Gravity theory on Poisson manifold with R-flux

Hisayoshi MURAKI (University of Tsukuba)

in collaboration with

Tsuguhiko ASAKAWA (Maebashi Institute of Technology)
Satoshi WATAMURA (Tohoku University)

References

Space-time Geometry probed with strings

String theory would describe quantum gravity

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Typical Example of T-duality

Compactifying on \( X^9 \sim X^9 + 2\pi R \) (periodic)

Contribution to KK tower (Mass spectrum of reduced theory)

\[ P_9 = \frac{K}{R} \]

\[ W_9 = \frac{1}{2\pi \alpha'} 2\pi N R = \frac{R}{\alpha'} N \]

\[ R \leftrightarrow \frac{\alpha'}{R} \]

Reduced theories are physically equivalent:

\textbf{T-duality} \hspace{1cm} [84 Kikkawa, Yamasaki]
Why Generalized Geometry?

T-duality implies a physical equivalence between two different background geometries (configurations of space-time metric and NSNS $B$-field) suggesting the appearances of:

- Strange metric (T-folds) [04 Hull], ...
- Non-geometric fluxes [05 Shelton, Taylor, Wecht], ...
- Exotic branes [10 de Boer, Shigemori], ...

➤ Machinery treating these as “geometry”

↔ (Poisson) Generalized Geometry
Plan of Today’s Talk

• Introduction & Motivations

• A Little Bit More on T-duality

• Generalized Geometry
  - Definitions & Properties
  - Generalized Riemannian Geometry

• Poisson Generalized Geometry
  - Definitions & Properties
  - Poisson Generalized Riemannian Geometry
A Little Bit More on T-duality: Buscher rule

**T-duality**: Physical equiv. between two backgrounds \((g, B) \sim (\tilde{g}, \tilde{B})\)

given by Buscher rule \(^{\text{[87 Buscher]}}\) \(\text{"0": isometry}\)

\[
\begin{align*}
\tilde{g}_{00} &= \frac{1}{g_{00}}, & \tilde{g}_{0i} &= \frac{B_{0i}}{g_{00}}, & \tilde{g}_{ij} &= g_{ij} - \frac{g_{0i}g_{0j} - B_{0i}B_{0j}}{g_{00}}, \\
\tilde{B}_{0i} &= \frac{g_{0i}}{g_{00}}, & \tilde{B}_{ij} &= B_{ij} - \frac{g_{0i}B_{0j} - g_{0j}B_{0i}}{g_{00}}.
\end{align*}
\]

**Metric** and **B-field** should be on **equal footing**
Generalized Geometry

Consider tangent and cotangent bundles at the same time!

- Canonical Inner Product

\[
\langle X + \xi, Y + \eta \rangle = \frac{1}{2} (i_X \eta + i_Y \xi)
\]

- Invariant under an action of O(D,D): T-duality transf.

- Operations corresponding to diffeo. + B-field gauge transf.

\[
[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)
\]

- $O(D,D)$-invariant inner product
  → Decompose in Positive-/Negative-definite subbundles
  \[ C_{\pm} = \{ X + (\pm g + B)(X) | X \in \Gamma(TM) \} \]

- Define a connection $\nabla$ on positive-def. subbundle $C_+$:
  Coefficients = Christoffel Symbol + $H$-flux ($B$-field's field strength)
  \[ \nabla_{\partial_i}(\partial_j)^+ = g^{lk}(2\Gamma_{kij} + H_{kij})(\partial_l)^+ \]
Gravity based on Gen. Geom.

- Curvature tensor:
  \[ R(X, Y)u := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]_C})u \]

- Ricci scalar:
  \[ \mathcal{R} - \frac{1}{4} H^i{}_{jk} H_{ijk} \]

- Einstein-Hilbert-like action:
  \[ S = \int d^D x \sqrt{g} \left( \mathcal{R} - \frac{1}{4} H^2 \right) \]

  ➢ This is the same as NSNS-sector of SUGRA

G.G. would be a good tool to “geometrize” string theory
Variant of Generalized Geometry

Slightly modifying structures of GG would be interesting:

- Vector + 1-form \( \xrightarrow{\text{unchanged}} \)
- Inner product (T-dual) \( \xrightarrow{\text{unchanged}} \)
- diffeo. + B-field gauge trnsf. \( \xrightarrow{\text{changed!}} \)

\[
[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)
\]

\( \xrightarrow{\text{Then how do we change it?}} \)
Hint: A chain of T-duality

String action \[ \mathcal{L} \sim g(\dot{X}^2 + X'^2) + B \dot{X} X' \]

\[ \Pi_i \sim g_{ij} \dot{X}^j + B_{ij} X'^j \]

Conjugate momenta:

\[ [p_i, p_j] \sim H_{ijk} w^k \]

T-duality:

\[ p_i \leftrightarrow w^i = X'^i \]

Fluxes associated with non-geom. background:

\[ [p_i, w^k] \sim F_{ij}^k w^j \]

\[ [p_i, w^k] \sim Q_{ij}^k p_j \]

\[ [w^i, w^j] \sim R_{ijk}^{i} p_k \]
Fluxes in Generalized Geometry

\[ H_{ijk} \leftrightarrow F^k_{ij} \leftrightarrow Q^j_k \leftrightarrow R^{ijk} \]

\[
\begin{align*}
[\partial_i, \partial_j] & \leftrightarrow [\partial_i, dx^k] \\
\sim H_{ijk} dx^k & \sim F^k_{ij} dx^j
\end{align*}
\]

Understood in terms of GG

\[
\begin{align*}
[X, Y] & \neq 0 & [\xi, \eta] & = 0 \\
[\xi, \eta] & \neq 0 & [X, Y] & = 0
\end{align*}
\]

Variant of GG interchanging roles of Vector and 1-form
Poisson Geometry

Poisson bi-vector \( \theta = \frac{1}{2} \theta^{ij} \partial_i \wedge \partial_j \)

- Poisson bracket \( \{ f, g \} = \theta^{ij} \partial_i f \partial_j g \)

- Jacobi id. \( \{ \{ f, g \}, h \} + \{ \{ g, h \}, f \} + \{ \{ h, f \}, g \} = 0 \)

\[ \Leftrightarrow \quad [\theta, \theta]_S = 0 \quad \text{Poisson cond.} \quad \theta^{l[i} \partial_l \theta^{j]k]} = 0 \]

Schouten bracket: e.g. \( [X \wedge Y, Z]_S = [X, Z] \wedge Y - [Y, Z] \wedge X \)

Extension of Lie br. to multi-vector

Lie algebra on 1-forms

Lie bracket: \( [\xi, \eta]_\theta = \mathcal{L}_{\theta(\xi)} \eta - \mathcal{L}_{\theta(\eta)} \xi + d(\theta(\eta, \xi)) \)
Cartan Algebra on Poly-Vector Fields

“Interior product” \( \tilde{\iota}_\xi \) \( \tilde{\iota}_\xi (X \wedge Y) = (i_X \xi) Y - (i_Y \xi) X \)

“Exterior derivative” \( d_\theta = [\theta, \cdot]_S \)
- Nilpotent \( d_\theta^2 = 0 \iff [\theta, \theta]_S = 0 \)

“Lie derivative” \( \overline{\mathcal{L}}_\xi f := \tilde{\iota}_\xi d_\theta f \)
\( \overline{\mathcal{L}}_\xi \xi := [\xi, \xi]_\theta \)
\( \overline{\mathcal{L}}_\xi X := (d_\theta \tilde{\iota}_\xi + \tilde{\iota}_\xi d_\theta) X \)

“Cartan algebra”: enables diff. calculus induced by 1-form
\[
\{\tilde{\iota}_\xi, \tilde{\iota}_\eta\} = 0, \quad \{d_\theta, \tilde{\iota}_\xi\} = \overline{\mathcal{L}}_\xi, \quad [\overline{\mathcal{L}}_\xi, \tilde{\iota}_\eta] = \tilde{\iota}_{[\xi, \eta]_\theta}, \\
[\overline{\mathcal{L}}_\xi, \overline{\mathcal{L}}_\eta] = \overline{\mathcal{L}}_{[\xi, \eta]_\theta}, \quad [d_\theta, \overline{\mathcal{L}}_\xi] = 0.
\]
Poisson Generalized Geometry 1408.2649

- Vector field and 1-form (same as GG)

- O(D,D)-invariant inner product (same as GG)

- Operations different from GG: Vector field $\leftrightarrow$ 1-form

\[
[X + \xi, Y + \eta] = [\xi, \eta]_{\theta} + \mathcal{L}_\xi Y - \mathcal{L}_\eta X - \frac{1}{2} d_{\theta}(\bar{\iota}_\xi Y - \bar{\iota}_\eta X)
\]

\[\text{cf.} \quad [X + \xi, Y + \eta]_{C} = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)\]
Physical intuition of **New Operation**

- **Operation in GG**:
  
  \[
  [X + \xi, Y + \eta]_C = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)
  \]

  \[\Pi_i = p_i + B_{ij} w^j\]

  **T-dual (cf. Marc’s talk)**

  \[\tilde{\Pi}^i = w^i + \beta^{ij} p_j\]

- **Operation in PGG**:

  \[
  [X + \xi, Y + \eta] = [\xi, \eta]_\theta + \mathcal{L}_\xi Y - \mathcal{L}_\eta X - \frac{1}{2} d_\theta(i_\xi Y - i_\eta X)
  \]
• $O(D,D)$-inv inner product
  ⇒ Decompose in Positive-/Negative-definite subbundles
  \[ C_\pm = \{ \xi + (\pm G + \beta)(\xi) | \xi \in \Gamma(T^* M) \} \]

• Define a connection $\nabla$ on positive-def. subbundle $C_+$:
  Coefficients $= \text{Contravariant Levi-Civita} + R$-flux
  
  \[ \nabla_{dx^i}(dx^j)^+ = (2\Gamma^i_{kj} + G_{lk}R^{kij})(dx^l)^+ \]

  \[ R = d\theta \beta = [\theta, \beta]_S = (\theta^l[i] \partial_l \beta^{jk} + \beta^l[i] \partial_l \theta^{jk}) \partial_i \wedge \partial_j \wedge \partial_k \]
Two characteristic tensors $G^{ij}$ & $\theta^{ij}$

\[
\begin{align*}
\bar{\Gamma}^{ij}_{k} &= \frac{1}{2} \left[ \theta^{mn} (\partial_m G^{ji}) - \theta^{mi} (\partial_m G^{jn}) \\
&\quad - \theta^{mj} (\partial_m G^{in}) - G^{jl} (\partial_l \theta^{in}) - G^{il} (\partial_l \theta^{jn}) \right] G_{nk} \\
\bar{\Gamma}^{[ij]}_{k} &= \frac{1}{2} (\partial_k \theta^{ij})
\end{align*}
\]

Contravariant Levi-Civita conn. $\mathcal{G}^{ij}$ & $\theta^{ij}$

\[
\nabla_{dx^k} G^{ij} = 0 \quad : \text{Compatible with metric}
\]

\[
\nabla_{dx^i} \theta^{jk} + (\text{cyclic}) = 0 \quad : \text{Respecting Poisson strc.}
\]

In particular, $\theta^{ij}$ is covariantly constant $\nabla_i \theta^{jk} = 0 \iff$ LC conn.

Extension of the Levi-Civita respecting Poisson structure !
Gravity based on PGG

- Curvature tensor:
  \[
  \overline{R}(\xi, \eta)u := (\overline{\nabla}_\xi \overline{\nabla}_\eta - \overline{\nabla}_\eta \overline{\nabla}_\xi - \overline{\nabla}_{[\xi,\eta]} )u
  \]

- Ricci scalar:
  \[
  \overline{\mathcal{R}} - \frac{1}{4} R^{ij} R_{ijk}^k
  \]
  \text{R-flux}
  \text{Curvature defined by Contra. LC}

- Einstein-Hilbert-like action:
  \[
  S = \int d^D x \sqrt{G} \left( \overline{\mathcal{R}} - \frac{1}{4} R^2 \right)
  \]
Summary

We gave

- a new geometric framework based on Poisson structure
  - T-dual counterpart of Generalized Geometry

- a well-defined formulation of $R$-flux
  - defined as a field strength of local bi-vector gauge potentials

- an Riemann geometry compatible with Poisson structure
Future directions

We established Riemann Geometry compatible with Poisson

- Well describable **Non-geometric background**?
- Extension to **quasi-Poisson, Nambu-Poisson** structures
- Poisson is semi-classical limit of **Non-commutativity**
  - “Riemann Geometry” on Non-commutative space
- etc..........