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On the moduli space of semilocal strings and lumps

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1. Introduction to (Non-Abelian) Semilocal Vortex
2. Analysis of Moduli Space by Moduli Matrix $\Longrightarrow$ A Duality in Bulk Theory
3. Worldsheet Effective Dynamics of Moduli
4. Summary and Discussion

Based on arXiv:0704.2218 [hep-th] (accpeted for publication in Phys. Rev. **D**) by

M. Eto, J. Evslin, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci and N. Y.

1 Introduction to (Non-Abelian) Semilocal Vortex

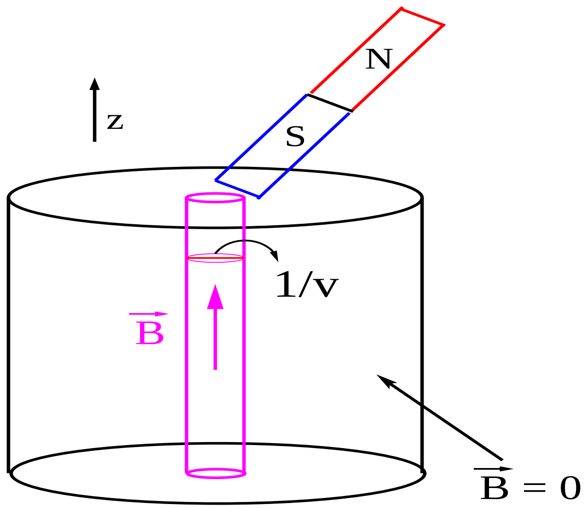
- Abrikosov-Nielsen-Olesen (ANO) Vortex in Abelian Higgs Model

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} |\mathcal{D}_\mu \Phi|^2 - \frac{\lambda}{8} \left(|\Phi|^2 - v^2 \right)^2 \right).$$

Finite Tension Soln. of Eq. of Motion : For BPS Case,

$$(\mathcal{D}_x + i\mathcal{D}_y)\,\Phi = 0, \qquad B + \frac{1}{2}\left(|\Phi|^2 - v^2\right) = 0.$$

★ ANO Vortex as Squeezed Magnetic Flux in Type II Superconductor



- Flux Energy is Proportional to Length

  $\Longrightarrow$ "Probe Monopoles" are Confined.

- Stability from Non Simply-Connected Vacuum Manifold

  $\Longleftarrow \quad \pi_1(S^1) = \pi_1(U(1)) = \mathbf{Z}.$

- Characterized by Only Positions on the plane.

- Extension to Multi-Flavor Case : $\Phi \longrightarrow \Phi_I$ $(I = 1, \cdots, N_f)$

  What Happens for the Simplest $N_f = 2$ Case ?

1. Vacuum Manifold Changes to S^3. ($\longleftarrow \Phi_I^\dagger \Phi_I = v^2$)

   ♢ $\pi_1(S^3) = 1$ (Trivial) $\Longrightarrow$ Stable Solution ?

   From Analysis of Perturbation (by Hindmarsh)

   ♢ For $\lambda/e^2 \leq 1$, Stable Solutions Do Exist and Classified by $\pi_1(U(1))$.

2. For $\lambda/e^2 = 1$, Vortex Solutions Have Another Kind of Parameters.

   ♢ These "Size" Moduli Determine Transverse Size of Vortex !

3. Large r Behavior is Quite Different from ANO $\Rightarrow$ "Lump" in Sigma Models.

These Vortex Solutions are Called Semilocal Vortex (or String) (Vachaspati-Achucarro).

★ Why This Semilocal Vortex is Interesting ?

● Another Extension of ANO Vortex from $U(1)$ to $U(N)$ Gauge Theory

In Such Extensions, There Exists Another Type of Vortex Solutions.

◇ Non-Abelian (NA) Vortex (Hanany-Tong, Auzzi-Bolognesi-Evslin-Konishi-Yung)

- NA-Vortex Has Also Another Kind of Parameters

$\Longrightarrow$ "Orientation" Moduli from $SU(N)$ Color-Flavor Diag. Symmetry.

We Have Studied Non-Abelian Ver. of Electric-Magnetic Duality and Confinement from

◇ Orientation Moduli of NA-Vortex. (Cf. Eto et. al., hep-th/0611313)

Quantum Mechanically, Our Analysis on NA-Duality Requires Certain Number of Flavors.

$\Longrightarrow$ NA-Vortex Becomes Semilocal and "Size" and "Orientation" Moduli Both Appear !

$\Downarrow$

Knowledge of Moduli Space for NA-Semilocal Vortex Gives New Hints for NA-Duality.

★ We Discuss the Moduli Space of NA-Semilocal Vortex Using Moduli Matrix Formalism.

2 Analysis of Moduli Space by Moduli Matrix

- Short Review of Local Non-Abelian Vortex

Consider the $U(N_c)$ Gauge Theory with Higgs Scalars

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - \frac{2}{g^2}\mathcal{D}_\mu\Phi^\dagger\,\mathcal{D}^\mu\Phi - \mathcal{D}_\mu H\,\mathcal{D}^\mu H^\dagger \right.$$
$$\left. -\lambda\left(\xi\,\mathbf{1}_{N_c} - H\,H^\dagger\right)^2\right] + \mathrm{Tr}\left[\,(H^\dagger\Phi - m\,H^\dagger)(\Phi\,H - m\,H)\,\right],$$

where Φ : Adjoint Higgs, H : N_f Fund. Higgs in $(N_c \times N_f)$ Matrix Form.

For Local $N_f = N_c$ Case, the Vacuum Becomes

$$\langle\Phi\rangle = m\,\mathbf{1}_{N_c}, \qquad \langle H\rangle = \sqrt{\xi}\,\mathbf{1}_{N_c}.$$

★ The Vacuum Preserves Color-Flavor Diagonal Sym. $SU(N_c)_{C+F}$.

- Eq. of Motion for BPS Vortex $\left(\lambda = g^2/4\right)$:

$$(\mathcal{D}_x + i\mathcal{D}_y)\ H = 0, \quad F_{xy} + \frac{g^2}{2}\left(\xi\,\mathbf{1}_N - H\,H^\dagger\right) = 0.$$

◇ "Non-Abelian" Zero Modes from the Breaking of $SU(N_c)_{C+F}$ by Vortex.

- Moduli Matrix Formalism for Non-Abelian Vortex (Eto-Isozumi-Nitta-Ohashi-Sakai)

Solutions for the Eq. of Motion ($z = x + i\,y$):

$$H = S^{-1}(z,\bar{z})\, H_0(z), \quad A_x + i\,A_y = -2\,i\,S^{-1}\,\bar{\partial}_z S(z,\bar{z}).$$

- $S(z,\bar{z})$ Satisfies a Nonlinear “Master Equation”:

$$\partial_z\,(\Omega^{-1}\partial_{\bar{z}}\,\Omega) = \frac{g^2}{4}\,(\xi\,1_N - \Omega^{-1}\,H_0\,H_0^\dagger). \quad (\Omega \equiv SS^\dagger)$$

- $H_0(z)$ is Moduli Matrix Encoding All Moduli Parameters up to the V-Transformation :

$$H_0(z) \longrightarrow V(z)H_0(z), \quad S(z,\bar{z}) \longrightarrow V(z)S(z,\bar{z}) \quad (V(z) \text{ is Hol. Matrix}).$$

- Vortex Number (Flux) k is Encoded in $\det H_0(z) \sim z^k$ at $z \longrightarrow \infty$.

- Another Construction of Moduli Space by Kähler Quotient

$$\frac{\{H_0(z)|\ \det H_0 \sim z^k\}}{\{V(z)|\, V \in GL(N_c;\mathrm{C})\}} \Longleftrightarrow \frac{\{\mathrm{Z},\Psi|\ (k\times k) \text{ and } (N_c\times k) \text{ Const. Matrix}\}}{\{U|\, U \in GL(k;\mathrm{C})\}},$$

where $Z \sim UZU^{-1}$ and $\Psi \sim \Psi U^{-1}$.

★ Simplest Example : 1-Vortex in $U(N)$ Theory $\longrightarrow$ $\mathcal{M} = \mathbb{C}P^{N-1}$.

- Moduli Space for Semilocal Non-Abelian Vortex with $N_f > N_c$
  - Non-Trivial Degenerate Higgs Vacua Appear:

$$\mathcal{V}_{\rm Higgs} \simeq \frac{SU(N_f)}{SU(N_c) \times SU(N_f - Nc) \times U(1)}$$

$$\Longrightarrow \quad SU(N_c)_{C+F} \times SU(N_f - N_c) \text{ Global Symmetry is Preserved.}$$

  - Moduli Matrix Becomes Rectangular : $H_0(z) = (D(z), Q(z))$,

    where $D(z)$: $N_c \times N_c$ Matrix and $Q(z)$: $N_c \times (N_f - N_c)$ Matrix.

$$\Longrightarrow \quad \text{Additional “Size” Moduli Appear from } Q(z).$$

  - Vortex No. $k \quad \Longleftarrow \quad \det H_0 H_0^\dagger \sim |z|^{2k} \qquad (|z| \sim \infty)$.

However, Kähler Quotient Construction Can be Applied to Semilocal Case :

$$\frac{\{H_0^{(k)}(z)\}}{\{V(z)\}} \Longleftrightarrow \frac{\{\mathbf{Z}, \Psi, \tilde{\Psi} |\ (k \times k), (N_c \times k), (k \times (N_c - N_f))\ \text{Matrix}\}}{\{U |\, U \in GL(k; \mathbf{C})\}},$$

where
$$\{Z, \Psi, \tilde{\Psi}\} \sim \{UZU^{-1}, \Psi U^{-1}, U\tilde{\Psi}\}$$

- Structure of the Quotient

From Construction with $H_0(z)$, $GL(k;\mathrm{C})$ Action Should be FREE on (Z,Ψ) :

$$\{UZU^{-1},\Psi U^{-1}\}=\{Z,\Psi\}\quad\Longrightarrow\quad U=1.$$

With this Condition, the Quotient $\{Z,\Psi,\widetilde{\Psi}\}/GL(k;\mathrm{C})$ is Equivalent to

$$\{(Z,\Psi,\widetilde{\Psi})|\ \ [Z^\dagger,Z]+\Psi^\dagger\Psi-\widetilde{\Psi}\widetilde{\Psi}^\dagger-r=0\,\}/U(k)\qquad (r>0).$$

$\Longrightarrow$ D-flat Conditions for Some 2-Dimensional Gauge Theory $\sim$ D-Brane Set-Up

- The Exchange Such That :

1. $N_c\longrightarrow N_f-N_c\equiv\widetilde{N_c}$
2. $GL(k;\mathrm{C})$ Free on $(\mathrm{Z},\Psi)\longrightarrow GL(k;\mathrm{C})$ Free on $(\mathrm{Z},\widetilde{\Psi})$
3. $r>0\longrightarrow r<0$.

Gives A Different Moduli Space of Vortex in $U(N_f-N_c)$ Gauge Theory.

★ These Moduli Spaces Corresspond to Two Different Reg. of A Parent Space !

Note : Vacuum of the Theory is Invariant under $N_c\longrightarrow\widetilde{N_c}$.

- Simplest Example of Moduli Space :

1-Vortex in $U(2)$ Gauge Theory with $N_f = 3$ $\quad (GL(1; \mathbf{C}) = U(1)^{\mathbf{C}} = \mathbf{C}^*)$

$$\left(\mathbf{Z}, \boldsymbol{\Psi}, \widetilde{\boldsymbol{\Psi}}\right) \sim \left(\mathbf{Z}, \lambda^{-1}\boldsymbol{\Psi}, \lambda\widetilde{\boldsymbol{\Psi}}\right), \qquad \lambda \in \mathbf{C}^*,$$

where Z, $\widetilde{\boldsymbol{\Psi}}$: Constant and $\boldsymbol{\Psi}$: 2-Vector.

Except for Position Moduli Z, Internal Moduli Space Appears to be

$$W\mathbb{C}P^2[1, 1, -1] \; : \; (y_1, y_2, y_3) \sim (\lambda\, y_1, \lambda\, y_2, \lambda^{-1}\, y_3) \quad (\neq (0, 0, 0)) \, .$$

$$\Downarrow$$

◊ This Space is NON-Hausdorff Space !

Because Two Distinct Points $(a, b, 0)$ and $(0, 0, 1)$ Has NO Disjoint Neighborhoods:

$$(\epsilon\, a, \epsilon\, b, 1) \; \sim \; (a, b, \epsilon), \qquad \text{where } \epsilon \text{ is Arbitrarily Small.}$$

In Order to Make the Space Hausdorff, We Should Eliminate Either Point:

★ Two "Regularizations" $\Longrightarrow$ Two Different Manifolds

This Corresponds to the Choice Between $U(2)$ Theory and "Dual" $U(1)$ Theory

- Two "Regularized" Spaces as Moduli Spaces of "Dual" Theories

  1. $W\mathbb{C}P^2[\underline{1,1},-1] \equiv W\mathbb{C}P^2[1,1,-1]-(0,0,1)$

     Moduli Space of $U(2)$ Theory $\Longrightarrow$ $\mathcal{M}_{2,3}=\widetilde{\mathbf{C}}^2$: Blow Up of $\mathbf{C}^2$

  2. $W\mathbb{C}P^2[1,1,\underline{-1}] \equiv W\mathbb{C}P^2[1,1,-1]-\mathbb{C}P^1$

     Moduli Space of $U(1)$ Theory $\Longrightarrow$ $\mathcal{M}_{1,3}=\mathbf{C}^2$

  ★ $GL(k,\mathbf{C})$ Free Condition $\Longleftrightarrow$ Eliminating "Irregular" Subspace.

- Self-Dual Case : 1-Vortex in $U(2)$ Theory with $N_f=4$:

  Parent Space is $W\mathbb{C}P^3[1,1,-1,-1]$ $\Longrightarrow$ Two Reg. Give Same Space.

  ★ Resolved Conifold : $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\rightarrow\mathbb{C}P^1$

- Generalization to $U(N_c)$ with N_f : Parent Space is $W\mathbb{C}P^{N_f-1}[1^{N_c},-1^{N_f-N_c}]$.

  1. $\mathcal{M}_{N_c,N_f}=W\mathbb{C}P^{N_f-1}[\underline{1^{N_c}},-1^{\widetilde{N}_c}]$: $\mathcal{O}(-1)^{\oplus\widetilde{N}_c}\rightarrow\mathbb{C}P^{N_c-1}$.

  2. $\mathcal{M}_{\widetilde{N}_c,N_f}=W\mathbb{C}P^{N_f-1}[1^{N_c},\underline{-1^{\widetilde{N}_c}}]$: $\mathcal{O}(-1)^{\oplus N_c}\rightarrow\mathbb{C}P^{\widetilde{N}_c-1}$.

- Lump Solution in Strong Coupling Limit

LEET of Strong Coupling Limit $\Longrightarrow$ Non-Linear Sigma Model on $\mathcal{V}_{\text{Higgs}}$.

This Sigma Model Has Codim. 2 Lump Solitons from $\pi_2(\mathcal{V}_{\text{Higgs}}) = \mathbf{Z}$.

$\Longrightarrow$ In the Strong Coupling Limit, Our Vortex Becomes this Lump Soliton.

Moduli Space of Smooth k-Lump Soliton is Also Determined by Moduli Matrix :

$$\begin{aligned}\mathcal{M}^{\text{lump}}_{N_c,N_f} &= \left\{(\mathbf{Z},\Psi,\widetilde{\Psi})|GL(k,\mathbf{C}) \text{ free on } (\mathbf{Z},\Psi) \text{ and } (\mathbf{Z},\widetilde{\Psi})\right\}/GL(k,\mathbf{C}) \\ &= \mathcal{M}_{N_c,N_f} \cap \mathcal{M}_{\widetilde{N}_c,N_f}.\end{aligned}$$

Finally, We Have the Following Diamond Diagram:

$$\begin{tikzcd}
 & \widehat{\mathcal{M}}_{N_c,N_f}\left(W\mathbb{C}P^2_{(1,1,-1)}\right) \arrow[dl] \arrow[dr] & \\
\mathcal{M}_{N_c,N_f}\left(\widetilde{\mathbf{C}}^2\right) \arrow[rr, dotted, leftrightarrow, "\text{Seiberg-like Duality}"] \arrow[dr, "g^2\to\infty"'] & & \mathcal{M}_{\widetilde{N}_c,N_f}\left(\mathbf{C}^2\right) \arrow[dl, "\tilde{g}^2\to\infty"] \\
 & \mathcal{M}^{\text{lump}}_{N_c,N_f}\left((\mathbf{C}^2)^*\right) &
\end{tikzcd}$$

3 Worldsheet Effective Dynamics of Moduli

- Worldsheet Effective Theory on Vortex

Possible to Obtain Eff. Theory by Promoting the Moduli to Slowly-Moving Fields

$$\Downarrow$$

2-Dim. Non-Linear Sigma Model on Our Moduli Space

In SUSY Context, Moduli Matrix Can Provide the Kähler Potential :

$$K = \mathrm{Tr}\int d^2z\left(\xi\ \log\Omega + \Omega^{-1}H_0H_0^\dagger + \mathcal{O}(1/g^2)\right).$$

Note : This Gives Standard $\mathbb{C}P^N$ Metric for Local NA-Vortex.

Crucial Difference from Local Vortex is

Existence of Non-Normalizable Moduli (such as "Size" Moduli).

Actually, Large r Behavior of Kähler Pot. Becomes (L : IR Cut-Off)

$$K \simeq 2\pi\xi\ \log L\ \mathrm{Tr}\left|\Psi\widetilde{\Psi}\right|^2 + \mathrm{const.} + \mathcal{O}(L^{-1}),$$

- Explicit Form of Kähler Potential for $U(2)$ Theory with $N_f = 3$:

$$K_{N_c=2,N_f=3} = \xi\pi|c|^2(1+|b|^2)\log\frac{L^2}{|c|^2(1+|b|^2)} + \mathcal{O}(L^0).$$

Replacement $(\tilde{c} = c,\ \tilde{b} = c\,b)$ Gives $K_{N_c=1,N_f=3}$ of $U(1)$ Dual Theory.

- Number of Normalizable Moduli ($\sim$ Dynamical Fields) Depends on

$\diamondsuit$ Rank of Matrix $\Psi\widetilde{\Psi}$.

$\Longrightarrow$ Number of Normalizable Moduli Does Change on Some Submanifold !

Simple Example : 1-Vortex in $U(N_c)$ Theory $\Longrightarrow$ Rank$(\Psi\widetilde{\Psi}) = \ell \leq 1$.

1. For $\ell = 1$ $(c \neq 0)$, Only Position Moduli is Normalizable $(\sim \mathbf{C})$.
2. For $\ell = 0$ $(c = 0)$, Space of Normalizable Moduli Becomes $\mathbf{C} \times \mathbb{C}P^{N_c-1}$.

★ Non-Trivial Moduli Enhancement Occurs !

Note : More Interesting Phenomena Occur in general k-Vortex Case.

4 Summary and Discussion

- Summary
  - We Have Discussed Aspects of the Moduli Space of Semilocal Non-Abelian Vortex in $U(N_c)$ Gauge Theory with $N_f > N_c$ by Using the Moduli Matrix Formalism.
  - We Have Found A Geometrical Correspondence of the Moduli Spaces of Vortex in the $U(N_c)$ Theory and $U(N_f - N_c)$ Theory.
  - We Have Also Studied Effective Theory of Moduli on the Worldsheet of Vortex.
- Discussion
  1. Understanding of Bulk Gauge Theory Dynamics Using Vortex Dynamics

     $\Longrightarrow$ Seiberg Duality from Semilocal NA-Vortex Moduli ?
  2. Understanding of the Dynamics for Moduli Enhancement.
  3. Dynamics of NON-BPS Semilocal NA-Vortex