YIITP Workshop 2010

Six-point gluon scattering amplitudes from $\mathbb{Z}_4$-symmetric integrable model

Yasuyuki Hatsuda (YITP)

Based on arXiv:1005.4487 [hep-th]

in collaboration with
K. Ito (TITECH), K. Sakai (Keio Univ.),
and Y. Satoh (Univ. of Tsukuba)

AdS/CFT Correspondence

- **Concrete example of gauge/string duality** — Maldacena '97

Type IIB Strings on $AdS_5 \times S^5$ vs. **$\mathcal{N}=4$ SU(N) Super Yang-Mills**

$$\boxed{\lambda \equiv g_{\rm YM}^2 N = 4\pi g_s N = \frac{R^4}{\alpha'^2}}$$

- **In the 't Hooft limit: $N \to \infty$ with λ held fixed, planar SYM/free string contribution is dominant**
- **The AdS/CFT correspondence is a strong/weak type duality**
  - **Gauge theory side: weak coupling analysis** $\lambda \ll 1$
  - **String theory side: strong coupling analysis** $\lambda \gg 1$

  ➡ **difficult to compare**

Integrability in AdS/CFT

- **Planar $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$ string theories have integrable structures**

  Minahan, Zarembo '02
  Bena, Polchinski, Roiban '03

- **Integrability is a power tool to analyze the spectrum of both theories**
- **Integrability also plays an important role in studying scattering amplitudes**

  ➡ **Thermodynamic Bethe ansatz (TBA) appears**

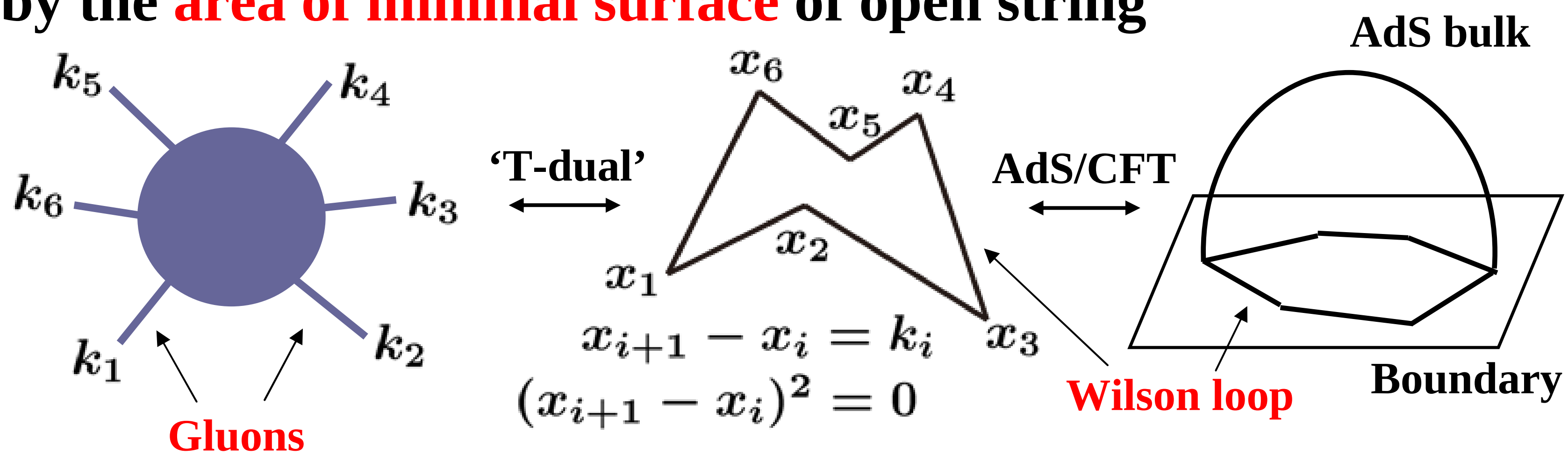
  Alday, Gaiotto, Maldacena '09

- **Motivation: With the help of integrability, we would like to find a formulation that connects weak and strong coupling analyses**

Alday-Maldacena Program

- How to compute gluon scattering amplitudes at **strong coupling** by using AdS/CFT

  Alday, Maldacena '07
- There is a duality between gluon amplitudes and expectation values of null polygonal Wilson loops
- The expectation value of Wilson loop can be computated by the **area of minimal surface** of open string



$$x_{i+1} - x_i = k_i$$
$$(x_{i+1} - x_i)^2 = 0$$

$$\boxed{\mathcal{A} \sim \langle W \rangle \sim e^{\frac{\sqrt{\lambda}}{2\pi}(\text{Area})}} \qquad \lambda \gg 1$$

- **Strategy**

Alday, Gaiotto, Maldacena '09

Start with classical strings in AdS

Solve equation of motion with null polygonal boundary

Substitute a solution into action

Area of minimal surface

Solve a set of integral equations (TBA equations)

Compute free energy

- **It is hard to construct solutions with polygonal boundaries**
- **Our goal is to know the area of minimal surface, not to construct solutions**
- **Alday, Gaiotto and Maldacena proposed a set of integral equations which determines the minimal area of the hexagonal Wilson loop in AdS(5)**

- **String sigma-model in AdS(5)**
- **Pohlmeyer reduction**

  EoMs + Virasoro constraints → Hitchin equations
- **Stokes phenomenon for solutions of Hitchin equations**
- **Consider a solution in each Stokes sector**
- **Define new functions $Y_j(\theta)$ from such solutions**
- **These *Y*'s satisfy some functional relations (Y-system)**
- **We can rewrite Y-system as a set of integral equations**
- **Such equations are of the form of Thermodynamic Bethe ansatz (TBA) equations**

$$\Downarrow$$

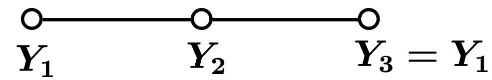
- **We studied the TBA equations for six-point case in detail**

Y-system and TBA equations

$$f^{\pm} \equiv f\left(\theta \pm \frac{\pi i}{4}\right)$$

- **Y-system**: $Y_1^+ Y_1^- = 1 + Y_2$

$$Y_2^+ Y_2^- = (1+\mu Y_1)(1+\mu^{-1} Y_1)$$



- **TBA equations**: $\epsilon(\theta) \equiv \log Y_1(\theta), \quad \tilde{\epsilon}(\theta) \equiv \log Y_2(\theta)$

Alday, Gaiotto, Maldacena '09

$$\epsilon(\theta) = 2|Z|\cosh\theta + \mathcal{K}_2 * \log(1+e^{-\tilde{\epsilon}}) + \mathcal{K}_1 * \log(1+\mu e^{-\epsilon})(1+\mu^{-1}e^{-\epsilon})$$

$$\tilde{\epsilon}(\theta) = 2\sqrt{2}|Z|\cosh\theta + 2\mathcal{K}_1 * \log(1+e^{-\tilde{\epsilon}}) + \mathcal{K}_2 * \log(1+\mu e^{-\epsilon})(1+\mu^{-1}e^{-\epsilon})$$

$$\mathcal{K}_1(\theta) = \frac{1}{2\pi\cosh\theta}, \quad \mathcal{K}_2(\theta) = \frac{\sqrt{2}\cosh\theta}{\pi\cosh 2\theta}$$

$$f * g = \int_{-\infty}^{\infty} d\theta' f(\theta-\theta') g(\theta')$$

Minimal Area

- **Although the area of the minimal surface is divergent, we can regularize it in a well understood way**

BDS conjecture **Bern, Dixon, Smirnov '05**

$$A = A_{\rm div} + A_{\rm BDS} - R$$

remainder function

$$R = R_1 - |Z|^2 - A_{\rm free}$$

$$R_1 = -\frac{1}{4}\sum_{k=1}^{3} {\rm Li}_2(1-U_k)$$

cross ratios $x_{ij} \equiv x_i - x_j$

$$U_1 = \frac{x_{14}^2 x_{36}^2}{x_{13}^2 x_{46}^2}, \; U_2 = \frac{x_{25}^2 x_{14}^2}{x_{24}^2 x_{15}^2}, \; U_3 = \frac{x_{36}^2 x_{25}^2}{x_{35}^2 x_{26}^2}$$

$$A_{\rm free} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta \Big(2|Z|\cosh\theta \log(1+\mu e^{-\epsilon(\theta)})(1+\mu^{-1}e^{-\epsilon(\theta)}) + 2\sqrt{2}|Z|\cosh\theta \log(1+e^{-\tilde{\epsilon}(\theta)}) \Big)$$

Goal

- Our goal is to know the remainder function as a function of the cross ratios
- Three cross ratios are related to the Y-function

$$U_k = 1 + Y_2\left(\frac{(2k-1)\pi i}{4} - i\varphi\right) \quad (k = 1, 2, 3)$$

- Thus we can relate the cross ratios to three parameters in TBA systems in principle

$$(U_1, U_2, U_3) \leftrightarrow (|Z|, \varphi, \mu)$$

- The TBA equations are easily solved numerically
- In some special cases, we can obtain analytical results

Exact Result at Massless Limit

- TBA equations can be solved in the massless limit $|Z| \to 0$

Alday, Gaiotto, Maldacena '09

- In this limit, Y-functions are independent of θ

  Functional relations → algebraic equations

$$Y_1^2 = 1 + Y_2, \quad Y_2^2 = (1+\mu Y_1)(1+\mu^{-1} Y_1)$$

$$\Rightarrow Y_1 = 2\cos\left(\frac{\phi}{3}\right), \; Y_2 = 1 + 2\cos\left(\frac{2\phi}{3}\right) \qquad \mu = e^{i\phi}$$

- The free energy is given by

$$A_{\text{free}} = \frac{1}{\pi}\left(\mathcal{L}_{\mu}(Y_1) + \mathcal{L}_{\mu^{-1}}(Y_1) + \mathcal{L}_1(Y_2)\right) = \frac{\pi}{6} - \frac{\phi^2}{3\pi}$$

$$\mathcal{L}_{\lambda}(x) \equiv \frac{1}{2}\left(\log x \log\left(1 + \frac{\lambda}{x}\right) - 2\,\mathrm{Li}_2\left(-\frac{\lambda}{x}\right)\right)$$

- **In this limit, three cross ratios are all equal**

$$U_1 = U_2 = U_3 = 4\cos^2\left(\frac{\phi}{3}\right)$$

- **We obtain the exact expression of the remainder function**

$$R(U,U,U) = -\frac{\pi}{6} + \frac{\phi^2}{3\pi} - \frac{3}{4}\,\mathrm{Li}_2(1-U)$$

$$U = 4\cos^2\left(\frac{\phi}{3}\right)$$

Analysis near Massless Limit

- **We can also obtain analytical expression near $|Z| \sim 0$ by using the CFT technique** YH, Ito, Sakai, Satoh, arXiv:1005.4487
- **Recall that wide classes of 2d massive integrable models can be regarded as mass deformations of CFTs** Zamolodchikov '87

$$S = S_{\mathrm{CFT}} + \lambda \int d^2x\, \varepsilon(x) \qquad \Delta_\varepsilon = \bar{\Delta}_\varepsilon = \frac{1}{3}$$

$\mathbb{Z}_4$ **parafermion CFT in our case (n = 6)**

- **The coupling constant is exactly related to the mass of TBA system** Fateev '94

$$(2\pi\lambda)^2 = \left[2\sqrt{\pi}\gamma\left(\frac{3}{4}\right)|Z|\right]^{8/3}\gamma\left(\frac{1}{6}\right), \quad \gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$$

$$\lambda = (0.44975388\ldots)|Z|^{4/3}$$

Free Energy near CFT point

- **Partition function**

evaluate by CFT action

$$\mathcal{Z} = \langle 1 \rangle = \left\langle \exp\left[-\lambda \int d^2x\, \varepsilon(x)\right] \right\rangle_0$$

- **The free energy is perturbatively expanded as**

$$A_{\mathrm{free}} = A_{\mathrm{free}}^{(\mathrm{CFT})} - |Z|^2 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n (2\pi)^{-\frac{4}{3}n+2}}{n!}$$

$$\times \int \prod_{j=2}^{n} d^2 z_j |z_j|^{-4/3} \langle V(0)\varepsilon(1)\varepsilon(z_2)\cdots\varepsilon(z_n)V(\infty)\rangle_{0,\mathrm{connected}}$$

- **Due to $\mathbb{Z}_2$-symmetry $\varepsilon \to -\varepsilon$, the terms with odd *n* vanish**
- **The first non-trivial correction is *n* = 2 case**

- **For n = 2, we can evaluate the correlation function exactly**

$$\langle V(0)\varepsilon(1)\varepsilon(z_2)V(\infty)\rangle_{0,\mathrm{connected}} = |1-z_2|^{-\frac{4}{3}}|z_2|^{\frac{2\phi}{3\pi}}$$

- **The first correction of the free energy:**

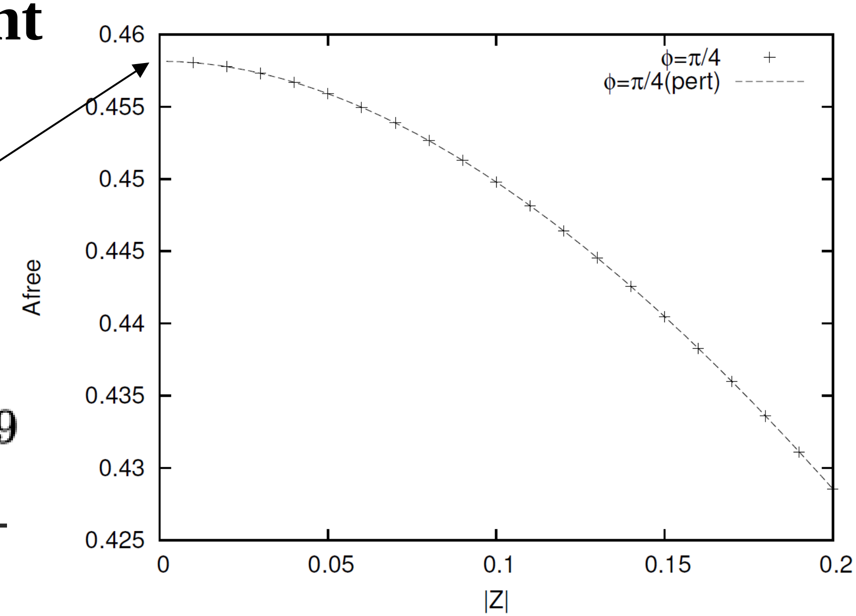
$$\delta A_{\mathrm{free}}^{(n=2)} = C\gamma\left(\frac{1}{3}+\frac{\phi}{3\pi}\right)\gamma\left(\frac{1}{3}-\frac{\phi}{3\pi}\right)|Z|^{8/3}$$

$$C = \frac{\pi}{2}\left[\frac{1}{\sqrt{\pi}}\gamma\left(\frac{3}{4}\right)\right]^{8/3}\gamma\left(\frac{1}{6}\right)\gamma\left(\frac{1}{3}\right) = 0.18461\ldots$$

- **This result is in good agreement with the numerical result!**

$$A_{\mathrm{free}}^{(\mathrm{CFT})}|_{\phi=\pi/4} = \frac{7\pi}{48}$$

$$\approx 0.458149$$



Remainder Function near CFT point

- To compute the remainder function, we need to know the behavior of R_1

$$R_1 = -\frac{1}{4}\sum_{k=1}^{3} \mathrm{Li}_2(1-U_k)$$

- Recall that

$$U_k = 1 + Y_2\left(\frac{(2k-1)\pi i}{4} - i\varphi\right) \quad (k = 1, 2, 3)$$

- We assume that the Y-function is expanded as

$$Y_2(\theta) = \sum_{n=0}^{\infty} \tilde{Y}_2^{(n)}(\theta, \phi)|Z|^{\frac{4}{3}n}$$

- **The first and second coefficients take the following forms**

$$\tilde{Y}_2^{(0)}(\theta,\phi) = 1 + 2\cos\left(\frac{2\phi}{3}\right)$$

$$\tilde{Y}_2^{(1)}(\theta,\phi) = y^{(1)}(\phi)\cosh\left(\frac{4\theta}{3}\right)$$

- **The perturbative expansion of R_1**

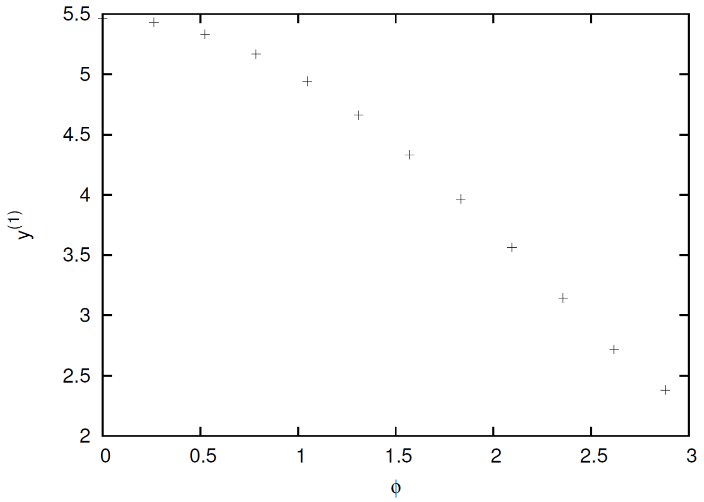
$$R_1 = \sum_{n=0}^{\infty} \tilde{R}_1^{(n)}(\varphi,\phi)|Z|^{\frac{4}{3}n}$$

$$\tilde{R}_1^{(0)}(\varphi,\phi) = -\frac{3}{4}\,\mathrm{Li}_2(1-4\beta^2) \qquad \beta = \cos\left(\frac{\phi}{3}\right)$$

$$\tilde{R}_1^{(1)}(\varphi,\phi) = 0$$

$$\tilde{R}_1^{(2)}(\varphi,\phi) = \frac{3(4\beta^2 - 1 + \log(4\beta^2))}{64\beta^2(4\beta^2-1)^2}y^{(1)}(\phi)^2$$

- **At present, we could not fix the function $y^{(1)}(\phi)$, but it can be evaluated numerically**
- **It should be fixed analytically**
- **In summary, the remainder function is expanded as**



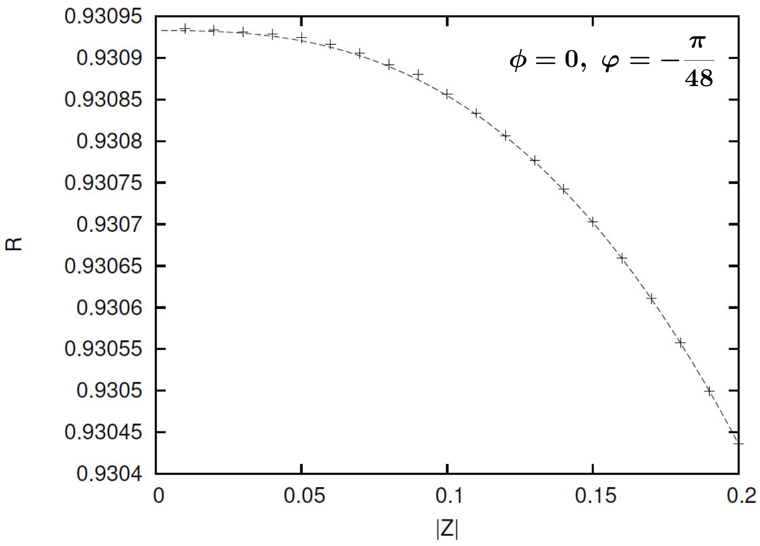
$$R = R^{(0)} + R^{(2)}|Z|^{\frac{8}{3}} + \mathcal{O}(|Z|^4)$$

$$R^{(0)} = -\frac{\pi}{6} + \frac{\phi^2}{3\pi} - \frac{3}{4}\mathrm{Li}_2(1-4\beta^2)$$

$$R^{(2)} = -C\gamma\left(\frac{1}{3}+\frac{\phi}{3\pi}\right)\gamma\left(\frac{1}{3}-\frac{\phi}{3\pi}\right)$$

$$+\frac{3(4\beta^2-1+\log(4\beta^2))}{64\beta^2(4\beta^2-1)^2}y^{(1)}(\phi)^2$$

$$\beta = \cos\left(\frac{\phi}{3}\right)$$



Comment on Large Mass Limit

- **Large mass case: $|Z| \gg 1$**
- **The TBA equations can be solved approximately**

$$\epsilon(\theta) = 2|Z|\cosh\theta + (\text{exponetial corrections})$$

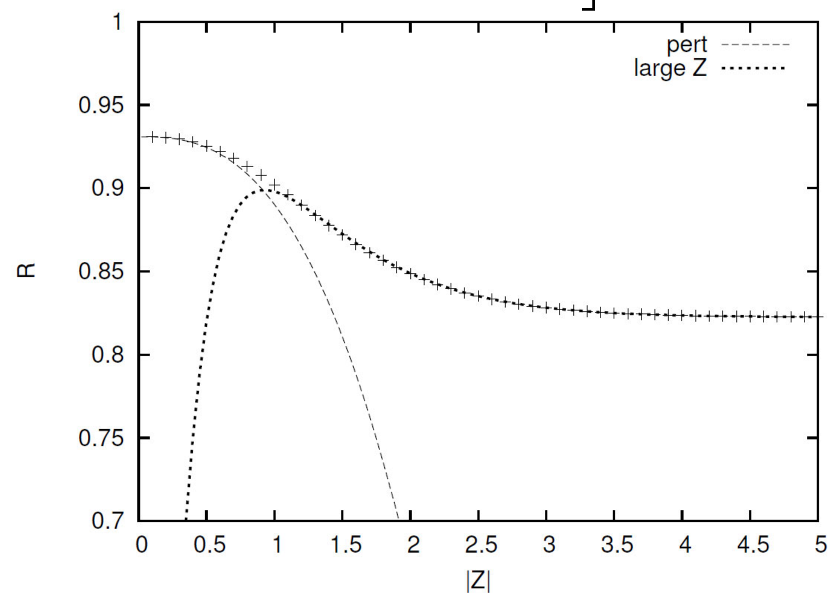
$$\tilde{\epsilon}(\theta) = 2\sqrt{2}|Z|\cosh\theta + (\text{exponetial corrections})$$

- **Free energy:**

modified Bessel fonction of the second kind

$$A_{\text{free}} \approx \frac{2|Z|}{\pi}\left[(\mu+\mu^{-1})K_1(2|Z|) + \sqrt{2}K_1(2\sqrt{2}|Z|)\right]$$

- **Similarly we can evaluate the remainder function**



Summary

- **Gluon scattering amplitude at strong coupling can be computed by the area of minimal surface with a null polygonal boundary**
- **The problem to determine the area of such minimal surface is mapped to a set of integral equations (TBA equations)**
- **We analyzed the TBA equations for six-point amplitudes in detail**
- **We obtained the analytical expression of the area up to an unknown function**
- **It is interesting to fix the analytic form of this unknown function**
- **Analysis of TBA equations for general *n*-point amplitudes**
- **Do TBA equations also appear if we consider α'-corrections?**