Compactifications on G_2 Manifolds

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- What?
- Why?
- How?

Plan

1. Introduction

2. G_2 Manifold

   - Classification, G_2 structure

3. G_2 CFT

   3.1 Worldsheet Theory

   3.2 G_2 CFT

4. Het/G_2

5. holonomy **の間の関係**

6. M-theory/G_2

   - Chiral Fermion (Proposal)

7. **まとめ**

1 Introduction

■ Motivation

M, String コンパクト化 ⇒ 興味ある物理

■ Minimal SUSY ($\mathcal{N} = 1$) Model

String 理論の理解 ⇒ ゲージ群、matter,· · ·

- M on G_2 ⇒ 4dim $\mathcal{N} = 1$ SUSY

Phenomenologically preferable

chiral fermion, non Abelian gauge theory, · · ·

⇐ SUGRA side でのアプローチ

- Het on G_2 ⇒ 3dim $\mathcal{N} = 1$ SUSY

Worldsheet Theory (exact CFT description)

⇒ 多様体の Geometry

- N=1 SCA +(付加的構造) ⇒ spacetime susy
- 多様体の間の関係?

$$K3 \leftrightarrow CY_3 \leftrightarrow G_2 \leftrightarrow Spin(7)$$

- G_2, Spin(7) 多様体の性質

G_2 → tri-critical Ising
Spin(7) → Ising

} 役割?

2 G_2 Manifold

■ string theory の Supersymmetric なコンパクト化

$$\mathbb{R}^{d-1,1} \times M^n$$

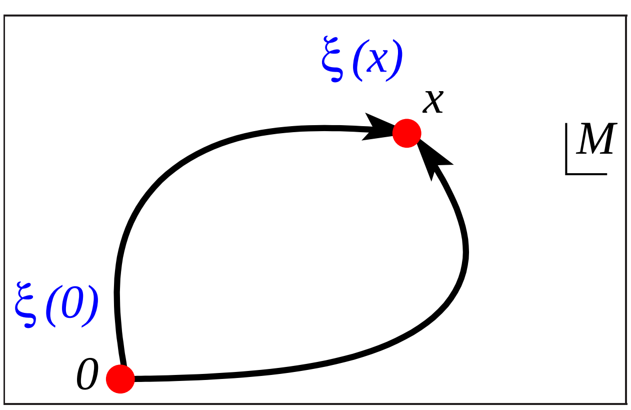
Supersymmetric $\Longleftrightarrow$ M^n : Special holonomy manifold

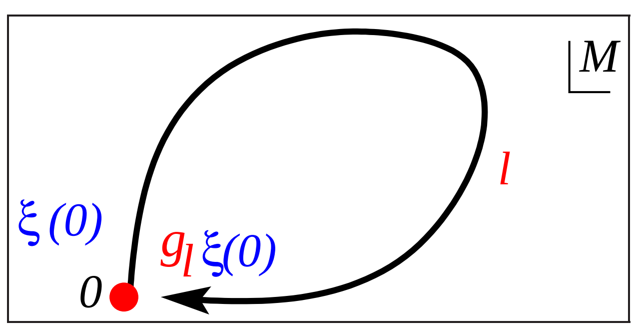
■ Special holonomy manifold

- gravitino $\psi_\mu(x)$ の susy 変換（パラメータ $\xi(x)$）

$$\delta\psi_\mu(x) = \nabla_\mu \xi(x) \quad (= 0 : \text{SUSY 不変})$$

いつこのような covariantly constant spinor があるか？





- holonomy group $G_{\rm hol}$

$$G_{\rm hol} = \{g_\ell | \ell : \text{loop}, \xi(0) \to g_\ell \xi(0)\} \subseteq SO(n)$$

- $G_{\rm hol} = SO(n) \Rightarrow$ cov. const. spinor はない。

Classification ; Berger (1955)

- M ; 単連結 n 次元多様体
- g ; irreducible, nonsymmetric リーマン計量

holonomy group	dimension	name
SO(n)	n	Riemann
U(m)	$n = 2m$	Kähler
SU(m)	$n = 2m$	Calabi-Yau
Sp(m)Sp(1)	$n = 4m$	quaternionic Kähler
Sp(m)	$n = 4m$	hyper Kähler
G_2	$n = 7$	G_2
Spin(7)	$n = 8$	Spin(7)

- special holonomy (Ricci-flat) $\cdots$ SU(m), Sp(m), G_2, Spin(7)
- exceptional holonomy $\cdots$ G_2, Spin(7)

holonomy group	dimension	name	♯ killing spinor
Sp(m)	$n = 4m$	hyper Kähler	(m+1,0)
SU(m)	$n = 2m$	Calabi-Yau	$(2, 0)$ ($m =$ even) $(1, 1)$ ($m =$ odd)
G_2	$n = 7$	G_2	1
Spin(7)	$n = 8$	Spin(7)	(1,0)

String のコンパクト化に使えるもの

- $G_{\mathrm{hol}} \subsetneq \mathrm{SO}(n)$ special holonomy

  $\Rightarrow$ G_{hol} 不変(singlet)な spinor ξ, $[\nabla_\mu, \nabla_\nu]\xi = R_{\mu\nu}\xi = 0$

  $\Rightarrow$ $\xi(x)$ はcov. const. spinor

- 残る susy の数 = G_{hol}-invariant killing spinor の数

- $n = 4$: Sp(1) = SU(2) holonomy — K3

- $n = 6$: SU(3) holonomy — Calabi-Yau 3-fold

- $n = 7$: G_2 holonomy

- $n = 8$: Spin(7), SU(4), Sp(2)の３種類。

K3, Calabi-Yau, は非常によく調べられているのでここでは、主に G_2 holonomy M^7 の多様体によるコンパクト化を調べる。

$$\mathrm{M}/\mathbb{R}^{3,1} \times M^7 \Rightarrow \text{4dim } \mathcal{N} = 1 \text{ SUSY}$$

$$\mathrm{Het}/\mathbb{R}^{2,1} \times M^7 \Rightarrow \text{3dim } \mathcal{N} = 1 \text{ SUSY}$$

What is G_2?

- G_2 Structure $G_2 \subset SO(7)$

  $\Leftrightarrow$ harmonic 3-form $\Phi = \frac{1}{3!} C_{abc} e^a \wedge e^b \wedge e^c$

  分解 $SO(7) \supset G_2$

  SO(7) generator M_{ab} $(\frac{1}{2} \cdot 7 \cdot 6 = 21)$

  $(*_7 C)_{abcd} \Rightarrow 21=14+7 \subset G_2$

  14 ; G_2 generator

Example ; barely G_2 Manifold

- $X = (CY_3 \times S^1)/\mathbb{Z}_2$

  $\mathbb{Z}_2 ; (\sigma, -1)\,, \quad S^1 \ni x \to -x$

- anti-holomorphic involution σ

  $\sigma^* J = -J\,, \quad \sigma^* \Omega = +\bar{\Omega}$

- G_2 structure Φ

  $\Phi = dx \wedge J + \mathrm{Re}\Omega$

- Betti Number

$$b_0 = b_7 = 1\,, \quad b_1 = b_6 = 0$$

$$b_2 = b_5\,, \quad b_3 = b_4$$

Het/G_2

3 G_2 CFT

3.1 Worldsheet Theory

■ World sheet 上の場の理論 — Conformal Field Theory

special holonomy manifold 上を動く string

- $\mathcal{N} = 1$ superconformal

+ special holonomy であることからくる余分な構造

Hyper Kähler	$\mathcal{N} = 4$ CFT	SU(2)
Calabi-Yau	$\mathcal{N} = 2$ CFT	U(1)
G_2	G_2 CFT	tricritical Ising
Spin(7)	Spin(7) CFT	Ising
		⇓ Spacetime SUSY

- G_2, Spin(7) の場合、余分な構造はもはや群ではない。

⇓

CFT による解析は有効。

heterotic string

- left-mover – worldsheet susy – spacetime susy
  right-mover – affine Lie algebra – gauge symmetry
- heterotic string のコンパクト化（最小の susy を持つもの）

時空の次元	内部の次元	内部空間（holonomy）	ゲージ群
6	4	K3 (SU(2))	E_7
4	6	CY_3 (SU(3))	E_6
3	7	(G_2)	?
2	8	(Spin(7))	?

ここでやりたいこと

- Het を G_2 多様体でコンパクト化

  ⇒ 3次元 $\mathcal{N} = 1$ susy,
  non-abelian gauge group

  - spacetime susy の素（spectral flow）
  - gauge 群？

- 様々な holonomy の間の関係

  Calabi-Yau コンパクト化 ⟺ G_2 コンパクト化

  間の関係

方針

worldsheet 上の理論（2次元の CFT）

⇒ 時空の理論の性質。

3.2 G_2 CFT

■ target G_2 **多様体の**2**次元** supersymmetric **シグマ模型** $=$ CFT

- **場** $(x^i, \psi^i),\ i = 1, \ldots, 7$, central charge $c = \frac{21}{2}$
- G_2 **シグマ模型の対称性**

$$(\mathcal{N} = 1 SCA) + \boxed{?} \quad = \quad \boxed{?????}$$

■ G_2 structure $\Longrightarrow$ 3-form Φ

$$\Phi = \Phi_{ijk}\psi^i\psi^j\psi^k \text{ ; operator}$$

■ Current

$$(T, G; \Phi, X; K, M)$$

$$(2, 3/2; 3/2, 2; 2, 5/2)$$

■ $c = 7/10\ N = 1$ SCA

$$(T^{\text{Tri}}, G^{\text{Tri}}) := (-\frac{1}{5}X, \frac{i}{\sqrt{15}}\Phi),$$

$T^{\text{Tri}} =$ $C = \frac{7}{10}$ **の** Virasoro **代数** (tricritical Ising)

- **全体の理論** $=$ (tri-Ising) $+$ (**残り**)

$$T = T^{\text{Tri}} + T^r \quad \Rightarrow \quad T^r T^{\text{Tri}} \sim (reg).$$

tricritical Ising の primary operator

operator	conf. dim.	sector
$[1, \epsilon'']$	$[0, 3/2]$	NS
$[\epsilon, \epsilon']$	$[1/10, 3/5]$	NS
σ	$3/80$	R
σ'	$7/16$	R

時空の SUSY

boson (NS) $\Leftrightarrow$ fermion (R)

— spectral flow operator

- 今の場合 (spectral flow operator)$= \sigma'$

Calabi-Yau との analogy

Calabi-Yau 3-fold	G_2 多様体
Kähler form holomorphic 3-form	G_2 invariant 3-form
$\mathcal{N}=2$ 代数 ($c=9$ 代数)	G_2 CFT 代数
$\cup$	$\cup$
affine U(1)	tri-Ising
$\downarrow$	$\downarrow$
spectral flow operator	spin operator σ'

4 Het/G_2

- $\mathbb{R}^{2,1} \times M^7$ 上の $E_8 \times E_8$ Het
- light-cone gauge $\mathbb{R}^{2,1} \longrightarrow \mathbb{R}^1$: (x^1, ψ^1) $c = 12, \bar{c} = 24$
  - ○ left ; $N = 1$ susy
  - ○ right ; $N = 0$ Lie algebra $E_8 \times E_8$

• E_8 = fermion 16 個 $\lambda^I, I = 1, \ldots, 16$

	$\mathbb{R}$	M^7	SO(9)	E_8
right	b	b $\times 7$ [f $\times 7$	f $\times 9$]	hidden E_8
		⇓	⇓	
		tri-Ising $c = \frac{7}{10}$	+ so(9) $c = \frac{9}{2}$	= ? $c = \frac{26}{5}$

■ $(c = \frac{26}{5}) \Rightarrow$ level 1 affine F_4

gauge 群 $= F_4$

■ F_4 adjoint **表現の** massless vector field $=$ F_4 **ゲージ場**

- massless $\Leftrightarrow$ $(h, \bar{h}) = (1/2, 1)$
- (F_4 adjoint 52) $=$ (so(9) adjoint 36) $+$ (so(9) spinor 16)
- so(9) adjoint $\quad (\psi^1)_{\text{left}} \otimes (\lambda^I \lambda^J)_{\text{right}}$

  $I, J = 1, \ldots, 9$
- so(9) spinor $\quad (\psi^1)_{\text{left}} \otimes (\underbrace{S^a}_{h=9/16} \quad \underbrace{\sigma'}_{h=7/16})_{\text{right}}$

  S^a : λ^I **から作ったスピン演算子**, $a = 1, \ldots, 16$

  σ' : tri-Ising **のスピン演算子**

■ **もっと精密な議論** (massive mode **も含めた**)

— Virasoro **代数、**affine Lie **代数の表現、**character

- tricritical Ising **の役割**?
- G_2 holonomy group **との関係**?

- レベル 1 の affine Lie algebra

group	center	bas	fun
G_2	$14/5$	0	$2/5$
F_4	$26/5$	0	$3/5$

group	center	bas	vec	spi
so(7)	7/2	0	$1/2$	$7/16$
so(9)	9/2	0	$1/2$	$9/16$

- tri-Ising (c=7/10 Virasoro)の表現 (6 種類)

$$h = 0, \frac{3}{2}, \frac{1}{10}, \frac{3}{5}, \frac{7}{16}, \frac{3}{80}$$

character

- holonomy group $G_2 \subset SO(7)$

$\mathrm{so}(7) \cong$ (tri-Ising) $\times G_2$ — $\mathrm{so}(7)/G_2 \cong$ (tri-Ising)

$$\begin{bmatrix} \chi^{\mathrm{so}(7)}_{\mathrm{bas}} \\ \chi^{\mathrm{so}(7)}_{\mathrm{vec}} \\ \chi^{\mathrm{so}(7)}_{\mathrm{spi}} \end{bmatrix} = \begin{bmatrix} \chi^{\mathrm{Tri}}_{0} & \chi^{\mathrm{Tri}}_{3/5} \\ \chi^{\mathrm{Tri}}_{3/2} & \chi^{\mathrm{Tri}}_{1/10} \\ \chi^{\mathrm{Tri}}_{7/16} & \chi^{\mathrm{Tri}}_{3/80} \end{bmatrix} \begin{bmatrix} \chi^{G_2}_{\mathrm{bas}} \\ \chi^{G_2}_{\mathrm{fun}} \end{bmatrix}$$

- gauge group $\mathrm{so}(9) \longrightarrow F_4 \Rightarrow \mathrm{so}(9) \times$ (tri-Ising) $\cong F_4$

$$\begin{bmatrix} \chi^{F_4}_{\mathrm{bas}} & \chi^{F_4}_{\mathrm{fun}} \end{bmatrix} = \begin{bmatrix} \chi^{\mathrm{so}(9)}_{\mathrm{bas}} & \chi^{\mathrm{so}(9)}_{\mathrm{vec}} & \chi^{\mathrm{so}(9)}_{\mathrm{spi}} \end{bmatrix} \begin{bmatrix} \chi^{\mathrm{Tri}}_{0} & \chi^{\mathrm{Tri}}_{3/5} \\ \chi^{\mathrm{Tri}}_{3/2} & \chi^{\mathrm{Tri}}_{1/10} \\ \chi^{\mathrm{Tri}}_{7/16} & \chi^{\mathrm{Tri}}_{3/80} \end{bmatrix}$$

5 holonomy の間の関係

$CY_3 \times S^1 \leftrightarrow G_2$ 対応

- G_2 : 3dim$\mathcal{N} = 1$ SUSY, F_4 gauge sym.
- $CY_3 \times S^1$: 3dim $\mathcal{N} = 2$ SUSY, E_6 gauge sym.

$G_2 \Rightarrow$SU(3)

G_2/SU(3)=(3-state Potts ($c = 4/5$))

- gauge **群の拡大** $F_4 \Rightarrow E_6$

$$(E_6) \cong (F_4) \times (\text{3-Potts})$$

- spacetime SUSY **の拡大** (R-sym. $\mathbb{Z}_2 \longrightarrow$U(1))

— CY$_3$ spectral flow operator

$\phi^{\text{Tri}}{}_{7/16}$ — **もともとあった。**

$\phi^{\text{Tri}}{}_{3/80}\ \phi^{\text{3-Potts}}{}_{2/5}$ — **新しく出てきた。**

- CY$_3$ **の** U(1)

$$(\text{tri-Ising}(c = 7/10)) \times (\text{3-Potts } (c = 4/5))$$
$$\cong (\text{Ising}(c = 1/2)) \times \text{U}(1)_6 (c = 1)$$

一般化。light-cone gauge の transverse 方向 8 次元。

			Spin(7)		G_2		CY_3		K3		
hol.	SO(8)	$\supset$	Spin(7)	$\supset$	G_2	$\supset$	SU(3)	$\supset$	SU(2)	$\supset$	$\{1\}$
		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$	
		Ising		Tri		3Potts		U(1)		SU(2)	
		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$	
gauge	SO(8)	$\subset$	SO(9)	$\subset$	F_4	$\subset$	E_6	$\subset$	E_7	$\subset$	E_8

- holonomy $\longrightarrow$ 小、gauge $\longrightarrow$ 大

  自由度の受渡し — 統計モデル

応用：SUSY identity (#boson) = (#fermion)

- flat のとき Jacobi abstruse identity

$$\chi_{\mathrm{vec}}^{\mathrm{so}(8)} = \chi_{\mathrm{spi}}^{\mathrm{so}(8)} \quad \longrightarrow \quad \theta_3^4 - \theta_4^4 - \theta_2^4 = 0$$

- special holonomy のとき

$$\chi_{\mathrm{vec},\Lambda}^{\mathrm{so}(8)/G_{\mathrm{hol}}} = \chi_{\mathrm{spi},\Lambda}^{\mathrm{so}(8)/G_{\mathrm{hol}}}, \qquad \Lambda : \ G_{\mathrm{hol}} \text{の表現}$$

$$\chi_\lambda^{\mathrm{so}(8)} = \sum_\Lambda \chi_{\lambda,\Lambda}^{\mathrm{so}(8)/G_{\mathrm{hol}}} \chi_\Lambda^{G_{\mathrm{hol}}}$$

- G_2 holonomy のとき

$$0 = F_0^{G_2} := \chi^{\text{Ising}}_{1/2}\chi^{\text{Tri}}_{0} + \chi^{\text{Ising}}_{0}\chi^{\text{Tri}}_{3/2} - \chi^{\text{Ising}}_{1/16}\chi^{\text{Tri}}_{7/16},$$
$$0 = F_1^{G_2} := \chi^{\text{Ising}}_{1/2}\chi^{\text{Tri}}_{3/5} + \chi^{\text{Ising}}_{0}\chi^{\text{Tri}}_{1/10} - \chi^{\text{Ising}}_{1/16}\chi^{\text{Tri}}_{3/80}.$$

- SU(3) holonomy のとき

$$0 = F_0^{\text{su}(3)} := \frac{1}{\eta^2}(\Theta_{6,6}\Theta_{0,2} + \Theta_{0,6}\Theta_{2,2} - 2\Theta_{3,6}\Theta_{1,2}),$$
$$0 = F_{\pm 1}^{\text{su}(3)} := \frac{1}{\eta^2}(\Theta_{2,6}\Theta_{0,2} + \Theta_{4,6}\Theta_{2,2} - \Theta_{1,6}\Theta_{1,2} - \Theta_{5,6}\Theta_{1,2}).$$

- **関係**

$$\begin{bmatrix} F_0^{\text{su}(3)} \\ F_1^{\text{su}(3)} \\ F_{-1}^{\text{su}(3)} \end{bmatrix} = \begin{bmatrix} C_0^{\text{3-Potts}} & C_{2/5}^{\text{3-Potts}} \\ C_{2/3}^{\text{3-Potts}} & C_{1/15}^{\text{3-Potts}} \\ C_{2/3}^{\text{3-Potts}} & C_{1/15}^{\text{3-Potts}} \end{bmatrix} \begin{bmatrix} F_0^{G_2} \\ F_1^{G_2} \end{bmatrix}$$

$C_h^{\text{3-Potts}}$: 3-state Potts の character

M / G_2

6 M-Theory/G_2

■ コンパクト化

- Singular G_2 ; G_2={H.K. over Q} [Acharya-Witten]

  ⇒ non abelian gauge symmetry with chiral fermions

○ Massless Fields ··· localized at singularities

- 1. non abelian gauge field

  codim4 singularity ; 11-4=7dim spacetime

  gauge field が$\mathbb{R}^4 \times \boldsymbol{Q}$を propagate

- 2. chiral fermion

  codim7 singularity ; 11-7=4dim spacetime

  chiral fermion が$\mathbb{R}^4 \times \mathbf{pt}$を propagate

  ($\mathbf{pt} \in \boldsymbol{Q}$; singularity)

  ○ singular pt$\in \boldsymbol{Q} \Leftrightarrow \sharp$ (chiral multiplet)=1

- **例** ; massless chiral field $\bar{5}$ of SU(5)$_G$

  M (fiber)

  generic pt$\in \boldsymbol{Q}$ $\longrightarrow$ A_4 singularity

  special pt$\in \boldsymbol{Q}$ $\longrightarrow$ A_5 singularity

  $\boldsymbol{Q}$; 3dim base space (G_2={H.K. fiber over $\boldsymbol{Q}$})

- **一般化** [Acharya,Witten], [Anguelova,Lazaroiu]

symmetry	chiral matter
$D_5 \longrightarrow A_4$	$\mathbf{10}$
$E_6 \longrightarrow D_5$	$\mathbf{16}$
$E_7 \longrightarrow E_6$	$\mathbf{27}$

- Anomally Cancellation [Witten]

  anomaly inflow

  $\longrightarrow \mathbb{R}^4 \times \mathrm{pt}$ **に** support **を持つ** charged chiral superfield

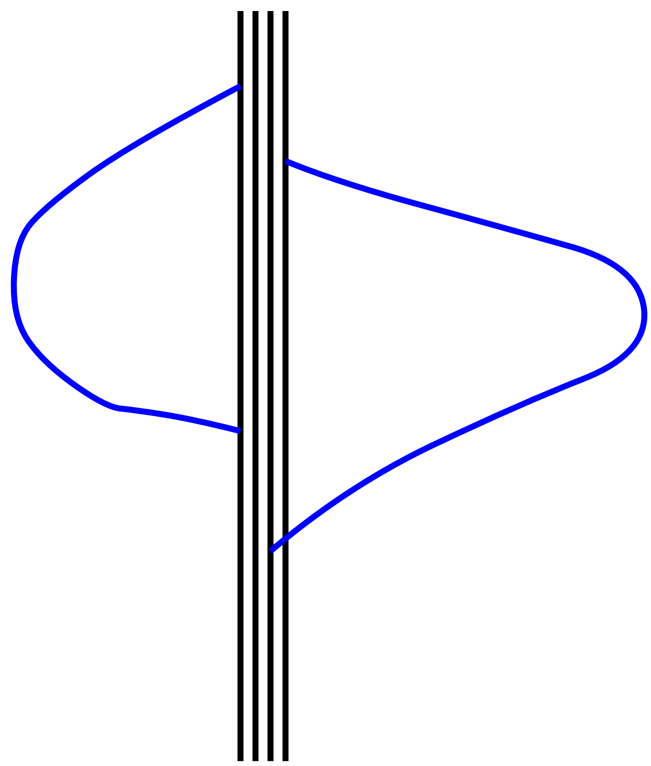
IIA D6-brane

- 1. codim4 singularity $\leftrightarrow$ codim3 in IIA

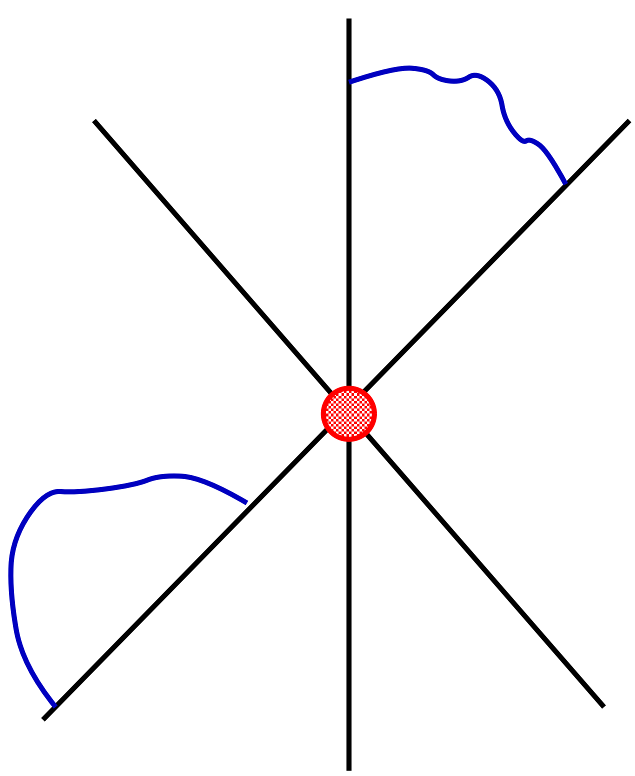
  7dim D6 worldvolume $\Rightarrow$ stacks of parallel D6-branes

- 2. codim7 singularity $\leftrightarrow$ codim6 in IIA

  $\Rightarrow$ intersecting D6-branes



parallel D6's



intersecting D6's

7 まとめ

○ Het on G_2 $\Rightarrow$ 3dim $\mathcal{N}=1$ SUSY

worldsheet theory (CFT)

- holonomy group so(7)/G_2 $\cong$(tri-Ising)
- gauge group (tri-Ising)$\times$so(9)$\cong$ F_4
- Manifolds

  Spin(7)$\leftrightarrow$ G_2 $\leftrightarrow$ $CY_3$$\leftrightarrow$ SU(2)

  (Ising), (tri-Ising), (3-Potts)

$\longrightarrow$ gauge group so(9), F_4, E_6, E_7

- Type II/G_2 ; G_2 =($CY_3\times S^1/\mathbb{Z}_2$)

  [Blumenhagen,Braun], [Roiban,Walcher], [Eguchi,Sugawara]
- coset G_2 [Sugiyama,Yamaguchi]
- super Lie algebra [Noyvert], [Naka]

○ M on G_2 $\Rightarrow$ 4dim $\mathcal{N}=1$ SUSY

non Abelian gauge theory, chiral fermion

- gauge symmetry $\leftrightarrow$ hyperkähler singularity (proposal)
- chiral matter $\leftrightarrow$ singular fiber
- singularity **の一般化**

  [Acharya,Witten], [Anguelova,Lazaroiu],
- anomaly cancellation [Witten],
- G_2 holonomy metric [Gibbons etc],$\cdots$

Tricritical Ising Model

空孔を持つ Ising Model

lattice **上の** spin site **の数が** fluctuate

$$E[\sigma_i, t_i] = -\sum_{\langle ij \rangle} t_i t_j (K + \delta_{\sigma_i, \sigma_j}) - \mu \sum_i t_i$$

$$t_i \equiv \sigma_i^2 = \begin{cases} 0 & (i\,;\ \text{vacant}) \\ 1 & (\text{otherwise}) \end{cases}$$

K ; energy (↑↓)

$K + 1$; energy (↑↑), energy (↓↓)

μ ; chemical potential

$(\beta, K, \mu) \Rightarrow$ 3 **つの** phase **が共存**