Exact Computations in Confining Phase using SUSY Localization

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Summary

Theory: 4d N=1 supersymmetric gauge theory
Method: localization with new choice of SUSY generator

We exactly compute gaugino condensation $\langle \text{Tr } (\lambda \lambda) \rangle$ in confining phase

Apply to: nonperturbative proof of Dijkgraaf-Vafa conjecture

Motivation

Analytic computations in QFT are important, and difficult

But, for SUSY field theory, we can!
Two major techniques:

1. Holomorphy
   Simple, But, Indirect

2. Localization
   Direct and Systematic, But, had not been applied to theory in confining phase

Apply localization technique to “dynamical object” (gaugino condensation) in confining phase!

Localization

For SUSY theory, take $\delta = \text{a SUSY generator}$

$I = \int (\delta \lambda)^\dagger \lambda$

$\langle \lambda \rangle$ is a fermion

Then, we found

$\int \delta I |_{\text{bosonic}} = \int (\delta \lambda)^\dagger \delta \lambda \geq 0$

$\delta^2 I (\phi) = 0$

Let us consider a correlator

$Z(t) = \int D\phi e^{-S(\phi) - 1/2 \delta I_{\lambda}(\phi) \delta I_{\lambda}(\phi)}$

where we assume $\delta S(\phi) = 0$

$\delta I_{\lambda}(\phi) = 0$

This is independent with $t$!

$\frac{dZ(t)}{dt} - \int D\phi \delta \left( e^{-S(\phi) - 1/2 \delta I_{\lambda}(\phi) \delta I_{\lambda}(\phi)} \right) = 0$

Taking $t \to \infty$, $Z(0) = Z(t \to \infty)$

In this limit, path-integral localized on $\delta I_{\lambda}(\phi) = 0$

$Z(0) = \int_{\text{saddle pt}} D\phi e^{\text{exponent}}$

$\left( e^{-S(\phi_0)} \times \text{ (1-loop determinant) } \right)$

Exactly computable!

What is done in this work

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<td>Seiberg-Witten curve</td>
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<td>N=1 SUSY</td>
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\[ 4D \, N = 1 \text{SUSY gauge theory on } R^3 \times S^1_R \]

**Fields (in Vector multiplet)**

\[
\begin{align*}
A_m & : \text{vector} \\
\lambda & : \text{chiral spinor} \\
\bar{\lambda} & : \text{anti-chiral spinor} \\
D & : \text{auxiliary field}
\end{align*}
\]

Adjoint representations of gauge group \( G \)

SU(2) \times SU(2) (\text{or } Spin(4)) \text{ SUSY}

\[
\begin{align*}
\delta A_m &= \frac{i}{2} (\epsilon \sigma_m \lambda - \bar{\epsilon} \sigma_m \lambda), \\
\delta \lambda &= \frac{i}{2} \sigma^m \epsilon F_{mn} - \epsilon D, \\
\delta \bar{\lambda} &= \frac{i}{2} \sigma^m \bar{\epsilon} F_{mn} - \bar{\epsilon} D, \\
\delta D &= -\frac{i}{2} \epsilon \sigma^m D_m \lambda - \frac{i}{2} \bar{\epsilon} \sigma^m D_m \lambda
\end{align*}
\]

**In Euclidian space, \( \epsilon \) and \( \bar{\epsilon} \) are independent**

**SUSY invariant Lagrangian:**

\[
L_g = \text{Tr} \left[ \frac{1}{2g^2} \left( \frac{1}{2} F_{mn} F^{mn} + D^2 + i \bar{\lambda} \sigma^m D_m \lambda \right) + i \frac{\theta}{16\pi^2} F \tilde{F} \right], \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}
\]

**For Localization, adding \( \delta I = \int d^4x \delta V \)**

where \( V = (\delta \lambda)^\dagger \lambda + (\delta \bar{\lambda})^\dagger \bar{\lambda} \) with a choice of \( \epsilon, \bar{\epsilon} \)

**Localization with \( \epsilon \neq 0, \quad \bar{\epsilon} \neq 0 \) (known choice)**

\[
\begin{align*}
(\delta \lambda)^\dagger \delta \lambda & \sim (F_{mn}^+ F^{-mn} + D^2)(\epsilon^\dagger \epsilon), \\
(\delta \bar{\lambda})^\dagger \delta \bar{\lambda} & \sim (F_{mn}^- F^{mn} + D^2)(\bar{\epsilon}^\dagger \bar{\epsilon})
\end{align*}
\]

\[
V|_{bosonic} = F_{mn} F^{mn} + D^2
\]

saddle point is trivial: \( F_{mn} = D = 0 \)

\[
L_g + t \delta V \sim \left( \frac{1}{2g^2} + t \right) (F^2 + D^2) + i \frac{\theta}{16\pi^2} F \tilde{F}
\]

Weak coupling in \( t \to \infty \) and No gauge couling dependence. This gives superconformal index

**Localization with \( \epsilon = 0, \quad \bar{\epsilon} \neq 0 \) (NEW choice)**

\[
V = F_{mn}^- F^{-mn} + D^2 = \frac{1}{2} F_{mn} F^{mn} + \frac{1}{2} F_{mn} \tilde{F}^{mn} + D^2
\]

saddle point is **instanton:** \( F_{mn}^- = D = 0 \)

\[
L_g + t \delta V \sim \left( \frac{1}{2g^2} + t \right) (F^2 + D^2) + i \frac{\theta}{16\pi^2} + t \right) F \tilde{F} \rightarrow i \frac{\tau}{8\pi} F \tilde{F}
\]

Weak coupling in \( t \to \infty \), **gauge coupling dependent.** Usual instanton analysis becomes exact.

Moreover, \( \text{Tr} (f(\lambda)) \) is an ”observable“ in localization, because \( \delta \lambda = 0 \)

**Gaugino condensation \( \langle \text{Tr} (\lambda \lambda) \rangle \) computation**

We need precisely 2 fermion zero modes for nonzero \( \langle \text{Tr} (\lambda \lambda) \rangle \)

1/\( N \) fractional instantons (=T-dual of BPS monopoles)

Usual 1-loop commutations around these give

the correct exact result ! \[
\frac{\text{Tr} \sqrt{\lambda^2}}{16\pi^2} = \Lambda^3 \omega \]

where \( \omega^N = 1 \) for \( G = SU(N) \)

\[
\Lambda^3 = \mu^3 \frac{1}{q^{1/2}(\mu)} \exp \frac{2\pi i \tau(\mu)}{N}
\]

\[
\text{Brane picture (SU(3) case)}
\]

This is \( R \) independent

**Localization for theory on \( R^4 \)?**

Theory DOESN’T become weak in \( t \to \infty \)

**Localization for theory on \( S^4 \)?**

We can NOT take \( t \to \infty \)

**General 4d \( N = 1 \) gauge theories (including chiral multiplets)**

Kinetic terms for chiral multiplets can be arbitrary large. Now gauge coupling can be weak also. Thus, we can integrate out them perturbatively! \( \rightarrow \) semi-classical monopole computations

Therefore, we can compute the correlators of observables(chiral ring) always in weak coupling

\( \rightarrow \) a Nonperturbative proof of Dijkgraaff-Vafa conjecture \( \text{(Marvelous Proof Which This Margin Is Too Narrow To Contain)} \)