Deformation of Super Yang-Mills Theories in R-R 3-form Background

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1. Introduction

non-perturbative properties of supersymmetric gauge theory from string theory

- Effective field theory on D-brane $\rightarrow U(N)$ super Yang-Mills
- We turn on the constant closed string backgrounds.
  (NSNS B-field, RR fields)
  - Nekrasov formula for $\mathcal{N}=2$ super Yang-Mills
  - Dijgraaf-Vafa theory for $\mathcal{N}=1$ supersymmetric gauge theory

  Closed string background plays an important role in these cases.

- Other effect of the closed string background → It deforms the structure of worldvolume spacetime (or superspace).

example 1.

constant NSNS B-field ⇒ noncommutative space(-time)

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu} = i(B^{-1})^{\mu\nu}.$$

example 2.

constant self-dual graviphoton field strength from RR 5-form
⇒ non(anti)commutative $\mathcal{N} = 1$ superspace

$$\{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = C^{\alpha\beta} = (2\pi\alpha')^{3/2}\mathcal{F}^{\alpha\beta}, \quad \mathcal{N} = 1 \to \mathcal{N} = 1/2.$$

What about other RR fields?

★ Here we will consider $\mathcal{N}=2,4$ super Yang-Mills theories in the background of RR 3-form (with fixed $(2\pi\alpha')^{1/2}\mathcal{F}$).

[Billo-Frau-Fucito-Lerda, 2006]

– Consider D3/D(-1) system in type IIB on $\mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2$ with constant $\mathcal{F}^{\alpha\beta}$ from RR 3-form. $(\mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2 \Rightarrow \mathcal{N}=2)$

– Calculate the disk amplitudes of D(-1) and mixed amplitude of D3/D(-1) with insertion of $\mathcal{F}^{\alpha\beta}$.

– Taking $\alpha' \to 0$ limit with fixed $(2\pi\alpha')^{1/2}\mathcal{F}^{\alpha\beta}$ and integrating out ADHM moduli, Nekrasov formula is obtained.

calculation in D3 side $\Rightarrow$ deformed action is obtained.

$$\begin{cases} \text{What is the property of the deformed action?} \\ \text{Extension to } \mathcal{N}=4 \text{ case} \end{cases}$$

2. Deformation in $\mathcal{N} = 4$ Super Yang-Mills

- undeformed part (ordinary $\mathcal{N} = 4$ super Yang-Mills)

  vertex operators (D3-branes in type IIB on $\mathbf{R}^4 \times \mathbf{R}^6$)

$$V_A, \quad V_\Lambda, \quad V_{\bar{\Lambda}}, \quad V_\varphi, \quad V_{H_{AA}}, \quad V_{H_{A\varphi}}, \quad V_{H_{\varphi\varphi}}$$

$(H_{AA})_{\mu\nu}$, $(H_{A\varphi})_{\mu a}$, $(H_{\varphi\varphi})_{ab}$: auxiliary fields to reduce higher order amplitudes to the lower ones

[Dine-Ichinose-Seiberg, 1987], [Atick-Dixon-Sen, 1987]

(ex.) $$-\frac{1}{4}[A_\mu, A_\nu]^2 \;\rightarrow\; (H_{AA})_{\mu\nu}(H_{AA})^{\mu\nu} + (H_{AA})^{\mu\nu}[A_\mu, A_\nu].$$

The disk amplitudes in $\alpha' \to 0$ limit give ($(H_{AA})_{\mu\nu} \to H_{\mu\nu}$, etc.)

$$\mathcal{L}_{\mathcal{N}=4} = \frac{-1}{g_{\rm YM}^2}\mathrm{Tr}\Big[\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\mu A^\nu + i\partial_\mu A_\nu[A^\mu, A^\nu]$$
$$+ \frac{1}{2}H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}H_{\mu\nu}[A^\mu, A^\nu] + \frac{1}{2}H_{ab}H^{ab} + \frac{1}{\sqrt{2}}H_{ab}[\varphi_a, \varphi_b]$$
$$+ \frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a + i\partial_\mu\varphi_a[A^\mu, \varphi_a] + \frac{1}{2}H_{\mu a}H^{\mu a} + H_{\mu a}[A^\mu, \varphi_a]$$
$$+ i\Lambda^A\sigma^\mu D_\mu\bar{\Lambda}_A - \frac{1}{2}(\Sigma^a)^{AB}\bar{\Lambda}_A[\varphi_a, \bar{\Lambda}_B] - \frac{1}{2}(\bar{\Sigma}^a)_{AB}\Lambda^A[\varphi_a, \Lambda^B]\Big].$$

After integrating out $H_{\mu\nu}$, $H_{\mu a}$, H_{ab}, the Lagrangian $\mathcal{L}_{\mathcal{N}=4}$ becomes the standard form of the Lagrangian of $\mathcal{N}=4$ super Yang-Mills theory.

"graviphoton" vertex operator

$$V_{\mathcal{F}}(z,\bar{z}) = (2\pi\alpha')\,\mathcal{F}^{\alpha\beta AB}\Big[S_\alpha(z)S_A(z)e^{-\frac{1}{2}\phi(z)}S_\beta(\bar{z})S_B(\bar{z})e^{-\frac{1}{2}\phi(\bar{z})}\Big].$$

$S_\alpha(z)$, $S_A(z)$: spin field, ϕ: bosonized superconformal ghost

- classification of the "graviphoton" field strength $\mathcal{F}^{\alpha\beta AB}$
  (S: symmetric, A: antisymmetric)

1. (S,S)-type (RR 5-form) $\mathcal{F}^{(\alpha\beta)(AB)} = \mathcal{F}^{\mu\nu abc}(\sigma_{\mu\nu})^{\alpha\beta}\big(\Sigma_{[a}\bar{\Sigma}_b\Sigma_{c]}\big)^{AB}$
2. (S,A)-type (RR 3-form) $\mathcal{F}^{(\alpha\beta)[AB]} = \mathcal{F}^{\mu\nu a}(\sigma_{\mu\nu})^{\alpha\beta}(\Sigma_a)^{AB}$
3. (A,S)-type (RR 3-form) $\mathcal{F}^{[\alpha\beta](AB)} = \mathcal{F}^{abc}\epsilon^{\alpha\beta}\big(\Sigma_{[a}\bar{\Sigma}_b\Sigma_{c]}\big)^{AB}$
4. (A,A)-type (RR 1-form) $\mathcal{F}^{[\alpha\beta][AB]} = \mathcal{F}^{a}\epsilon^{\alpha\beta}(\Sigma_a)^{AB}$

Here $(\Sigma_a)^{AB}$, $(\bar{\Sigma}_a)_{AB}$: six-dimensional gamma-matrices

Note. $\mathcal{F}^{\mu\nu a}$ and $\mathcal{F}^{abc}$ satisfy the self-dual condition.

$$\mathcal{F}^{\mu\nu a} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\rho\sigma}{}^{a}, \quad \mathcal{F}^{abc} = \frac{i}{3!}\epsilon^{abcdef}\mathcal{F}_{def}.$$

- (S,A)-deformation up to second order

contribution from $\langle\langle V_A V_\varphi V_\mathcal{F}\rangle\rangle$, $\langle\langle V_{H_{AA}} V_\varphi V_\mathcal{F}\rangle\rangle$ and $\langle\langle V_\Lambda V_\Lambda V_\mathcal{F}\rangle\rangle$ in $\alpha' \to 0$ limit with fixed $(2\pi\alpha')^{1/2}\mathcal{F}^{(\alpha\beta)[AB]}$

$$\mathcal{L}_{(\mathrm{S,A})} =$$
$$\frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}\left[C^{\mu\nu a}\left(i\varphi_a F_{\mu\nu} + \frac{1}{2}(\bar{\Sigma}_a)_{AB}\Lambda^A\sigma_{\mu\nu}\Lambda^B\right) + \frac{1}{2}C^{\mu\nu a}C_{\mu\nu}{}^{b}\varphi_a\varphi_b\right].$$

Here

$$C^{\mu\nu a} = C^{(\alpha\beta)[AB]}(\sigma^{\mu\nu})_{\alpha\beta}(\bar{\Sigma}^a)_{AB}, \; C^{(\alpha\beta)[AB]} = -2\pi(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[AB]}.$$

deformed SUSY transformation

$$\begin{aligned}
\delta A_\mu &= i(\xi^A\sigma_\mu\bar{\Lambda}_A + \bar{\xi}_A\bar{\sigma}_\mu\Lambda^A), \\
\delta\Lambda^A &= \sigma^{\mu\nu}\xi^A(F_{\mu\nu} - iC_{\mu\nu}{}^a\varphi_a) \\
&\qquad + (\Sigma_a)^{AB}\sigma^\mu\bar{\xi}_B D_\mu\varphi_a - i(\Sigma_{ab})^A{}_B\xi^B[\varphi_a,\varphi_b], \\
\delta\bar{\Lambda}_A &= \bar{\sigma}^{\mu\nu}\bar{\xi}_A F_{\mu\nu} + (\bar{\Sigma}_a)_{AB}\bar{\sigma}^\mu\xi^B D_\mu\varphi_a - i(\bar{\Sigma}_{ab})_A{}^B\bar{\xi}_B[\varphi_a,\varphi_b], \\
\delta\varphi_a &= i(\xi^A(\bar{\Sigma}_a)_{AB}\Lambda^B + \bar{\xi}_A(\Sigma_a)^{AB}\bar{\Lambda}_B),
\end{aligned}$$

and we have to require the condition:

$$\varepsilon_{ABCD}C^{(\alpha\beta)[BC]}\xi^D_\beta = 0, \quad C^{(\alpha\beta)[AB]}\bar{\xi}_{\dot{\alpha}B} = 0.$$

– The deformed SUSY transf. contains only the term linear in C.
(In non(anti)commutative case, the SUSY transformation contains higher order terms of C.)
[Araki-Ito-Ohtsuka, 2004], [Ito-H.N, 2005]

– In generic deformation, all of SUSY are broken.
(non(anti)commutative case $\Rightarrow \mathcal{N} = 1/2$ SUSY)

number of unbroken SUSY (only $C^{(\alpha\beta)[12]}$, $C^{(\alpha\beta)[34]} \neq 0$)

		rank of $C^{(\alpha\beta)[12]}$		
		0	1	2
rank of $C^{(\alpha\beta)[34]}$	0	$\mathcal{N} = (2,2)$	$\mathcal{N} = (3/2,1)$	$\mathcal{N} = (1,1)$
	1	$\mathcal{N} = (3/2,1)$	$\mathcal{N} = (1,0)$	$\mathcal{N} = (1/2,0)$
	2	$\mathcal{N} = (1,1)$	$\mathcal{N} = (1/2,0)$	$\mathcal{N} = (0,0)$

- (A,S)-deformation up to second order

contribution from $\langle\langle V_{H_{\varphi\varphi}} V_{\varphi} V_{\mathcal{F}} \rangle\rangle$ and $\langle\langle V_{\bar{\Lambda}} V_{\bar{\Lambda}} V_{\mathcal{F}} \rangle\rangle$ in $\alpha' \to 0$ limit

$$\mathcal{L}_{(\mathrm{A,S})} =$$

$$\frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}\left[C^{(AB)}\left[(\bar{\Sigma}^{abc})_{AB}\,\varphi_a\varphi_b\varphi_c + 2\bar{\Lambda}_A\bar{\Lambda}_B\right] + \frac{1}{4}C^{abc}C^{abd}\varphi_c\varphi_d\right],$$

where

$$C^{(AB)} = -\pi i(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{[\alpha\beta](AB)}\epsilon_{\alpha\beta},$$
$$C^{abc} = C^{(AB)}(\bar{\Sigma}^{abc})_{AB} = C^{(AB)}\left(\bar{\Sigma}^{[a}\Sigma^{b}\bar{\Sigma}^{c]}\right)_{AB}.$$

– "mass" and superpotential deformation

– SUSY condition $\Rightarrow \mathcal{N} = (1/2, 0)$ or $(0,0)$

- second-order deformation

  – We insert only one graviphoton vertex operator.

  – second-order deformation $\Leftarrow$ integration of auxiliary fields
    Are there other contributions to the second-order term?

second-order amplitude $\langle\!\langle V_{\mathcal{F}} V_{\mathcal{F}} \cdots \rangle\!\rangle$ $\begin{cases} \text{reducible} & \text{(reduced to first-order ampl.)} \\ \text{irreducible} & \end{cases}$

possible form of the irreducible amplitude at second order

  – (S,A)-type: $\langle\!\langle V_{H_{\varphi\varphi}} V_{\mathcal{F}} V_{\mathcal{F}} \rangle\!\rangle$, but probably no contribution
    (conjecture from consistency with $\mathcal{N} = 2$ theory)

  – (A,S)-type: no irreducible amplitude

- fuzzy sphere in (S,A)-deformation

  scalar potential

$$V(\varphi) = -\frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}\left[\frac{1}{4}[\varphi_a, \varphi_b]^2 + \frac{1}{2}(C^{\mu\nu a}\varphi_a)^2\right].$$

stationary condition and fuzzy sphere ansatz

$$\left[\varphi_b, [\varphi_a, \varphi_b]\right] + C_{\mu\nu a}C^{\mu\nu b}\varphi_b = 0, \quad [\varphi_a, \varphi_b] = i f_{abc}\varphi_c.$$

relation between f_{abc} and $C^{\mu\nu a}$

$$f_{abc}f_{bcd} = C_{\mu\nu a}C^{\mu\nu}{}_d.$$ Killing metric is given by $C^{\mu\nu a}$.

self-dual condition of $C^{\mu\nu a}$
$\Rightarrow$ (Killing metric) $=$ diag$(X, Y, Z, 0, 0, 0)$ $\Rightarrow$ fuzzy $\mathbf{S}^2$.

3. Non-abelian Chern-Simons term

D-brane action $\Rightarrow$ DBI action $+$ CS term (due to RR fields)

DBI action $\rightarrow$ super Yang-Mills action in $\alpha' \rightarrow 0$ limit

CS term $\rightarrow$ (first-order) deformation term in $\alpha' \rightarrow 0$ limit

- CS term [Myers, 1999]

$$\frac{1}{\lambda^2 g_{\mathrm{YM}}^2} \mathrm{STr} \int_{\mathcal{M}_4} P\big[\exp(i\lambda \mathrm{i}_\varphi^2)\lambda^{1/2}\mathcal{A}\big] \wedge \exp(\lambda F), \quad \lambda = 2\pi\alpha'.$$

STr: symmetrized trace, $P[\cdots]$: pullback, i_φ: interior product,
$\mathcal{A}$: formal sum of RR potential, F: gauge field strength on D-brane

• (S,A)-deformation

4-dim. self-dual condition: $*_4\mathcal{F}^{\mu\nu a} = \mathcal{F}^{\mu\nu a} \longrightarrow *_{10}\mathcal{F}^{\mu\nu a} = *_6\mathcal{F}^{\mu\nu a}$

$\Rightarrow$ We have to consider dual 7-form in addition to 3-form.

lowest contribution from 3-form (potential: $\mathcal{A}_{\mu a}$, $\mathcal{A}_{\mu\nu}$)

$$\frac{\lambda^{-1/2}}{g_{\mathrm{YM}}^2}\mathrm{STr}\int P[\mathcal{A}]\wedge F = \frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}\int d^4x\,\lambda^{1/2}\underbrace{(\partial_a\mathcal{A}_{\mu\nu} + \partial_{[\mu}\mathcal{A}_{\nu]a})}_{=\mathcal{F}_{\mu\nu a}}\varphi_a F^{\mu\nu}.$$

Then under $\alpha' \to 0$ limit: $\lambda^{1/2}\mathcal{F}^{\mu\nu a} \sim C^{\mu\nu a} =$ fixed, this contribution gives the deformation term $\mathrm{Tr}(C^{\mu\nu a}\varphi_a F_{\mu\nu})$.

contribution from 7-form $\Rightarrow$ higher order of α' then it disappears in $\alpha' \to 0$ limit.

• (A,S)-deformation

6-dim. self-dual condition: $*_6\mathcal{F}^{abc}=i\mathcal{F}^{abc}\longrightarrow *_{10}\mathcal{F}^{abc}=i*_4\mathcal{F}^{abc}$

$\Rightarrow$ We have to consider dual 7-form in addition to 3-form.

contribution from 3-form $\Rightarrow$ higher order of α' then it disappears in $\alpha'\to 0$ limit.

lowest contribution from 7-form (potential: $\mathcal{A}_{\mu\nu\rho\sigma ab}$, $\mathcal{A}_{\mu\nu\rho abc}$)

$$\frac{1}{g_{\rm YM}^2}\mathrm{Tr}\int d^4x\,\lambda^{1/2}\underbrace{(\partial_{[a}\mathcal{A}_{bc]\mu\nu\rho\sigma}+\partial_{[\mu}\mathcal{A}_{\nu\rho\sigma]abc})}_{=\mathcal{F}_{\mu\nu\rho\sigma abc}=\epsilon_{\mu\nu\rho\sigma}\mathcal{F}_{abc}}\epsilon^{\mu\nu\rho\sigma}(\varphi_a\varphi_b\varphi_c).$$

Then under $\alpha'\to 0$ limit: $\lambda^{1/2}\mathcal{F}^{abc}\sim C^{abc}=$ fixed, this contribution gives the deformation term $\mathrm{Tr}(C^{abc}\varphi_a\varphi_b\varphi_c)$.

4. Reduction to $\mathcal{N}=2$ Super Yang-Mills

- Orbifolding in (S,A)-deformation

$\mathcal{N}=4$: type IIB on $\mathbf{R}^4 \times \mathbf{R}^6 \Rightarrow$

$\mathcal{N}=2$: type IIB on $\mathbf{R}^4 \times \mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2$

We put N fractional D3-branes at the singularity of the orbifold $\mathbf{R}^4/\mathbf{Z}_2$. In terms of the fields, Orbifold projection is expressed by

$$\Lambda_\alpha^A = 0 \text{ for } A=3,4, \quad \varphi_a = 0 \text{ for } a=3,4,5,6,$$

and only $C^{\mu\nu[12]}$ and $C^{\mu\nu[34]}$ are nonzero.

Under the reduction, $\mathcal{L}_{\mathcal{N}=4}$ becomes $\mathcal{L}_{\mathcal{N}=2}$.

$$\mathcal{L}_{\mathcal{N}=2} = \frac{1}{g_{\rm YM}^2}\mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - D_\mu\varphi D^\mu\bar{\varphi} - \frac{1}{2}[\varphi,\bar{\varphi}]^2 \right.$$
$$\left. - i\Lambda^{i\alpha}(\sigma^\mu)_{\alpha\dot{\beta}}D_\mu\bar{\Lambda}_i{}^{\dot{\beta}} - \frac{i}{\sqrt{2}}\Lambda^i[\bar{\varphi},\Lambda_i] + \frac{i}{\sqrt{2}}\bar{\Lambda}_i[\varphi,\bar{\Lambda}^i]\right].$$

deformation term ([Billo-Frau-Fucito-Lerda] for $\bar{C}=0$ case)

$$\mathcal{L}_{(\mathrm{S,A})} =$$
$$\frac{1}{g_{\rm YM}^2}\mathrm{Tr}\left[i(C^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi)F_{\mu\nu} - \frac{1}{\sqrt{2}}\bar{C}^{\mu\nu}\Lambda^i\sigma_{\mu\nu}\Lambda_i + \frac{1}{2}(C^{\mu\nu}\bar{\varphi} + \bar{C}^{\mu\nu}\varphi)^2\right].$$

Here $C^{\mu\nu}$ and $\bar{C}^{\mu\nu}$ are defined by

$$C^{\mu\nu} = 2\sqrt{2}iC^{\mu\nu[12]}, \quad \bar{C}^{\mu\nu} = -2\sqrt{2}iC^{\mu\nu[34]}.$$

$C^{\mu\nu}$: VEV of graviphoton field strength in $\mathcal{N}=2$ SUGRA

$\bar{C}^{\mu\nu}$: coming from vector multiplet in $\mathcal{N}=2$ SUGRA

SUSY condition

$$\bar{C}^{(\alpha\beta)}\xi^i_\beta = 0, \qquad \bar{\xi}_i = 0 \text{ or } C^{(\alpha\beta)} = 0.$$

number of unbroken SUSY

		rank of $C^{(\alpha\beta)}$		
		0	1	2
rank of $\bar{C}^{(\alpha\beta)}$	0	$\mathcal{N}=(1,1)$	$\mathcal{N}=(1,0)$	$\mathcal{N}=(1,0)$
	1	$\mathcal{N}=(1/2,1)$	$\mathcal{N}=(1/2,0)$	$\mathcal{N}=(1/2,0)$
	2	$\mathcal{N}=(0,1)$	$\mathcal{N}=(0,0)$	$\mathcal{N}=(0,0)$

- case of (A,S)-deformation

  orbifold projection

$$C^{(AB)} = \frac{1}{2}\begin{pmatrix} C^{(ij)} & 0 \\ 0 & \bar{C}^{(\hat{i}\hat{j})} \end{pmatrix}, \quad i,j = 1,2, \quad \hat{i},\hat{j} = 3,4.$$

  deformation term

$$\mathcal{L}_{(\mathrm{A,S})} = \frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}\Big[C^{(ij)}\bar{\Lambda}_{\dot{\alpha}i}\bar{\Lambda}^{\dot{\alpha}}_{j} - C^{(ij)}C_{(ij)}\bar{\varphi}^2 - \bar{C}^{(\hat{i}\hat{j})}\bar{C}_{(\hat{i}\hat{j})}\varphi^2\Big].$$

  – “mass” deformation

  – SUSY condition $\Rightarrow C^{(ij)}\bar{\xi}_j = 0 \Rightarrow \mathcal{N} = (1,0)$ or $(1,1/2)$

5. Summary and Outlook

Summary

1. We construct $\mathcal{N} = 2, 4$ super Yang-Mills theory with (S,A) and (A,S)-type deformation up to the second order of the R-R 3-form background.

2. In both case of (S,A) and (A,S)-type deformation, the number of unbroken supersymmetry depends on the rank of deformation parameter. We also find the fuzzy sphere configuration.

3. (S,A) and (A,S)-type deformations are consistent with the CS coupling in D-brane effective action.

Outlook

1. instanton calculus with (S,A)-deformation
   - $\mathcal{N} = 2$ case $\longrightarrow$ Nekrasov formula can be obtained.

     [Billo-Frau-Fucito-Lerda]
   - $\mathcal{N} = 4$ case $\longrightarrow$???
2. physical and geometrical meaning of deformation
   - (A,S)-type $\longrightarrow$ "mass" and superpotential deformation
   - (S,A)-type $\longrightarrow$ correspondence to Ω-background?
3. instanton effects in string theory (D(-1)-side), AdS/CFT, . . .