It is well known that instantons in gauge theories play important roles in the study of non-perturbative effects. The gauge instantons in four dimensions are defined by configurations such that the gauge field strength 2-form $F$ satisfies the self-duality relation $F \wedge F = \hat{a} F$. A salient feature of the self-dual instantons in four dimensions is its systematic construction of solutions, known as ADHM construction.

**ADHM construction in $\mathbb{R}^4$**

In higher dimensions, there are two types of generalization of the ASD equations in $\mathbb{R}^4$.

First type is called as "secular type", and second type is called as "self-dual type". In this paper, we define the eight dimensional "instanton" such that the field strength satisfies the self-duality relation in $\mathbb{R}^8$. Furthermore, we expect that the eight dimensional instanton has non-zero topological charge given by the 4th Chern number $C^{(4)} = \int F \wedge F \wedge F \wedge F$, $F \wedge F = \hat{a} F$.

The generalization of the ASD equations in $\mathbb{R}^8$ is given by

\[
F + F = \hat{a} F \wedge F \quad \text{and} \quad ADHM construction.
\]

II: Secular type

\[
F_{\mu\nu\rho\sigma} \pm 2 \xi_{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} = \hat{a} F_{\mu\nu\rho\sigma} \wedge 1, \ldots, 8
\]

ADHM like construction

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**ADHM construction in $\mathbb{R}^8$**

The first step of constructing the ADHM construction in $\mathbb{R}^8$ is to find appropriate algebra basis $\epsilon_{\mu\nu}$, which constructing "ASD tensor" $\sum_{\mu\nu} \epsilon_{\mu\nu} F_{\mu\nu\rho\sigma}$. Here, an "ASD tensor" means that a tensor is satisfying the ASD equations, and "ASD algebra basis" means algebra basis which construct the ASD tensor.

In four dimension, $\epsilon_{\mu\nu}$ are the standard coordinates on $\mathbb{R}^4$ and indices $\mu, \nu, \rho, \sigma = 1, 2, 3, 4$. The ASD equation in $\mathbb{R}^4$ is given by

\[
F = \hat{a} F \wedge F \quad \implies \quad F_{\mu\nu} = \frac{1}{2} \xi_{\mu\nu} F_{\rho\sigma}
\]

where indices in square brackets \{\} are to be antisymmetrized, $\xi_{\mu\nu} = (\delta_{\mu\nu} - \eta_{\mu\nu})$.

**I: THE EIGHT DIMENSIONAL ADHM CONSTRUCTION**

The first step of constructing the ADHM construction in $\mathbb{R}^8$ is to find appropriate algebra basis $\epsilon_{\mu\nu}$ which constructing "ASD tensor" $\sum_{\mu\nu} \epsilon_{\mu\nu} F_{\mu\nu\rho\sigma}$. Here, an "ASD tensor" means that a tensor is satisfying the ASD equations, and "ASD algebra basis" means algebra basis which construct the ASD tensor.

In eight dimension, $\epsilon_{\mu\nu}$ are the standard coordinates on $\mathbb{R}^8$ and indices $\mu, \nu, \rho, \sigma = 1, 2, 3, 4$ and $i, j, k, l = 1, 2, 3, 7$. The ASD equation in $\mathbb{R}^8$ is given by

\[
F = \hat{a} F \wedge F \quad \implies \quad F_{\mu\nu\rho\sigma} = \frac{1}{2} \xi_{\mu\nu\rho\sigma} F_{\rho\sigma}
\]

where indices in square brackets \{\} are to be antisymmetrized, $\xi_{\mu\nu\rho\sigma} = (\delta_{\mu\nu\rho\sigma} - \eta_{\mu\nu\rho\sigma})$.

**I-i: ASD tensor**

The ASD tensor in $\mathbb{R}^8$ is given by

\[
\begin{align*}
&\epsilon_{\mu\nu} = \pm \frac{1}{2} \sqrt{\xi_{\mu\nu}} \epsilon_{\mu\nu}
\end{align*}
\]

By analogy of the ASD tensor in $\mathbb{R}^4$ (i.e., $\hat{a}$-Hocht tensor), we define

\[
\begin{align*}
&\sum_{\mu\nu} \epsilon_{\mu\nu} = \epsilon_{\mu\nu}, \epsilon_{\mu\nu}
\end{align*}
\]

and $\hat{a}$-Hocht tensor $\sum_{\mu\nu}$ satisfy the four dimensional ASD equations

\[
\sum_{\mu\nu} \epsilon_{\mu\nu} \epsilon_{\mu\nu} = \pm \frac{1}{2} \sqrt{\xi_{\mu\nu}} \epsilon_{\mu\nu} \epsilon_{\mu\nu}.
\]

**II: Secular type solutions**

\[
F_{\mu\nu\rho\sigma} = \frac{1}{2} \xi_{\mu\nu\rho\sigma} F_{\rho\sigma}
\]

\[
\Delta \Delta = 1 : 1
\]

**Nahm transform**

A scheme of to obtain the eight dimensional instanton's gauge field from Weyl operator is analogy by four dimensional ones. So

\[
\text{Weyl equations: } \Delta V(x) = 0 \quad \text{with } V(x) \text{ in } (k + 1) \times (k + 1) \text{ matrix column vector with basis from } \epsilon_{\mu\nu}, \text{ } V(x) \text{ satisfy normalization: } V' V = 1.
\]

We obtain the gauge field $A_\mu(x)$ of eight dimensional instantons as

\[
A_\mu(x) = V(x) A_\mu(x), \quad \text{where } A_\mu(x) = -i \partial_\mu V(x) V(x).
\]

**Introduce ADHM constraint in $\mathbb{R}^8$**

Next we calculate field strength $F_{\mu\nu}$ from Eq.10 to introduce the eight dimensional ADHM constraint.

\[
F_{\mu\nu} = \partial_\mu A_\nu - A_\mu A_\nu - (\mu \leftrightarrow \nu)
\]

\[
\partial_\mu V(x) = -i V(x) V'(x) \partial_\mu V(x)
\]

\[
\Delta V(x) = 0 = \Delta V = V'(x) V(x)
\]

\[
\Delta \Delta = 1 : 1
\]

Here we use the completeness relation $1 = V(\Delta) V'(x) = 1 = \Delta V(x) V'(x)$, Eq. 40 can write as

\[
\Delta V = V'(x) V(x)
\]

\[
\Delta \Delta = 1 : 1
\]

Now we can find eight dimensional ASD algebra with reference to construction of quaternions.

**ADHM construction in $\mathbb{R}^8$**

In this poster, we show that ADHM construction in $\mathbb{R}^8$ and reproduces the known one

\[
\epsilon_{\mu\nu} = \pm \frac{1}{2} \sqrt{\xi_{\mu\nu}} \epsilon_{\mu\nu} \epsilon_{\mu\nu}
\]

Some properties of ASD algebra basis $\epsilon_{\mu\nu}$

\[
\epsilon_{\mu\nu} + \epsilon_{\mu\nu} = \epsilon_{\mu\nu} + \epsilon_{\mu\nu} = \epsilon_{\mu\nu} + \epsilon_{\mu\nu} = 2 \lambda_{\mu\nu}
\]

and ASD tensor $\sum_{\mu\nu}$ need satisfy the eight dimensional ASD equations

\[
\sum_{\mu\nu} \epsilon_{\mu\nu} \epsilon_{\mu\nu} = \pm \frac{1}{2} \sqrt{\xi_{\mu\nu}} \epsilon_{\mu\nu} \epsilon_{\mu\nu}.
\]

The generalization of the ASD equations in $\mathbb{R}^8$ is given by

\[
F = \hat{a} F \wedge F \wedge F \wedge F \quad \implies \quad F_{\mu\nu\rho\sigma} = \frac{1}{2} \xi_{\mu\nu\rho\sigma} F_{\rho\sigma}
\]

where indices in square brackets \{\} are to be antisymmetrized, $\xi_{\mu\nu\rho\sigma} = (\delta_{\mu\nu\rho\sigma} - \eta_{\mu\nu\rho\sigma})$.

**II: Secular type solutions**

\[
F_{\mu\nu\rho\sigma} = \frac{1}{2} \xi_{\mu\nu\rho\sigma} F_{\rho\sigma} \wedge 1, \ldots, 8
\]

ADHM like construction

E. Corrigan, P. Goddard and A. Kent

Here we require that commutativity of $C^iV^jC^k$, that is
$$c_{ij}(C^iV^jC^k) = (C^iV^jC^k)c_{ij}$$
and then Eq. (8) is
$$\mathcal{P}_{\text{fermions}} = \frac{1}{4} F_{\mu\nu\rho\sigma} F^\mu_{\nu\rho\sigma}$$
$$\implies V'(\Delta'\Lambda') = \frac{1}{2} \sum_{i=1}^{4} (C^iV^jC^k) = (C^iV^jC^k)c_{ij}.$$ 
Since Eq. (2), $\sum_{i=1}^{4} c_{ij} = 0$, we rewrite Eq. (9) using completeness relation $V = \sum_{i=1}^{4} C^iV^jC^k$, that
$$c_{ij}C^i = C^i_c c_{ij} = \left(\Delta'\Lambda'\Lambda'\Delta'\right)c_{ij}.$$ 
The first condition $c_{ij}C^i = C^i_c c_{ij}$ is included condition Eq. (6), so we omit this condition. On the other hand, second condition is to simplify using Eq. (6) to $c_{ij}C^i = C^i_c c_{ij}$. And this condition is also include condition Eq. (6):
$$c_{ij}(\Delta'\Lambda')^{-1} = (\Delta'\Lambda')^{-1}c_{ij} \iff \Delta'\Lambda' = I_k \otimes E,$$ 
where $E$ is $k \times k$ matrix.

The eight dimensional ADHM constraint:

ADHM constraint in $\mathbb{R}^8$: $\Delta'\Lambda' = (I_k \otimes I_k \otimes I_k) \otimes E = I_k \otimes E$. Where $E$ is $k \times k$ matrix.

The eight dimensional ADHM equations with canonical form

The eight dimensional ADHM data $C,D$ are canonical form, such that
$$C = \begin{pmatrix} I_k & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & I_k \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & I_k \\ 0 & 0 & I_k & 0 \\ \end{pmatrix},$$
$$D = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \\ \end{pmatrix},$$
where $|$ means matrix size.

We assume that $S_{12} = s_{12} S_{21} = s_{21}$, then

$R^8$ ADHM-eq.: $\Delta'\Lambda' = I_k \otimes E_{|k|} \implies$
$$[7^7, 7^7] - [7^7, 7^7] + \frac{1}{2} \left( [5^7, 5^7] - [5^7, 5^7] \right) = 0,$$
$$[7^7, 7^7] + [7^7, 7^7] + \frac{1}{2} \left( [5^7, 5^7] + [5^7, 5^7] \right) = 0,$$
$$[7^7, 7^7] + [7^7, 7^7] + \frac{1}{2} \left( [5^7, 5^7] + [5^7, 5^7] \right) = 0,$$
$$[7^7, 7^7] + [7^7, 7^7] + \frac{1}{2} \left( [5^7, 5^7] + [5^7, 5^7] \right) = 0,$$
$$[7^7, 7^7] + [7^7, 7^7] + \frac{1}{2} \left( [5^7, 5^7] + [5^7, 5^7] \right) = 0,$$
$$[7^7, 7^7] + [7^7, 7^7] + \frac{1}{2} \left( [5^7, 5^7] + [5^7, 5^7] \right) = 0,$$
$$[7^7, 7^7] + [7^7, 7^7] + \frac{1}{2} \left( [5^7, 5^7] + [5^7, 5^7] \right) = 0,$$
$$[7^7, 7^7] + [7^7, 7^7] + \frac{1}{2} \left( [5^7, 5^7] + [5^7, 5^7] \right) = 0,$$
$$[7^7, 7^7] + [7^7, 7^7] + \frac{1}{2} \left( [5^7, 5^7] + [5^7, 5^7] \right) = 0,$$

II: THE EIGHT DIMENSIONAL ADHM DATA

In the eight dimensions, instanton charge $Q$ is 4th Chern number

$$Q = \int_{\Sigma} \text{Tr}(F \wedge F \wedge F \wedge F) = \int_{\Sigma} \text{Tr}(\mathcal{P}_{\text{fermions}} F \wedge F \wedge F \wedge F) = N \int_{\Sigma} \text{d}^8x Q = N \int_{\Sigma} \text{d}^8x,$$

where $N$ is normalization constant. However it is difficult to use this equations directly for calculation of charge density. So we use formula to calculate the charge density form ADHM data.

Formula to calculate charge density: $Q = 16 \text{Tr}(V' (\Delta' \Lambda' \Lambda' \Delta') V')$.

Il-i: BPST type 1-instanton

We extend the eight dimensional BPST instanton ADHM data to the eight dimensional ones

$$C = \begin{pmatrix} I_k & 0 \\ I_k & 0 \\ 0 & I_k \\ 0 & I_k \\ \end{pmatrix},$$
$$D = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix} \implies \Delta = (A_0-k)^2 $$

where $A_0$ is moduli, $\Delta := (x^a - a_x)^2$ and $a_x$ is position moduli.

Since $\Delta' = (I_1, \Delta')$, indeed this ADHM data satisfy the eight dimensional ADHM constraints

$$\Delta'(\Delta' \Lambda')(\Delta' \Lambda')(\Delta' \Lambda')(\Delta' \Lambda')(\Delta' \Lambda')(\Delta' \Lambda')(\Delta' \Lambda')(\Delta' \Lambda') = I_k \otimes \left( \Delta' + \frac{1}{2} (\Delta')^2 \right)$$

Thus we have established the eight dimensional generalization of the BPST instanton, i.e. eight dimensional ADHM construction.

Il-ii: Hopf type $k$-instanton

$\gamma$ Hopf type ADHM data are

$$C = \begin{pmatrix} I_k & 0 \\ I_k & 0 \\ 0 & I_k \\ 0 & I_k \\ \end{pmatrix},$$
$$D = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1k} \\ S_{21} & S_{22} & \cdots & S_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ S_{k1} & S_{k2} & \cdots & S_{kk} \\ \end{pmatrix},$$

where $S_{ij}$ is matrix size moduli.

$$\Delta = (I_k \otimes I_k) \otimes \left( \Delta' + \frac{1}{2} (\Delta')^2 \right).$$

Therefore the normalization constant $N_{\text{is}}$ is $N \sim 105^{16^{14}}$.

III: SUMMARY AND FURTHER WORKS

We have established the general scheme to construct the eight dimensional instantons, i.e. eight dimensional ADHM construction.

We have also shown the explicit form of the higher charge solutions based on the 't Hooft ansatz.

Is there the eight dimensional generalization of the Osborn's formula?

Can we establish the eight dimensional noncommutative ADHM construction?

Can we establish the seven/eight dimensional Nahm construction of monopole/caloron?

Can we establish more general dimensional ADHM construction?

How to relate between the eight dimensional ADHM and D-brane systems? etc...