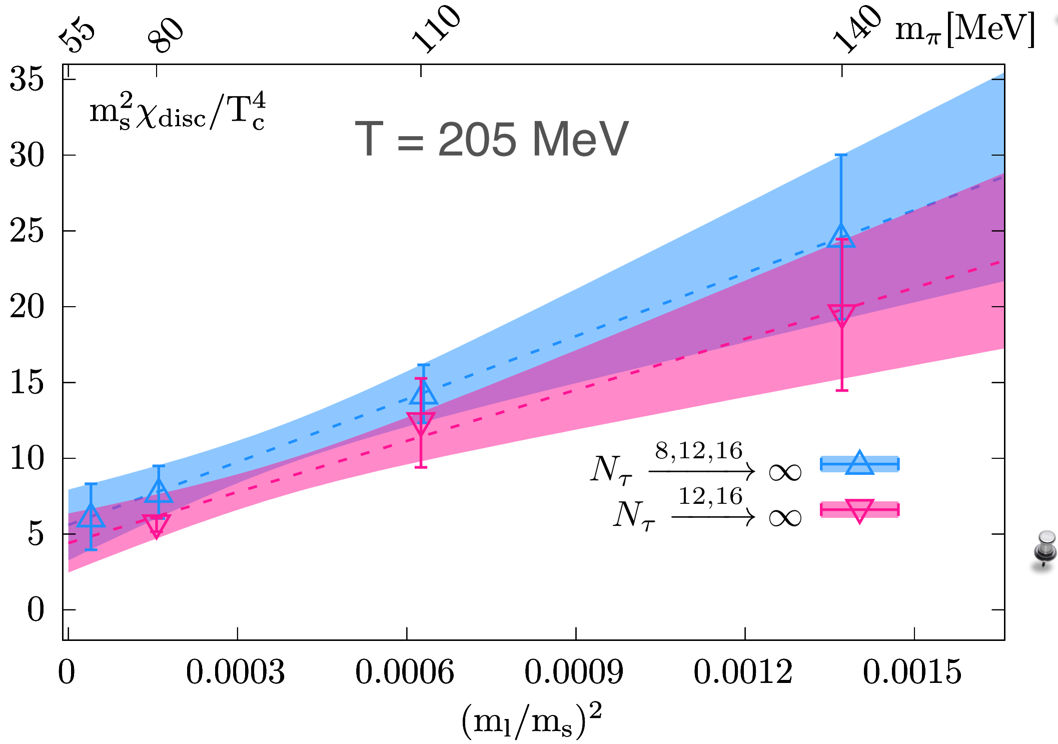
Due to the restoration of Z(2) subgroup of $SU_L(2)\times SU_R(2)$ symmetry, the partition function is an even function of m_l

	χ^2/dof	
N_τ	Linear fits	Quadratic fits
8	0.43	13972.7
12	4.4	1504.0
16	0.1	198.5

$$\langle\bar\psi\psi\rangle \propto m_l \text{ as } m_l \to 0$$

$$\chi_{\rm disc} \propto m_l^2 \text{ as } m_l \to 0$$

Continuum & chiral extrapolations of $\chi_{\rm disc}$



- **Joint fit:** simultaneous fits

  Continuum: $c_0 + c_1/N_\tau^2 + c_2/N_\tau^4$

  Chiral: quadratic in quark mass

  Value at $N_\tau \to \infty$ and $m_l \to 0$:
  5.6 ± 2.3

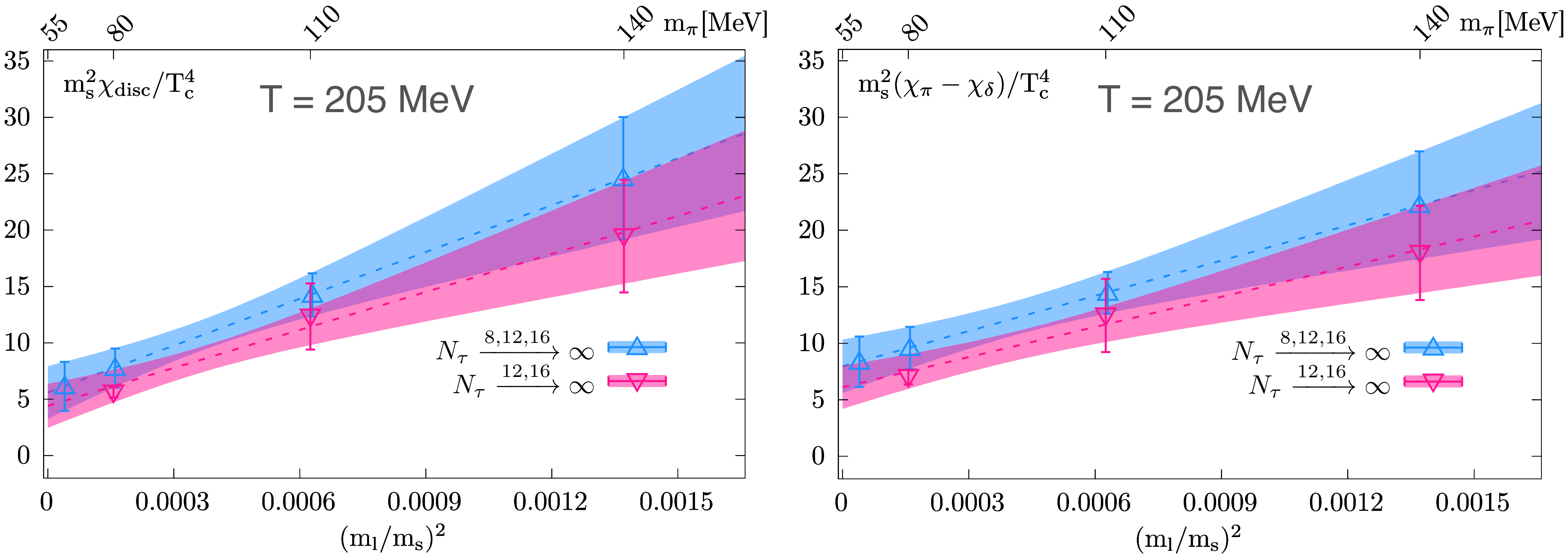
- **Sequential fit:** first continuum and then chiral extrapolations

  Continuum: quadratic in $1/N_\tau$ with N_τ=12&16

  Chiral: quadratic in quark mass

  Value at $N_\tau \to \infty$ and $m_l \to 0$:
  4.4 ± 1.9

Continuum & chiral extrapolations of two $U_A(1)$ measures



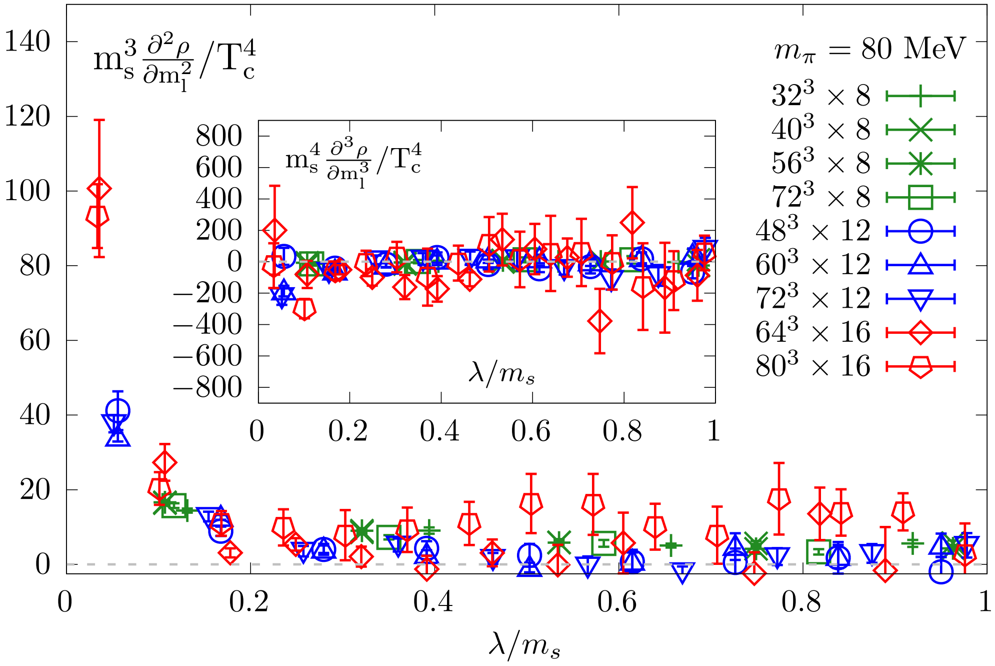
$N_\tau \to \infty$ and $m_l \to 0$	$m_s^2\chi_{disc}/T_c^4$	$m_s^2(\chi_\pi-\chi_\delta)/T_c^4$
Joint fit	5.6 ± 2.3	8.0 ± 2.4
Sequential fit	4.4 ± 1.9	6.1 ± 1.9

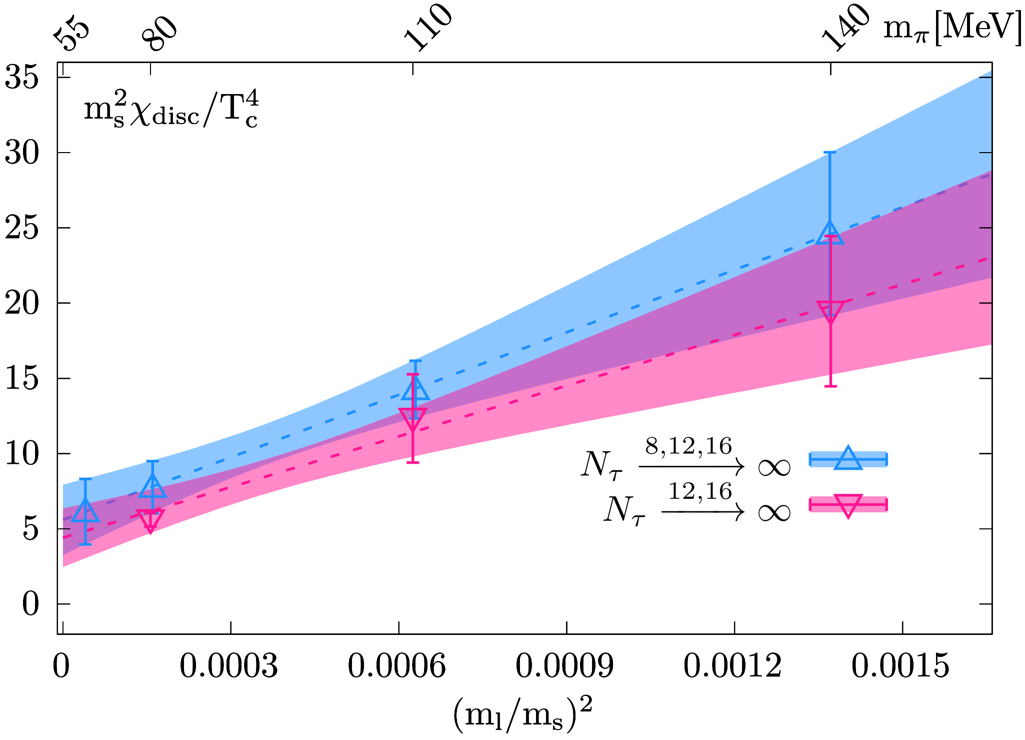
Axial anomaly remains manifested at T~1.6T_c
in the chiral limit at 2-3 sigma level

Summary & Conclusions

- ☑ We established novel relations between $\partial^n \rho / \partial^n m_l$ & C_{n+1}

In N_f=2+1 QCD at T~1.6T_c





Summary & Conclusions

Our study suggests:

- At T~1.6T_c the microscopic origin of axial anomaly is driven by the weakly interacting instanton gas motivated $\rho(\lambda \to 0, m_l \to 0) \propto m_l^2 \delta(\lambda)$
- N_f=2+1 QCD: 2nd order chiral phase transition belongs to 3-d O(4) universality class

Outlook:

- ☐ The methodology would be useful for other discretization schemes

Backup

Calculation of eigenvalue spectrum

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues

- Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number

mode number:

$$\boxed{\bar{n}[s,t] \approx \frac{1}{N_r}\sum_{r=1}^{N_r}\sum_{j=0}^{p}\boxed{g_j^p}\,\boxed{\gamma_j}\langle\xi_r^\dagger \boxed{T_j}(A)\xi_r\rangle}$$

$$\Downarrow$$

Spectrum:

$$\boxed{\rho(\lambda,\delta) = \frac{1}{V}\,\frac{\bar{n}[\lambda-\delta/2, \lambda+\delta/2]}{\delta} \qquad (\lambda \geq \delta/2)}$$

T_j: Chebyshev polynomial

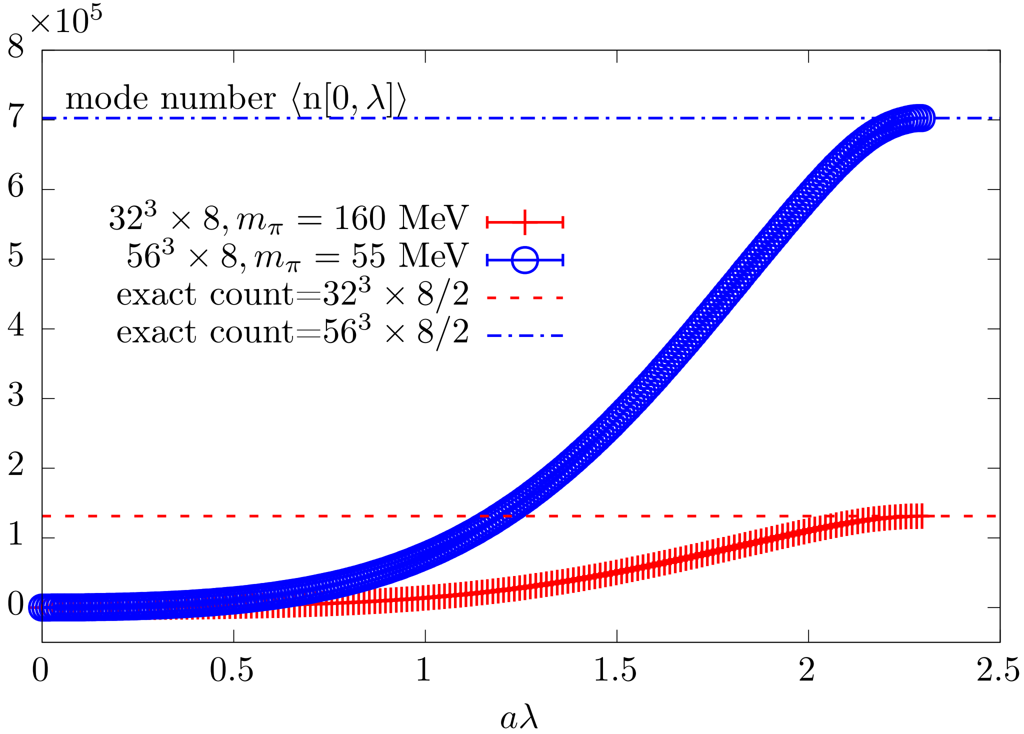
γ_j: coefficient

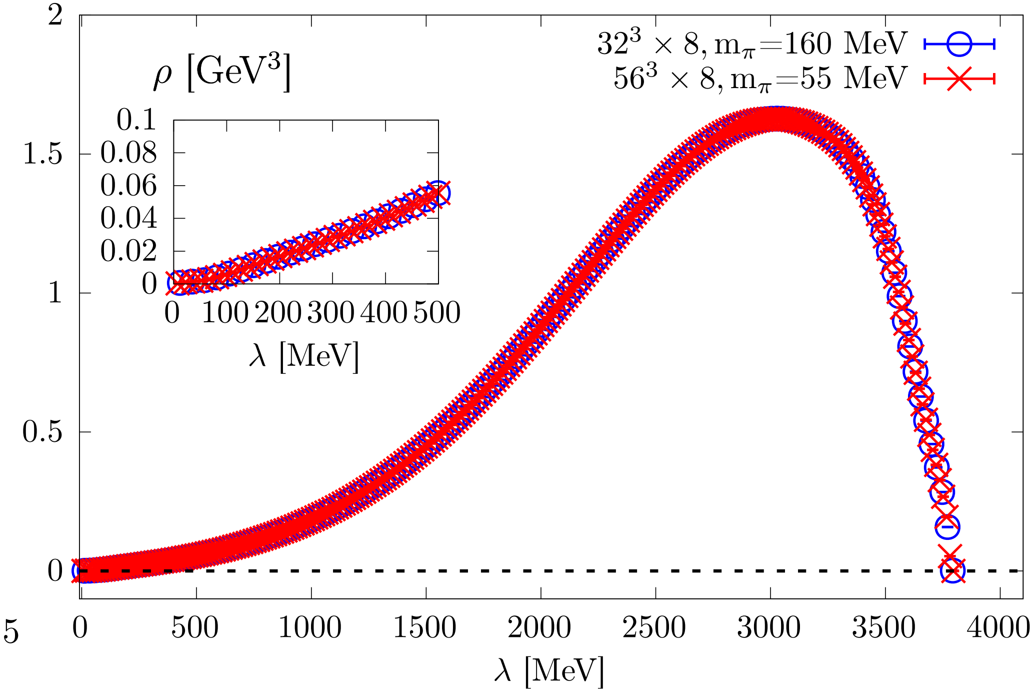
p: polynomial order

g_j^p: Jackson’s dumping factor

YuZhang, Lattice19',arXiv:2001.05217
Giusti and Luscher, arXiv:0812.3638
A.Patela, arXiv:1204.432
DiNapoli et al.,arXiv: 1308.4275
Itou et al, arXiv:1411.1155
Fodor et al.,arXiv:1605.08091
Cossu et al.,arXiv:1601.00744

Mode number and Complete ρ



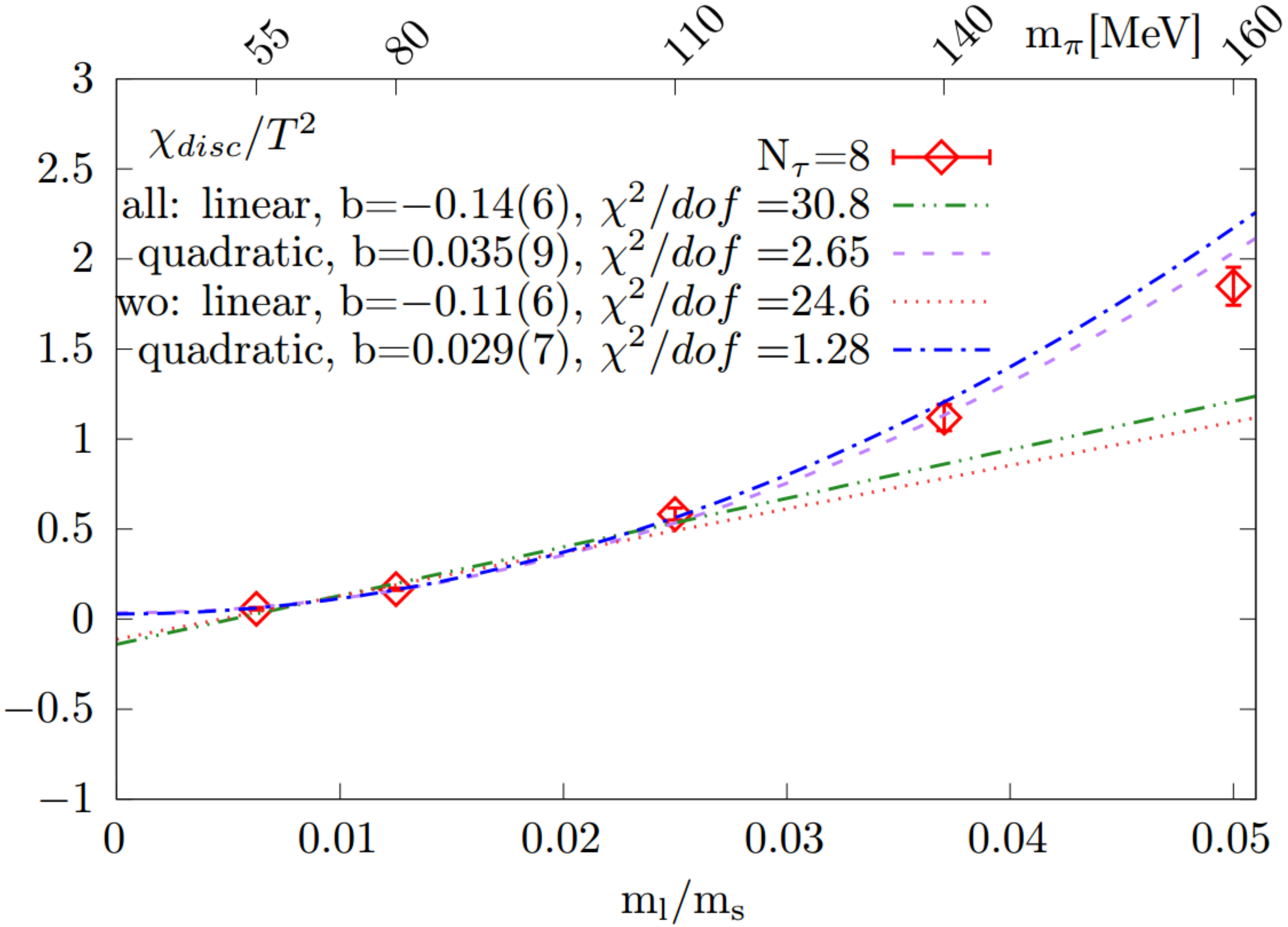


- Converges to the exact count

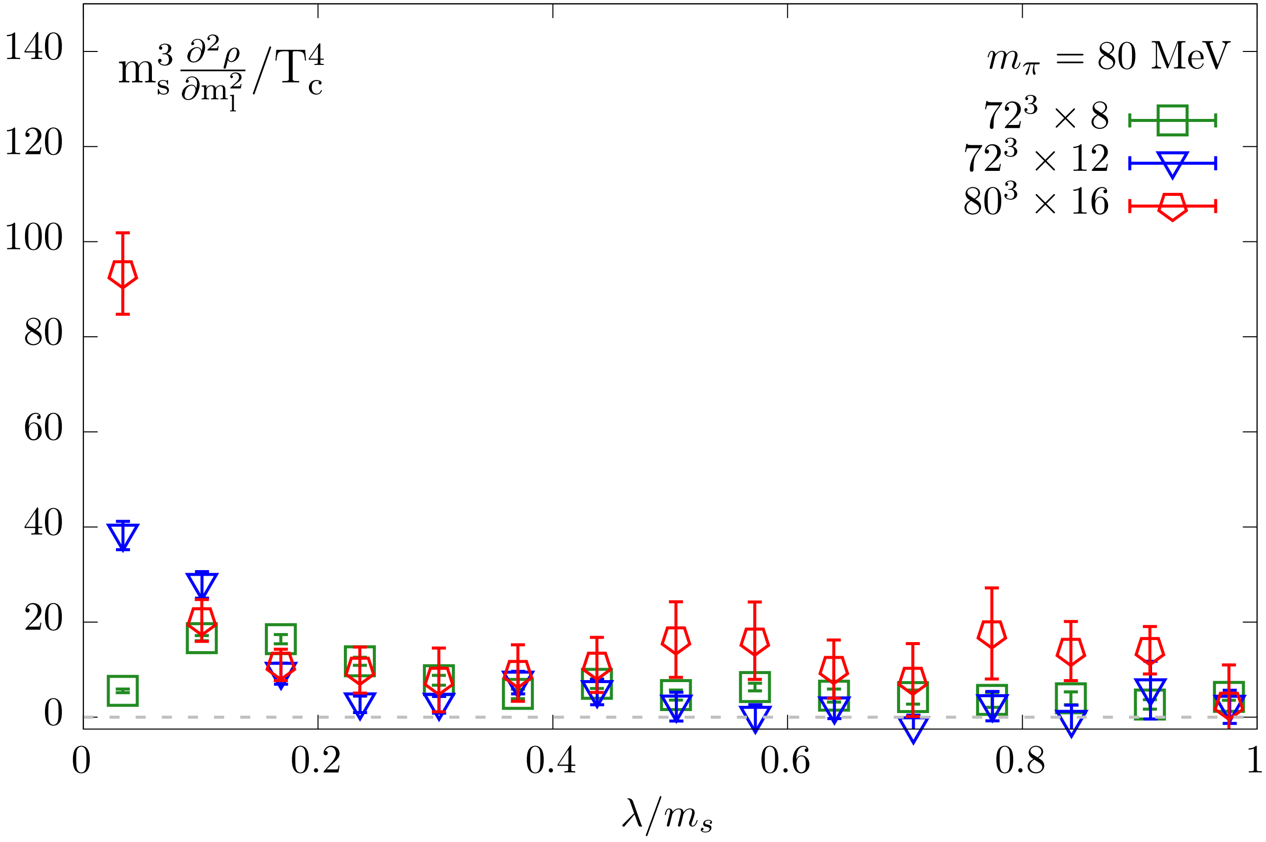
- Mass dependence can be hardly observed from ρ directly

Utilize the Chebyshev filtering technique combined with a stochastic estimate of the mode number

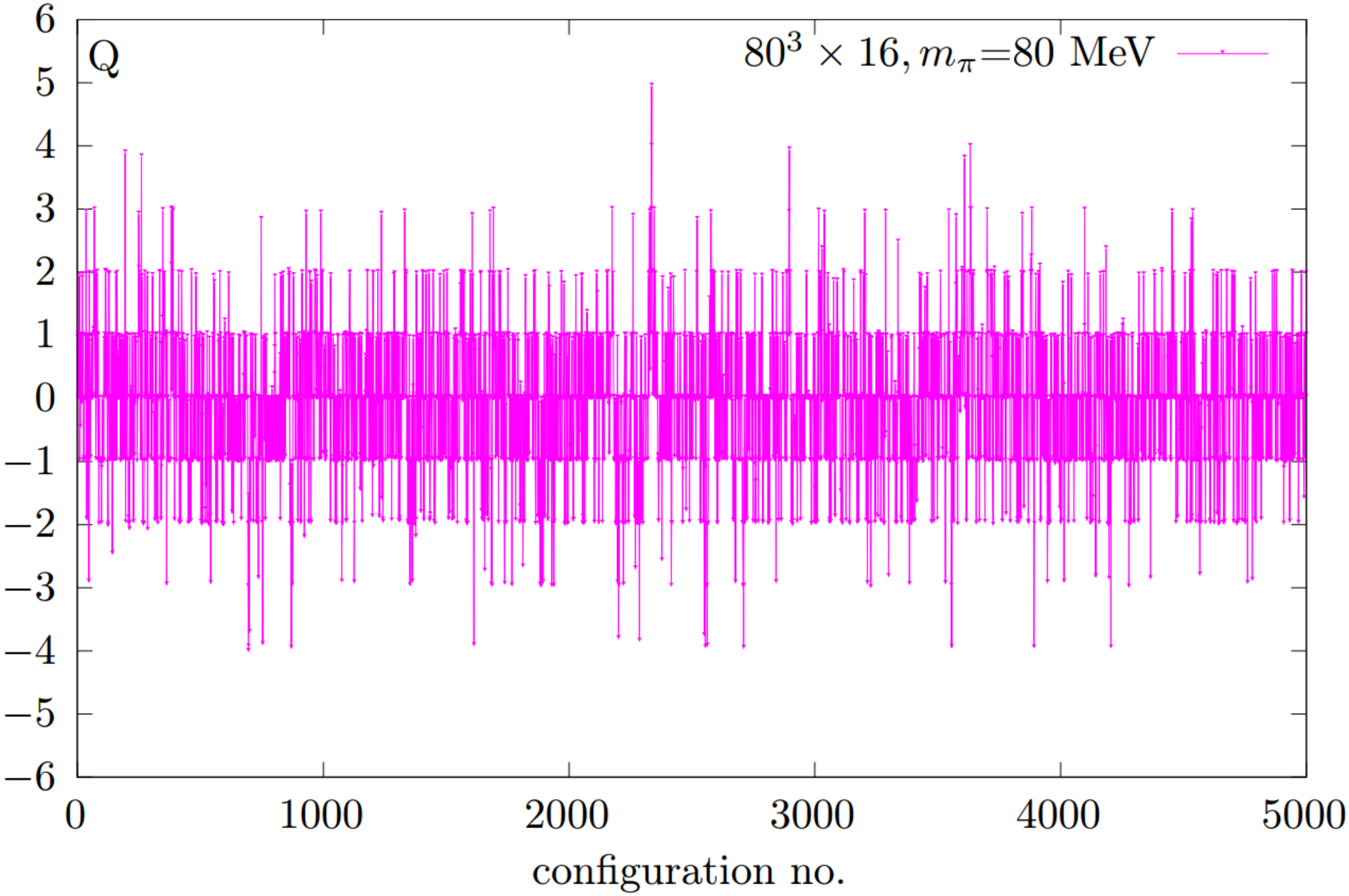
Giusti and Luscher, JHEP03(2009)013, Patella PRD86(2012)025006, Cossu et al., PTEP 2016(2016)093B06
Fodor et al., arXiv:1605.08091, de Forcrand & Jäger, arXiv: 1710.07305, YuZhang, Lattice19', arXiv:2001.05217


χ_{disc}/T^2
N$_\tau$=8
all: linear, b=−0.14(6), χ^2/dof =30.8
quadratic, b=0.035(9), χ^2/dof =2.65
wo: linear, b=−0.11(6), χ^2/dof =24.6
quadratic, b=0.029(7), χ^2/dof =1.28
m$_\pi$[MeV]
55
80
110
140
160
3
2.5
2
1.5
1
0.5
0
−0.5
−1
0
0.01
0.02
0.03
0.04
0.05
m$_l$/m$_s$

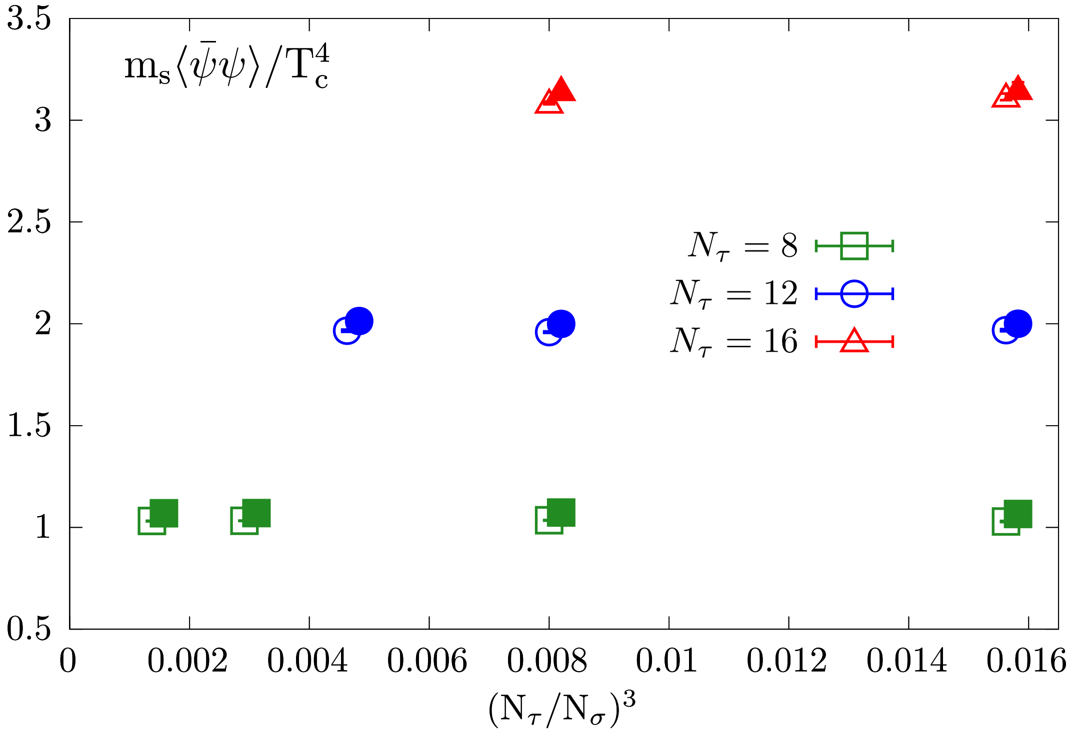
2^{nd} mass derivative of ρ



Time history of the topological charge



Chiral condensate

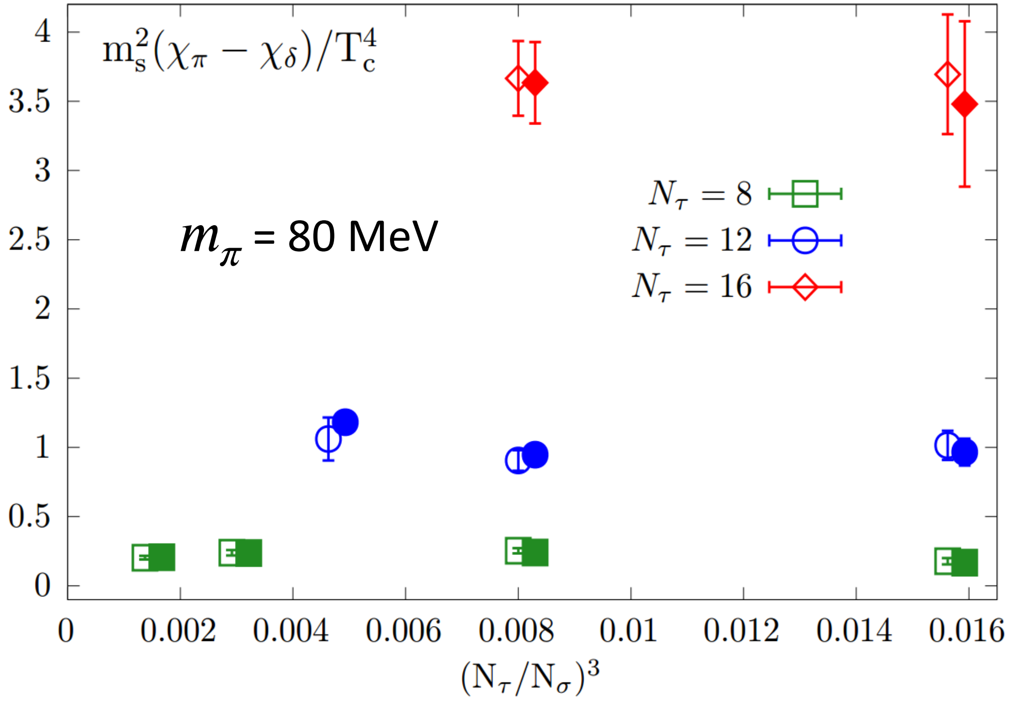


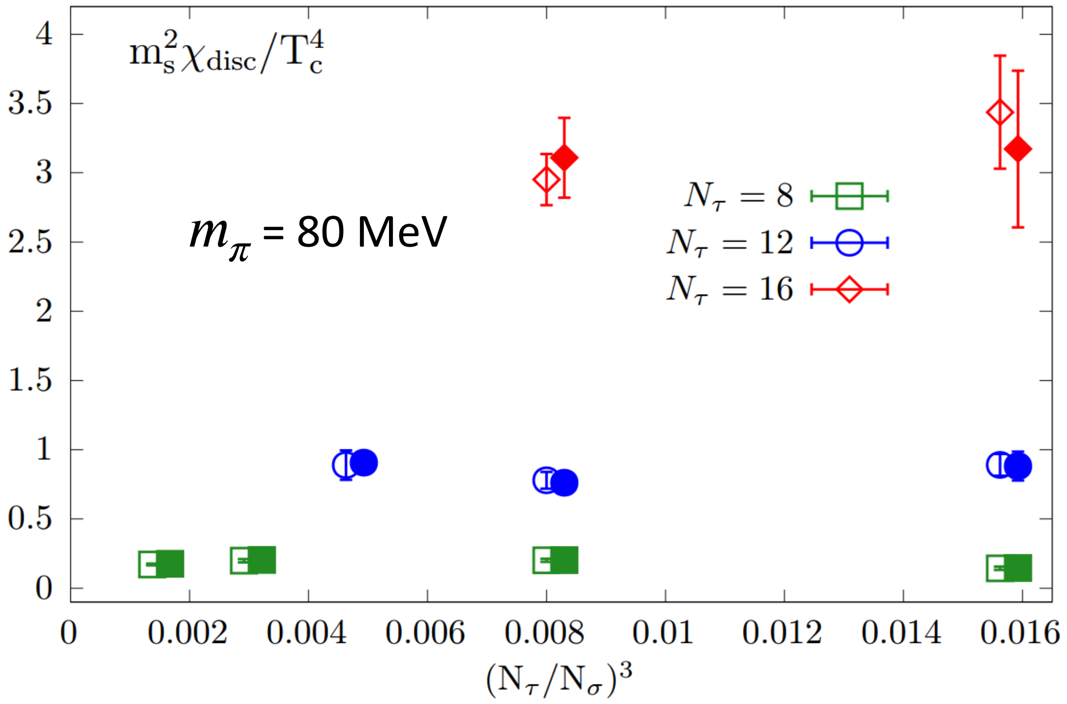
$$\langle\bar\psi\psi\rangle = \int_0^\infty \mathrm{d}\lambda \frac{4m_l\rho}{\lambda^2+m_l^2} + \frac{2T}{V}\frac{\langle|Q_{\mathrm{top}}|\rangle}{m}$$

$$\langle|Q_{\mathrm{top}}|\rangle \propto \sqrt{V}$$

Zero mode contribution vanishes in the thermodynamical limit

Volume dependence of two $U_A(1)$ measures





Volume dependences is very small

Signatures of symmetry restorations in ρ

$$\langle\bar{\psi}\psi\rangle = \int_0^\infty \mathrm{d}\lambda \frac{4 m_l \rho}{\lambda^2 + m_l^2} \qquad \chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda \frac{8 m_l^2 \rho}{(\lambda^2 + m_l^2)^2}$$

- Restoration of $SU_L(2)\times SU_R(2)$ symmetry :
  - $\rho(0) = 0$ as from Banks-casher relation: $\lim_{m_l \to 0} \langle\bar{\psi}\psi\rangle = \pi\rho(0)$
  - Partition function is an even function in m_l due to the Z(2) subgroup

Banks and Casher,
NPB 169 (1980) 103

- Effective restoration of $U_A(1)$ symmetry :
  - A sizable gap in the near-zero modes, i.e. $\rho(\lambda < \lambda_c) = 0$
  - If ρ is analytic in m_l^2 and λ, $U_A(1)$ breaking is absent in up to 6 point correlation functions of π and δ

Cohen, nucl-th/980106

Aoki, Fukaya and Taniguchi, PRD 86 (2012) 114512

Possible behaviors of ρ making $SU_L(2)\times SU_R(2)$ restored but not $U_A(1)$

- Dilute instanton gas approximation $\rho \sim m^2\delta(\lambda)$ will lead to $U_A(1)$ breaking even in the chiral limit

Gross, Yaffe & Pisarski, RMP 81’

- LQCD: At high T for the physical m_l , the T dependence of χ_t follows dilute instanton gas approximation prediction

See a recent review, Lombardo & Trunin,
IJMPA 35 (2020) 2030010

Due to $\rho \sim m^2\delta(\lambda)$? what happens for $m_l \to 0$?

Poisson distribution

$$
\begin{aligned}
C_2(\lambda,\lambda') &= \langle \rho_u(\lambda)\rho_u(\lambda')\rangle - \langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle \\
&= (\frac{1}{V})^2 \left\langle \sum_{k=1}^{N} \delta(\lambda-\lambda_k) \sum_{l=1}^{N} \delta(\lambda'-\lambda_l) \right\rangle - \langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle \\
&= (\frac{1}{V})^2 \left\langle \sum_{k=1}^{N} \delta(\lambda-\lambda_k)\delta(\lambda'-\lambda_k) \right\rangle + (\frac{1}{V})^2 \left\langle \sum_{k=1}^{N} \delta(\lambda-\lambda_k) \sum_{l\neq k} \delta(\lambda'-\lambda_l) \right\rangle - \langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle^{(1)} \\
&\boxed{= \frac{1}{V}\langle \rho_u(\lambda)\rangle \delta(\lambda-\lambda') + (\frac{1}{V})^2 \left\langle \sum_{k=1}^{N} \delta(\lambda-\lambda_k) \right\rangle \left\langle \sum_{l\neq k} \delta(\lambda'-\lambda_l) \right\rangle - \langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle}
\end{aligned}
$$

$$
\frac{1}{V}\left\langle \sum_{l\neq k} \delta(\lambda'-\lambda_l) \right\rangle = \frac{N-1}{N}\langle \rho_u(\lambda')\rangle \qquad (N = V/2)
$$

$$
\boxed{C_2(\lambda,\lambda') = \frac{1}{V}\langle \rho_u(\lambda)\rangle \delta(\lambda-\lambda') - \frac{1}{N}\langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle}
$$

Signatures of symmetry restorations in ρ

Chiral symmetry restoration: $\chi_\pi - \chi_\delta = \chi_{\mathrm{disc}}$

$$\chi_\pi - \chi_\delta = \int_0^\infty \mathrm{d}\lambda \frac{8m_l^2 \rho}{(\lambda^2 + m_l^2)^2}$$

$$\chi_{\mathrm{disc}} = \int_0^\infty \mathrm{d}\lambda \frac{4m_l \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$

$$\begin{tikzcd}
\chi_{5,\mathrm{con}} & \pi : \bar{q}\gamma_5\frac{\tau}{2}q \arrow[r, leftrightarrow, "SU_L(2)\times SU_R(2)"] \arrow[d, leftrightarrow, "U_A(1)"'] & \sigma : \bar{q}q \arrow[d, leftrightarrow, "U_A(1)"] & \chi_{\mathrm{con}} + \chi_{\mathrm{disc}} \\
\chi_{\mathrm{con}} & \delta : \bar{q}\frac{\tau}{2}q \arrow[r, leftrightarrow, "SU_L(2)\times SU_R(2)"'] & \eta : \bar{q}\gamma_5 q & \chi_{5,\mathrm{con}} - \chi_{5,\mathrm{disc}}
\end{tikzcd}$$

Toublan and Verbaarschot, NPB603 (2001) 343
HotQCD PRD 86 (2012) 094503
Kanazawa & Yamamoto, JHEP 01 (2016) 141

If eigenvalues are uncorrelated, they obey the Poisson statistics:

$$C_n^{\mathrm{Po}}(\lambda_1, ..., \lambda_n) = \delta(\lambda_1 - \lambda_2)...\delta(\lambda_n - \lambda_{n-1}) \langle \rho_U(\lambda_1)\rangle + \mathcal{O}(1/N)$$

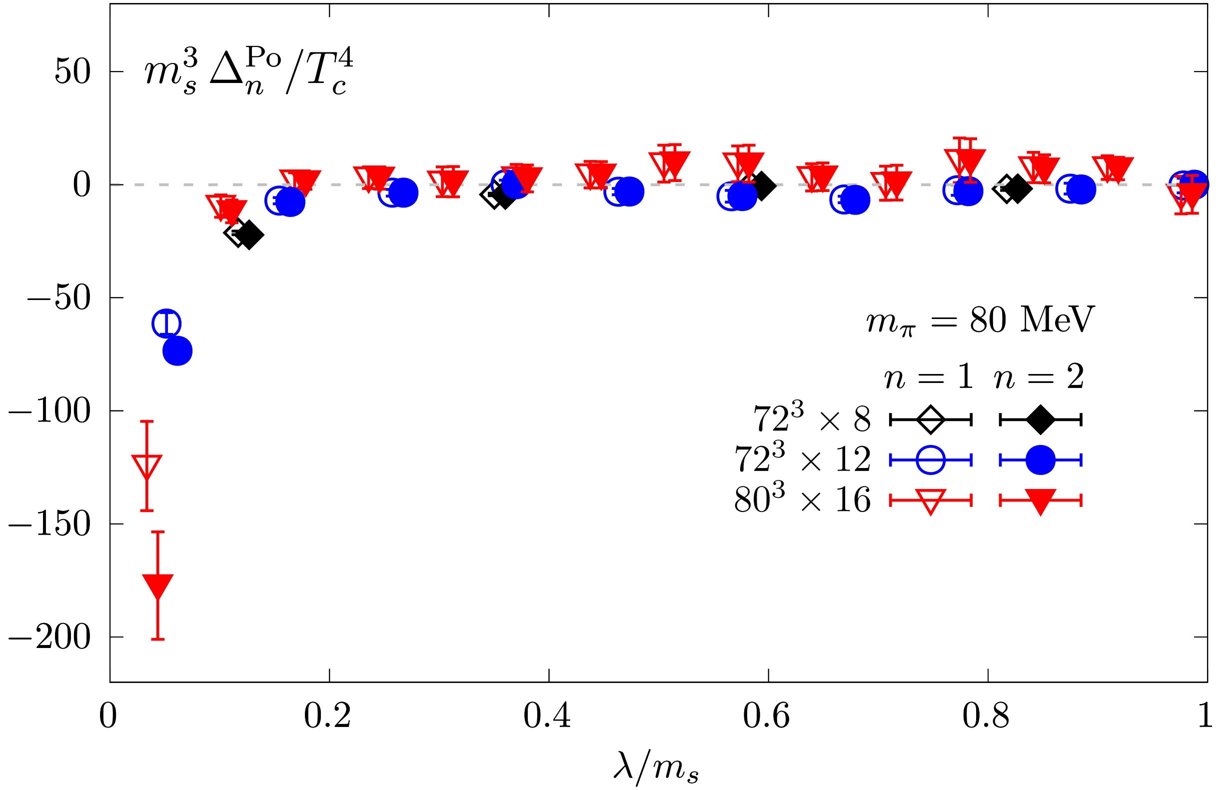
$$\left(\frac{\partial\rho}{\partial m_l}\right)^{\mathrm{Po}} = \frac{4m_l\rho}{\lambda^2 + m_l^2} - \frac{V\rho}{TN}\langle\bar{\psi}\psi\rangle \Longrightarrow \chi_{\mathrm{disc}}^{\mathrm{Po}} = 2(\chi_\pi - \chi_\delta)$$

Non-Poisson correlation among eigenvalues are needed for chiral symmetry restoration if $\chi_\pi - \chi_\delta \neq 0$

Kanazawa & Yamamoto,
JHEP 01 (2016) 141

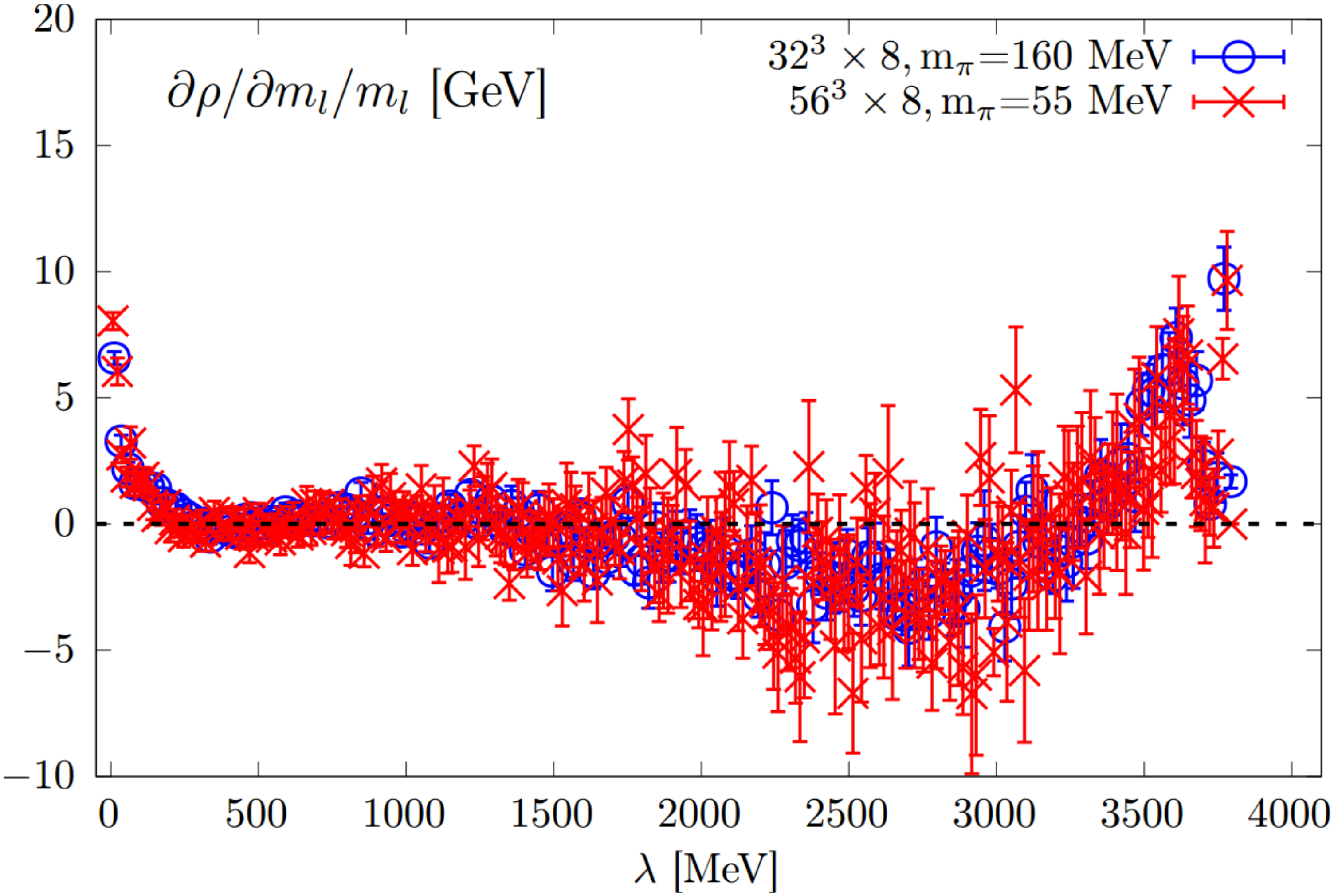
Non-Poisson correlations

$$\Delta_n^{\rm Po} = m_l^{n-2}[\partial^n\rho/\partial m_l^n - (\partial^n\rho/\partial m_l^n)^{\rm Po}]$$

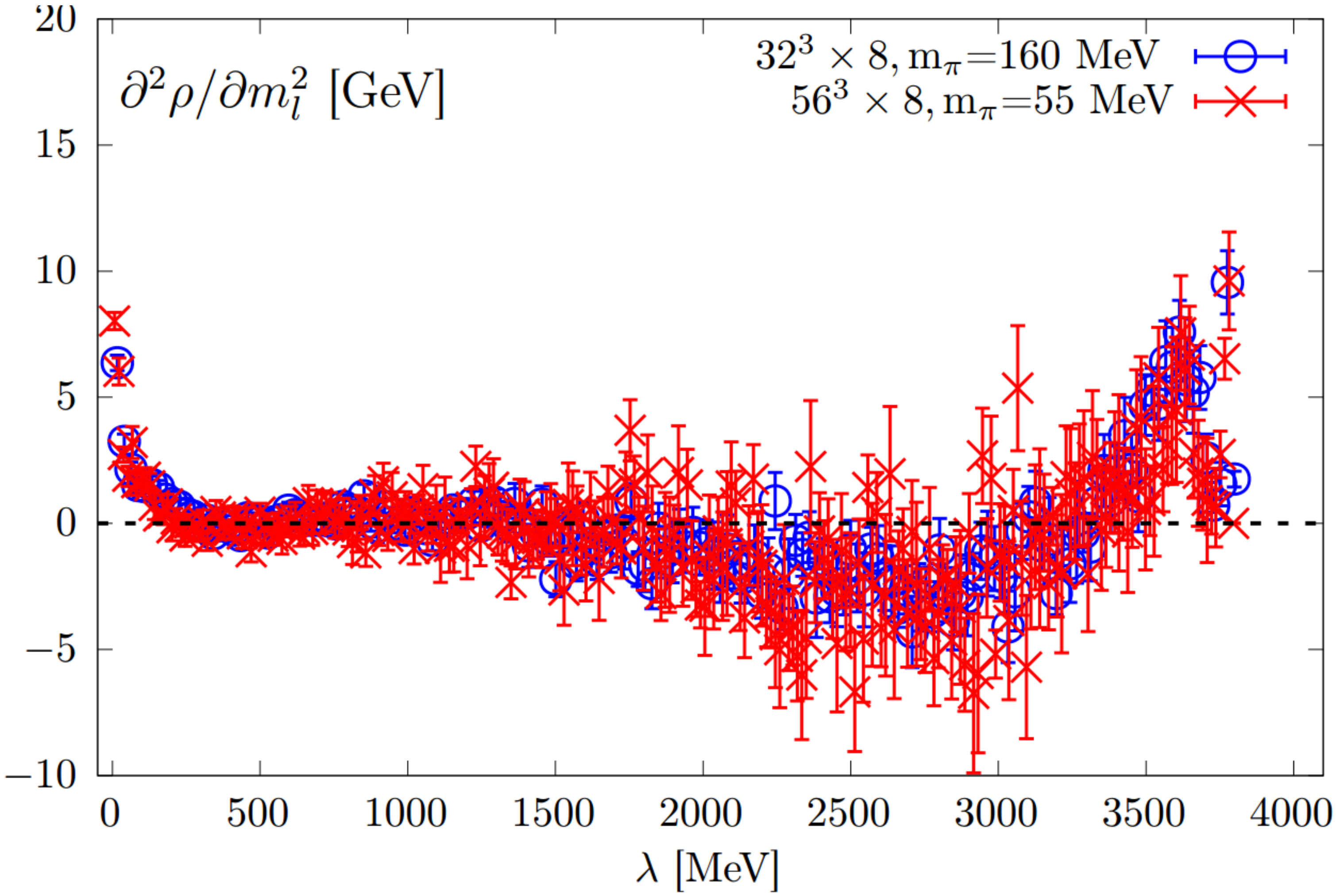


Repulsive non-Poisson correlation at small λ range gives rise to the $\rho(\lambda \to 0)$ peak

Quark mass dependence of $m_l^{-1}\left(\partial\rho/\partial m_l\right)$



Quark mass dependence of $\partial^2\rho/\partial m_l^2$



Symmetries of QCD

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \sum_{q \in u,d,s,c,b,t} \bar{q}[i\gamma^\mu(\partial_\mu - igA_\mu) - m_q]q$$

$$\boxed{SU_L(N_f) \times SU_R(N_f)} \times U_V(1) \times \boxed{U_A(1)} \quad (m_q = 0)$$

☆ $\boxed{SU_L(N_f) \times SU_R(N_f)}$ chiral symmetry

- SSB in the vacuum: $SU_L(N_f) \times SU_R(N_f) \to SU_V(N_f)$
- Restored at $\mathrm{T} \geq \mathrm{T_c}$

☆ $\boxed{U_A(1)}$ symmetry

- Broken on the quantum level due to ABJ anomaly

$$\partial_\mu j_5^\mu = \frac{g^2 N_f}{16\pi^2} F_a^{\mu\nu} \tilde{F}_{\mu\nu}^a \neq 0 \quad (\tilde{F}_{\mu\nu}^a \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} F_a^{\lambda\rho})$$