Entanglement negativity of a free massless Dirac fermion on 2d torus

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Abstract:

- Entanglement negativity is a computable entanglement measure for mixed states.
- We calculate the entanglement negativity by bosonization.

1. Entanglement measures

Entanglement is an important nonlocal order.

e.g. topologically ordered phase
confinement/deconfinement phase

separable state
\[ \sum_i p_i \rho_i \otimes \rho_i \]
entangled state
\[ \sum_i p_i \rho_i \otimes \rho_i \neq \sum_i p_i \rho_i \otimes \rho_i \]

In quantum information theory, many entanglement measures are defined to classify the entangled states and the phases.

- Entanglement entropy
- Mutual information
- Entanglement negativity
- Distillable entanglement
- Entanglement cost

2. Entanglement negativity


Hilbert space basis
\[ \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \]

partial transpose density matrix
\[ \rho^{T_B} = \sum_i p_i^{(A)} |e_i^{(A)} \rangle \langle e_i^{(B)}| \]

logarithmic negativity
\[ \mathcal{E} = \ln \text{Tr} |\rho^{T_B}| \]

Example: two-level spin system

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle + |1\rangle \right) \]
\[ \rho = |\psi\rangle \langle \psi| \]
\[ \rho^{T_B} = \left( \begin{array}{cccc} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{array} \right) \]

eigenvalues: [1, 0, 0, 0]

\[ \mathcal{E} = \ln \text{Tr} |\rho^{T_B}| = \ln \left( \frac{1}{2} |0\rangle \langle 0| + |1\rangle \langle 1| + |1\rangle \langle 0| - |0\rangle \langle 1| \right) = \ln 2 \]

3. Advantage of entanglement negativity

1. Good entanglement measure for mixed states
- Entanglement measure of separable state is zero.
- Entanglement measure doesn’t increase under LOCC (local operations and classical communication).

Entanglement entropy of mixed states doesn’t satisfy the properties, but entanglement negativity satisfies them.

2. Computable

Definition of distillable entanglement
\[ E_D(\rho) = \sup \left\{ r : \lim_{n \to \infty} \left[ \frac{1}{n} \text{Tr} \{ \rho^{(n)} \} - \Phi(2^n) \right] = 0 \right\} \]

Computation of entanglement negativity is relatively easy.

4. Entanglement negativity of QFT

In QFT, entanglement negativity can be calculated by path integral.

But, direct calculation of \( \text{Tr} (\rho^{T_B}) \) is difficult.

So, we use a trick:

- even \( n \):
  \[ \text{Tr}(\rho^{T_B})^{(n)} = \sum_{\lambda} \lambda^{2n} = \sum_{\lambda > 0} |\lambda|^{2n} + \sum_{\lambda < 0} |\lambda|^{2n} \]

- odd \( n \):
  \[ \text{Tr}(\rho^{T_B})^{(n)} = \sum_{\lambda} \lambda^{2n} = \sum_{\lambda > 0} |\lambda|^{2n} - \sum_{\lambda < 0} |\lambda|^{2n} \]

\[ \mathcal{E} = \lim_{n \to \infty} \text{Tr} (\rho^{T_B})^{(n)} = \lim_{n \to \infty} \ln \text{Tr} (\rho^{T_B})^{(n)} \]

Generally, it dependence of \( \text{Tr} (\rho^{T_B})^{(n)} \) is different in the case of even \( n \) and odd \( n \), and we can get nonzero logarithmic negativity.

5. Replica trick

Consider two-dimensional field theory.

\[ \tau = 0 \quad \text{A} \quad \text{B} \]

\[ \sum_{3} \]

\[ \text{Tr}(\rho^{T_B})^{3} \]

Product of \( \rho^{T_B} \) corresponds to connecting three plane at \( \tau = 0 \) and partial transposition corresponds to reversal of the connection way at the red and blue lines.

We can calculate \( \mathcal{E} = \lim_{n \to \infty} \ln \text{Tr} (\rho^{T_B})^{(n)} \) by path integral on \( n \).

6. Twist fields

To calculate path integral on \( \Sigma_1 \), we introduce the twist fields.

path integral on \( \Sigma_n \) with \( \phi \)

\[ \approx \]

Example of the twist fields \( T \)

\[ \]

From correlation functions of the twist fields, we can calculate entanglement negativity like

\[ \mathcal{E} = \lim_{n \to \infty} \ln \text{Tr} (T_2^{T_A(L)} T_2^{T_A(0)}) \].
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- The result of a pure state is consistent with other results.
- We can get the result of mixed states naively, but its consistency and interpretation are unclear.

7 Free massless Dirac fermion on 2d torus

[T. Azeyanagi, T. Nishoika, T. Takayanagi, 2007]

2d torus

\[ \begin{align*}
    z &\sim z + 1 \\
    z &\sim z + i \beta \\
    \text{bosonization} & \\
    \hat{\psi} &\sim e^{i \phi} \\
    \text{Dirac fermion} &\sim \text{scalar} \phi
\end{align*} \]

By introducing the replica fields \( \psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(n)} \), we can calculate the logarithmic negativity

\[ \mathcal{E} = \lim_{n \to 1} \ln \left( \mathcal{T}_{n}^{2}(L, L) \mathcal{T}_{n}^{-2}(0, 0) \right) \]

(We fix the spin structure.)

8 Explicit form of the twist fields

Boundary conditions for \( \mathcal{T}_{n}^{2}(L, L) \)

\[ \hat{\psi}^{(m)}(z) = e^{2\pi i (2m \pi - (n-1))} \hat{\psi}^{(m)}(z) \]

In free theory, we can define \( \sigma_{m}^{2} \) as \( \mathcal{T}_{m}^{2} = \Pi_{m} \sigma_{m}^{2} \).

sum of conformal weight \( \Delta_{m} + \Delta_{m} \) of \( \sigma_{m}^{2} \)

\[ \sum_{m=0}^{n-1} (\Delta_{m} + \Delta_{m}) = \frac{2}{n} \sum_{m=0}^{n-1} (2m + 2) = (n/2 - 2/n)6 \]

It is consistent with the result of

[P. Calabrese, J. Cardy, E. Tonni, 2012]

9 Useful formula

[\[ P. D. Francesco, P. Mathieu, D. Senechal, "Conformal Field Theory"]

\[ (\Omega_{n,w}(z), \Omega_{n,w}(0)) = \left( \frac{2\pi z}{\pi} \right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \sigma_{m}^{2} \rho_{m}^{2} e^{\frac{2\pi i m}{n}} \]

To fix the spin structure, we change the summation like

\[ \sum_{m=1}^{n} q^{2m-1} \rho_{m}^{2} \approx \frac{\rho_{n}(0)^{2}}{2} \]

10 Entanglement negativity of the pure state

\[ \langle \sigma_{m}^{2}(L, L) \rangle^{2}(0, 0) \rangle \]

\[ \nu = 3 \]

\[ \mathcal{E} = \lim_{n \to 1} \ln \left( \mathcal{T}_{n}^{2}(L, L) \mathcal{T}_{n}^{-2}(0, 0) \right) \]

\[ \nu = 3 \]

\[ \mathcal{E} = \frac{1}{2} \left[ \ln \left( \mathcal{T}_{n}^{2}(L, L) \mathcal{T}_{n}^{-2}(0, 0) \right) \right] \]

\[ \nu = 3 \]

\[ \mathcal{E} = \frac{1}{2} \left[ \ln \left( \mathcal{T}_{n}^{2}(L, L) \mathcal{T}_{n}^{-2}(0, 0) \right) \right] \]

\[ \nu = 3 \]

Red part is consistent with the result of \( \beta \to \infty \).

(We introduce a cutoff \( \alpha \).)

11 Consistency check

From the geometry,

\[ \langle \mathcal{T}_{n}^{2}(L, L) \mathcal{T}_{n}^{-2}(0, 0) \rangle \] must hold.

\[ \mathcal{E} = \lim_{n \to 1} \ln \left( \mathcal{T}_{n}^{2}(L, L) \mathcal{T}_{n}^{-2}(0, 0) \right) \]

We can check

\[ \langle \mathcal{T}_{n}^{2}(L, L) \mathcal{T}_{n}^{-2}(0, 0) \rangle \] explicitly.

12 Naive calculation of mixed states

By assuming that we can use same twist fields for mixed states, we can calculate the entanglement negativity of mixed states naively.

\[ \mathcal{E} = \lim_{n \to 1} \ln \left( \mathcal{T}_{n}^{2}(L, L) \mathcal{T}_{n}^{-2}(0, 0) \right) \]

(Non-conservation of charge)

\[ \mathcal{E} = \lim_{n \to 1} \ln \left( \mathcal{T}_{n}^{2}(L, L) \mathcal{T}_{n}^{-2}(0, 0) \right) \]

Is this result true?

What is the interpretation of this result?