Hagedorn Strings and Corrsponding Principle in AdS$_3$

Dan Tomino

(National Taiwan Normal University)

with

Feng-Li Lin, Toshihiro Matsuo

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1. Introduction

String theory in finite temperature:
Old and interesting problem

Even in gas,
Statistical/ thermodynamic treatment draws much more non-trivial aspects than what we expect from free theory.

Many body effects in string theory:
Some of them may be observed through the finite temperature system
.

a fundamental question:

What happens when strings are in high temperature environment?

Known interesting behavior of thermal string gas in flat space is

Hagedorn Singularity/Transition

Canonical partition function $=$ ∞/non-analytic

If $\beta \leq \beta_H$: Hagedorn inverse temperature

String density of states in high energy:

$$\Omega(E) \ \sim \ e^{\beta_H E}$$

Canonical partition function:

$$Z_{can}(\beta) \ = \ \int \ \Omega(E) \, e^{-\beta E} \, dE$$

non-analytic behavior in $\beta \leq \beta_H$.

There have been many speculations about this phenomenon:

- Ends of thermal ensemble?
- Phase transition to black hole?

It still be an interesting quastion to ask what really happens to strings above the Hagedorn temperature.

Why AdS_3?

- Solvable example of curved space string
- Application to strings/BH correspondence, AdS/CFT (in future)

Plan of the talk

$\surd$1. Introduction
2. Strings in AdS_3
3. Thermal partition function
    Hagedorn behavior, (Density of states)
4. Application to BH
5. Summary

2. Strings in AdS_3

AdS_3 space metric:

$$ds^2/k \;=\; -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\theta^2.$$

$k = l_{AdS}^2/l_{string}^2$.

Use another coordinate for the target space:

$V = \sqrt{k}\sinh\rho e^{i\theta}, \quad \bar{V} = \sqrt{k}\sinh\rho e^{-i\theta}, \quad \Phi = \sqrt{k}(t - \log\cosh\rho).$

AdS$_3$ string action: $SL(2,R)$ WZW action

$$S = \frac{1}{\pi}\int d^2z \left[-\partial\Phi\bar{\partial}\Phi + \left(\partial\bar{V} + \frac{\partial\Phi\bar{V}}{\sqrt{k}}\right)\left(\bar{\partial}V + \frac{\bar{\partial}\Phi V}{\sqrt{k}}\right)\right].$$

central charge: $c_{SL} = 3 + \frac{6}{k-2}$.

For consistency, we need some "internal space" s.t. $AdS_3 \times \mathcal{M}$. Request is

$$c_{SL} + c_{int} = 26 \quad \Rightarrow \quad c_{int} = 23 - \frac{6}{k-2}.$$

We set internal CFT as free boson's one (technical reason).

2-kinds of string in AdS_3 **(Maldacena, Ooguri '00)**

- Short string
  Localize around $r \sim 0$ with discreat spectrum

$$E_{short} = 1 + q + \bar{q} + 2w + \sqrt{1 + 4(k-2)\left(N_w + h - 1 - \frac{1}{2}w(w+1)\right)}$$

$q, \bar{q}, h, N_w, w$ are integer.

- Long string
  Stretch over whole space with continuous stectrum

$$E_{long} = \frac{k}{2}w + \frac{1}{w}\left(\frac{2s^2 + \frac{1}{2}}{k-2} + \tilde{N} + h + \bar{\tilde{N}} + \bar{h} - 2\right)$$

s is continuous.

3. Thermal Partition Function

AdS_3 part: 1-loop determinant of the Euclidean WZW action ($t \Rightarrow t_E = it$) with

$$\Phi(z+2\pi) = \Phi(z) + \beta n, \qquad \Phi(z+2\pi\tau) = \Phi(z) + \beta m,$$
$$V(z+2\pi) = V(z), \qquad V(z+2\pi\tau) = V(z),$$

β: thermal period, (m,n): thermal winding numbers,
τ: moduli parameter of string world sheet (torus).

Evaluated by **Gawedzki '86**.

Total partition function:

$$
\begin{aligned}
\mathcal{Z}(\beta) &= \int_{F_0} \frac{d^2\tau}{4\tau_2} \mathcal{Z}_{gh} \mathcal{Z}_{int} \mathcal{Z}_{AdS} \\
&= V_{int} \int_R \frac{d^2\tau}{4\tau_2} \frac{\beta(1-2/k)^{1/2} |\eta(\tau)|^{4-2c_{int}}}{(4\pi^2\alpha'\tau_2)^{(c_{int}+1)/2}} \\
&\quad \sum_{m=-\infty}^{\infty} \frac{e^{-\beta^2|m-n\tau|^2/4\pi\alpha'\tau_2 + 2\pi(\mathrm{Im}U_{n,m})^2/\tau_2}}{|\vartheta_1(\tau, U_{n,m})|^2} \Bigg|_{n=0},
\end{aligned}
$$

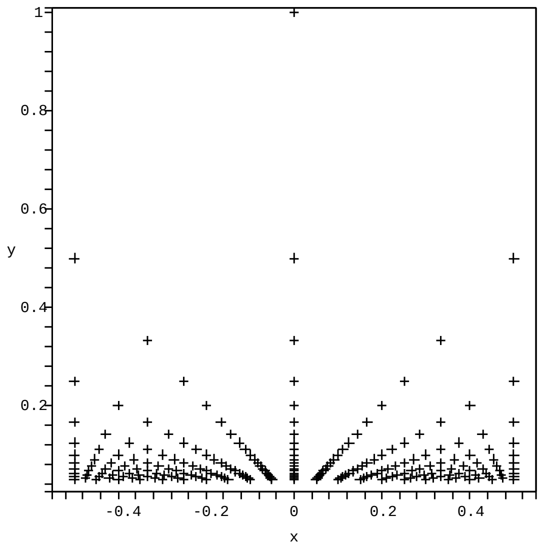
$R:$ strip domain $|\tau_1| \leq \frac{1}{2}, \tau_2 \geq 0.$

Dominant contribution:

$(m,n) = (\pm 1, 0), \quad Im(\tau) = \tau_2 \to 0.$

Note (I) : **Many divergent points!**

The points $2\pi\tau = (U \pm l)/w$ for integer (l, w) give pole.



We need regularization.

Note (II) :

$$\mathcal{Z}(\beta) = \sum_{E_{short}} e^{-\beta E_{short}} + \int dE_{long}(s)\ \rho(s) e^{-\beta E_{long}(s)},$$

$$\rho(s) = \frac{1}{\pi}\mathbf{log}\,\boldsymbol{\epsilon} + \frac{1}{2\pi i}\frac{d}{2ds}\log\left(\frac{\Gamma(\frac{1}{2} - is - m(w,s,N,h))\Gamma(\frac{1}{2} - is + \tilde{m}(w,s,\tilde{N},\tilde{h}))}{\Gamma(\frac{1}{2} + is - m(w,s,N,h))\Gamma(\frac{1}{2} + is + \tilde{m}(w,s,\tilde{N},\tilde{h}))}\right)$$

$\log\epsilon$ divergence.

ϵ: regularization parameter

Derived by **Maldacena, Ooguri and Son '00**.

- Hagedorn behavior

Asymptotic form of partition function in $|\tau| \to 0$:

$$\mathcal{Z} \simeq \frac{\sqrt{\pi} V_{int} (k-2)^{1/2}}{4(4\pi^2)^{(c_{int}+1)/2}}$$

$$\lim_{\epsilon \to 0} \sum_{w=w_{min}}^{\infty} \int_{w+\epsilon}^{w+1-\epsilon} \frac{dy}{y} \left(\frac{\beta}{2\pi\sqrt{k}y} \right)^{(c_{int}+1)/2} \frac{\exp\left[-\frac{\beta^2 - \beta_{AdS}^2}{2\beta} \sqrt{k} y \right]}{|\sin(\pi y)|}.$$

comes from the domain $\frac{-2\pi\sqrt{k}}{\beta} \leq \frac{\tau_1}{\tau_2} \leq \frac{2\pi\sqrt{k}}{\beta}$.

(No Hagedorn behavior in other domains.)

β_{AdS}^{-1} : Hagedorn temperature of string gas in $AdS_3 \times \mathcal{M}$

$$\boxed{\beta_{AdS} = \frac{4\pi}{l_s}\sqrt{\frac{k-\frac{9}{4}}{k-2}}}$$

(Main result)

- unitarity of internal CFT: $k > 2 + \frac{6}{23} (> \frac{9}{4}) \;\Rightarrow\; \beta_{AdS} > 0$
- universality if internal CFT is unitary and compact

• Density of states

Further evaliation:

$$\mathcal{Z}(\beta) \;=\; |\ln\epsilon|\,\frac{V_{int}(k-2)^{1/2}}{2\sqrt{\pi}}\left(\frac{\beta}{8\pi^3\sqrt{k}}\right)^{\frac{c_{int}+1}{2}} \sum_{w=w_{min}}^{\infty} w^{-\frac{c_{int}+3}{2}}\exp\left[-\frac{\beta^2-\beta_{AdS}^2}{2\beta}\sqrt{k}w\right]+O(\epsilon^0).$$

Can be identified with the long string contribution.

$|\log\epsilon|$: related IR divergence due to infinite vol. of AdS space.

Maxwell-Boltzmann gas approximation:

$$Z \simeq e^{\mathcal{Z}}$$

Free energy:

$$F \ \simeq \ -h(\beta)(\beta - \beta_{AdS})^{(c_{int}+1)/2} + \text{regular part} \quad (\beta \to \beta_{AdS}).$$

Non-analyticity is similar to the one of flat space case.

Inverse Laplace transformation:

$$\begin{aligned}\Omega(E) &= \oint \frac{d\beta}{2\pi i}\, e^{\beta E} Z(\beta) \\ &= CV\,\frac{e^{\beta_{AdS}E+\gamma_0 V}}{E^{(c_{int}+1)/2+1}}\,(1+O(1/E))\end{aligned}$$

- $\log\Omega(E) \sim \beta_{AdS}E$
- Implies single long string dominance of distribution
- Break down of thermodynamics (Famous in flat space)
  $E<0$, $C=\frac{\partial E}{\partial T}<0$ (above β_{AdS}^{-1}) in microcanonical ensemble
- Introducing winding strings in $\mathcal{M}$ may cure this pathology

4. Application to Black Hole

Bekenstein-Hawking entropy of AdS_{d+1} Schwarzschild black hole

$$S_{BH} = \frac{d-2}{d-1}\frac{1+\frac{d}{d-2}\frac{r_+^2}{l_{AdS}^2}-\mu^2}{1+\frac{r_+^2}{l_{AdS}^2}+\mu^2}\beta_{BH}M,$$

$$\beta_{BH} = \frac{4\pi l_{AdS}^2 r_+}{(d-2)(1-\mu^2)l_{AdS}^2+dr_+^2}, \qquad \mu = \frac{w_{d+1}Q}{2r_+^{d-2}}$$

$$S_{BH} \propto \beta_{BH}(M)M \quad \Rightarrow \quad \rho(M) \sim e^{\beta_{BH}(M)M}$$

$$(S_{string} \sim \beta_{st}M, \quad \Omega(E) \sim e^{\beta_{st}E})$$

Strings/BH correspondence in AdS space

Strings may correspond to BH

$$\text{if} \quad \beta_{st} \sim \beta_{BH} \quad \text{then} \quad S_{string} \sim S_{BH}$$

Let us examine AdS_3 : $\beta_{BH}(r_+) = 2\pi \frac{l_{AdS}^2}{r_+}$

$$\frac{r_+}{l_s} = \frac{k}{2}\sqrt{\frac{k-2}{k-9/4}}$$

$l_{AdS} >> l_s$, then r_+ large(**stable, different from flat space**).
$l_{AdS} \sim l_s$, then r_+ has minimum.

5. Summary

- String gas in $AdS_3 \times \mathcal{M}$

  Evaluated 1-loop partition function of SL(2,R) WZW and internal+ghost CFT

- Hagedorn temperature $\beta_{AdS}(k)$, Density of states $\Omega(E)$.

  Single long string is dominant configuration in $\beta \sim \beta_{AdS}$.

- Strings/BH correspondence in AdS_3.

  Stable BH with large size is possible. Minimum size of BH?

Outlook

- String gas in $AdS_3 \times \mathcal{M}$

  Effect of gravity, discrepancy from MB gas ....

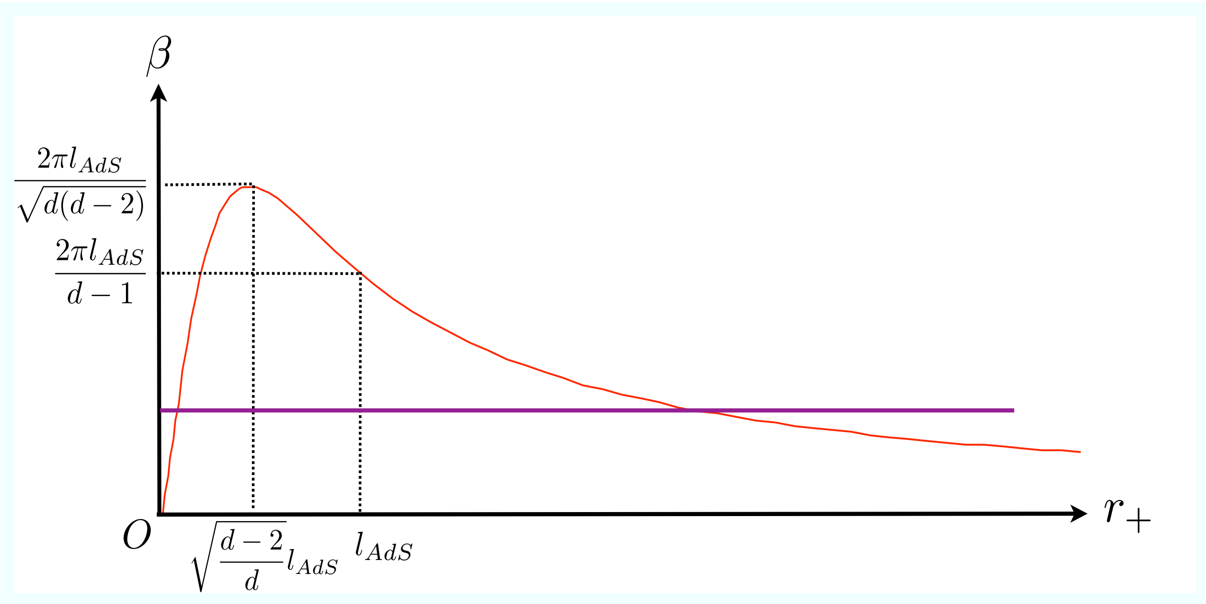
- Hagedorn temperature $\beta_{AdS}(k)$, Density of states $\Omega(E)$.

  Introducing chemical potentioal, winding string, short string effect ....

- Strings/BH correspondence in AdS_3.

  More verification, prediction ....

(END)



Fiuger of the relation between β and AdS BH radius

Discussion: Stirngs in BTZ BH

AdS_3 and BTZ BH are related by coordinate transformation.

Moduli parameter of boundary torus:

$$\tau_{AdS} = \frac{i\beta}{2\pi\sqrt{k}}, \quad \tau_{BTZ} = \frac{i\beta_{BTZ}}{2\pi\sqrt{k}}$$

Relation :

$$\tau = -1/\tau_{BTZ}.$$

The string partition function on BTZ

$$\mathcal{Z}_{BTZ}(\beta_{BTZ}) = \mathcal{Z}(\frac{4\pi^2 k}{\beta}).$$

Hagedorn divergence if $\beta_{BTZ} \geq \frac{4\pi^2 k}{\beta_{AdS}} \equiv \beta_{BTZ}^{Hag}$.

It may implies minimum size of BTZ: $r_+^{min} = \frac{\beta_{AdS}}{2\pi}$.

String α' correction makes BTZ unstable.