Integrable Bootstrap for Structure Constants in N=4 SYM

Shota Komatsu
(Perimeter Institute)

• We have no satisfactory understanding of AdS/CFT.

• It is important to study in detail how the building blocks of the two theories are related with each other.

• For conformal field theories, the building blocks $\rightarrow$ 2- and 3-point functions.

\[ \langle O_1 O_2 \rangle \quad \langle O_1 O_2 O_3 \rangle \]
AdS$_5$/CFT$_4$ correspondence

\[
\begin{align*}
N=4 & \quad U(N) \\
\text{SYM in 4d} & \quad \leftrightarrow \quad \text{String Theory on} \\
& \quad \text{AdS}_5 \times S^5
\end{align*}
\]

Goal of this talk:

Non-perturbative framework to compute 3pt-functions at finite \('t\) Hooft coupling in the large N limit.

How?

Map the problem to 2d system and use Integrability.
Interesting Observation: [Eden Heslop Korchemsky Sokatchev]

\[ \mathcal{O}_1 : \text{tr}(\tilde{Z}D^2\tilde{Z}) + \cdots \quad \text{non-BPS twist 2} \]

\[ \mathcal{O}_2 : \text{tr}(Z^3) \quad \text{BPS length 3} \]

\[ \mathcal{O}_3 : \text{tr}(\bar{Z}^3) \quad \text{BPS length 3} \]

\[ (C_{123})^2 = \frac{1}{6} - 2g^2 + 28g^4 + \cdots \]
Interesting Observation:

\[ \mathcal{O}_1 : \text{tr}(\tilde{Z} D^2 \tilde{Z}) + \cdots \] 
non-BPS twist 2

\[ \mathcal{O}_2 : \text{tr} (Z^3) \] 
BPS length 3

\[ \mathcal{O}_3 : \text{tr} (\bar{Z}^3) \] 
BPS length 3

\[
(C_{123})^2 = \frac{1}{6} - 2g^2 + 28g^4 + \cdots
\]

\[ \mathcal{O}_1 : \text{tr}(\tilde{Z} D^2 \tilde{Z}) + \cdots \] 
non-BPS twist 2

\[ \mathcal{O}_2 : \text{tr} (Z^2) \] 
BPS length 2

\[ \mathcal{O}_3 : \text{tr} (\bar{Z}^2) \] 
BPS length 2

\[
(C_{123})^2 = \frac{1}{6} - 2g^2 + (28 + 12\zeta(3))g^4 + \cdots
\]
• Why do they agree up to 1-loop?
• Why do they start to differ at 2-loop?
• How does zeta function come about?
• Why do they agree up to 1-loop?
• Why do they start to differ at 2-loop?
• How does zeta function come about?

Interesting physical mechanism behind!
1. Two-point functions

2. Perturbative computation of 3pt functions

3. 3pt from Hexagons (Asymptotic Part)

4. 3pt from Hexagons (“Wrapping” Effects)

5. Outlook
1. Two-point functions
2-point functions

\[ \langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \rangle = \frac{\delta_{ij}}{|x_1 - x_2|^{\Delta_i}} \]

\[ \Delta = \Delta_0 + \gamma \text{ anomalous dimension} \]

Single trace operators:

\[ \mathcal{O}_i = \text{tr} \left( ZXZZD_\mu Z \psi \cdots \right) \text{ etc.} \]

\[ Z = \phi_1 + i\phi_2, \quad X = \phi_3 + i\phi_4 \]

\[ D_\mu : \text{ covariant derivative} \]

\[ \psi : \text{ fermion} \]
Relation to spin chain

\[ \text{tr} \left( ZXZZX \cdots \right) \leftrightarrow | \uparrow \downarrow \uparrow \downarrow \cdots \rangle \]

Diagonalization of \( \Gamma_{IJ}^{1\text{-loop}} \) \leftrightarrow \text{Diagonalization of } H_{\text{Heisenberg}}

\( \gamma_{1\text{-loop}} \leftrightarrow E_{\text{Heisenberg}} \)

One can efficiently compute the 1-loop anomalous dimension by solving \textbf{Bethe equation}. 
Bethe equation

General spin-chain state

\[ |p_1, p_2, \ldots, p_M \rangle = \]

Bethe equation (periodicity condition)

\[ e^{ip_j L} \prod_{k \neq j} S_{jk} = 1 \iff = 1 \]

Free propagation \quad 2 \text{ to } 2 \text{ scattering}

Energy:

\[ \gamma_{1\text{-loop}} = \sum_j E(p_j) \]

Dispersion:

\[ E(p) = 2g^2 (\sin p/2)^2 \quad (g^2 = \lambda/4\pi^2) \]

S-matrix:

\[ S(p, q) = \frac{\cot p/2 - \cot q/2 + 2i}{\cot p/2 - \cot q/2 - 2i} \]
• One can repeat the same analysis for 2-loop.

• At higher loops, this approach is less effective simply because the computation of the mixing matrix becomes hard.
Use of symmetry

- Consider an infinitely long BPS operator made up of Z’s

\[ \text{tr} \left( \cdots ZZZZZZ \cdots \right) \]
Use of symmetry [Beisert]

• Consider an infinitely long BPS operator made up of $Z$'s

\[ \cdots ZZZZ \cdots \]

spin-chain vacuum

• Non BPS operators can be constructed by putting magnons on top of this vacuum:

\[ \cdots ZD_\mu ZZ \cdots \]
Use of symmetry

• Consider an infinitely long BPS operator made up of Z’s
  \[ \cdots ZZZZZ \cdots \]
  spin-chain vacuum

• Non BPS operators can be constructed by putting magnons on top of this vacuum:
  \[ \cdots ZD_\mu ZZ \cdots \]

• Symmetry preserved by the “vacuum” is
  \[ U(1) \times PSU(2|2)_L \times PSU(2|2)_R \subset PSU(2, 2|4) \]

• Magnons belong to the bifundamental irrep of PSU(2 | 2)^2

\[
\begin{align*}
\Psi_{A\dot{A}} & \quad A = (a|\alpha) \\
\dot{A} & \quad \dot{A} = (\dot{a}|\dot{\alpha}) \\
\Psi_{\alpha\dot{\alpha}} & \quad = D_{\alpha\dot{\alpha}} \\
\Psi_{a\dot{a}} & \quad = \Phi_{a\dot{a}} \\
\Psi_{a\dot{a}}, \Psi_{\alpha\dot{\alpha}} & \quad : \text{fermion}
\end{align*}
\]
In addition, the vacuum is invariant under two central charges:

\[
C_1 : \cdots \Psi \cdots \mapsto \cdots [Z, \Psi] \cdots \quad \leftarrow \{Q, Q\}
\]

\[
C_2 : \cdots \Psi \cdots \mapsto \cdots [Z^{-1}, \Psi] \cdots \quad \leftarrow \{S, S\}
\]

* These generators add or subtract one unit of $Z$. This is the symmetry of the vacuum because the chain is infinite.

2 to 2 magnon $S$-matrix is determined up to a phase by this centrally-extended symmetry.

\[
S_{12} = S_{12}^{0} \cdot S_{L} \times S_{R}
\]

Phase can be determined by requiring the crossing symmetry of the $S$-matrix.
• Assuming the factorizability of multi-particle S-matrix, one can write down the finite-coupling version of Bethe eq,

Assumption:

\[ S_{\text{multi}} = S_{2 \rightarrow 2} \]

Asymptotic Bethe Ansatz:

\[ "e^{ip_j L} \prod_{k \neq j} S_{jk} = 1" \]

Only schematic (actual equations are more complicated)

Energy:

\[ \gamma = \sum_j E(p_j) \]

Dispersion:

\[ E(p) = \sqrt{1 + 2g^2 (\sin p/2)^2} \]
“Rapidity” parametrization:

\[ E(p) = \sqrt{1 + 2g^2 \sin \frac{p}{2}^2} \]

\[ p(u) = \frac{1}{i} \log \frac{x^+(u)}{x^-(u)} \]

\[ E(u) = 1 + 2g \left( \frac{i}{x^+(u)} - \frac{i}{x^-(u)} \right) \]

\[ S_{12}^0 = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^-} \frac{1}{\sigma_{12}^2} \]

\[ x(u) = \frac{u + \sqrt{u^2 - 4g^2}}{2g} \]

\[ x^\pm(u) = x(u \pm i/2) \]
Rapidity torus

\[ p(u) = \frac{1}{i} \log \frac{x^+(u)}{x^-(u)} \quad E(u) = 1 + 2g \left( \frac{i}{x^+(u)} - \frac{i}{x^-(u)} \right) \]

\[ x(u) = \frac{u + \sqrt{u^2 - 4g^2}}{2g} \]

- cut for \( x^- \)
- cut for \( x^+ \)
Finite size correction

• For a finite-size operator, there are corrections coming from virtual particles going around the chain and scattering physical particles.

\[ e^{-LE_{\text{virtual}}} \sim O(g^{2L}) \]
Virtual particle from “mirror transformation”

\[ E^2 = 1 + 4g^2 (\sin p/2)^2 \]

\[ \mapsto \quad p_{\text{mirror}}^2 = -1 + 4g^2 \left( \sinh E_{\text{mirror}}/2 \right)^2 \]

Mirror dispersion is obtained by the analytic continuation in the u-space

\[ E \rightarrow ip_{\text{mirror}} \]

\[ p \rightarrow iE_{\text{mirror}} \]
Mirror dispersion is obtained by the analytic continuation in the u-space

\[
E^2 = 1 + 4g^2(\sin p/2)^2
\]

\[\mapsto p_{\text{mirror}}^2 = -1 + 4g^2(\sinh E_{\text{mirror}}/2)^2\]

\[E \to ip_{\text{mirror}}\]

\[p \to iE_{\text{mirror}}\]
Lessons from 2pt functions

- First study infinitely long operators.
- Make use of (centrally extended) symmetry.
- Finite size corrections from virtual particles.
- One can move a particle from one edge to the other by the “mirror transformation”.
2. Perturbative computation of 3pt functions
3-point functions

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3) \rangle = \frac{C_{ijk}}{|x_1 - x_2|^{\Delta_i + \Delta_j - \Delta_k} |x_2 - x_3|^{\Delta_j + \Delta_k - \Delta_i} |x_3 - x_1|^{\Delta_k + \Delta_i - \Delta_j}}$$
Perturbative computation

Tree-level:

[Okuyama-Tseng] [Roiban-Volovich][Alday-David-Narain-Gava]
[Escobedo-Gromov-Sever-Vieira] [Foda] [Kazama-Nishimura-S.K.]

\[ \ell_{12} = \frac{L_1 + L_2 - L_3}{2} \]
bridge length

\[ \mathcal{O}_1 : \text{tr} \left( ZZ \bar{X} \bar{X} \right) + \cdots \]
Perturbative computation

Result (tree-level SL(2) sector):

\[ \mathcal{O}_1 : \{ u \}, \quad \mathcal{O}_2 : \{ \}, \quad \mathcal{O}_3 : \{ \} \]

\[ C_{123} = \sum_{\alpha \cup \bar{\alpha} = \{ u \}} (-1)^{\bar{\alpha}} e^{i p_{\bar{\alpha}} \ell_{12}} \prod_{s \in \alpha, t \in \bar{\alpha}} f(s, t) \]

\[ f(s, t) = \frac{u - v - i}{u - v} \]

\[ \frac{f(v, u)}{f(u, v)} = S^{1\text{-loop}}(u, v) \]

square-root of S-matrix
Perturbative computation

1-loop:

The actual computation is much more complicated.
Perturbative computation

Result (one-loop SL(2) sector):

\[ \mathcal{O}_1 : \{ u \}, \quad \mathcal{O}_2 : \{ \}, \quad \mathcal{O}_3 : \{ \} \]

\[ C_{123} = \sum_{\alpha \cup \bar{\alpha} = \{ u \}} (-1)^{\bar{\alpha}} e^{ip\bar{\alpha}l_{12}} \prod_{s \in \alpha, t \in \bar{\alpha}} f(s, t) \]

\[ f(s, t) = \frac{u - v - i}{u - v} \left( 1 - \frac{g^2(u - v - i)}{(u^2 + 1/4)(v^2 + 1/4)} \right) \]

\[ \frac{f(v, u)}{f(u, v)} = S^{2-\text{loop}}(u, v) \]
Lessons from perturbative 3pt

- $3pt = \text{sum over partition of magnons}$
- Building block = “square-root” of the S-matrix.
3. 3pt from Hexagons
3pt

\[ O_1 \]

\[ O_2 \]

\[ O_3 \]
$3 \text{pt} = \text{Hexagon}^2$
More precisely...

\[ S(u_1, u_2)e^{ip_1\ell} \times u_1 \times \ + e^{i(p_1+p_2)\ell} \times u_2 \times + \]

propagation

\[ u_1u_2 \times \ + e^{ip_2\ell} \times u_1 \times \ + \]

countour

scattering
More precisely...

Sum over partitions!

\[
S(u_1, u_2)e^{ip_1 \ell} \times + e^{ip_2 \ell} \times + e^{i(p_1 + p_2) \ell} \times
\]

propagation

scattering
Building block = Hexagon form factor

Severely constrained by the symmetry
(+ Integrable bootstrap equations)
Use of symmetry

\[ \text{BPS 3pt} = \langle \text{tr} \tilde{Z}^{L_1}(x_1) \text{ tr} \tilde{Z}^{L_2}(x_2) \text{ tr} \tilde{Z}^{L_3}(x_3) \rangle \]

\[ \tilde{Z}(x) \equiv e^{\mathcal{T}^a \cdot Z(0)} \]

\[ = ((1 + a^2) \phi_1 + i(1 - a^2) \phi_2 + ia\phi_3) (0, a, 0, 0) \]

\[ \mathcal{T} = -i\epsilon_{\alpha\dot{\alpha}} P^{\alpha\dot{\alpha}} + \epsilon_{a\dot{a}} R^{a\dot{a}} \]

Twisted translation

Residual symmetry =

\[ \text{PSU}(2|2) \times \text{PSU}(2|2) \to \text{PSU}(2|2)_D \]

\[ \bigcup \]

\[ O(3) \times O(3) + 8(Q + S) \]
Use of symmetry

One magnon form factor

\[ \mathcal{h}_{AA'} \propto \epsilon_{AA'} \begin{pmatrix} \epsilon_{a\dot{a}} & 0 \\ 0 & \epsilon_{\alpha\dot{\alpha}} \end{pmatrix} \]
Use of symmetry

Two magnon form factor

\[
\begin{align*}
\mathcal{S}_{12} &= S_{12}^0 \cdot S_L \times S_R \\
&= h_{12} \times \epsilon_{A'C'} \epsilon_{B'D'} S_{AB}^{C'D'}
\end{align*}
\]

“square-root” of S-matrix
Use of symmetry

Multi-magnon form factor

\[ A\dot{A} \quad B\dot{B} \quad C\dot{C} = \prod_{i<j} h_{ij} \]

SU(2|2) S-matrices
Bootstrap eq. for $h_{12}$

Watson eq.:

$$h_{12} = S^0_{12} h_{21}$$
Bootstrap eq. for $h_{12}$

Crossing eq.:

$$h(v^{2\gamma}, u)h(v, u) = \frac{x^-(v) - x^-(u)}{x^-(v) - x^+(u)} \frac{1 - 1/x^+(v)x^-(u)}{1 - 1/x^+(v)x^+(u)}$$
Solution:

$$ h_{12} = \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^+} \frac{1}{\sigma_{12}} $$

(Not unique but this choice is the simplest and correctly reproduces the weak-coupling result.)
All-loop prediction

\[ C_{123} \propto \sum_{\alpha \cup \bar{\alpha} = \{u\}} (-1)^{\bar{\alpha}} e^{ip_{\bar{\alpha}} \cdot \ell_{12}} \prod_{s \in \alpha, t \in \bar{\alpha}} \frac{1}{h(s, t)} \]

\[ \frac{h(u, v)}{h(v, u)} = S_{12}^0(u, v) \]
Bridge-length 2

\[ (C_{123})^2 = \frac{1}{6} - 2g^2 + 28g^4 + \cdots \]

Matches with the OPE decomposition of 4pt functions of BPS ops.

[Sokatchev et al.]
Bridge-length 1

\[ (C_{123})^2 = \frac{1}{6} - 2g^2 + (28 + 12\zeta(3))g^4 + \cdots \]

Perturbation result contains a zeta-function part, which cannot be reproduced by the sum over partitions.
4. Finite size correction to Hexagons
Finite size correction

In addition to sum over partitions, we should include the virtual-particle corrections.
Virtual particle corrections

\[ \alpha \times \bar{\alpha} = \int d\omega_B \int d\omega_L \int d\omega_R \text{Integrand} \]

\[ \text{Integrand} = e^{-E(\omega_B)l_L} e^{-E(\omega_L)l_L} e^{-E(\omega_R)l_L} \times \ldots \]

Suppression coming from the propagation of the virtual particles

\[ \mathcal{O} \left( g^2(n_B l_B + n_L l_L + n_R l_R) \right) \]
Virtual particle corrections from mirror transformation

\[ \alpha \]

\[ W_L \quad W_R \quad W_B \]
Virtual particle corrections from mirror transformation

$\alpha \ w_{R}^{-\gamma} \ w_{B}^{-3\gamma} \ w_{L}^{-5\gamma}$
Virtual particle corrections from mirror transformation

\[ \alpha \quad w_{R}^{-\gamma} \quad w_{B}^{-3\gamma} \quad w_{L}^{-5\gamma} \]

\[ h(\alpha, w_{R}^{-\gamma})h(\alpha, w_{B}^{-3\gamma})h(\alpha, w_{L}^{-5\gamma}) \]
Full expression for the integrand

\[ \text{Integrand} = \mu(\mathbf{w}_B^\gamma)\mu(\mathbf{w}_L^\gamma)\mu(\mathbf{w}_R^\gamma)e^{-E(\mathbf{w}_B)}l_B e^{-E(\mathbf{w}_L)}l_L e^{-E(\mathbf{w}_R)}l_R T(\mathbf{w}_B^\gamma)T(\mathbf{w}_L^{-\gamma})T(\mathbf{w}_R^{-\gamma}) \]

\[ \times h^\#(\mathbf{w}_B^\gamma, \mathbf{w}_B^\gamma)h^\#(\mathbf{w}_L^\gamma, \mathbf{w}_L^\gamma)h^\#(\mathbf{w}_R^\gamma, \mathbf{w}_R^\gamma)h(\mathbf{w}_L^{-\gamma}, \mathbf{w}_R^{-5\gamma})h(\mathbf{w}_R^{-\gamma}, \mathbf{w}_L^{-5\gamma}) \]

\[ \times h(u, \mathbf{w}_B^{-3\gamma}) \sum_{\alpha \cup \bar{\alpha} = u} (-1)^{|\bar{\alpha}|} e^{i p_{\bar{\alpha}} l_R} \frac{h(\alpha, \mathbf{w}_L^{-5\gamma})h(\alpha, \mathbf{w}_R^{-\gamma})h(\bar{\alpha}, \mathbf{w}_L^{-\gamma})h(\bar{\alpha}, \mathbf{w}_R^{-5\gamma})}{h(\alpha, \bar{\alpha})} \]

Measure: \( \mu(u) = \text{Res}_{u=v} \frac{1}{h(u, v)} \)

Transfer matrix: \( T(u) \)
(comes from matrix structure)
Virtual particle corrections

Tree-level and 1-loop:

2-loop:

\[(C_{123})^2 = \frac{1}{6} - 2g^2 + (28 + 12\zeta(3))g^4 + \cdots\]

Zeta function indeed comes from the mirror particle.
Virtual particle corrections

Tree-level and 1-loop:

2-loop:

3-loop:
Virtual particle corrections

\[ \mathcal{O}_1 \]
\[ \mathcal{O}_2 \]
\[ \mathcal{O}_3 \]

Tree-level and 1-loop:

2-loop: No new contributions.

3-loop:
Virtual particle corrections

Tree-level and 1-loop:

2-loop: No new contributions.

3-loop:

All our predictions agree with the recent 3-loop results [Chicherin, Drummond, Heslop, Sokatchev] [Eden] (see also [Eden, Sfondrini])
Summary

- Non-perturbative approach to study 3pt functions: $3pt = \text{Hexagon}^2$
- Complete agreement with 3-loop data.
- Agreement with the strong coupling result (minimal surface in AdS) [Kazama, SK]
Future directions

1. 4-loop

2. 4-point function from hexagons?

3. Resumming virtual particles?
   Reproducing the strong coupling result? TBA, QSC for 3pt?

4. Other theories?
   ABJM? 4d N=2?
   [Pomoni, Mitev]