Extended Supersymmetric σ-Model Based on the SO(2N+1) Lie Algebra of the Fermion Operators

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Extended supersymmetric σ-model is proposed, basing on the $SO(2N+1)$ Lie algebra of fermion operators composed of creation-annihilation operators and pair operators. The canonical transformation, the extension of the $SO(2N)$ Bogoliubov transformation to the $SO(2N+1)$ group, is introduced. Embedding the $SO(2N+1)$ group into an $SO(2N+2)$ group and using $\frac{SO(2N+2)}{U(N+1)}$ coset variables, we investigate a new aspect of the supersymmetric σ-model on the Kähler manifold of the symmetric space $\frac{SO(2N+2)}{U(N+1)}$. We construct a Killing potential which is the extension of the Killing potential in the $\frac{SO(2N)}{U(N)}$ coset space given by van Holten et al. to that in the $\frac{SO(2N+2)}{U(N+1)}$ coset space. The Killing potential plays an important role to see behaviour of the vacuum expectation value of the σ-model fields. Bosonization of the $SO(2N+1)$ Lie operators is made. The vacuum functions for these bosons are expressed in terms of the corresponding Kähler potential and a $U(1)$ phase.

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Plan of the Talk

- 1. Introduction
- 2. The SO(2N+1) Lie algebra of fermion operators and the Bogoliubov transformation
- 3. Embedding into an SO(2N+2) group
- 4. σ-model on the SO(2N+2)/U(N+1) coset manifold
- 5. Expression for SO(2N+2)/U(N+1) Killing potential
- 6. Discussions and concluding remarks
- 7. Appendix
  A. Bosonization of SO(2N+2) Lie operators
  B. Vacuum function for bosons
  C. Differential forms for bosons
  over SO(2N+2)/U(N+1) coset space

Supersymmetric σ-Model

- 1. The supersymmetric extension of nonlinear mdoels was first given by Zumino, by introducing scalar fields in a complex Kähler manifold. [1].
- 2. The higher dimensional nonlinear σ-models defined on symmetric spaces and on hyper Kähler manifolds have been itensively studied [2, 3, 4, 5].
- 3. van Holten et al. have discussed a supersymmetric σ-models on the Kähler coset spaces. They have presented a way of constructing the Killing potentials on the coset spaces $\frac{SO(2N)}{U(N)}$[2].
- 4. Higashijima et al. have given Ricci-flat metrics on compact Kähler manifolds, $\frac{SU(N)}{[SU(N-M)\times U(M)]}$, $\frac{SO(2N)}{U(N)}$ and $\frac{Sp(N)}{U(N)}$ [3].

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1 Introduction

The set of the fermion operators composed of annihilation-creation and pair operators forms a larger Lie algebra, Lie algebra of the $SO(2N+1)$ group.

→ Group extension of the $SO(2N)$ Bogoliubov transformation for fermions

The fermion Lie operators are mapped into the regular representation of the $SO(2N+1)$ group and are represented by **Bose operators**.

The **bose images** of the fermion Lie operators are expressed by closed first order differential forms.

We give an extended supersymmetric σ-model on Kähler coset space $\frac{G}{H} = \frac{SO(2N+2)}{U(N+1)}$, basing on the $SO(2N+1)$ Lie algebra of the fermion operators. Embedding the $SO(2N+1)$ group into an $SO(2N+2)$ group and using the $\frac{SO(2N+2)}{U(N+1)}$ coset variables. we investigate a new aspect of the supersymmetric σ-model on the Kähler manifold of the symmetric space $\frac{SO(2N+2)}{U(N+1)}$.

We construct Killing potential which is the extension of Killing potential in the $\frac{SO(2N)}{U(N)}$ coset space given by van Holten et al. to that in the $\frac{SO(2N+2)}{U(N+1)}$ coset space:

→ **Killing potential is equivalent with**

Generalized density matrix

Its diagonal-block part : A reduced scalar potential with

Fayet-Ilipoulos term

The reduced scalar potential is optimized to see behaviour of the vacuum expectation value of the σ-model fields.

2 The SO(2N+1) Lie algebra of fermion operators and the Bogoliubov transformation

c_α **and** $c^\dagger_\alpha$**,** $\alpha = \mathbf{1}, \cdots, N$ **: Annihilation and creation operators of the fermion**

$$\boxed{\{c_\alpha,\ c^\dagger_\beta\} = \delta_{\alpha\beta}, \quad \{c_\alpha,\ c_\beta\} = \{c^\dagger_\alpha,\ c^\dagger_\beta\} = 0.} \tag{2.1}$$

The set of fermion operators consisting of annihilation-creation operators and pair operators:

$$\boxed{\left.\begin{array}{ccc} c_\alpha, & c^\dagger_\alpha, & \\ E^\alpha{}_\beta = c^\dagger_\alpha c_\beta - \frac{1}{2}\delta_{\alpha\beta}, & E^{\alpha\beta} = c^\dagger_\alpha c^\dagger_\beta, & E_{\alpha\beta} = c_\alpha c_\beta, \\ E^{\alpha\dagger}_{\ \beta} = E^\beta{}_\alpha, & E^{\alpha\beta} = E^\dagger_{\beta\alpha}, & E_{\alpha\beta} = -E_{\beta\alpha}. \end{array}\right\} \quad (2.2)}$$

The set of fermion operators (2.2) form an $SO(2N+1)$ **Lie algebra.**

As a consequence of the anti-commutatin relation (2.1), the commutation relations for the fermion operators (2.2) in the $SO(2N+1)$ Lie algebra:

$$[E^\alpha{}_\beta,\ E^\gamma{}_\delta] = \delta_{\gamma\beta} E^\alpha{}_\delta - \delta_{\alpha\delta} E^\gamma{}_\beta, \quad (U(N)\ \text{algebra}) \tag{2.3}$$

$$\left.\begin{array}{l} [E^\alpha{}_\beta,\ E_{\gamma\delta}] = \delta_{\alpha\delta} E_{\beta\gamma} - \delta_{\alpha\gamma} E_{\beta\delta}, \\ [E^{\alpha\beta},\ E_{\gamma\delta}] = \delta_{\alpha\delta} E^\beta{}_\gamma + \delta_{\beta\gamma} E^\alpha{}_\delta - \delta_{\alpha\gamma} E^\beta{}_\delta - \delta_{\beta\delta} E^\alpha{}_\gamma, \\ [E_{\alpha\beta},\ E_{\gamma\delta}] = 0, \end{array}\right\} \tag{2.4}$$

$$\left.\begin{array}{l}[c^\dagger_\alpha,\ c_\beta]=2E^\alpha{}_\beta,\ [c_\alpha,\ c_\beta]=2E_{\alpha\beta},\\[2ex] [c_\alpha,\ E^\beta{}_\gamma]=\delta_{\alpha\beta}c_\gamma,\ [c_\alpha,\ E_{\beta\gamma}]=0,\\[2ex] [c_\alpha,\ E^{\beta\gamma}]=\delta_{\alpha\beta}c^\dagger_\gamma-\delta_{\alpha\gamma}c^\dagger_\beta.\end{array}\right\} \tag{2.5}$$

n**: fermion number operator** $n\!=\!c^\dagger_\alpha c_\alpha$:

$$\{c_\alpha,\ (-1)^n\}=\{c^\dagger_\alpha,\ (-1)^n\}=0. \tag{2.6}$$

Operator Θ defined by $\Theta\!\equiv\!\theta_\alpha c^\dagger_\alpha-\bar\theta_\alpha c_\alpha$: Due to the relation $\Theta^2\!=\!-\bar\theta_\alpha\theta_\alpha$,

$$\left.\begin{array}{l}e^\Theta=Z+X_\alpha c^\dagger_\alpha-\bar X_\alpha c_\alpha,\quad \bar X_\alpha X_\alpha+Z^2=1,\\[2ex] Z=\cos\theta,\quad X_\alpha=\dfrac{\theta_\alpha}{\theta}\sin\theta,\quad \theta^2=\bar\theta_\alpha\theta_\alpha.\end{array}\right\} \tag{2.7}$$

From (2.1), (2.6) and (2.7), we have

$$\left.\begin{array}{l}e^\Theta(c_\alpha,c^\dagger_\alpha,\tfrac{1}{\sqrt2})(-1)^n e^{-\Theta}=(c_\beta,c^\dagger_\beta,\tfrac{1}{\sqrt2})(-1)^n G_X,\\[2ex] G_X\equiv\left[\begin{array}{ccc}\delta_{\beta\alpha}-\bar X_\beta X_\alpha & \bar X_\beta\bar X_\alpha & -\sqrt2 Z\bar X_\beta\\ X_\beta X_\alpha & \delta_{\beta\alpha}-X_\beta\bar X_\alpha & \sqrt2 ZX_\beta\\ \sqrt2 ZX_\alpha & -\sqrt2 Z\bar X_\alpha & 2Z^2-1\end{array}\right].\end{array}\right\} \tag{2.8}$$

From (2.8) and the commutability of $U(g)$ with $(-1)^n$,

$$U(G)(c_\alpha,c^\dagger_\alpha,\frac{1}{\sqrt2})(-1)^n U^\dagger(G)\!=\!(c_\beta,c^\dagger_\beta,\frac{1}{\sqrt2})(-1)^n\begin{bmatrix}A_{\beta\alpha} & \bar B_{\beta\alpha} & -\dfrac{\bar x_\beta}{\sqrt2}\\ B_{\beta\alpha} & \bar A_{\beta\alpha} & \dfrac{x_\beta}{\sqrt2}\\ \dfrac{y_\alpha}{\sqrt2} & -\dfrac{\bar y_\alpha}{\sqrt2} & z\end{bmatrix}, \tag{2.9}$$

$$
\left.\begin{aligned}
&A_{\alpha\beta} = a_{\alpha\beta} - \bar{X}_\alpha Y_\beta = a_{\alpha\beta} - \frac{\bar{x}_\alpha y_\beta}{2(1+z)}, \\
&B_{\alpha\beta} = b_{\alpha\beta} + X_\alpha Y_\beta = b_{\alpha\beta} + \frac{x_\alpha y_\beta}{2(1+z)}, \\
&x_\alpha = 2ZX_\alpha,\ y_\alpha = 2ZY_\alpha,\ z = 2Z^2 - 1.
\end{aligned}\right\} \tag{2.10}
$$

$$
\boxed{U(G)(c, c^\dagger, \tfrac{1}{\sqrt{2}})U^\dagger(G) = (c, c^\dagger, \tfrac{1}{\sqrt{2}})(z-\rho)G, \qquad (2.11)}
$$

$$
\boxed{G \equiv \begin{bmatrix} A & \bar{B} & -\dfrac{\bar{x}}{\sqrt{2}} \\ B & \bar{A} & \dfrac{x}{\sqrt{2}} \\ \dfrac{y}{\sqrt{2}} & -\dfrac{\bar{y}}{\sqrt{2}} & z \end{bmatrix},\ G^\dagger G = GG^\dagger = 1_{2N+1},\ \ \det G = 1, (2.12)}
$$

$$
\boxed{\left.\begin{aligned}
&U(G)U(G') = U(GG'),\ \ U(G^{-1}) = U^{-1}(G) = U^\dagger(G), \\
&U(1_{2N+1}) = \mathbb{I}_G,
\end{aligned}\right\} (2.13)}
$$

$(c,\ c^\dagger,\ \frac{1}{\sqrt{2}})$: $(2N+1)$-dimensional row vector $((c_\alpha),\ (c_\alpha^\dagger),\ \frac{1}{\sqrt{2}})$
$A = (A^\alpha{}_\beta)$ and $B = (B_{\alpha\beta})$: $N \times N$ matrices.
$U(G)$: nonlinear transformation with a q-number gauge $z - \rho$:
$\rho = x_\alpha c_\alpha^\dagger - \bar{x}_\alpha c_\alpha$ and $\rho^2 = -\bar{x}_\alpha x_\alpha = z^2 - 1$
The matrix G is a matrix belonging to the $SO(2N+1)$ group.

$\boxed{\text{When } z = 1,}$ the G becomes an $SO(2N)$ matrix g.
$SO(2N+1)$ WF $|\,G\!>\,=\,U(G)|\,0\!>$:

$$\boxed{\begin{aligned} &|\,G\!>\,=\,<\!0\,|U(G)|\,0\!>\!(1\!+\!r_\alpha c_\alpha^\dagger)\exp(\tfrac{1}{2}\cdot q_{\alpha\beta}c_\alpha^\dagger c_\beta^\dagger)|\,0\!>, \\ &r_\alpha = \frac{1}{1+z}(x_\alpha + q_{\alpha\beta}\bar{x}_\beta), \quad q\!=\!ba^{-1}, \end{aligned}} \tag{2.14}$$

$$\boxed{<\!0\,|\,U(G)\,|\,0\!>\,=\,\bar{\Phi}_{00}(G)\!=\!\sqrt{\frac{1\!+\!z}{2}}\left[\det(1_N\!+\!q^\dagger q)\right]^{-\frac{1}{4}}e^{i\frac{\tau}{2}}.} \tag{2.15}$$

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3 Embedding into an SO(2N+2) group

Projection operator $P_{\pm}$ onto the sub-spaces of even and odd fermion numbers

$$\boxed{P_{\pm} = \frac{1}{2}(1 \pm (-1)^n), \quad P_{\pm}^2 = P_{\pm}, \quad P_+ P_- = 0,} \tag{3.1}$$

Operators with the supurious index 0:

$$\boxed{\left.\begin{array}{l} E^0{}_0 = -\frac{1}{2}(-1)^n = \frac{1}{2}(P_- - P_+), \\[2ex] E^\alpha{}_0 = c^\dagger_\alpha P_- = P_+ c^\dagger_\alpha, \quad E^{\alpha 0} = -c^\dagger_\alpha P_+ = -P_- c^\dagger_\alpha. \end{array}\right\}} \tag{3.2}$$

Indices $p, \ q \ \cdots$ running over $N+1$ values $0, 1, \cdots \ N$.

Unified notation : $E^p{}_q, \ E_{pq}$ and E^{pq}.

The $SO(2N+1)$ group is embedded into an $SO(2N+2)$ group. The embedding leads us to an unified formulation of the $SO(2N+1)$ regular representation in which paired and unpaired modes are treated in an equal way.

$(N+1) \times (N+1)$ matrices $\mathcal{A}$ and $\mathcal{B}$:

$$\boxed{\mathcal{A} = \begin{bmatrix} A & -\frac{\bar{x}}{2} \\ \frac{y}{2} & \frac{1+z}{2} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B & \frac{x}{2} \\ -\frac{y}{2} & \frac{1-z}{2} \end{bmatrix}, \quad y = x^{\mathrm{T}} a - x^\dagger b.} \tag{3.3}$$

Imposition of the ortho-normalization of the G

$\longrightarrow$

Matrices $\mathcal{A}$ and $\mathcal{B}$ satisfy the ortho-normalization condition

$$\mathcal{G} = \begin{bmatrix} \mathcal{A} & \bar{\mathcal{B}} \\ \mathcal{B} & \bar{\mathcal{A}} \end{bmatrix}, \quad \mathcal{G}^\dagger \mathcal{G} = \mathcal{G}\mathcal{G}^\dagger = 1_{2N+2}, \quad \det \mathcal{G} = 1, \tag{3.4}$$

$$\mathcal{A}^\dagger \mathcal{A} + \mathcal{B}^\dagger \mathcal{B} = 1_{N+1}, \ \mathcal{A}^{\mathrm{T}} \mathcal{B} + \mathcal{B}^{\mathrm{T}} \mathcal{A} = 0, \ \mathcal{A}\mathcal{A}^\dagger + \bar{\mathcal{B}}\mathcal{B}^{\mathrm{T}} = 1_{N+1}, \ \bar{\mathcal{A}}\mathcal{B}^{\mathrm{T}} + \mathcal{B}\mathcal{A}^\dagger = 0. \tag{3.5}$$

Decomposition of the matrices $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A}=\begin{bmatrix} 1_N-\dfrac{\bar{x}r^{\mathrm{T}}}{2} & -\dfrac{\bar{x}}{2} \\ \\ \dfrac{(1+z)r^{\mathrm{T}}}{2} & \dfrac{1+z}{2} \end{bmatrix}\begin{bmatrix} a & 0 \\ \\ 0 & 1 \end{bmatrix}, \ \mathcal{B}=\begin{bmatrix} 1_N+\dfrac{xr^{\mathrm{T}}q^{-1}}{2} & \dfrac{x}{2} \\ \\ -\dfrac{(1+z)r^{\mathrm{T}}q^{-1}}{2} & \dfrac{1-z}{2} \end{bmatrix}\begin{bmatrix} b & 0 \\ \\ 0 & 1 \end{bmatrix}, \tag{3.6}$$

$$\mathcal{A}^{-1}=\begin{bmatrix} a^{-1} & 0 \\ \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1_N & \dfrac{\bar{x}}{1+z} \\ -r^{\mathrm{T}} & 1 \end{bmatrix}. \tag{3.7}$$

$\frac{SO(2N+2)}{U(N+1)}$ coset variable with the $N+1$-th component:

$$\boxed{\mathcal{Q}=\mathcal{B}\mathcal{A}^{-1}=\begin{bmatrix} q & r \\ -r^{\mathrm{T}} & 0 \end{bmatrix}=-\mathcal{Q}^{\mathrm{T}}.} \tag{3.8}$$

$SO(2N+1)$ variables $q_{\alpha\beta}$ and r_α : **Independent variables** of the $\frac{SO(2N+2)}{U(N+1)}$ coset space.

4 σ-model on the SO(2N+2)/U(N+1) coset manifold

Matrix elements of $\mathcal{Q}$ and $\bar{\mathcal{Q}}$: Co-ordinates on the $\frac{SO(2N+2)}{U(N+1)}$ coset manifold, on which the real line element can be well defined by a hermitian metric tensor on the coset manifold

$$ds^2 = \mathcal{G}_{pq\ \underline{rs}} d\mathcal{Q}^{pq} d\bar{\mathcal{Q}}^{\underline{rs}} \ (\mathcal{Q}^{pq} = \mathcal{Q}_{pq} \text{ and } \bar{\mathcal{Q}}^{\underline{rs}} = \bar{\mathcal{Q}}_{rs}). \tag{4.1}$$

The hermitian metric tensor $\mathcal{G}_{pq\ \underline{rs}}$ is locally given through a real scalar function,

Kähler potential:

$$\boxed{\mathcal{K}(\mathcal{Q}^\dagger,\ \mathcal{Q}) = \ln\det\left(1_{N+1} + \mathcal{Q}^\dagger\mathcal{Q}\right),} \tag{4.2}$$

Expression for the components of the metric tensor

$$\mathcal{G}_{pq\ \underline{rs}} = \frac{\partial^2\mathcal{K}(\mathcal{Q}^\dagger,\ \mathcal{Q})}{\partial\mathcal{Q}^{pq}\partial\bar{\mathcal{Q}}^{\underline{rs}}} = \left\{\left(1_{N+1} + \mathcal{Q}\mathcal{Q}^\dagger\right)^{-1}\right\}_{sp}\left\{\left(1_{N+1} + \mathcal{Q}^\dagger\mathcal{Q}\right)^{-1}\right\}_{qr} \tag{4.3}$$
$$-(r \leftrightarrow s) - (p \leftrightarrow q) + (p \leftrightarrow q,\ r \leftrightarrow s).$$

In two/four-dimensional space-time, the simplest representation of $\mathcal{N} = 1$ supersymmetry is a scalar multiplet $\phi = \{\mathcal{Q}, \psi_L, H\}$ where $\mathcal{Q}$ and H are complex scalars and $\psi_L \equiv \frac{1}{2}(1+\gamma_5)\psi$ is a left-handed chiral spinor defined through a Majorana spinor:

$$\boxed{\phi = \mathcal{Q} + \bar{\theta}_R\psi_L + \bar{\theta}_R\theta_L H.} \tag{4.4}$$

General theory of the supersymmetric σ-model can be constructed from the $[N]$ scalar multiplets $\phi^{[\alpha]} = \{\mathcal{Q}^{[\alpha]}, \psi_L^{[\alpha]}, H^{[\alpha]}\}([\alpha] = 1,\ \cdots,\ [N])$.

Supersymmetry transformations :

$$\delta\mathcal{Q}^{[\alpha]} = \bar{\varepsilon}_R\psi_L^{[\alpha]},\ \ \delta\psi_L^{[\alpha]} = \frac{1}{2}(\partial\!\!\!/\mathcal{Q}^{[\alpha]}\varepsilon_R + H^{[\alpha]}\varepsilon_L),\ \ \delta H^{[\alpha]} = \bar{\varepsilon}_L\partial\!\!\!/\psi_L^{[\alpha]}, \tag{4.5}$$

ε : Majorana spinor parameter

Following Zumino [1] and van Holten et al. [2],
Lagrangian of a supersymmetric σ-model : Complex scalar fields : $\mathcal{Q}^{[\alpha]}([\alpha]=1,\cdots,\frac{N(N+1)}{2}\ (=[N]))$, and Spinors $\psi_L^{[\alpha]}$ and $\bar{\psi}_L^{[\alpha]}$:

$$\begin{aligned}\mathcal{L}_{\text{chiral}} = &-\mathcal{G}_{[\alpha][\underline{\beta}]}\left(\partial_\mu \bar{\mathcal{Q}}^{[\underline{\beta}]}\partial_\mu \mathcal{Q}^{[\alpha]} + \bar{\psi}_L^{[\underline{\beta}]}\overleftrightarrow{\not{\boldsymbol{D}}}\psi_L^{[\alpha]}\right)\\ &+W_{;[\alpha][\beta]}\bar{\psi}_R^{[\beta]}\psi_L^{[\alpha]} + \bar{W}_{;[\underline{\alpha}][\underline{\beta}]}\bar{\psi}_L^{[\underline{\beta}]}\psi_R^{[\underline{\alpha}]}\\ &-\mathcal{G}_{[\alpha][\underline{\alpha}]}\bar{W}_{;[\underline{\alpha}]}W_{;[\alpha]} + \frac{1}{2}\boldsymbol{R}_{[\alpha][\underline{\beta}][\gamma][\underline{\delta}]}\bar{\psi}_L^{[\underline{\beta}]}\gamma_\mu\psi_L^{[\alpha]}\bar{\psi}_L^{[\underline{\delta}]}\gamma_\mu\psi_L^{[\gamma]},\end{aligned} \tag{4.6}$$

and a Kähler covariant derivative is $\boldsymbol{D}_\mu\psi_L^{[\alpha]} = \partial_\mu\psi_L^{[\alpha]} + \Gamma^{[\alpha]}_{[\beta][\gamma]}\psi_L^{[\beta]}\partial_\mu\mathcal{Q}^{[\gamma]}$.

Auxiliary fields $H^{[\alpha]}$ are eliminated through their field equations

$$H^{[\alpha]} = \Gamma^{[\alpha]}_{[\beta][\gamma]}\bar{\psi}_R^{[\beta]}\psi_L^{[\gamma]} + \mathcal{G}^{[\alpha][\underline{\alpha}]}\bar{W}_{,[\underline{\alpha}]}. \tag{4.7}$$

Right-handed chiral spinor ψ_R : $\psi_R = C\bar{\psi}_L^{\mathrm{T}}$

5 Expression for SO(2N+2)/U(N+1) Killing potential

$SO(2N+2)$ infinitesimal left transformation of an $SO(2N+2)$ matrix $\mathcal{G}$ to $\mathcal{G}'$:

$$
\begin{aligned}
\mathcal{G}' = (1_{2N+2} + \delta\mathcal{G})\mathcal{G} &= \begin{bmatrix} 1_N + \delta\mathcal{A} & \delta\bar{\mathcal{B}} \\ \delta\mathcal{B} & 1_N + \delta\bar{\mathcal{A}} \end{bmatrix} \mathcal{G} \\
&= \begin{bmatrix} \mathcal{A} + \delta\mathcal{A}\mathcal{A} + \delta\bar{\mathcal{B}}\delta\mathcal{B} & \bar{\mathcal{B}} + \delta\mathcal{A}\bar{\mathcal{B}} + \delta\bar{\mathcal{B}}\bar{\mathcal{A}} \\ \mathcal{B} + \delta\bar{\mathcal{A}}\mathcal{B} + \delta\mathcal{B}\mathcal{A} & \bar{\mathcal{A}} + \delta\bar{\mathcal{A}}\bar{\mathcal{A}} + \delta\mathcal{B}\bar{\mathcal{B}} \end{bmatrix}.
\end{aligned} \tag{5.1}
$$

Define a $\frac{SO(2N+2)}{U(N+1)}$ coset variable $\mathcal{Q}'(=\mathcal{B}'\mathcal{A}'^{-1})$ in the $\mathcal{G}'$ frame.
$\mathcal{Q}'$ is calculated infinitesimally as

$$
\boxed{
\begin{aligned}
\mathcal{Q}' = \mathcal{B}'\mathcal{A}'^{-1} &= \left(\mathcal{B} + \delta\bar{\mathcal{A}}\mathcal{B} + \delta\mathcal{B}\mathcal{A}\right)\left(\mathcal{A} + \delta\mathcal{A}\mathcal{A} + \delta\bar{\mathcal{B}}\delta\mathcal{B}\right)^{-1} \\
&= (\mathcal{Q} + \delta\bar{\mathcal{A}}\mathcal{Q} + \delta\mathcal{B})(1_{N+1} + \delta\mathcal{A} + \delta\bar{\mathcal{B}}\mathcal{Q})^{-1} \\
&= \mathcal{Q} + \delta\mathcal{B} - \mathcal{Q}\delta\mathcal{A} + \delta\bar{\mathcal{A}}\mathcal{Q} - \mathcal{Q}\delta\bar{\mathcal{B}}\mathcal{Q}.
\end{aligned}
} \tag{5.2}
$$

The Kähler metrics admit a set of holomorphic isometries, **Killing vectors**, $\mathcal{R}^{i[\alpha]}(\mathcal{Q})$ and $\bar{\mathcal{R}}^{i[\underline{\alpha}]}(\bar{\mathcal{Q}})$ $(i=1, \cdots, \dim \mathcal{G})$,

$$
\mathcal{R}^i{}_{[\underline{\beta}]}(\mathcal{Q})_{,\,[\alpha]} + \bar{\mathcal{R}}^i{}_{[\alpha]}(\mathcal{Q})_{,\,[\underline{\beta}]} = 0, \quad \mathcal{R}^i{}_{[\underline{\beta}]}(\mathcal{Q}) = \mathcal{G}_{[\alpha][[\underline{\beta}]}\mathcal{R}^{i[\alpha]}(\mathcal{Q}). \tag{5.3}
$$

These isometries define infinitesimal symmetry transformations :
$\delta\mathcal{Q} = \mathcal{Q}' - \mathcal{Q} = \mathcal{R}(\mathcal{Q})$ and $\delta\bar{\mathcal{Q}} = \bar{\mathcal{R}}(\bar{\mathcal{Q}})$ such that $\mathcal{G}'(\mathcal{Q}, \bar{\mathcal{Q}}) = \mathcal{G}(\mathcal{Q}, \bar{\mathcal{Q}})$.
Killing equation (5.3) is the necessary and sufficient condition for an infinitesimal co-ordinate transformation

$$
\boxed{
\begin{aligned}
&\delta\mathcal{Q}^{[\alpha]} = \left(\delta\mathcal{B} - \delta\mathcal{A}^{\mathrm{T}}\mathcal{Q} - \mathcal{Q}\delta\mathcal{A} + \mathcal{Q}\delta\mathcal{B}^{\dagger}\mathcal{Q}\right)^{[\alpha]} = \xi_i\mathcal{R}^{i[\alpha]}(\mathcal{Q}), \\
&\delta\bar{\mathcal{Q}}^{[\underline{\alpha}]} = \xi_i\bar{\mathcal{R}}^{i[\underline{\alpha}]}(\bar{\mathcal{Q}}).
\end{aligned}
} \tag{5.4}
$$

ξ_i : infinitesimal and global group parameter. Due to the Killing equation, the Killing vectors $\mathcal{R}^{i[\alpha]}(\mathcal{Q})$ and $\bar{\mathcal{R}}^{i[\underline{\alpha}]}(\bar{\mathcal{Q}})$ can be written locally as the gradient of some real scalar function, the Killing potentials $\mathcal{M}^i(\mathcal{Q}, \bar{\mathcal{Q}})$ such that

$$\boxed{\mathcal{R}^i{}_{\underline{[\alpha]}}(\mathcal{Q}) = -i\mathcal{M}^i{}_{,\underline{[\alpha]}}, \quad \bar{\mathcal{R}}^i{}_{[\alpha]}(\bar{\mathcal{Q}}) = i\mathcal{M}^i{}_{,[\alpha]}.} \tag{5.5}$$

According to van Holten et al. and using the infinitesimal $SO(2N+2)$ matrix $\delta\mathcal{G}$ (A.3), **the Killing potential $\mathcal{M}_\sigma$** :

$$\left.\begin{array}{l}
\mathcal{M}_\sigma\!\left(\delta\mathcal{A}, \delta\mathcal{B}, \delta\mathcal{B}^\dagger\right) = \mathrm{Tr}\!\left(\delta\mathcal{G}\widetilde{\mathcal{M}}_\sigma\right) = \mathrm{tr}\!\left(\delta\mathcal{A}\mathcal{M}_{\sigma\delta\mathcal{A}} + \delta\mathcal{B}\mathcal{M}_{\sigma\delta\mathcal{B}^\dagger} + \delta\mathcal{B}^\dagger\mathcal{M}_{\sigma\delta\mathcal{B}}\right), \\[2ex]
\widetilde{\mathcal{M}}_\sigma \equiv \begin{bmatrix} \widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}} & \widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}^\dagger} \\[2ex] -\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}} & -\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}^{\mathrm{T}}} \end{bmatrix}, \quad
\begin{array}{l} \mathcal{M}_{\sigma\delta\mathcal{A}} = \widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}} + \left(\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}^{\mathrm{T}}}\right)^{\mathrm{T}}, \\[2ex] \mathcal{M}_{\sigma\delta\mathcal{B}} = \widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}}, \quad \mathcal{M}_{\sigma\delta\mathcal{B}^\dagger} = \widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}^\dagger}. \end{array}
\end{array}\right\} \tag{5.6}$$

Introduce $(N+1)$-dimensional matrices $\mathcal{R}(\mathcal{Q};\delta\mathcal{G})$, $\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})$ and $\mathcal{X}$:

$$\left.\begin{array}{l}
\mathcal{R}(\mathcal{Q};\delta\mathcal{G}) = \delta\mathcal{B} - \delta\mathcal{A}^{\mathrm{T}}\mathcal{Q} - \mathcal{Q}\delta\mathcal{A} + \mathcal{Q}\delta\mathcal{B}^\dagger\mathcal{Q}, \quad \mathcal{R}_T(\mathcal{Q};\delta\mathcal{G}) = -\delta\mathcal{A}^{\mathrm{T}} + \mathcal{Q}\delta\mathcal{B}^\dagger, \\[2ex]
\mathcal{X} = (1_{N+1} + \mathcal{Q}\mathcal{Q}^\dagger)^{-1} = \mathcal{X}^\dagger.
\end{array}\right\} \tag{5.7}$$

Killing potential $\mathcal{M}_\sigma$:

$$\left.\begin{array}{rl}
-i\mathcal{M}_\sigma\left(\mathcal{Q}, \bar{\mathcal{Q}}; \delta\mathcal{G}\right) & = -\mathrm{tr}\Delta\left(\mathcal{Q}, \bar{\mathcal{Q}}; \delta\mathcal{G}\right), \\[2ex]
\Delta\left(\mathcal{Q}, \bar{\mathcal{Q}}; \delta\mathcal{G}\right) & \equiv \mathcal{R}_T(\mathcal{Q};\delta\mathcal{G}) - \mathcal{R}(\mathcal{Q};\delta\mathcal{G})\mathcal{Q}^\dagger\mathcal{X} \\[2ex]
& = \left(\mathcal{Q}\delta\mathcal{A}\mathcal{Q}^\dagger - \delta\mathcal{A}^{\mathrm{T}} - \delta\mathcal{B}\mathcal{Q}^\dagger + \mathcal{Q}\delta\mathcal{B}^\dagger\right)\mathcal{X}.
\end{array}\right\} \tag{5.8}$$

$$\boxed{-i\mathcal{M}_{\sigma\delta\mathcal{B}} = -\mathcal{X}\mathcal{Q}, \quad -i\mathcal{M}_{\sigma\delta\mathcal{B}^\dagger} = \mathcal{Q}^\dagger\mathcal{X}, \quad -i\mathcal{M}_{\sigma\delta\mathcal{A}} = 1_{N+1} - 2\mathcal{Q}^\dagger\mathcal{X}\mathcal{Q}.} \tag{5.9}$$

Their components :

$$\boxed{\left.\begin{array}{l} -i\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}} = -\mathcal{X}\mathcal{Q}, \;\; -i\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{B}^\dagger} = \mathcal{Q}^\dagger\mathcal{X}, \\ \\ -i\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}} = -\mathcal{Q}^\dagger\mathcal{X}\mathcal{Q}, \;\; -i\widetilde{\mathcal{M}}_{\sigma\delta\mathcal{A}^{\mathrm{T}}} = \mathcal{Q}. \end{array}\right\}} \qquad (5.10)$$

Introduction of a $(2N+2) \times (N+1)$ **isometric matrix $\mathcal{U}$** by

$$\boxed{\mathcal{U}^{\mathrm{T}} = \left[\, \mathcal{B}^{\mathrm{T}}, \; \mathcal{A}^{\mathrm{T}} \,\right], \quad (\mathcal{U}^\dagger\mathcal{U} = 1_{N+1}).} \qquad (5.11)$$

To make clear meaning of the Killing potential, introduce a $(2N+2)\times(2N+2)$ matrix:

$$\boxed{\mathcal{W} = \mathcal{U}\mathcal{U}^\dagger = \begin{bmatrix} \mathcal{R} & \mathcal{K} \\ \\ -\bar{\mathcal{K}} & 1_{N+1} - \bar{\mathcal{R}} \end{bmatrix} = \mathcal{W}^\dagger, \quad \begin{array}{l} \mathcal{R} = \mathcal{B}\mathcal{B}^\dagger, \\ \\ \mathcal{K} = \mathcal{B}\mathcal{A}^\dagger, \end{array}} \qquad (5.12)$$

which satisfies **the idempotency relation**: $\boxed{\mathcal{W}^2 = \mathcal{W}.}$

The matrix $\mathcal{W}$ is a natural extension of the generalized density matrix in the $SO(2N)$ CS rep to the $SO(2N+2)$ CS rep.

$$\mathcal{A} = (1_{N+1} + \mathcal{Q}^\dagger\mathcal{Q})^{-\frac{1}{2}}\,\overset{\circ}{\mathcal{U}}, \;\; \mathcal{B} = \mathcal{Q}(1_{N+1} + \mathcal{Q}^\dagger\mathcal{Q})^{-\frac{1}{2}}\,\overset{\circ}{\mathcal{U}}, \;\; \overset{\circ}{\mathcal{U}} \in U(N+1), \quad (5.13)$$

The Killing potential $-i\widetilde{\mathcal{M}}_\sigma$ is given by the generalized density matrix as

$$\boxed{-i\widetilde{\mathcal{M}}_\sigma = \begin{bmatrix} -\bar{\mathcal{R}} & -\bar{\mathcal{K}} \\ \\ \mathcal{K} & -(1_{N+1} - \mathcal{R}) \end{bmatrix} \rightarrow -i\overline{\widetilde{\mathcal{M}}}_\sigma = \begin{bmatrix} \mathcal{R} & \mathcal{K} \\ \\ -\bar{\mathcal{K}} & 1_{N+1} - \bar{\mathcal{R}} \end{bmatrix}.} \qquad (5.14)$$

To our great surprise, the expression for the Killing potential just becomes equivalent with the generalized density matrix.

The inverse matrix $\mathcal{X}$ leads to the form

$$\boxed{\mathcal{X} = \begin{bmatrix} \mathcal{Q}_{qq^\dagger} & \mathcal{Q}_{qr} \\ \mathcal{Q}^\dagger_{qr} & \mathcal{Q}_{r^\dagger r} \end{bmatrix}, \quad \chi = (1_N + qq^\dagger)^{-1} = \chi^\dagger,} \tag{5.15}$$

$$\boxed{\mathcal{Q}_{qq^\dagger} = \chi - \frac{1+z}{2}\chi(rr^\dagger - q\bar{r}r^{\mathrm{T}}q^\dagger)\chi, \;\; \mathcal{Q}_{q\bar{r}} = \frac{1+z}{2}\chi q\bar{r}, \;\; \mathcal{Q}_{r^\dagger r} = \frac{1+z}{2}.} \tag{5.16}$$

Introduction of **auxiliary function**:

$\boxed{\lambda = rr^\dagger - q\bar{r}r^{\mathrm{T}}q^\dagger}$,

Killing potential $\mathcal{M}_\sigma$ expressed in terms of $q,\ r$ and $1+z = 2Z^2$ as,

$$-i\mathcal{M}_{\sigma\delta\mathcal{B}} = \begin{bmatrix} -\chi q + Z^2\left(\chi\lambda\chi q + \chi q\bar{r}r^{\mathrm{T}}\right) & -\chi r + Z^2\chi\lambda\chi r \\ -Z^2\left(r^{\mathrm{T}}q^\dagger\chi q - r^{\mathrm{T}}\right) & -Z^2 r^{\mathrm{T}}q^\dagger\chi r \end{bmatrix}, \tag{5.17}$$

$$\boxed{\begin{aligned} &-i\mathcal{M}_{\sigma\delta\mathcal{A}} = \\ &\begin{bmatrix} 1_N - 2q^\dagger\chi q + 2Z^2\left(q^\dagger\chi\lambda\chi q + q^\dagger\chi q\bar{r}r^{\mathrm{T}} + \bar{r}r^{\mathrm{T}}q^\dagger\chi q - \bar{r}r^{\mathrm{T}}\right) & -2q^\dagger\chi r + 2Z^2\left(q^\dagger\chi\lambda\chi r + \bar{r}r^{\mathrm{T}}q^\dagger\chi r\right) \\ -2r^\dagger\chi q + 2Z^2\left(r^\dagger\chi\lambda\chi q + r^\dagger\chi q\bar{r}r^{\mathrm{T}}\right) & 1 - 2r^\dagger\chi r + 2Z^2 r^\dagger\chi\lambda\chi r \end{bmatrix}. \end{aligned}} \tag{5.18}$$

Identities and relations:

$$r^{\mathrm{T}}q^\dagger\chi r = 0, \quad r^\dagger\chi q\bar{r} = 0, \quad r^\dagger\chi r = \frac{1-Z^2}{Z^2}, \quad r^\dagger\chi\lambda\chi r = \left(\frac{1-Z^2}{Z^2}\right)^2, \tag{5.19}$$

$$1 - 2r^\dagger\chi r + 2Z^2 r^\dagger\chi\lambda\chi r = 2Z^2 - 1, \tag{5.20}$$

$$\chi\lambda\chi r = \frac{1-Z^2}{Z^2}\chi r, \quad r^\dagger\chi\lambda\chi = \frac{1-Z^2}{Z^2}r^\dagger\chi, \quad q^\dagger\chi q = 1_N - \bar{\chi}. \tag{5.21}$$

We get compact forms of the Killing potential $\mathcal{M}_\sigma$ as,

$$-i\mathcal{M}_{\substack{\sigma\delta\mathcal{B}\\ \left(\sigma\delta\mathcal{B}^\dagger\right)}} = \begin{bmatrix} \begin{array}{c} -\chi q + Z^2\left(\chi r r^\dagger \chi q + \chi q \bar{r} r^{\mathrm{T}} \bar{\chi}\right) \\ \left(q^\dagger \chi - Z^2\left(q^\dagger \chi r r^\dagger \chi + \bar{\chi}\bar{r} r^{\mathrm{T}} q^\dagger \chi\right)\right) \end{array} & \begin{array}{c} -Z^2\chi r \\ \left(-Z^2 \bar{\chi}\bar{r}\right) \end{array} \\ & \\ \begin{array}{c} Z^2 r^{\mathrm{T}} \bar{\chi} \\ \left(Z^2 r^\dagger \chi\right) \end{array} & \begin{array}{c} 0 \\ (0) \end{array} \end{bmatrix}, \quad (5.22)$$

$$\boxed{-i\mathcal{M}_{\sigma\delta\mathcal{A}} = \begin{bmatrix} 1_N - 2q^\dagger \chi q + 2Z^2\left(q^\dagger \chi r r^\dagger \chi q - \bar{\chi}\bar{r} r^{\mathrm{T}} \bar{\chi}\right) & -2Z^2 q^\dagger \chi r \\ & \\ -2Z^2 r^\dagger \chi q & 2Z^2 - 1 \end{bmatrix}.} \quad (5.23)$$

Introduction of a gauge covarint derivative:

$$\left.\begin{aligned} \boldsymbol{D}_\mu \mathcal{Q}^{[\alpha]} &= \partial_\mu \mathcal{Q}^{[\alpha]} - g_i A^i{}_\mu \mathcal{R}^{i[\alpha]}(\mathcal{Q}), \\ \boldsymbol{D}_\mu \psi_L^{[\alpha]} &= \partial_\mu \psi_L^{[\alpha]} - g_i A^i{}_\mu \mathcal{R}^{i[\alpha]}{}_{,[\beta]}(\mathcal{Q})\psi_L^{[\beta]} + \Gamma^{[\alpha]}_{[\beta][\gamma]}\psi_L^{[\beta]}\partial_\mu \mathcal{Q}^{[\gamma]}. \end{aligned}\right\} \quad (5.24)$$

Introduction of gauge fields in Lagrangian, via the gauge covarint derivatives, the σ-model is no longer invariant under the supersymmetry transformations.

To restore the supersymmetry, it is necessary to add the terms

$$\begin{aligned} \Delta\mathcal{L}_{\text{chiral}} = {} & 2\mathcal{G}_{[\alpha][\underline{\alpha}]}\left(\mathcal{R}^i{}_{[\underline{\alpha}]}(\mathcal{Q})\bar{\psi}_L^{[\underline{\alpha}]}\lambda_R^i + \bar{\mathcal{R}}^i{}_{[\alpha]}(\mathcal{Q})\bar{\lambda}_R^i \psi_L^{[\alpha]}\right) \\ & - g_i \mathrm{tr}\left\{D^i(\mathcal{M}^i + \xi^i)\right\}, \end{aligned} \quad (5.25)$$

where ξ_i **are Fayet-Ilipoulos parameters :**

Full Lagrangian :

$$\boxed{\mathcal{L} = -\mathrm{tr}\left\{\tfrac{1}{4}\mathcal{F}^i_{\mu\nu}\mathcal{F}^i_{\mu\nu} + \tfrac{1}{2}\bar{\lambda}^i \not{D}\lambda^i - \tfrac{1}{2}D^i D^i\right\} + \mathcal{L}_{\text{chiral}}(\partial_\mu \to D_\mu) + \Delta\mathcal{L}_{\text{chiral}}.} \quad (5.26)$$

Eliminating the auxiliary field D^i by $D^i = -g_i(\mathcal{M}^i + \xi^i)$,

Scalar potential:

$$\boxed{V_{\rm SC} = -\frac{1}{2} g_i^2 \mathrm{tr}\left\{(\mathcal{M}^i + \xi^i)^2\right\}.} \tag{5.27}$$

A reduced scaler potential arising from the gauging of $SU(N+1) \times U(1)$ including a Fayet-Ilipoulos term with parameter ξ is of the special interest:

$$\boxed{V_{\rm redSC} = \frac{g_{U(1)}^2}{2(N+1)} \left(\xi - i\mathcal{M}_Y\right)^2 + \frac{g_{SU(N+1)}^2}{2} \mathrm{tr}\left(-i\mathcal{M}_t\right)^2.} \tag{5.28}$$

New quantities $\mathrm{tr}\left(-i\mathcal{M}_t\right)^2$ and $-i\mathcal{M}_Y$ are defined below.

$$\left.\begin{aligned}
\mathrm{tr}(-i\mathcal{M}_t)^2 =& \mathrm{tr}(-i\mathcal{M}_{\sigma\delta\mathcal{A}})^2 - \frac{1}{N+1}(-i\mathcal{M}_Y)^2, \\
-i\mathcal{M}_Y =& \mathrm{tr}\left(-i\mathcal{M}_{\sigma\delta\mathcal{A}}\right), \\
\mathrm{tr}\left(-i\mathcal{M}_{\sigma\delta\mathcal{A}}\right) =& -N + 2\mathbf{tr}(\chi) + 2Z^2\mathbf{tr}(\chi r r^\dagger) - 4Z^2\mathbf{tr}(\chi r r^\dagger \chi) \\
& +2Z^2 - 1, \\
\mathrm{tr}\left(-i\mathcal{M}_{\sigma\delta\mathcal{A}}\right)^2 =& N - 4\mathbf{tr}(\chi) + 4\mathbf{tr}(\chi\chi) + 12Z^2\mathbf{tr}(\chi r r^\dagger \chi) \\
& -16Z^2\mathbf{tr}(\chi\chi r r^\dagger \chi) \\
& -4Z^4 r^\dagger \chi\chi r \cdot \mathbf{tr}(\chi r r^\dagger) + 8Z^4 r^\dagger \chi\chi r \cdot \mathbf{tr}(\chi r r^\dagger \chi) \\
& +1 - 4Z^4 r^\dagger \chi\chi r,
\end{aligned}\right\} \tag{5.29}$$

Calculate approximately the quantities $r^\dagger \chi\chi r$ and $\mathbf{tr}(rr^\dagger)$ as

$$r^\dagger \chi\chi r = \frac{1}{4Z^4}(x^\dagger + x^{\mathrm{T}} q^\dagger)\chi\chi(x + q\bar{x}) = \frac{1}{4Z^4} x^\dagger \chi x$$

$$\approx \frac{1}{4Z^4}\left\{\frac{1}{N}(N + \mathbf{tr}(q^\dagger q))\right\}^{-1} x^\dagger x = \frac{1-Z^2}{Z^2}\left\{\frac{1}{N}(N + <q^\dagger q>)\right\}^{-1} \tag{5.30}$$

$$\equiv \frac{1-Z^2}{Z^2} <\chi>,$$

$$\begin{aligned} \mathbf{tr}(rr^\dagger) =& r^\dagger r = \frac{1}{4Z^4}(x^\dagger + x^{\mathrm{T}} q^\dagger)(x + q\bar{x}) = \frac{1}{4Z^4} x^\dagger \chi^{-1} x \\ \approx& \frac{1-Z^2}{Z^2}\frac{1}{<\chi>} \equiv <rr^\dagger> . \end{aligned} \tag{5.31}$$

Approximating $\mathbf{tr}(\chi rr^\dagger)$ as $<\chi>\mathbf{tr}(rr^\dagger)$, $\mathrm{tr}\,(-i\mathcal{M}_{\sigma\delta\mathcal{A}})$ and $\mathrm{tr}\,(-i\mathcal{M}_{\sigma\delta\mathcal{A}})^2$ are computed as

$$\left.\begin{aligned} \mathrm{tr}\,(-i\mathcal{M}_{\sigma\delta\mathcal{A}}) =& 1 - N + 2(2Z^2-1)<\chi>, \\ \mathrm{tr}\,(-i\mathcal{M}_{\sigma\delta\mathcal{A}})^2 =& 1 + N - 4(2Z^2-1)<\chi>+4(2Z^4-1)<\chi>^2 . \end{aligned}\right\} \tag{5.32}$$

Final form of the reduced scalar potential:

$$\boxed{\begin{aligned} V_{\mathrm{redSC}} =& \frac{g^2_{U(1)}}{2(N+1)}\left\{\xi + 1 - N + 2(2Z^2-1)<\chi>\right\}^2 \\ &+2\frac{g^2_{SU(N+1)}}{N+1}\Big[N - 2(2Z^2-1)<\chi> \\ &+\left\{2(N-1)Z^4 + 4Z^2 - (N+2)\right\}<\chi>^2\Big] . \end{aligned}} \tag{5.33}$$

To see behaviour of the vacuum expectation value of the σ-fields, it is very important to analyze the form of the reduced scaler potential.

Variation of the reduced scalar potential with respect to Z and $<\chi>$:

$$\boxed{\begin{aligned} & g^2_{U(1)}\left\{\xi+1-N+2(2Z^2-1)<\chi>\right\} \\ & -2g^2_{SU(N+1)}\left\{1-((N-1)Z^2+1)<\chi>\right\}=0, \end{aligned}} \tag{5.34}$$

$$\boxed{\begin{aligned} & g^2_{U(1)}\left\{\xi+1-N+2(2Z^2-1)<\chi>\right\}(2Z^2-1) \\ & -2g^2_{SU(N+1)}\left[2Z^2-1-\left\{2(N-1)Z^4+4Z^2-(N+2)\right\}<\chi>\right]=0. \end{aligned}} \tag{5.35}$$

g^2-independent relation:

$$\boxed{\begin{aligned} & \left\{1-((N-1)Z^2+1)<\chi>\right\}(2Z^2-1) \\ & -\left[2Z^2-1-\left\{2(N-1)Z^4+4Z^2-(N+2)\right\}<\chi>\right]=0, \end{aligned}} \tag{5.36}$$

which reads

$$\boxed{(N+1)(Z^2-1)<\chi>=0 \quad \longrightarrow \quad Z^2=1.} \tag{5.37}$$

To find proper solutions for the extended supersymmetric σ-model, rescaling Goldstone fields $\mathcal{Q}$ by mass parameter, introduce $(N+1)$-dimensional matrices $\mathcal{R}_f(\mathcal{Q}_f;\delta\mathcal{G})$, $\mathcal{R}_{fT}(\mathcal{Q}_f;\delta\mathcal{G})$ and $\mathcal{X}_f$:

$$\boxed{\left.\begin{aligned}
&\mathcal{R}_f(\mathcal{Q}_f;\delta\mathcal{G})=\frac{1}{f}\delta\mathcal{B}-\delta\mathcal{A}^{\mathrm{T}}\mathcal{Q}_f-\mathcal{Q}_f\delta\mathcal{A}+f\mathcal{Q}_f\delta\mathcal{B}^{\dagger}\mathcal{Q}_f,\\
&\mathcal{R}_{fT}(\mathcal{Q}_f;\delta\mathcal{G})=-\delta\mathcal{A}^{\mathrm{T}}+f\mathcal{Q}_f\delta\mathcal{B}^{\dagger},\\
&\mathcal{X}_f=(1_{N+1}+f^2\mathcal{Q}_f\mathcal{Q}_f^{\dagger})^{-1}=\mathcal{X}^{\dagger},\ \mathcal{Q}_f=\begin{bmatrix} q & \frac{1}{f}r_f\\ -\frac{1}{f}r_f^{\mathrm{T}} & 0\end{bmatrix},\\
&r_f=\frac{1}{2Z^2}\left(x+fq\bar{x}\right),\ f\equiv\frac{1}{m_\sigma}.
\end{aligned}\right\}} \tag{5.38}$$

Due to the rescaling, the Killing potential $\mathcal{M}_\sigma$ is deformed as

$$\left.\begin{aligned}
-i\mathcal{M}_{f\sigma}\left(\mathcal{Q}_f,\bar{\mathcal{Q}}_f;\delta\mathcal{G}\right)&=-\mathrm{tr}\Delta_f\left(\mathcal{Q}_f,\bar{\mathcal{Q}}_f;\delta\mathcal{G}\right),\\
\Delta_f\left(\mathcal{Q}_f,\bar{\mathcal{Q}}_f;\delta\mathcal{G}\right)&\equiv\mathcal{R}_{fT}(\mathcal{Q}_f;\delta\mathcal{G})-\mathcal{R}_f(\mathcal{Q}_f;\delta\mathcal{G})f^2\mathcal{Q}_f^{\dagger}\mathcal{X}_f\\
&=\left(f^2\mathcal{Q}_f\delta\mathcal{A}\mathcal{Q}_f^{\dagger}-\delta\mathcal{A}^{\mathrm{T}}-f\delta\mathcal{B}\mathcal{Q}_f^{\dagger}+f\mathcal{Q}_f\delta\mathcal{B}^{\dagger}\right)\mathcal{X}_f.
\end{aligned}\right\} \tag{5.39}$$

A f-deformed Killing potential $\mathcal{M}_{f\sigma}$:

$$\boxed{\left.\begin{aligned}
&-i\mathcal{M}_{f\sigma\delta\mathcal{B}}=-f\mathcal{X}_f\mathcal{Q}_f,\quad -i\mathcal{M}_{f\sigma\delta\mathcal{B}^{\dagger}}=f\mathcal{Q}_f^{\dagger}\mathcal{X}_f,\\
&-i\mathcal{M}_{f\sigma\delta\mathcal{A}}=1_{N+1}-2f^2\mathcal{Q}_f^{\dagger}\mathcal{X}_f\mathcal{Q}_f.
\end{aligned}\right\}} \tag{5.40}$$

The **inverse matrix** $\mathcal{X}_f$ leads to

$$\boxed{\mathcal{X}_f = \begin{bmatrix} \mathcal{Q}_{fqq^\dagger} & \mathcal{Q}_{fqr} \\ \mathcal{Q}^\dagger_{fqr} & \mathcal{Q}_{fr^\dagger r} \end{bmatrix}, \quad \chi_f = (1_N + f^2 qq^\dagger)^{-1} = \chi_f^\dagger,} \tag{5.41}$$

$$\mathcal{Q}_{fqq^\dagger} = \chi_f - Z^2\chi_f(r_f r_f^\dagger - f^2 q\bar{r}_f r_f^{\mathrm{T}} q^\dagger)\chi_f, \tag{5.42}$$

$$\mathcal{Q}_{q\bar{r}} = fZ^2\chi_f q\bar{r}_f, \quad \mathcal{Q}_{r^\dagger r} = Z^2. \tag{5.43}$$

Introduce **f-deformed auxiliary function:** $\boxed{\lambda_f = rr^\dagger - f^2 q\bar{r}r^{\mathrm{T}}q^\dagger = \lambda_f^\dagger}$,

$$\boxed{\begin{gathered} -i\mathcal{M}_{f\sigma\delta\mathcal{A}} = \\ \begin{bmatrix} \begin{array}{l} 1_N - 2q^\dagger\chi_f q + 2Z^2\big(q^\dagger\chi_f\lambda_f\chi_f q + q^\dagger\chi_f q\bar{r}_f r_f^{\mathrm{T}} \\ \quad + \bar{r}_f r_f^{\mathrm{T}} q^\dagger\chi_f q - \frac{1}{f^2}\bar{r}_f r_f^{\mathrm{T}}\big) \end{array} & \begin{array}{l} -2\frac{1}{f}q^\dagger\chi_f r_f + 2\frac{1}{f}Z^2\big(q^\dagger\chi_f\lambda_f\chi_f r_f \\ \quad + \bar{r}_f r_f^{\mathrm{T}} q^\dagger\chi_f r_f\big) \end{array} \\ \begin{array}{l} -2\frac{1}{f}r_f^\dagger\chi_f q + 2\frac{1}{f}Z^2\big(r_f^\dagger\chi_f\lambda_f\chi_f q \\ \quad + r_f^\dagger\chi_f q\bar{r}_f r_f^{\mathrm{T}}\big) \end{array} & 1 - 2\frac{1}{f^2}r_f^\dagger\chi_f r_f + 2\frac{1}{f^2}Z^2 r_f^\dagger\chi_f\lambda_f\chi_f r_f \end{bmatrix} \end{gathered}} . \tag{5.44}$$

Identities and relations:

$$\left.\begin{aligned} & r_f^{\mathrm{T}} q^\dagger\chi_f r_f = 0, \quad r_f^\dagger\chi_f q\bar{r}_f = 0, \\ & r_f^\dagger\chi_f r_f = \frac{1-Z^2}{Z^2}, \quad r_f^\dagger\chi_f\lambda_f\chi_f r_f = \left(\frac{1-Z^2}{Z^2}\right)^2, \end{aligned}\right\} \tag{5.45}$$

$$1 - 2\frac{1}{f^2}r_f^\dagger\chi_f r_f + 2\frac{1}{f^2}Z^2 r_f^\dagger\chi_f\lambda_f\chi_f r_f = \frac{1}{f^2}(2Z^2-1) + 1 - \frac{1}{f^2}, \tag{5.46}$$

$$\chi_f\lambda_f\chi_f r_f = \frac{1-Z^2}{Z^2}\chi_f r_f, \; r_f^\dagger\chi_f\lambda_f\chi_f = \frac{1-Z^2}{Z^2}r_f^\dagger\chi_f, \; q^\dagger\chi_f q = \frac{1}{f^2}(1_N - \bar{\chi}_f). \tag{5.47}$$

Compact form of **the f-deformed Killing potential** $\mathcal{M}_{f\sigma\delta\mathcal{A}}$:

$$\boxed{\begin{aligned}&-i\mathcal{M}_{f\sigma\delta\mathcal{A}}\\&=\begin{bmatrix}1_N-2q^\dagger\chi_f q+2Z^2\left(q^\dagger\chi_f r_f r_f^\dagger\chi_f q-\frac{1}{f^2}\bar{\chi}_f\bar{r}_f r_f^{\mathrm{T}}\bar{\chi}_f\right) & -2\frac{1}{f}Z^2 q^\dagger\chi_f r_f\\ -2\frac{1}{f}Z^2 r_f^\dagger\chi_f q & \frac{1}{f^2}(2Z^2-1)+1-\frac{1}{f^2}\end{bmatrix}.\end{aligned}} \tag{5.48}$$

A f-deformed reduced scalar potential:

$$\boxed{\left.\begin{aligned}&V_{f_{\mathrm{redSC}}}=\frac{g^2_{U(1)}}{2(N+1)}\left(\xi-i\mathcal{M}_{fY}\right)^2+\frac{g^2_{SU(N+1)}}{2}\mathrm{tr}\left(-i\mathcal{M}_{ft}\right)^2,\\&\mathrm{tr}(-i\mathcal{M}_{ft})^2=\mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})^2-\frac{1}{N+1}(-i\mathcal{M}_{fY})^2,\\&-i\mathcal{M}_{fY}=\mathrm{tr}\left(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}\right),\end{aligned}\right\}} \tag{5.49}$$

$$\begin{aligned}\mathrm{tr}\left(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}\right)=&-N+2\mathbf{tr}(\chi_f)+2\frac{1}{f^2}Z^2\mathbf{tr}(\chi_f r_f r_f^\dagger)\\&-4\frac{1}{f^2}Z^2\mathbf{tr}(\chi_f r_f r_f^\dagger\chi_f)+\frac{1}{f^2}(2Z^2-1)+1-\frac{1}{f^2},\end{aligned} \tag{5.50}$$

$$
\mathrm{tr}\,(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})^2 = N - 4\frac{1}{f^2}(1-\frac{1}{f^2})N - 4\frac{1}{f^4}\mathbf{tr}(\chi_f) + 4\frac{1}{f^4}\mathbf{tr}(\chi_f\chi_f)
$$

$$
+4\frac{1}{f^2}(1-\frac{1}{f^2})Z^2\mathbf{tr}(\chi_f r_f r_f^\dagger) + 12\frac{1}{f^4}Z^2\mathbf{tr}(\chi_f r_f r_f^\dagger \chi_f)
$$

$$
-16\frac{1}{f^4}Z^2\mathbf{tr}(\chi_f\chi_f r_f r_f^\dagger\chi_f) \tag{5.51}
$$

$$
-4\frac{1}{f^4}Z^4 r_f^\dagger\chi_f\chi_f r_f\cdot\mathbf{tr}(\chi_f r_f r_f^\dagger) + 8\frac{1}{f^4}Z^4 r_f^\dagger\chi_f\chi_f r_f\cdot\mathbf{tr}(\chi_f r_f r_f^\dagger\chi_f)
$$

$$
+\frac{1}{f^4} + 2(1-\frac{1}{f^2})(2Z^2-1) + (1-\frac{1}{f^2})^2 - 4\frac{1}{f^4}Z^4 r_f^\dagger\chi_f\chi_f r_f.
$$

Identity :

$$
r_f^\dagger\chi_f r_f = \frac{1}{4Z^4}(x^\dagger\chi_f x + x^{\mathrm{T}}q^\dagger\chi_f q\bar{x}) = \frac{1}{4Z^4}x^{\mathrm{T}}\bar{x} = \frac{1-Z^2}{Z^2}, \tag{5.52}
$$

Approximate formulas for the quantities $r_f^\dagger\chi_f\chi_f r_f$ and $\mathbf{tr}(r_f r_f^\dagger)$:

$$
\begin{aligned}
r_f^\dagger\chi_f\chi_f r_f =& \frac{1}{4Z^4}(x^\dagger + f x^{\mathrm{T}}q^\dagger)\chi_f\chi_f(x+fq\bar{x}) = \frac{1}{4Z^4}x^\dagger\chi_f x \\
\approx& \frac{1}{4Z^4}\left\{\frac{1}{N}(N+f^2\mathbf{tr}(q^\dagger q))\right\}^{-1}x^\dagger x \equiv \frac{1-Z^2}{Z^2} <\chi_f>,
\end{aligned} \tag{5.53}
$$

$$
\begin{aligned}
\mathbf{tr}(r_f r_f^\dagger) =& r_f^\dagger r_f = \frac{1}{4Z^4}(x^\dagger + f x^{\mathrm{T}}q^\dagger)(x+fq\bar{x}) = \frac{1}{4Z^4}x^\dagger\chi_f^{-1}x \\
\approx& \frac{1-Z^2}{Z^2}\frac{1}{<\chi_f>},
\end{aligned} \tag{5.54}
$$

$\mathbf{tr}(\chi_f r_f r_f^\dagger) \approx <\chi_f> \mathbf{tr}(r_f r_f^\dagger)$, $\mathrm{tr}\,(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})$ and $\mathrm{tr}\,(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})^2$ are given as

$$\left.\begin{aligned}
\mathrm{tr}\,(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}) =& 1-N+2\left\{\frac{1}{f^2}(2Z^2-1)+1-\frac{1}{f^2}\right\}<\chi_f>, \\
\mathrm{tr}\,(-i\mathcal{M}_{\sigma\delta\mathcal{A}})^2 =& 1+N-4\frac{1}{f^2}\left(1-\frac{1}{f^2}\right)N+2\left(1-\frac{1}{f^2}\right)^2(2Z^2-1) \\
& -4\frac{1}{f^2}\left\{\frac{1}{f^2}(2Z^2-1)-\left(1-\frac{1}{f^2}\right)\right\}<\chi_f> \\
& +4\frac{1}{f^4}(2Z^4-1)<\chi_f>^2 .
\end{aligned}\right\} \quad (5.55)$$

Final form of f-deformed reduced scalar potential:

$$\boxed{\begin{aligned}
V_{f_{\mathrm{redSC}}} =& \frac{g_{U(1)}^2}{2(N+1)}\left[\xi+1-N+2\left\{\frac{1}{f^2}(2Z^2-1)+1-\frac{1}{f^2}\right\}<\chi_f>\right]^2 \\
& +2\frac{g_{SU(N+1)}^2}{N+1}\left[N-\frac{1}{f^2}\left\{1+\frac{1}{f^2}-\left(1-\frac{1}{f^2}\right)N\right\}(2Z^2-1)<\chi_f>\right. \\
& +\frac{1}{f^4}\left\{2(N-1)Z^4+4Z^2-(N+2)\right\}<\chi_f>^2 \\
& -\frac{1}{f^2}\left(1-\frac{1}{f^2}\right)N(N+1)+\frac{1}{2}\left(1-\frac{1}{f^2}\right)^2(N+1)(2Z^2-1) \\
& -\left(1-\frac{1}{f^2}\right)\left\{1-\frac{1}{f^2}-\left(1+\frac{1}{f^2}\right)N\right\}<\chi_f> \\
& \left.-\left(1-\frac{1}{f^2}\right)\left\{2\frac{1}{f^2}(2Z^2-1)+1-\frac{1}{f^2}\right\}<\chi_f>^2\right].
\end{aligned}} \quad (5.56)$$

Variation of f-deformed reduced scalar potential with respect to Z and $<\chi_f>$,

$$\boxed{\begin{aligned}&g_{U(1)}^2\left[\xi+1-N+2\left\{\frac{1}{f^2}(2Z^2-1)+1-\frac{1}{f^2}\right\}<\chi_f>\right]\\&-2g_{SU(N+1)}^2\left[\frac{1}{2}\left\{1+\frac{1}{f^2}-\left(1-\frac{1}{f^2}\right)N\right\}\right.\\&-\frac{1}{f^2}\left\{(N-1)Z^2+1\right\}<\chi_f>\\&\left.-\left(1-\frac{1}{f^2}\right)<\chi_f>+\frac{1}{4}f^2\left(1-\frac{1}{f^2}\right)^2(N+1)\frac{1}{<\chi_f>}\right]=0,\end{aligned}} \tag{5.57}$$

$$\boxed{\begin{aligned}&g_{U(1)}^2\left[\xi+1-N+2\left\{\frac{1}{f^2}(2Z^2-1)+1-\frac{1}{f^2}\right\}<\chi_f>\right]\\&\left\{\frac{1}{f^2}(2Z^2-1)+1-\frac{1}{f^2}\right\}\\&-2g_{SU(N+1)}^2\left[\frac{1}{2}\frac{1}{f^2}\left\{1+\frac{1}{f^2}-\left(1-\frac{1}{f^2}\right)N\right\}(2Z^2-1)\right.\\&-\frac{1}{f^4}\left\{2(N-1)Z^4+4Z^2-(N+2)\right\}<\chi_f>\\&+\frac{1}{2}\left(1-\frac{1}{f^2}\right)\left\{1-\frac{1}{f^2}-\left(1+\frac{1}{f^2}\right)N\right\}\\&\left.+\left(1-\frac{1}{f^2}\right)\left\{2\frac{1}{f^2}(2Z^2-1)+1-\frac{1}{f^2}\right\}<\chi_f>\right]=0.\end{aligned}} \tag{5.58}$$

g^2-indepedent relation:

$$\boxed{\begin{aligned}&\left[-\frac{1}{f^4}(N+1)(Z^2-1)+3\frac{1}{f^2}\left(1-\frac{1}{f^2}\right)(2Z^2-1)\right.\\&\left.-\frac{1}{f^2}\left(1-\frac{1}{f^2}\right)\{(N-1)Z^2+1\}+2\left(1-\frac{1}{f^2}\right)^2\right]<\chi_f>\\&=1-\frac{1}{f^4}-\left(1-\frac{1}{f^2}\right)^2N+\frac{1}{4}f^2\left(1-\frac{1}{f^2}\right)^2(N+1)\\&\left\{\frac{1}{f^2}(2Z^2-1)+1-\frac{1}{f^2}\right\}\frac{1}{<\chi_f>},\end{aligned}} \tag{5.59}$$

which reads, a proper solution for the Z^2.

$$\boxed{\begin{aligned}&\left[8\frac{1}{f^2}\left\{\left(1-\frac{1}{f^2}\right)-\frac{1}{4}\right\}<\chi_f>-\frac{1}{2}\left(1-\frac{1}{f^2}\right)^2(N+1)\frac{1}{<\chi_f>}\right]Z^2\\&=1-\frac{1}{f^4}-\left(1-\frac{1}{f^2}\right)^2N-\left\{\frac{1}{f^4}(N+1)+2\left(1-\frac{1}{f^2}\right)\right\}<\chi_f>\\&-\frac{1}{4}f^2\left(1-\frac{1}{f^2}\right)^3(N+1)\frac{1}{<\chi_f>}.\end{aligned}} \tag{5.60}$$

Solution of the $\frac{SO(2N+2)}{U(N+1)}$ supersymmetric σ-model. $f=1$, (5.60) $\longrightarrow$ a simple solution (5.37).

Another equation for Z^2:

$$
\boxed{\begin{aligned}
&\left\{2g_{U(1)}^2+(N-1)g_{SU(N+1)}^2\right\} <\chi_f> Z^2\\
&=-\frac{1}{2}g_{U(1)}^2\left[f^2(\xi+1-N)-2f^2\left\{\frac{1}{f^2}-\left(1-\frac{1}{f^2}\right)\right\}<\chi_f>\right]\\
&+g_{SU(N+1)}^2\left[\frac{1}{2}f^2\left\{1+\frac{1}{f^2}-\left(1-\frac{1}{f^2}\right)N\right\}-f^2<\chi_f>\right.\\
&\left.+\frac{1}{4}f^4\left(1-\frac{1}{f^2}\right)^2(N+1)\frac{1}{<\chi_f>}\right].
\end{aligned}} \tag{5.61}
$$

Ultimate goal of determining $<\chi_f>$**:**

$$
\boxed{\begin{aligned}
&\left\{2g_{U(1)}^2+(N-1)g_{SU(N+1)}^2\right\}\left\{\frac{1}{f^2}(N-1)+2\right\}<\chi_f>^3\\
&+\left[g_{U(1)}^2\left[2-f^2-2\left\{1-\frac{1}{f^4}-\left(1-\frac{1}{f^2}\right)^2N\right\}\right]\right.\\
&\left.+g_{SU(N+1)}^2\left[f^2-(N-1)\left\{1-\frac{1}{f^4}-\left(1-\frac{1}{f^2}\right)^2N\right\}\right]\right]<\chi_f>^2\\
&-\frac{1}{2}\left[g_{U(1)}^2f^2\left\{\xi+1-N-\left(1-\frac{1}{f^2}\right)^3(N+1)\right\}\right.\\
&\left.+g_{SU(N+1)}^2f^2\left\{1+\frac{1}{f^2}-\left(1-\frac{1}{f^2}\right)N-\frac{1}{2}\left(1-\frac{1}{f^2}\right)^3(N^2-1)\right\}\right]<\chi_f>\\
&-\frac{1}{4}g_{SU(N+1)}^2f^4\left(1-\frac{1}{f^2}\right)^2(N+1)=0.
\end{aligned}} \tag{5.62}
$$

6 Discussions and concluding remarks

To approach an approxmate solution for $<\chi_f>$, put $g^2_{U(1)} = g^2_{SU(N+1)}$ and neglect the terms $\left(2 - \frac{1}{f^2}\right)^n$, $(n = 2, 3)$, since we consider a small fluctuation of f around 1 :

$$\boxed{\begin{aligned} &(N+1)\left\{\frac{1}{f^2}(N-1)+2\right\} <\chi_f>^2 \\ &-\left\{\left(1-\frac{1}{f^4}\right)(N+1)-2\right\} <\chi_f> \\ &-\frac{1}{2}f^2\left\{\xi+2+\frac{1}{f^2}-\left(2-\frac{1}{f^2}\right)N\right\}=0, \end{aligned}} \tag{6.1}$$

$$\boxed{\begin{aligned} &8\frac{1}{f^2}\left\{\left(1-\frac{1}{f^2}\right)-\frac{1}{4}\right\} <\chi_f> Z^2 \\ &=1-\frac{1}{f^4}-\left\{\frac{1}{f^4}(N+1)+2\left(1-\frac{1}{f^2}\right)\right\} <\chi_f> . \end{aligned}} \tag{6.2}$$

Equation (6.1) is easily solved as

$$\boxed{\left.\begin{aligned} &<\chi_f>=\frac{1}{2}\frac{f^2}{N^2-1+2f^2(N+1)}\left\{\left(1-\frac{1}{f^4}\right)(N+1)-2\pm\sqrt{D_{<\chi_f>}}\right\}, \\ &D_{<\chi_f>}\equiv 2\left\{N^2-1+2f^2(N+1)\right\}\xi+\left\{\left(1-\frac{1}{f^4}\right)(N+1)-2\right\}^2 \\ &\qquad +2\left\{N^2-1+2f^2(N+1)\right\}\left\{2+\frac{1}{f^2}-\left(2-\frac{1}{f^2}\right)N\right\}. \end{aligned}\right\}} \tag{6.3}$$

$\boxed{N=5, \text{The case I: } f=1.01 \text{ and } f=0.99}$

$$\left.\begin{aligned} <\chi_f>=&\begin{cases}\left(28.0\times\sqrt{18.612\xi-38.677}-25.0\right)\times 10^{-3}, & (f=1.01),\\ \left(27.0\times\sqrt{17.880\xi-32.353}-30.0\right)\times 10^{-3}, & (f=0.99),\end{cases}\\ Z^2=&\begin{cases}3.220-\dfrac{1}{1.295\times\sqrt{18.612\xi-38.677}-1.156}, & (f=1.01),\\ 2.980+\dfrac{1}{13.635\times\sqrt{17.880\xi-32.353}-14.140}, & (f=0.99).\end{cases}\end{aligned}\right\} \tag{6.4}$$

Noticing $0<Z^2<1$, after a fine tuning for the parameter ξ,

$$\left.\begin{aligned} Z^2&=0.448, \quad <\chi_f>=7.844\times 10^{-3}, \quad (f=1.01,\ \xi=2.152),\\ Z^2&=0.629, \quad <\chi_f>=-8.481\times 10^{-3}, \quad (f=0.99,\ \xi=1.845),\end{aligned}\right\} \tag{6.5}$$

$\boxed{N=5, \text{The case II: } f=1.001 \text{ and } f=0.999}$

$$\left.\begin{aligned} <\chi_f>=&\begin{cases}\left(28.0\times\sqrt{18.012\xi-35.256}-27.0\right)\times 10^{-3}, & (f=1.001),\\ \left(27.0\times\sqrt{17.988\xi-34.736}-28.0\right)\times 10^{-3}, & (f=0.999),\end{cases}\\ Z^2=&\begin{cases}3.020-\dfrac{1}{13.860\times\sqrt{18.012\xi-35.256}-13.365}, & (f=1.001),\\ 2.980+\dfrac{1}{13.635\times\sqrt{17.988\xi-34.736}-14.140}, & (f=0.999),\end{cases}\end{aligned}\right\} \tag{6.6}$$

$$\left.\begin{aligned} Z^2&=0.763, \quad <\chi_f>=0.916\times 10^{-3}, \quad (f=1.001,\ \xi=2.0125),\\ Z^2&=0.264, \quad <\chi_f>=-0.730\times 10^{-3}, \quad (f=0.999,\ \xi=1.9880).\end{aligned}\right\} \tag{6.7}$$

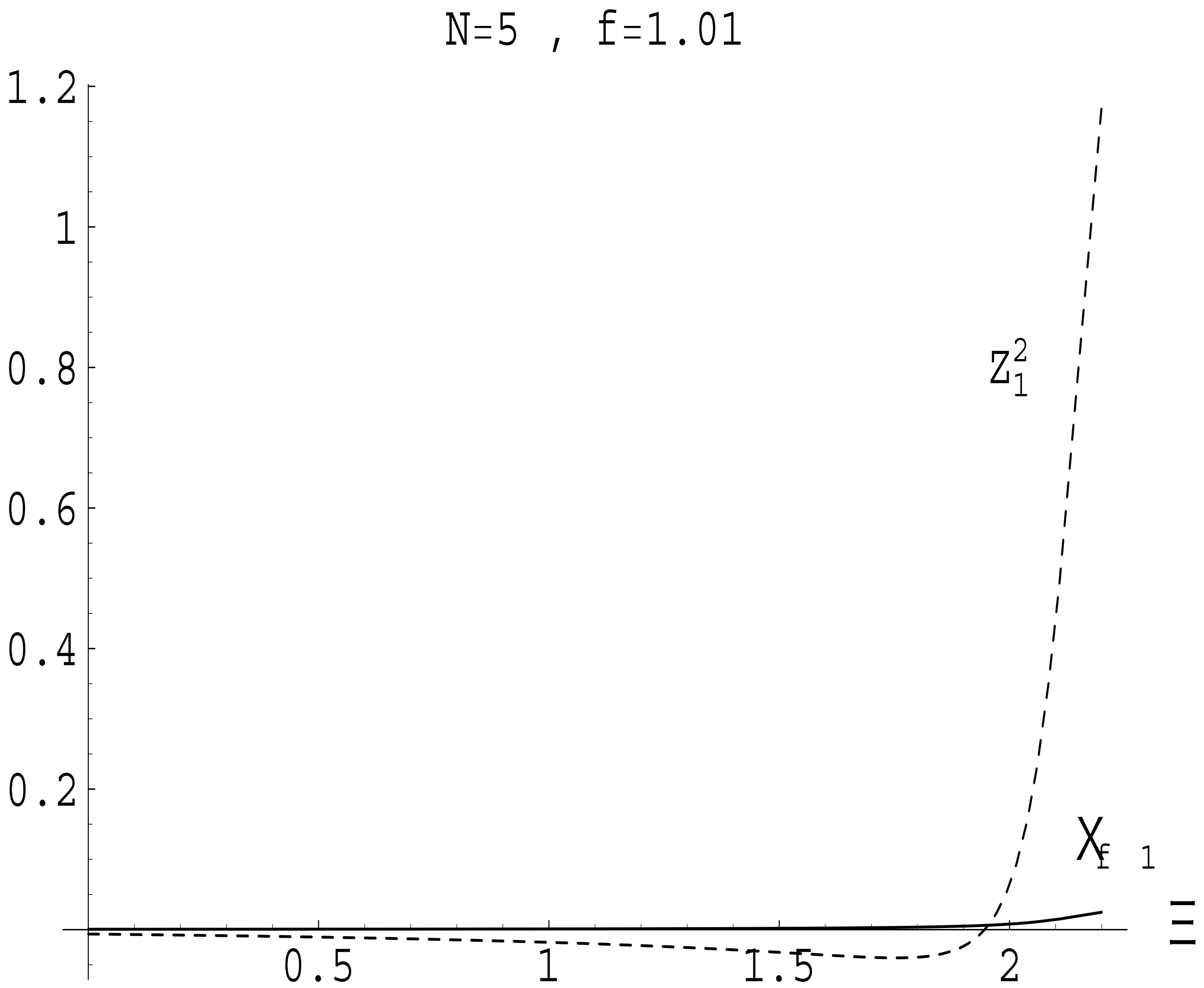

Figure 1: We have used the symbols $<\chi_f>_1$, $<\chi_f>_2$ and $<\chi_f>_3$ to denote the three solutions of equation (5.62). The solutions $<\chi_f>_2$ and $<\chi_f>_3$ are always negative. On the other hand, $<\chi_f>_1$ is always positive. We have used the symbols Z_1^2, Z_2^2 and Z_3^2 to denote the values of the Z^2 parameter associated with the solutions $<\chi_f>_1$, $<\chi_f>_2$ and $<\chi_f>_3$, respectively, according to equation (5.60). For f=1.01 we can always find an interval for ξ where the conditions $<\chi_f>_1 > 0$ and $0 < Z_1^2 < 1$ are both satisfied. However, for f=5 and f=10 this seems to be not possible anymore.

We have given an extended supersymmetric σ-model on Kähler coset space $\frac{G}{H} = \frac{SO(2N+2)}{U(N+1)}$, basing on the $SO(2N+1)$ Lie algebra of the fermion operators. Embedding the $SO(2N+1)$ group into an $SO(2N+2)$ group and using the $\frac{SO(2N+2)}{U(N+1)}$ coset variables, we have investigated a new aspect of the supersymmetric σ-model on the Kähler manifold of the symmetric space $\frac{SO(2N+2)}{U(N+1)}$.

We have constructed a Killing potential which is just the extension of the Killing potential in the $\frac{SO(2N)}{U(N)}$ coset space to that in the $\frac{SO(2N+2)}{U(N+1)}$ coset space. To our great surprise, it has been shown that the Killing potential is equivalent with the generalized density matrix which is an important clue to fermion many-body problems. Its diagonal-block matrix is related to a reduced scalar potential with the Fayet-Ilipoulos term. The reduced scalar potential has been optimized to see behaviour of the vacuum expectation value of the σ-model fields. We have got, however, a too simple solution $Z^2 = 1$. .

To find proper solutions for the extended supersymmetryic σ-mdoel, after rescaling Goldstone fields by a mass parameter, minimization of the reduced scalar potential has been made. Fayet-Ilipoulos term makes a crucial role to acquire proper solutions for Z^2 . To get proper solutions for wide range of a rescaling parameter f, we have solved the cubic equation for $<\chi_f>$.

We have given bosonization of the $SO(2N+2)$ Lie operators, vacuum functions and differential forms for their bosons expressed in terms of the $\frac{SO(2N+2)}{U(N+1)}$ coset variables, a $U(1)$ phase and the Kähler potential. This provides a powerful tool of describing the Goldstone bosons but accompanying fermionic modes. The effectiveness of $\frac{SO(2N+2)}{U(N+1)}$ Kähler manifold is expected to open a new field for exploration of low-energy elementary particle physics by the supersymmetric σ-model.

Appendix

A Bosonization of SO(2N+2) Lie operators

Fermion state vector $|\,\Psi>$ corresponding to a function $\Psi(\mathcal{G})$ in $\mathcal{G}\in SO(2N+2)$:

$$|\,\Psi> = \int U(\mathcal{G})|\,0><0\,|U^{\dagger}(\mathcal{G})|\,\Psi>d\mathcal{G} = \int U(\mathcal{G})|\,0>\Psi(\mathcal{G})d\mathcal{G}. \tag{A.1}$$

The $\mathcal{G}$ is given by (3.3) and (3.4) and the $d\mathcal{G}$ is an invariant group integration. When an infinitesimal operator $\mathbb{I}_{\mathcal{G}}+\delta\widehat{\mathcal{G}}$ and a corresponding infinitesimal unitary operator $U(1_{2N+2}+\delta\mathcal{G})$ is operated on $|\,\Psi>$,
using $U^{-1}(1_{2N+2}+\delta\mathcal{G})=U(1_{2N+2}-\delta\mathcal{G})$, it transforms $|\,\Psi>$ as

$$\begin{aligned}
&U(1_{2N+2}-\delta\mathcal{G})|\,\Psi> = (\mathbb{I}_{\mathcal{G}}-\delta\widehat{\mathcal{G}})|\,\Psi> \\
&= \int U(\mathcal{G})|\,0><0\,|U^{\dagger}((1_{2N+2}+\delta\mathcal{G})\mathcal{G})|\,\Psi>d\mathcal{G} \\
&= \int U(\mathcal{G})|\,0>\Psi((1_{2N+2}+\delta\mathcal{G})\mathcal{G})d\mathcal{G} = \int U(\mathcal{G})|\,0>(1_{2N+2}+\delta\boldsymbol{\mathcal{G}})\Psi(\mathcal{G})d\mathcal{G},
\end{aligned} \tag{A.2}$$

$$\left.\begin{aligned}
&1_{2N+2}+\delta\mathcal{G} = \begin{bmatrix} 1_{N+1}+\delta\mathcal{A} & \delta\bar{\mathcal{B}} \\ \delta\mathcal{B} & 1_{N+1}+\delta\bar{\mathcal{A}} \end{bmatrix}, \\
&\delta\mathcal{A}^{\dagger}=-\delta\mathcal{A},\ \mathrm{tr}\delta\mathcal{A}=0,\ \delta\mathcal{B}=-\delta\mathcal{B}^{\mathrm{T}}, \\
&\delta\widehat{\mathcal{G}} = \delta\mathcal{A}^{p}{}_{q}E^{q}{}_{p}+\frac{1}{2}\left(\delta\mathcal{B}_{pq}E^{qp}+\delta\bar{\mathcal{B}}_{pq}E_{qp}\right), \\
&\delta\boldsymbol{\mathcal{G}} = \delta\mathcal{A}^{p}{}_{q}\boldsymbol{\mathcal{E}}^{\boldsymbol{q}}{}_{\boldsymbol{p}}+\frac{1}{2}\left(\delta\mathcal{B}_{pq}\boldsymbol{\mathcal{E}}^{\boldsymbol{qp}}+\delta\bar{\mathcal{B}}_{pq}\boldsymbol{\mathcal{E}}_{\boldsymbol{qp}}\right).
\end{aligned}\right\} \tag{A.3}$$

The operation of $\mathbb{I}_{\mathcal{G}}-\delta\widehat{\mathcal{G}}$ on the $|\,\Psi>$ in the fermion space corresponds to the left multiplication by $1_{2N+2}+\delta\mathcal{G}$ for the variable of the $\mathcal{G}$ of the function $\Psi(\mathcal{G})$.

For a small parameter ϵ, Representation on the $\Psi(\mathcal{G})$:

$$\rho(e^{\epsilon\delta\mathcal{G}})\Psi(\mathcal{G}) = \Psi(e^{\epsilon\delta\mathcal{G}}\mathcal{G}) = \Psi(\mathcal{G}+\epsilon\delta\mathcal{G}\mathcal{G}) = \Psi(\mathcal{G}+d\mathcal{G}), \tag{A.4}$$

which leads us to a relation $d\mathcal{G} = \epsilon\delta\mathcal{G}\mathcal{G}$.

$$
\left.
\begin{array}{l}
d\mathcal{G} = \begin{bmatrix} d\mathcal{A} & d\bar{\mathcal{B}} \\ d\mathcal{B} & d\bar{\mathcal{A}} \end{bmatrix} = \epsilon \begin{bmatrix} \delta\mathcal{A}\mathcal{A} + \delta\bar{\mathcal{B}}\mathcal{B} & \delta\mathcal{A}\bar{\mathcal{B}} + \delta\bar{\mathcal{B}}\bar{\mathcal{A}} \\ \delta\mathcal{B}\mathcal{A} + \delta\bar{\mathcal{A}}\mathcal{B} & \delta\bar{\mathcal{A}}\bar{\mathcal{A}} + \delta\mathcal{B}\bar{\mathcal{B}} \end{bmatrix}, \\[3em]
d\mathcal{A} = \epsilon\dfrac{\partial\mathcal{A}}{\partial\epsilon} = \epsilon(\delta\mathcal{A}\mathcal{A} + \delta\bar{\mathcal{B}}\mathcal{B}), \quad d\mathcal{B} = \epsilon\dfrac{\partial\mathcal{A}}{\partial\epsilon} = \epsilon(\delta\mathcal{B}\mathcal{A} + \delta\bar{\mathcal{A}}\mathcal{B}).
\end{array}
\right\} \quad \text{(A.5)}
$$

Differential representation of $\rho(\delta\mathcal{G})$, $d\rho(\delta\mathcal{G})$;

$$
\begin{aligned}
d\rho(\delta\mathcal{G})\Psi(\mathcal{G}) =& \left[\frac{\partial\mathcal{A}^p{}_q}{\partial\epsilon}\frac{\partial}{\partial\mathcal{A}^p{}_q} + \frac{\partial\mathcal{B}_{pq}}{\partial\epsilon}\frac{\partial}{\partial\mathcal{B}_{pq}} + \frac{\partial\bar{\mathcal{A}}^p{}_q}{\partial\epsilon}\frac{\partial}{\partial\bar{\mathcal{A}}^p{}_q} + \frac{\partial\bar{\mathcal{B}}_{pq}}{\partial\epsilon}\frac{\partial}{\partial\bar{\mathcal{B}}_{pq}} \right] \\
& \Psi(\mathcal{G}).
\end{aligned} \quad \text{(A.6)}
$$

Explicit forms of the differential representation: $d\rho(\delta\mathcal{G})\Psi(\mathcal{G}) = \delta\boldsymbol{\mathcal{G}}\Psi(\mathcal{G})$. Each operator in $\delta\boldsymbol{\mathcal{G}}$ is expressed in a differential form :

$$
\boxed{
\left.
\begin{array}{l}
\boldsymbol{\mathcal{E}^p{}_q} = \bar{\mathcal{B}}_{pr}\dfrac{\partial}{\partial\bar{\mathcal{B}}_{qr}} - \mathcal{B}_{qr}\dfrac{\partial}{\partial\mathcal{B}_{pr}} - \bar{\mathcal{A}}^q{}_r\dfrac{\partial}{\partial\bar{\mathcal{A}}^p{}_r} + \mathcal{A}^p{}_r\dfrac{\partial}{\partial\mathcal{A}^q{}_r}, \\[1.5em]
\boldsymbol{\mathcal{E}_{pq}} = \bar{\mathcal{A}}^p{}_r\dfrac{\partial}{\partial\bar{\mathcal{B}}_{qr}} - \mathcal{B}_{qr}\dfrac{\partial}{\partial\mathcal{A}^p{}_r} - \bar{\mathcal{A}}^q{}_r\dfrac{\partial}{\partial\bar{\mathcal{B}}_{pr}} + \mathcal{B}_{pr}\dfrac{\partial}{\partial\mathcal{A}^q{}_r}.
\end{array}
\right\}
} \quad \text{(A.7)}
$$

Definition of the boson operators $\boldsymbol{\mathcal{A}^p{}_q}$ and $\boldsymbol{\bar{\mathcal{A}}^p{}_q}$, etc.:

$$
\boxed{
\left.
\begin{array}{l}
\boldsymbol{\mathcal{A}} = \dfrac{1}{\sqrt{2}}\left(\mathcal{A} + \dfrac{\partial}{\partial\bar{\mathcal{A}}}\right), \quad \boldsymbol{\mathcal{A}}^\dagger = \dfrac{1}{\sqrt{2}}\left(\bar{\mathcal{A}} - \dfrac{\partial}{\partial\mathcal{A}}\right), \\[1.5em]
\boldsymbol{\bar{\mathcal{A}}} = \dfrac{1}{\sqrt{2}}\left(\bar{\mathcal{A}} + \dfrac{\partial}{\partial\mathcal{A}}\right), \quad \boldsymbol{\mathcal{A}}^{\mathrm{T}} = \dfrac{1}{\sqrt{2}}\left(\mathcal{A} - \dfrac{\partial}{\partial\bar{\mathcal{A}}}\right), \\[1.5em]
[\boldsymbol{\mathcal{A}},\ \boldsymbol{\mathcal{A}}^\dagger] = 1, \quad [\boldsymbol{\bar{\mathcal{A}}},\ \boldsymbol{\mathcal{A}}^{\mathrm{T}}] = 1, \\[1em]
[\boldsymbol{\mathcal{A}},\ \boldsymbol{\bar{\mathcal{A}}}] = [\boldsymbol{\mathcal{A}},\ \boldsymbol{\mathcal{A}}^{\mathrm{T}}] = 0, \quad [\boldsymbol{\mathcal{A}}^\dagger,\ \boldsymbol{\bar{\mathcal{A}}}] = [\boldsymbol{\mathcal{A}}^\dagger,\ \boldsymbol{\mathcal{A}}^{\mathrm{T}}] = 0.
\end{array}
\right\}
} \quad \text{(A.8)}
$$

The differential operators (A.7) can be converted into a boson operator representation

$$\boxed{\begin{aligned}\boldsymbol{\mathcal{E}}^{p}{}_{q} &= \boldsymbol{\mathcal{B}}^{\dagger}_{pr}\boldsymbol{\mathcal{B}}_{qr} - \boldsymbol{\mathcal{B}}^{\mathrm{T}}_{qr}\bar{\boldsymbol{\mathcal{B}}}_{pr} - \boldsymbol{\mathcal{A}}^{q\dagger}_{r}\boldsymbol{\mathcal{A}}^{p}{}_{r} + \boldsymbol{\mathcal{A}}^{p\mathrm{T}}_{r}\bar{\boldsymbol{\mathcal{A}}}^{q}{}_{r} = \boldsymbol{\mathcal{B}}^{\dagger}_{p\tilde{r}}\boldsymbol{\mathcal{B}}_{q\tilde{r}} - \boldsymbol{\mathcal{A}}^{q\dagger}_{\tilde{r}}\boldsymbol{\mathcal{A}}^{p}{}_{\tilde{r}}, \\ \boldsymbol{\mathcal{E}}_{pq} &= \boldsymbol{\mathcal{A}}^{p\dagger}_{r}\boldsymbol{\mathcal{B}}_{qr} - \boldsymbol{\mathcal{B}}^{\mathrm{T}}_{qr}\bar{\boldsymbol{\mathcal{A}}}^{p}{}_{r} - \boldsymbol{\mathcal{A}}^{q\dagger}_{r}\boldsymbol{\mathcal{B}}_{pr} + \boldsymbol{\mathcal{B}}^{\mathrm{T}}_{pr}\bar{\boldsymbol{\mathcal{A}}}^{q}{}_{r} = \boldsymbol{\mathcal{A}}^{p\dagger}_{\tilde{r}}\boldsymbol{\mathcal{B}}_{q\tilde{r}} - \boldsymbol{\mathcal{A}}^{q\dagger}_{\tilde{r}}\boldsymbol{\mathcal{B}}_{p\tilde{r}},\end{aligned}}\ \Bigg\} \quad \text{(A.9)}$$

by using the notation $\boldsymbol{\mathcal{A}}^{p\mathrm{T}}_{r+N} = \boldsymbol{\mathcal{B}}^{\dagger}_{pr}$ and $\boldsymbol{\mathcal{B}}^{\mathrm{T}}_{pr+N} = \boldsymbol{\mathcal{A}}^{p\dagger}_{r}$ to use a suffix $\tilde{r}$, $(\tilde{r} = 0,\ 1,\ \cdots\ 2N)$. Then we have the **boson images** of the fermion $SO(2N+1)$ Lie operators as

$$\boxed{\begin{aligned}\boldsymbol{E}^{\alpha}{}_{\beta} &= \boldsymbol{\mathcal{B}}^{\dagger}_{\alpha\tilde{r}}\boldsymbol{\mathcal{B}}_{\beta\tilde{r}} - \boldsymbol{\mathcal{A}}^{\beta\dagger}_{\tilde{r}}\boldsymbol{\mathcal{A}}^{\alpha}{}_{\tilde{r}}, \\ \boldsymbol{E}_{\alpha\beta} &= \boldsymbol{\mathcal{A}}^{\alpha\dagger}_{\tilde{r}}\boldsymbol{\mathcal{B}}_{\beta\tilde{r}} - \boldsymbol{\mathcal{A}}^{\beta\dagger}_{\tilde{r}}\boldsymbol{\mathcal{B}}_{\alpha\tilde{r}}, \\ \boldsymbol{c}_{\alpha} &= \boldsymbol{\mathcal{A}}^{\alpha\dagger}_{\tilde{r}}(\boldsymbol{\mathcal{A}}^{0}{}_{\tilde{r}} - \boldsymbol{\mathcal{B}}_{0\tilde{r}}) + (\boldsymbol{\mathcal{A}}^{0\dagger}_{\tilde{r}} - \boldsymbol{\mathcal{B}}^{\dagger}_{0\tilde{r}})\boldsymbol{\mathcal{B}}_{\alpha\tilde{r}} = \boldsymbol{\mathcal{A}}^{\alpha\dagger}_{\tilde{r}}\boldsymbol{\mathcal{Y}}_{\tilde{r}} + \boldsymbol{\mathcal{Y}}^{\dagger}{}_{\tilde{r}}\boldsymbol{\mathcal{B}}_{\alpha\tilde{r}}.\end{aligned}}\ \Bigg\} \quad \text{(A.10)}$$

This representation involves, in addition to the original $\boldsymbol{A}^{\alpha}{}_{\beta}$ and $\boldsymbol{B}_{\alpha\beta}$ bosons, their complex conjugate bosons and the $\boldsymbol{\mathcal{Y}}_{\tilde{r}}$ bosons. The complex conjugate bosons arise from the use of matrix $\mathcal{G}$ as the variables of representation and the $\boldsymbol{\mathcal{Y}}_{\tilde{r}}$ bosons arise from extension of algebra from $SO(2N)$ to $SO(2N+1)$ and embedding of the $SO(2N+1)$ into $SO(2N+2)$.

$$\frac{\partial}{\partial \mathcal{A}^{p}{}_{q}} \det \mathcal{A} = (\mathcal{A}^{-1})_{p}{}^{q} \det \mathcal{A}, \qquad \frac{\partial}{\partial \mathcal{A}^{p}{}_{q}} (\mathcal{A}^{-1})_{s}{}^{r} = -(\mathcal{A}^{-1})_{s}{}^{q} (\mathcal{A}^{-1})_{p}{}^{r}, \quad \text{(A.11)}$$

we get the relations which are valid when operated onto functions on the right coset$\frac{SO(2N+2)}{SU(N+1)}$

$$\left.\begin{aligned}\frac{\partial}{\partial \mathcal{B}_{pq}} &= \textstyle\sum_{p>q} (\mathcal{A}^{-1})_{r}{}^{q} \dfrac{\partial}{\partial \mathcal{Q}_{pr}}, \\ \frac{\partial}{\partial \mathcal{A}^{p}{}_{q}} &= -\textstyle\sum_{r>s} Q_{rp} (\mathcal{A}^{-1})_{s}{}^{q} \dfrac{\partial}{\partial \mathcal{Q}_{sr}} - \dfrac{i}{2} (\mathcal{A}^{-1})_{p}{}^{q} \dfrac{\partial}{\partial \tau}.\end{aligned}\right\} \quad \text{(A.12)}$$

B Vacuum function for bosons

The function $\Phi_{00}(\mathcal{G})$ in $\mathcal{G} \in SO(2N+2)$ corresponds to the free fermion vacuum function in the physial fermion space.

$$\boxed{\left(\boldsymbol{\mathcal{E}^{p}{}_{q}} + \tfrac{1}{2}\delta_{pq}\right)\Phi_{00}(\mathcal{G}) = \boldsymbol{\mathcal{E}_{pq}}\Phi_{00}(\mathcal{G}) = 0, \quad \Phi_{00}(1_{2N+2}) = 1.} \tag{B.1}$$

The vacuum function $\Phi_{00}(\mathcal{G})$ to satisfy (B.1) is given by $\Phi_{00}(\mathcal{G})=[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}$,

The proof:

$$\begin{aligned}
&\left(\boldsymbol{\mathcal{E}^{p}{}_{q}} + \frac{1}{2}\delta_{pq}\right)[\det(\bar{\mathcal{A}})]^{\frac{1}{2}} \\
&= \frac{1}{2}\delta_{pq}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}} \\
&+ \left(\bar{\mathcal{B}}_{pr}\frac{\partial}{\partial\bar{\mathcal{B}}_{qr}} - \mathcal{B}_{qr}\frac{\partial}{\partial\mathcal{B}_{pr}} - \bar{\mathcal{A}}^{q}{}_{r}\frac{\partial}{\partial\bar{\mathcal{A}}^{p}{}_{r}} + \mathcal{A}^{p}{}_{r}\frac{\partial}{\partial\mathcal{A}^{q}{}_{r}}\right)[\det(\bar{\mathcal{A}})]^{\frac{1}{2}} \\
&= \frac{1}{2}\delta_{pq}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}} - \bar{\mathcal{A}}^{q}{}_{r}\frac{\partial}{\partial\bar{\mathcal{A}}^{p}{}_{r}}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}} \\
&= \frac{1}{2}\delta_{pq}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}} - \frac{1}{2}\frac{1}{[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}}\bar{\mathcal{A}}^{q}{}_{r}\frac{\partial}{\partial\bar{\mathcal{A}}^{p}{}_{r}}\det(\bar{\mathcal{A}}) \\
&= \frac{1}{2}\delta_{pq}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}} - \frac{1}{2}\frac{1}{[\det(\bar{\mathcal{A}})]^{\frac{1}{2}}}(\mathcal{A}\bar{\mathcal{A}}^{-1})_{qp}\det(\bar{\mathcal{A}}) = 0,
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
&\boldsymbol{\mathcal{E}_{pq}}[\det(\bar{\mathcal{A}})]^{\frac{1}{2}} \\
&= \left(\bar{\mathcal{A}}^{p}{}_{r}\frac{\partial}{\partial\bar{\mathcal{B}}_{qr}} - \mathcal{B}_{qr}\frac{\partial}{\partial\mathcal{A}^{p}{}_{r}} - \bar{\mathcal{A}}^{q}{}_{r}\frac{\partial}{\partial\bar{\mathcal{B}}_{pr}} + \mathcal{B}_{pr}\frac{\partial}{\partial\mathcal{A}^{q}{}_{r}}\right)[\det(\bar{\mathcal{A}})]^{\frac{1}{2}} = 0.
\end{aligned} \tag{B.3}$$

The vacuum functions $\Phi_{00}(G)$, $G \in SO(2N+1)$ and $\Phi_{00}(g)$, $g \in SO(2N)$ satisfy

$$\boxed{\boldsymbol{c_{\alpha}} \Phi_{00}(G) = \left(\boldsymbol{E^{\alpha}_{\beta}} + \tfrac{1}{2}\delta_{\alpha\beta} \right) \Phi_{00}(G) = \boldsymbol{E_{\alpha\beta}} \Phi_{00}(G) = 0, \quad \Phi_{00}(1_{2N+1}) = 1,} \tag{B.4}$$

$$\boxed{\left(\boldsymbol{e^{\alpha}_{\beta}} + \tfrac{1}{2}\delta_{\alpha\beta} \right) \Phi_{00}(g) = \boldsymbol{e_{\alpha\beta}} \Phi_{00}(g) = 0, \quad \Phi_{00}(1_{2N}) = 1.} \tag{B.5}$$

By using the $SO(2N+2)$ Lie operators E^{pq}, the expression for the $SO(2N+1)$ WF $|\, G>$ is converted to a form quite similar to the $SO(2N)$ WF $|\, g>$ as

$$|\, G> \, = \, <0\,|U(G)|\,0> \exp\left(\frac{1}{2} \cdot \mathcal{Q}_{pq} E^{pq} \right) |\,0>, \tag{B.6}$$

where we have used the nilpotency relation $(E^{\alpha 0})^2 = 0$. Equation (B.6) leads to the property $U(G)|\,0> = U(\mathcal{G})|\,0>$. On the other hand,

$$\det \mathcal{A} = \frac{1+z}{2} \det a, \quad \det \mathcal{B} = 0. \tag{B.7}$$

Vacuum function $\Phi_{00}(\mathcal{G})$ expressed in terms of the Kähler potential:

$$\boxed{\overline{<0\,|U(\mathcal{G})|\,0>} = \Phi_{00}(\mathcal{G}) = \left[\det(\bar{\mathcal{A}}) \right]^{\frac{1}{2}} = e^{-\frac{1}{4}\mathcal{K}(\mathcal{Q},\, \mathcal{Q}^{\dagger})} e^{-i\frac{\tau}{2}},} \tag{B.8}$$

$$\boxed{\Phi_{00}(\mathcal{G}) = \Phi_{00}(G) = \sqrt{\frac{1+z}{2}} \left[\det(\bar{a}) \right]^{\frac{1}{2}} = \sqrt{\frac{1+z}{2}} e^{-\frac{1}{4}\mathcal{K}(q,\, q^{\dagger})} e^{-i\frac{\tau}{2}}.} \tag{B.9}$$

C Differential forms for bosons over SO(2N+2)/U(N+1) coset space

The boson images of the fermion $SO(2N+2)$ Lie operators $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}{}_{\boldsymbol{q}}$ etc. can be represented by the closed first order differential forms over the $\frac{SO(2N+2)}{U(N+1)}$ coset space in terms of the $\frac{SO(2N+2)}{U(N+1)}$ coset variables Q_{pq} and the phase variable $\tau \left(= \frac{i}{2} \ln \left[\frac{\det(A^*)}{\det(A)} \right] \right)$ of the $U(N+1)$ as

$$
\boxed{\left.
\begin{aligned}
&\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}{}_{\boldsymbol{q}} = \bar{Q}_{pr} \frac{\partial}{\partial \bar{Q}_{qr}} - Q_{qr} \frac{\partial}{\partial Q_{pr}} - i\delta_{pq} \frac{\partial}{\partial \tau}, \\
&\boldsymbol{\mathcal{E}}_{\boldsymbol{pq}} = Q_{pr} Q_{sq} \frac{\partial}{\partial Q_{rs}} - \frac{\partial}{\partial \bar{Q}_{pq}} - iQ_{pq} \frac{\partial}{\partial \tau}, \quad \boldsymbol{\mathcal{E}}^{\boldsymbol{pq}} = \bar{\boldsymbol{\mathcal{E}}}_{\boldsymbol{pq}},
\end{aligned}
\right\}}
\tag{C.1}
$$

The phase variable τ is identical with the phase variable $\tau \left(= \frac{i}{2} \ln \left[\frac{\det(a^*)}{\det(a)} \right] \right)$ of the $U(N)$, due to the first equation of (B.7).

The images of the fermion $SO(2N+1)$ Lie operators:

$$
\left.
\begin{aligned}
&\boldsymbol{E}^{\alpha}{}_{\beta} = \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}, \quad \boldsymbol{E}_{\alpha\beta} = \boldsymbol{\mathcal{E}}_{\boldsymbol{\alpha\beta}}, \quad \boldsymbol{E}^{\alpha\beta} = \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha\beta}}, \\
&\boldsymbol{c}_{\boldsymbol{\alpha}} = \boldsymbol{\mathcal{E}}_{\boldsymbol{0\alpha}} - \boldsymbol{\mathcal{E}}^{\boldsymbol{0}}{}_{\boldsymbol{\alpha}}, \quad \boldsymbol{c}^{\dagger}_{\boldsymbol{\alpha}} = \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha 0}} - \boldsymbol{\mathcal{E}}^{\boldsymbol{\alpha}}{}_{\boldsymbol{0}}.
\end{aligned}
\right\}
\tag{C.2}
$$

The representations of the $SO(2N+1)$ Lie operators in terms of the variables $q_{\alpha\beta}$ and r_{α}:

$$
\boxed{\left.
\begin{aligned}
&\boldsymbol{E}^{\alpha}{}_{\beta} = \boldsymbol{e}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} + \bar{r}_{\alpha} \frac{\partial}{\partial \bar{r}_{\beta}} - r_{\beta} \frac{\partial}{\partial r_{\alpha}}, \quad \boldsymbol{e}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} = \bar{q}_{\alpha\gamma} \frac{\partial}{\partial \bar{q}_{\beta\gamma}} - q_{\beta\gamma} \frac{\partial}{\partial q_{\alpha\gamma}} - i\delta_{\alpha\beta} \frac{\partial}{\partial \tau}, \\
&\boldsymbol{E}_{\alpha\beta} = \boldsymbol{e}_{\boldsymbol{\alpha\beta}} + (r_{\alpha} q_{\beta\xi} - r_{\beta} q_{\alpha\xi}) \frac{\partial}{\partial r_{\xi}}, \quad \boldsymbol{e}_{\boldsymbol{\alpha\beta}} = q_{\alpha\gamma} q_{\delta\beta} \frac{\partial}{\partial q_{\gamma\delta}} - \frac{\partial}{\partial \bar{q}_{\alpha\beta}} - iq_{\alpha\beta} \frac{\partial}{\partial \tau},
\end{aligned}
\right\}}
\tag{C.3}
$$

$$
\boxed{\boldsymbol{c}_{\boldsymbol{\alpha}} = \frac{\partial}{\partial \bar{r}_{\alpha}} + \bar{r}_{\xi} \frac{\partial}{\partial \bar{q}_{\alpha\xi}} + (r_{\alpha} r_{\xi} - q_{\alpha\xi}) \frac{\partial}{\partial r_{\xi}} - q_{\alpha\xi} r_{\eta} \frac{\partial}{\partial q_{\xi\eta}} + ir_{\alpha} \frac{\partial}{\partial \tau}, \quad \boldsymbol{c}^{\dagger}_{\boldsymbol{\alpha}} = -\bar{\boldsymbol{c}}_{\boldsymbol{\alpha}}.}
\tag{C.4}
$$

The vacuum function $\Phi_{00}(G)$ in $G\in SO(2N+1)$ is given in (B.4).

$$\boxed{\boldsymbol{c_\alpha}\Phi_{00}(G)=0, \quad \boldsymbol{c_\alpha^\dagger}\Phi_{00}(G)=\bar{r}_\alpha\Phi_{00}(G), \quad \boldsymbol{c_\alpha^\dagger}=-\boldsymbol{\bar{c}_\alpha},} \tag{C.5}$$

and **the property** $\boxed{U(\mathcal{G})|\,0\!>\;=U(G)|\,0\!>}$

Exact identities:

$$\left.\begin{aligned} &\boldsymbol{c_\alpha}U(\mathcal{G})|\,0\!> = \left(-r_\alpha + r_\alpha r_\xi c_\xi^\dagger - q_{\alpha\xi}c_\xi^\dagger\right)\cdot U(G)|\,0\!>, \\ &\boldsymbol{c_\alpha^\dagger}U(\mathcal{G})|\,0\!> = -c_\alpha^\dagger\cdot U(G)|\,0\!>. \end{aligned}\right\} \tag{C.6}$$

Succesively using these identities, on the $U(\mathcal{G})|\,0\!>$, operators $\boldsymbol{c_\alpha}$ and $\boldsymbol{c_\alpha^\dagger}$ are shown to satisfy exactly the anti-commutation relations of the fermion annihilation-creation operators:

$$\boxed{\left(\boldsymbol{c_\alpha^\dagger c_\beta}+\boldsymbol{c_\beta c_\alpha^\dagger}\right)U(\mathcal{G})|\,0\!> = \delta_{\alpha\beta}\cdot U(\mathcal{G})|\,0\!>,} \tag{C.7}$$

$$\boxed{(\boldsymbol{c_\alpha c_\beta}+\boldsymbol{c_\beta c_\alpha})\,U(\mathcal{G})|\,0\!> = (\boldsymbol{c_\alpha^\dagger c_\beta^\dagger}+\boldsymbol{c_\beta^\dagger c_\alpha^\dagger})U(\mathcal{G})|\,0\!> = 0.} \tag{C.8}$$

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