Thermodynamic Geometry of Yang-Mills Gauge Theory

Bhupendra Nath Tiwari

University of Information Science and Technology,
"St. Paul the Apostle", Partizanska Str. bb 6000 Ohrid,
Republic of Macedonia

[a]INFN-Laboratori Nazionali di Frascati,
Via E. Fermi 40, 00044 Frascati, Italy

Talk @
"YITP Workshop, Strings and Fields 2018,
July 30 (Mon) - August 3 (Fri), 2018"

In Collaboration With:
Prof. Stefano Bellucci[a]

Abstract

We study vacuum fluctuation properties of an ensemble of SU(N) gauge theory configurations in the limit of a large number of colors. We explore statistical properties of moduli fluctuations by analyzing the critical behavior and geometric invariants at a given vacuum parameter. Further, we discuss the nature of long-range correlations, interacting/ noninteracting domains, and associated phase transitions. Finally, we provide possible directions towards its phenomenological developments.

Keywords: Intrinsic Geometry; String Theory; Yang-Mills Gauge Theory; Black Hole Physics; Statistical aspects of black holes, Higher-dimensional black holes, black strings, and related objects in the light of Statistical Fluctuation and Flow (In)stabilities.

Plan of the Talk

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1 Introduction

14. In this talk, we study statistical properties of the charged anticharged black hole configurations in string theory. Specifically, we illustrate that the components of the vacuum fluctuations define a set of local pair correlations against the parameters, *e.g.*, charges, anticharges, mass and angular momenta, if any.

15. Our consideration follows from the notion of the thermodynamic geometry, mainly introduced by Weinhold [1] and Ruppeiner [2, 3]. Importantly, our framework provides a simple platform to geometrically understand the nature of local statistical pair correlations and underlying global statistical structures pertaining to the vacuum phase transitions.

16. In diverse contexts, this perspective offers an understanding of the phase structures of mixture of gases, black hole configurations [4, 5], strong interactions, e.g., hot QCD [6], quarkonium configurations [7] and some other systems as well.

17. The main purpose of the present talk is to determine the state-space properties of extremal and non-extremal black hole configurations in string theory, in general.

18. String theory, as the most promising framework to understand all possible fundamental interactions, celebrates the physics of black holes, both at the zero and non-zero temperature domains. Our consideration hereby plays a crucial role in understanding the possible phases and statistical stability of the string theory vacua.

2 Motivations from String Theory

19. $\mathcal{N} = 2$ SUGRA is a low energy limit of the Type II string theory, admitting extremal black hole solutions with the zero Hawking temperature and a non-zero macroscopic entropy.

20. The entropy depends on a large number of scalar moduli arising from the compactification of the 10 dimension theory down to the 4 dimensional physical spacetime.

21. This involves a 6 dimensional compactifying manifold. Interesting string theory compactifications involve T^6, $K_3 \times T^2$ and Calabi-Yau manifold.

22. The macroscopic entropy exhibits a fixed point behavior under the radial flow of the scalar fields. The attractor mechanism, as introduced by Ferrara-Kallosh-Strominger attractor mechanism, requires a validity from the microscopic/ statistical basis of the entropy.

23. In this talk, we shall explore attractor fixed point structures in relation with the statistical properties and intrinsic state-space configurations.

24. We shall provide the statistical understanding of attractor mechanism, moduli space geometry and explain the vacuum fluctuations of black branes.

3 Motivations from Gauge Theory

25. We study vacuum fluctuation properties of an ensemble of $SU(N)$ gauge theory configurations.

26. In the limit of large number of colors, *viz.* $N_c \to \infty$, we explore statistical nature of the topological susceptibility.

27. We analyzing its critical behavior at a nonzero vacuum parameter θ and temperature T.

28. We find that the system undergoes a vacuum phase transition at the chiral symmetry restoration temperature as well as at an absolute value of the vacuum angle θ.

29. The long range correlation length solely depends on θ for the theories having critical exponent $e = 2$ or $T = T_d + 1$, where T_d is the decoherence temperature.

30. the unit critical exponent vacuum configuration corresponds to a noninteracting statistical basis pertaining to a constant mass of η'.

4 Definition of State-space Geometry

31. For any thermodynamic system, there exist equilibrium thermodynamic states given by the maxima of the entropy. These states may be represented by points on the state-space.

32. Along with the laws of the equilibrium thermodynamics, the theory of fluctuations leads to the intrinsic Riemannian geometric structure on the space of equilibrium states, [*Ruppeiner, PRD 1978*].

33. The invariant distance between two arbitrary equilibrium states is inversely proportional to the fluctuations connecting the two states. In particular, "less probable fluctuation" means "states are far apart".

34. For a given set of such states $\{X_i\}$, the state-space metric tensor is defined by

$$g_{ij}(X) \quad = \quad -\partial_i \partial_j S(X_1, X_2, \ldots, X_n) \tag{1}$$

35. A physical proof of Eq.(1) can be given as follows:

36. Up to the second order, the Taylor expansion of the entropy $S(X_1, X_2, \dots, X_n)$ gives

$$S - S_0 = -\frac{1}{2}\sum_{i=1}^{n} g_{ij}\Delta X^i \Delta X^j, \tag{2}$$

where

$$g_{ij} := -\frac{\partial^2 S(X_1, X_2, \dots, X_n)}{\partial X^i \partial X^j} = g_{ji} \tag{3}$$

is called extended Ruppenier state-space metric tensor. As the limit, the relative co-ordinates ΔX^i are defined as $\Delta X^i := X^i - X^i_0$, for given $\{X^i_0\} \in M_n$.

37. The probability distribution in the Gaussian approximation has the form:

$$P(X_1, X_2, \dots, X_n) = A\ exp(-\frac{1}{2} g_{ij}\Delta X^i \Delta X^j) \quad \text{“??”} \tag{4}$$

38. With the normalization:

$$\int \prod_i dX_i P(X_1, X_2, \dots, X_n) = 1, \tag{5}$$

we examine the nature of

$$P(X_1, X_2, \dots, X_n) = \frac{\sqrt{g(X)}}{(2\pi)^{n/2}} exp(-\frac{1}{2} g_{ij} dX^i \otimes dX^j), \tag{6}$$

where g_{ij} is defined as the inner product $g(\frac{\partial}{\partial X^i}, \frac{\partial}{\partial X^j})$ on the tangent space $T(M_n) \times T(M_n)$ with

$$g(X) := \|g_{ij}\| \tag{7}$$

as the determinant of the corresponding matrix $[g_{ij}]_{n\times n}$. For a given state-space M_n, we shall think $\{dX^i\}_{i=1}^n$ as the basis of the cotangent space $T^\star(M_n)$.

39. In the sequel, we chose a neutral vacuum with $X^i_0 = 0$ while studying black holes.

5 State-space Surface: Review

5.1 Black Hole Entropy

40. As a first exercise, we have illustrated the thermodynamic state-space geometry for the two charge extremal black holes with electric charge q and magnetic charge p.

41. As the maxima of their macroscopic entropy $S(q,p)$, the next step is to examine the statistical fluctuations about attractor fixed point configuration of the extremal black hole.

42. Later on, we shall analyze the state-space geometry of non-extremal counterparts. We have shown that the state-space correlations now modulate relatively swiftly to an equilibrium statistical basis than the corresponding extremal solutions.

5.2 Statistical Fluctuations

43. The Ruppenier metric on the state-space (M_2, g) of two charge black hole is defined by

$$g_{qq} = -\frac{\partial^2 S(q,p)}{\partial q^2}, \quad g_{qp} = -\frac{\partial^2 S(q,p)}{\partial q \partial p}, \quad g_{pp} = -\frac{\partial^2 S(q,p)}{\partial p^2} \tag{8}$$

44. The Christoffel connections on the (M_2, g) are defined by

$$\Gamma_{ijk} = g_{ij,k} + g_{ik,j} - g_{jk,i} \tag{9}$$

45. The only non-zero component of the Riemann curvature tensor is

$$R_{qpqp} = \frac{N}{D}, \tag{10}$$

where

$$\begin{aligned} N \;:=\; & S_{pp}S_{qqq}S_{qpp} + S_{qp}S_{qqp}S_{qpp} \\ & + S_{qq}S_{qqp}S_{ppp} - S_{qp}S_{qqq}S_{ppp} \\ & - S_{qq}S_{qpp}^2 - S_{pp}S_{qqp}^2 \end{aligned} \tag{11}$$

and

$$D := (S_{qq}S_{pp} - S_{qp}^2)^2 \tag{12}$$

46. The scalar curvature and corresponding R_{ijkl} of the two dimensional intrinsic state-space manifold $(M_2(R), g)$ is given by

$$R(q,p) = \frac{2}{\|g\|} R_{qpqp}(q,p) \tag{13}$$

5.3 Stability Conditions

47. For a given set of state-space variables $\{X^1, X^2, \ldots, X^n\}$, the local stability condition of the underlying statistical configuration demands

$$\{g_{ii}(X^i) > 0;\ \forall i = 1, 2, \ldots, n\} \tag{14}$$

48. The principle components of the state-space metric tensor $\{g_{ii}(X^i) \mid i = 1, 2, \ldots, n\}$ signify a set of definite heat capacities (or the related comprehensibilities) whose positivity apprises that the black hole solution comply an underlying locally equilibrium statistical configuration.

49. The positivity of the principle components of the state-space metric tensor is not sufficient to insure the global stability of the chosen configuration, and thus one may only achieves a locally equilibrium statistical system.

50. Global stability condition constraint over allowed domain of the parameters of black hole configurations requires that all the principle components and all the principle minors of the metric tensor must be strictly positive definite, [*Ruppeiner, RMP 1995*].

51. This condition implies that the following set of simultaneous equations be satisfied

$$\begin{aligned}
p_0 &:= 1, \\
p_1 &:= g_{11} > 0, \\
p_2 &:= \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix} > 0, \\
p_3 &:= \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{vmatrix} > 0, \\
&\vdots \\
p_n &:= \|g\| > 0
\end{aligned} \tag{15}$$

5.4 Long Range Correlations

52. The thermodynamic scalar curvature of the state-space manifold is proportional to the correlation volume. Physically, the scalar curvature signifies an existence of interaction(s) in the underlying statistical system.

53. Ruppenier has in particular noticed for the black holes in general relativity that the scalar curvature

$$R(X) \sim \xi^d, \tag{16}$$

where d is spatial dimension of the statistical system and the ξ fixes the physical scale, [*Ruppeiner, RMP 1995*].

54. The limit $R(X) \longrightarrow \infty$ indicates existence of certain critical points or phase transitions in the underlying statistical system.

55. "All the statistical degrees of freedom of a black hole live on the black hole event horizon" signifies that the scalar curvature indicates an average number of correlated Plank areas on the event horizon of the black hole, [*Ruppeiner, PRD 1978*].

56. Ruppenier has further conjectured that

    (a) The zero state-space scalar curvature indicates certain bits of information on the event horizon, fluctuating independently of the each other.

    (b) The diverging scalar curvature signals a phase transition, indicating an ensemble of highly correlated pixels of informations.

6 Some Physical Motivations

6.1 Extremal Black Holes

57. State-space of extremal (supersymmetric) black holes is a reduced phase-space comprising of the states respecting the extremality (BPS) condition.

58. The state-spaces of the extremal black holes possess an intrinsic geometric description.

59. Our intrinsic geometric analysis offers a possible zero temperature characterization of the limiting extremal black brane attractors.

60. From the perspective gauge/ gravity correspondence, we may think that the existence of state-space geometry could be relevant to the boundary gauge theories, namely, an ensemble of CFT states is parametrized by finitely many charges.

6.2 Non-extremal Black Holes

61. We shall analyze the state-space geometry of non-extremal black holes by an addition of the anti-brane charge(s) to the entropy of the corresponding extemal black holes.

62. To interrogate the stability of a chosen black hole system, we shall investigate the question that the underlying metric $g_{ij}(X_i) = -\partial_i \partial_j S(X_1, X_2, \ldots, X_n)$ should be a non-degenerate state-space manifold.

63. The exact dependence varies from case to case. In the next section, we shall proceed our analysis with an increasing number of the brane charges and antibrane charges.

6.3 Chemical Geometry

64. The thermodynamic configurations of non-extremal black holes in string theory with small statistical fluctuations in a "canonical" ensemble are stable if

$$\|\partial_i \partial_j S(X_1, X_2, \ldots, X_n)\| < 0 \tag{17}$$

65. The thermal fluctuations of non-extremal black holes, when considered in the canonical ensemble, give a closer approximation to the microcanonical entropy

$$S = S_0 - \frac{1}{2}\ln(CT^2) + \cdots \tag{18}$$

66. In Eq. (18), the S_0 is the entropy in "canonical" ensemble and C is the specific heat of black hole statistical configuration.

67. At low temperature, the quantum effects dominate and the above expansion does not hold any more. For example, for the BTZ-CS black holes, we notice that the large entropy limit is the stability bound, beyond which quantum effects dominate, [*Solodukhin Phys. Rev. D74 024015, 2006*].

68. The stability condition of canonical ensemble is just $C > 0$. In other words, the Hessian of the internal energy w. r. t. the chemical variable *viz.* $\{x_1, x_2, \ldots, x_n\}$ remains positive definite

$$\|\partial_i \partial_j E(x_1, x_2, \ldots, x_n)\| > 0 \tag{19}$$

69. The state-space co-ordinates $\{X^i\}$ and intensive chemical variables $\{x_i\}$ are conjugate to each other. In particular, the $\{X^i\}$ are defined as Legendre transform of $\{x_i\}$

$$X^i := \frac{\partial S(x)}{\partial x_i} \tag{20}$$

6.4 Physics of Correlation

70. Geometrically, the positivity of the heat capacity $C > 0$ turns out to be the condition that $g_{ij} > 0$. In many cases, this restriction on the parameters corresponds to the situation away from the extremality condition $r_+ = r_-$.

71. Far from the extremality condition, even at the zero antibrane charge (or angular momentum), we find that there is a finite value of the state-space scalar curvature, unlike the non-rotating or only brane charged extremal configurations.

72. The Ruppenier geometry of the two charge extremal configurations turns out to be flat. So, the Einstein-Hilbert contributions lead to a non interacting statistical system. Some two derivative black hole configurations turn out to be ill-defined, as well.

73. The determinant(s) of the state-space tensor should be positive definite. If not, the configuration requires further higher order corrections $\in \{stringy,\ quantum\}$.

74. For non-extremal black branes, the global effects arise from the nature of the state-space scalar curvature $R(S(X_1, X_2, \ldots, X_n))$, and in fact, the statistical signature is kept intact under the limit of the extremality.

75. Given a non-extremal configuration, we find that $R(S(X_1, X_2, \ldots, X_n))|_{no\ antichagre} \neq 0$ gives statistical stability bound(s), and thus the state-space analysis offers sensible domain for the parameters of the black hole(s).

7 Black Holes in String Theory:

[*S. Bellucci and B.N.T.: Phys. Rev. D (2010)*] and [*Entropy (2010)*].

7.1 State-space Geometry of Extremal Black Holes

76. At the two derivative Einstein-Hilbert level, Ref. [*Strominger and Vafa: arXiv:hep-th/9601029v2*] shows that the leading order entropy of the three charge D_1-D_5-P extremal black holes is

$$S_{micro} = 2\pi\sqrt{n_1 n_5 n_p} = S_{macro} \tag{21}$$

77. The components of state-space metric tensor are

$$\begin{aligned} g_{n_1 n_1} &= \frac{\pi}{2n_1}\sqrt{\frac{n_5 n_p}{n_1}},\ g_{n_1 n_5} = -\frac{\pi}{2}\sqrt{\frac{n_p}{n_1 n_5}} \\ g_{n_1 n_p} &= -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1 n_p}},\ g_{n_5 n_5} = \frac{\pi}{2n_5}\sqrt{\frac{n_1 n_p}{n_5}} \\ g_{n_5 n_p} &= -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5 n_p}},\ g_{n_p n_p} = \frac{\pi}{2n_p}\sqrt{\frac{n_1 n_5}{n_p}} \end{aligned} \tag{22}$$

78. For distinct $i, j \in \{1, 5\}$ and p, list of relative correlation functions follow scalings

$$\begin{aligned} \frac{g_{ii}}{g_{jj}} &= (\frac{n_j}{n_i})^2,\ \frac{g_{ii}}{g_{pp}} = (\frac{n_p}{n_i})^2,\ \frac{g_{ii}}{g_{ij}} = -(\frac{n_j}{n_i}) \\ \frac{g_{ii}}{g_{ip}} &= -(\frac{n_p}{n_i}),\ \frac{g_{ip}}{g_{jp}} = (\frac{n_j}{n_i}),\ \frac{g_{ii}}{g_{jp}} = -(\frac{n_j n_p}{n_i^2}) \\ \frac{g_{ip}}{g_{pp}} &= -(\frac{n_p}{n_i}),\ \frac{g_{ij}}{g_{ip}} = (\frac{n_p}{n_j}),\ \frac{g_{ij}}{g_{pp}} = -(\frac{n_p^2}{n_i n_j}) \end{aligned} \tag{23}$$

79. The local stabilities along the lines and on two dimensional surfaces of the state-space manifold are simply measured by

$$p_1 = \frac{\pi}{2n_1}\sqrt{\frac{n_5 n_p}{n_1}},\ p_2 = -\frac{\pi^2}{4n_1 n_5^2 n_p}(n_p^2 n_1 + n_5^3) \tag{24}$$

80. Local stability of the entire equilibrium phase-space configurations of the D_1-D_5-P extremal black holes are determined by the $p_3 := g$ determinant of the state-space metric tensor

$$\|g\| = -\frac{1}{2}\pi^3 (n_1 n_5 n_p)^{-1/2} \tag{25}$$

81. The universal nature of statistical interactions and the other properties concerning MSW rotating black branes are elucidated by the state-space scalar curvature

$$R(n_1, n_5, n_p) = \frac{3}{4\pi\sqrt{n_1 n_5 n_p}} \tag{26}$$

82. The constant entropy (or scalar curvature) curve defining state-space manifold is higher dimensional hyperbola

$$n_1 n_5 n_p = c^2, \tag{27}$$

where c takes respective value of $(c_S, c_R) = (S_0/2\pi, 3/4\pi R_0)$.

83. Similar state-space results hold for the four charge tree level extremal black holes.

7.2 State-space Geometry of Non-extremal Black Holes

[*S. Bellucci and B.N.T.: Phys. Rev. D (2010)*] and [*Entropy (2010)*].

84. Let us examine the state-space configuration of the four charg-anticharge black holes.

85. Such a configuration is the non-extremal D_1-D_5 black hole with non-zero momenta along the clockwise and anticlockwise directions of Kaluza-Klein compactification circle S^1, [*C. G. Callan, J. M. Maldacena: arXiv:hep-th/9602043v2*].

86. For given brane charges and Kaluza-Klein momenta, the microscopic entropy and macroscopic entropy match with

$$S_{micro} = 2\pi\sqrt{n_1 n_5}(\sqrt{n_p} + \sqrt{\overline{n_p}}) = S_{macro} \tag{28}$$

87. State-space covariant metric tensor is defined as negative Hessian matrix of entropy with respect to number of D_1, D_5 branes $\{n_i \mid i = 1, 5\}$ and clockwise-anticlockwise Kaluza-Klein momentum charges $\{n_p, \overline{n_p}\}$.

88. The components of the metric tensor are

$$\begin{aligned}
g_{n_1 n_1} &= \frac{\pi}{2}\sqrt{\frac{n_5}{n_1^3}}(\sqrt{n_p} + \sqrt{\overline{n_p}}),\ g_{n_1 n_5} = -\frac{\pi}{2\sqrt{n_1 n_5}}(\sqrt{n_p} + \sqrt{\overline{n_p}}) \\
g_{n_1 n_p} &= -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1 n_p}},\ g_{n_1 \overline{n_p}} = -\frac{\pi}{2}\sqrt{\frac{n_5}{n_1 \overline{n_p}}} \\
g_{n_5 n_5} &= \frac{\pi}{2}\sqrt{\frac{n_1}{n_5^3}}(\sqrt{n_p} + \sqrt{\overline{n_p}}),\ g_{n_5 n_p} = -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5 n_p}} \\
g_{n_5 \overline{n_p}} &= -\frac{\pi}{2}\sqrt{\frac{n_1}{n_5 \overline{n_p}}},\ g_{n_p n_p} = \frac{\pi}{2}\sqrt{\frac{n_1 n_5}{n_p^3}} \\
g_{n_p \overline{n_p}} &= 0,\ g_{\overline{n_p}\,\overline{n_p}} = \frac{\pi}{2}\sqrt{\frac{n_1 n_5}{\overline{n_p}^3}}
\end{aligned} \tag{29}$$

89. For distinct $i, j \in \{1, 5\}$, and $k, l \in \{p, \overline{p}\}$ describing four charge non-extremal D_1-D_5-P-$\overline{P}$ black holes, the statistical pair correlations consist of the following scaling relations

$$\begin{aligned}
\frac{g_{ii}}{g_{jj}} &= (\frac{n_j}{n_i})^2,\ \frac{g_{ii}}{g_{kk}} = \frac{n_k}{n_i^2}\sqrt{n_k}(\sqrt{n_p} + \sqrt{\overline{n_p}}),\ \frac{g_{ii}}{g_{ij}} = -\frac{n_j}{n_i} \\
\frac{g_{ii}}{g_{ik}} &= -\frac{\sqrt{n_k}}{n_i}(\sqrt{n_p} + \sqrt{\overline{n_p}}),\ \frac{g_{ik}}{g_{jk}} = \frac{n_j}{n_i},\ \frac{g_{ii}}{g_{jk}} = -\frac{n_j}{n_i^2}\sqrt{n_k}(\sqrt{n_p} + \sqrt{\overline{n_p}}) \\
\frac{g_{ik}}{g_{kk}} &= -\frac{n_k}{n_i},\ \frac{g_{ij}}{g_{ik}} = \frac{\sqrt{n_k}}{n_j}(\sqrt{n_p} + \sqrt{\overline{n_p}}),\ \frac{g_{ij}}{g_{kk}} = -\frac{n_k}{n_i n_j}\sqrt{n_k}(\sqrt{n_p} + \sqrt{\overline{n_p}})
\end{aligned} \tag{30}$$

90. The list of other mix relative correlation functions concerning the non-extremal D_1-D_5-P-$\overline{P}$ black holes are

$$\begin{aligned} \frac{g_{ik}}{g_{il}} &= \sqrt{\frac{n_l}{n_k}},\ \frac{g_{ik}}{g_{jl}} = \frac{n_j}{n_i}\sqrt{\frac{n_l}{n_k}},\ \frac{g_{kl}}{g_{ij}} = 0 \\ \frac{g_{kl}}{g_{ii}} &= 0,\ \frac{g_{kk}}{g_{ll}} = (\frac{n_l}{n_k})^{3/2},\ \frac{g_{kl}}{g_{kk}} = 0 \end{aligned} \tag{31}$$

91. Local stability criteria on possible surfaces and hyper-surfaces of underlying state-space configuration are determined by the positivity of

$$\begin{aligned} p_0 &= 1,\ p_1 = \frac{\pi}{2}\sqrt{\frac{n_5}{n_1^3}}(\sqrt{n_p} + \sqrt{\overline{n_p}}) \\ p_2 &= 0,\ p_3 = -\frac{1}{2n_p}\frac{\pi^3}{\sqrt{n_1 n_5}}(\sqrt{n_p} + \sqrt{\overline{n_p}}) \end{aligned} \tag{32}$$

92. Complete local stability of full non-extremal D_1-D_5 black brane state-space configuration is acertained by positivity of the determinant of state-space metric tensor

$$g(n_1, n_5, n_p, \overline{n_p}) = -\frac{1}{4}\frac{\pi^4}{(n_p\overline{n_p})^{3/2}}(\sqrt{n_p} + \sqrt{\overline{n_p}})^2 \tag{33}$$

93. Global state-space properties concerning four charge non-extremal D_1-D_5 black holes are determined by the regularity of the state-space scalar curvature invariant

$$R(n_1, n_5, n_p, \overline{n_p}) = \frac{9}{4\pi\sqrt{n_1 n_5}}(\sqrt{n_p} + \sqrt{\overline{n_p}})^{-6} f(n_p, \overline{n_p}), \tag{34}$$

where the function $f(n_p, \overline{n_p})$ of two momenta $(n_p, \overline{n_p})$ running in opposite directions of the KK circle S^1 has been defined as

$$f(n_p, \overline{n_p}) := n_p^{5/2} + 10 n_p^{3/2}\overline{n_p} + 5 n_p^{1/2}\overline{n_p}^2 + 5 n_p^2 \overline{n_p}^{1/2} + 10 n_p \overline{n_p}^{3/2} + \overline{n_p}^{5/2} \tag{35}$$

94. Large charge non-extremal D_1-D_5 black branes have non-vanishing scalar curvature function on the state-space manifold (M_4, g), and thus imply an almost everywhere weakly interacting statistical basis.

95. The constant entropy curve is non-standard curve is

$$\frac{c^2}{n_1 n_5} = (\sqrt{n_p} + \sqrt{\overline{n_p}})^2 \tag{36}$$

96. As in the case of two charge D_0-D_4 extremal black holes and D_1-D_5-P extremal black holes, the constant c takes the same value of $c := S_0^2/4\pi^2$.

97. For given state-space scalar curvature k, the constant state-space curvature curves take the form of

$$f(n_p, \overline{n_p}) = k\sqrt{n_1 n_5}(\sqrt{n_p} + \sqrt{\overline{n_p}})^6 \tag{37}$$

98. Similar results hold for the six and eight charge-anticharge non-extremal black holes.

8 SU(N) Gauge Theory Configurations

99. To explore the Wienhold's chemical geometry towards high energy physics, lets recall that

100. Yang-Mills gauge theory has opened interesting avenues in understanding the strong nuclear processes and decay reactions [E. Witten, NPB 1979].

101. CP breaking, duality principle between large N gauge theories and string theory that describes a set of adjacent vacua are separated by domain walls [E. Witten, PRL 1998].

102. Given the vacuum angle x, temperature y and decoherence temperature d, the free energy [Kharzeev, Pisarski, Tytgat 1998] undermining the deconfining phase transition can be represented as the expression

$$F(x,y) = \left(1 + cx^2\right)(d-y)^{2-e}, \tag{38}$$

where c is the coefficient of the anomaly and e is the critical exponent.

103. Herewith, we see that the flow components of free energy fluctuations are

$$\begin{aligned} F_x(x,y) &= 2cx(d-y)^{2-e}, \\ F_y(x,y) &= -(2-e)(1+cx^2)(d-y)^{1-e} \end{aligned} \tag{39}$$

104. The metric tensor g on the chemical surface $\mathcal{M}_2(\mathbb{R})$ - - defined via the Hessian matrix $Hess(F(x,y))$ of $F(x,y)$ - - reduce as

$$\begin{aligned} F_{xx} &= 2c(d-y)^{2-e}, \\ F_{xy} &= -2cx(2-e)(d-y)^{1-e}, \\ F_{yy} &= (1-e)(2-e)\left(1+cx^2\right)(d-y)^{-e} \end{aligned} \tag{40}$$

105. For a given vacuum bubble system, the principle components of the metric tensor $\{F_{xx}, F_{yy}\}$, which signify self pair correlations, remain positive definite functions over a range of the vacuum angle x and temperature y.

106. It is not difficult to see that the determinant of the metric tensor reduces as the expression

$$\begin{aligned} ||g||(x,y) &= -4c^2(2-e)^2x^2(d-y)^{2-2e} \\ &+ 2c(1-e)(2-e)(1+cx^2)(d-y)^{2-2e} \end{aligned} \quad (41)$$

107. Thus, the parity odd bubble configuration corresponds to a degenerate statistical ensemble for either an absolute value of the vacuum angle

$$|x| = \left(\frac{e-1}{e-3}\right)c^{-1/2} \quad (42)$$

or the temperature y stays fixed at the decoherence temperature, or the critical exponent takes a fixed value $e = 2$.

108. For examining global nature of fluctuations, we need calculate non-trivial Christoffel connections that are the third derivative of the free energy as

$$\begin{aligned} F_{111} &= 0, \\ F_{112} &= -2c(2-e)(d-y)^{1-e}, \\ F_{122} &= 2c(1-e)(2-e)x(d-y)^{-e}, \\ F_{222} &= (1-e)(2-e)e(1+cx^2)(d-y)^{-(1+e)} \end{aligned} \quad (43)$$

109. As per this notion of thermodynamic geometry [Ruppeiner, RMP 1995], the global nature of phase transition curves can be examined over the range of vacuum angle x and QCD temperature y describing a fluctuating ensemble of parity odd bubbles.

110. In this case, we find that the scalar curvature reads as

$$R(x, y) = \frac{k(e-1)(d-y)^{e-2}}{\Big(1 - e + c(e-3)x^2\Big)^2} \tag{44}$$

111. This shows that the statistical analysis of the YM free energy vacuum fluctuations renders the following limiting correlation area

$$\tilde{A} \propto lim_{x \to 0,\ y \to 0}\ R(x, y) \ = - \frac{T_d^{e-2}}{e-1}, \tag{45}$$

112. As a result, the $SU(N)$ gauge theory vacuum configuration corresponds to an interacting statistical ensemble, even in the limit of zero temperature and zero vacuum parameter.

 In addition, for the systems with critical exponent $e = 2$, we observe that the scalar curvature becomes independent of the temperature y, which as a function of the vacuum parameter x reduces as the following singly peaked squared Lorentzian function

$$R(x) = \frac{k}{\Big(1 + cx^2\Big)^2} \tag{46}$$

113. In conclusion, we find that the intrinsic scalar curvature as given in Eqn.(44) doubly diverges for the following absolute value of the vacuum angle

$$|x| = \pm\sqrt{\frac{1}{c}(\frac{e-1}{e-3})}, \tag{47}$$

 whenever the system possesses a critical exponent $e \notin \{1, 2, 3\}$ and $y \neq d$, as the phenomenon of decoupling happens at the decoherence temperature.

9 Multi-centered D_6-D_4-D_2-D_0 Black Branes

[*S. Bellucci and B.N.T.: Phys. Rev. D (2010)*] and [*Entropy (2010)*].

114. We have explicated the state-space manifolds containing both the single center and double center four charge black brane configurations.

115. A charge $\Gamma = \sum_i \Gamma_i$ obtained by wrapping the constituent D branes around various cycles of the compactifying space X.

116. Multi-centered solutions ([*F. Denef, G. W. Moore: arXiv:0705.2564v1, arXiv:hep-th/0702146v2*]) are analyzed by considering the type IIA string theory compactified on the product $X := T_1^2 \times T_2^2 \times T_3^2$.

117. Entropy as a function the charge Γ corresponding to $p0$ D_6 branes on X, p D_4 branes on $(T_1^2 \times T_2^2) + (T_2^2 \times T_3^2) + (T_3^2 \times T_1^2)$, q D_2 branes on $(T_1^2 + T_2^2 + T_3^2)$ and $q0$ D_0 branes is

$$S(\Gamma) := \pi\sqrt{-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2} \tag{48}$$

118. The D_6, D_4, D_2, D_0 brane charges $\Gamma_i := (p_i^\Lambda, q_{\Lambda,i})$ form local co-ordinates on the intrinsic state-space manifold (M_4, g).

119. The components of the covariant metric tensor are given by

$$\begin{aligned}
g_{p0p0} &= -4\pi\frac{-3p^2q^2q0^2 + 3pq^4q0 - q^6 + p^3q0^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{p0p} &= 6\pi\frac{-p^3q0^2q + 2p^2q0q^3 + p^2q0^3p0 - pq^5 - 2pq^2p0q0^2 + p0q^4q0}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{p0q} &= -12\pi\frac{2p^3q^2q0 + p^2qq0^2p0 - 2pq^3q0p0 - q^4p^2 + q^5p0 - p^4q0^2}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{p0q0} &= -\pi\frac{-6p^4qq0 + 3p^2q^2q0p0 - 9pqq0^2p0^2 + 5q^3p^3 - 6q^4p0p + 6q^3p0^2q0 + 6p0q0^2p^3 + p0^3q0^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{pp} &= -12\pi\frac{p^4q0^2 - p^3q^2q0 - 3p^2qq0^2p0 + 4pq^3q0p0 - p0^2q0^2q^2 + p0^2q0^3p - q^5p0}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{pq} &= 3\pi\frac{2p^4qq0 - 2p0q0^2p^3 + 3p^2q^2q0p0 - 3q^3p^3 + 2q^4p0p - pqq0^2p0^2 - 2q^3p0^2q0 + (p0q0)^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{pq0} &= -12\pi\frac{p^5q0 - 2p^3q0p0q - p^4q^2 + 2p^2q^3p0 + pq^2p0^2q0 - p0^2q^4}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{qq} &= -12\pi\frac{4p^3q0p0q - p^2q^3p0 - p^2q0^2p0^2 - 3pq^2p0^2q0 + p0^2q^4 - p^5q0 + p0^3qq0^2}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{qq0} &= 6\pi\frac{-p^5q + 2p^3q^2p0 - 2p^2qp0^2q0 + p0p^4q0 - p0^2q^3p + p0^3q^2q0}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}} \\
g_{q0q0} &= -4\pi\frac{-p^6 + 3p^4p0q - 3p0^2p^2q^2 + p0^3q^3}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{3/2}}
\end{aligned} \tag{49}$$

120. Define a charge vector $X_a = (p0, p, q, q0)$ with a set of notations $1 \leftrightarrow p_0, 2 \leftrightarrow p, 3 \leftrightarrow q, 4 \leftrightarrow q_0$.

121. The local stability condition of the underlying statistical configuration under the Gaussian fluctuations requires that all the principle components of the fluctuations should be positive definite, i.e. for given set of state-space variables $\Gamma_i := (p_i^\Lambda, q_{\Lambda,i})$ one must demands that $\{g_{ii}(\Gamma_i) > 0;\ \forall i = 1, 2\}$. The concerned state-space metric constraints are thus defined by

$$g_{ii}(X_a) \ > \ 0 \ \forall \ i \in \{1, 2, 3, 4\} \mid m_{ii} < 0 \tag{50}$$

where

$$\begin{aligned} m_{11} &:= -3p^2q^2q0^2 + 3pq^4q0 - q^6 + p^3q0^3 \\ m_{22} &:= p^4q0^2 - p^3q^2q0 - 3p^2qq0^2p0 + 4pq^3q0p0 \\ &\quad -p0^2q0^2q^2 + p0^2q0^3p - q^5p0 \\ m_{33} &:= 4p^3q0p0q - p^2q^3p0 - p^2q0^2p0^2 - 3pq^2p0^2q0 \\ &\quad +p0^2q^4 - p^5q0 + p0^3qq0^2 \\ m_{44} &:= -p^6 + 3p^4p0q - 3p0^2p^2q^2 + p0^3q^3 \end{aligned} \tag{51}$$

122. For distinct $i, j, k, l \in \{1, 2, 3, 4\}$, the admissible statistical pair correlations are consisting of diverse scaling properties. The set of nontrivial relative correlations signifying possible scaling relations of state-space correlations are nicely depicted by

$$\begin{aligned} C_r \ = \ & \left\{ \frac{g_{11}}{g_{12}}, \frac{g_{11}}{g_{13}}, \frac{g_{11}}{g_{14}}, \frac{g_{11}}{g_{22}}, \frac{g_{11}}{g_{23}}, \frac{g_{11}}{g_{24}}, \frac{g_{11}}{g_{33}}, \frac{g_{11}}{g_{34}}, \frac{g_{11}}{g_{44}}, \frac{g_{12}}{g_{13}}, \frac{g_{12}}{g_{14}}, \frac{g_{12}}{g_{22}}, \frac{g_{12}}{g_{23}}, \frac{g_{12}}{g_{24}}, \frac{g_{12}}{g_{33}}, \right. \\ & \frac{g_{12}}{g_{34}}, \frac{g_{12}}{g_{44}}, \frac{g_{13}}{g_{14}}, \frac{g_{13}}{g_{22}}, \frac{g_{13}}{g_{23}}, \frac{g_{13}}{g_{24}}, \frac{g_{13}}{g_{33}}, \frac{g_{13}}{g_{34}}, \frac{g_{13}}{g_{44}}, \frac{g_{14}}{g_{22}}, \frac{g_{14}}{g_{23}}, \frac{g_{14}}{g_{24}}, \frac{g_{14}}{g_{33}}, \frac{g_{14}}{g_{34}}, \frac{g_{14}}{g_{44}}, \\ & \left. \frac{g_{22}}{g_{23}}, \frac{g_{22}}{g_{24}}, \frac{g_{22}}{g_{33}}, \frac{g_{22}}{g_{34}}, \frac{g_{22}}{g_{44}}, \frac{g_{23}}{g_{24}}, \frac{g_{23}}{g_{33}}, \frac{g_{23}}{g_{34}}, \frac{g_{23}}{g_{44}}, \frac{g_{24}}{g_{33}}, \frac{g_{24}}{g_{34}}, \frac{g_{24}}{g_{44}}, \frac{g_{33}}{g_{34}}, \frac{g_{33}}{g_{44}}, \frac{g_{34}}{g_{44}} \right\} \end{aligned} \tag{52}$$

123. The local stability condition constraint the allowed domain of the parameters of black hole configurations, and requires positivity of the following simultaneous equations

$$\begin{aligned} p_1 &= -4\pi \frac{(-3p^2q^2q0^2 + 3pq^4q0 - q^6 + q0^3p^3)}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - p0^2q0^2)^{-3/2}} \\ p_2 &= -12\pi^2 \frac{(q0^4p^4 - 4q^2q0^3p^3 + 6q^4q0^2p^2 - 4q^6q0p + q^8)}{(4p^3q0 - 3p^2q^2 - 6p0pqq0 + 4p0q^3 + p0^2q0^2)^{-2}} \\ p_3 &= -36\pi^3 \frac{(-3p^2q^2q0^2 + 3pq^4q0 - q^6 + q0^3p^3)}{(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - p0^2q0^2)^{-3/2}} \end{aligned} \tag{53}$$

124. The stability of full state-space configuration is determined by computing the determinant of the metric tensor

$$\|g\| = 9\pi^4 \tag{54}$$

125. The determinant of the metric tensor takes positive definite value, and thus there exist positive definite volume form on the state-space manifold (M_4, g) of concerned leading order multi-centered D_6-D_4-D_2-D_0 black brane configurations.

126. Conclusive nature of the state-space interaction and the other global properties of the statistical configurations are analyzed by determining state-space scalar curvature invariant

$$R(\Gamma) = \frac{8}{3\pi}(-4p^3q0 + 3p^2q^2 + 6p0pqq0 - 4p0q^3 - (p0q0)^2)^{-1/2} \tag{55}$$

127. For some given constant charge Γ_0, both the constant entropy and constant scalar curvature curves are again defined as

$$4p^3q0 - 3p^2q^2 - 6p0pqq0 + 4p0q^3 + (p0q0)^2 = c, \tag{56}$$

where the respective real constants $c := (c_S, c_R)$ are given by

$$\begin{aligned} c_S &:= -(\frac{S(\Gamma_0)}{\pi})^2 \ for\ constant\ entropy \\ c_R &:= -(\frac{8}{3\pi R(\Gamma_0)})^2 \ for\ constant\ scalar\ curvature \end{aligned} \tag{57}$$

9.1 Single Center D_6-D_4-D_2-D_0 Configurations

9.1.1 State-space Correlations

128. For the charges, $p0 := 0$; $p := 6\Lambda$; $q := 0$; $q0 := -12\Lambda$; describe single center configurations *[Denef and Moore: arXiv:0705.2564v1, hep-th/ 0702146v2]*.

129. The above state-space correlation functions reduce to

$$\begin{aligned} g_{11} &= \pi\sqrt{2},\ g_{13} = \frac{3}{2}\pi\sqrt{2} = -g_{22} \\ g_{24} &= \frac{3}{4}\pi\sqrt{2} = -g_{33},\ g_{44} = \frac{1}{8}\pi\sqrt{2} \\ g_{12} &= 0 = g_{14} = g_{23} = g_{34} \end{aligned} \tag{58}$$

130. For all Λ, the concerned state-space metric constraints are

$$\begin{aligned} g_{ii}(X_a) &> 0\ \forall\ i = 1, 3 \\ g_{jj}(X_a) &< 0\ \forall\ j = 2, 4 \end{aligned} \tag{59}$$

131. The relative correlations defined as $c_{ijkl} := g_{ij}/g_{kl}$ reduce to the following three set of constant values.

132. There are only 15 non vanishing finite ratios defining the relative state-space correlation functions

$$\begin{aligned}
c_{1113} &= \frac{2}{3} = -c_{1122},\ c_{1124} = \frac{4}{3} = -c_{1133} \\
c_{1144} &= 8,\ c_{1322} = -1 = c_{2433} \\
c_{1324} &= 2 = -c_{1333},\ c_{2233} = 2 = -c_{2224} \\
c_{1344} &= 12 = -c_{2244},\ c_{2444} = 6 = -c_{3344}
\end{aligned} \tag{60}$$

133. The set of vanishing ratios of relative correlation functions is

$$\begin{aligned}
C_R^0 &:= \{c_{1213}, c_{1224}, c_{1222}, c_{1224}, c_{1233}, c_{1244}, c_{1422}, \\
&\quad c_{1424}, c_{1433}, c_{1444}, c_{2324}, c_{2333}, c_{2344}, c_{3444}\} \\
&= \{0\}
\end{aligned} \tag{61}$$

134. The limiting ill-defined relative correlations are characterized by the set

$$\begin{aligned}
C_R^\infty &:= \{c_{1112}, c_{1114}, c_{1123}, c_{1134}, c_{1223}, c_{1234}, c_{1314}, c_{1323}, \\
&\quad c_{1334}, c_{1423}, c_{1434}, c_{2223}, c_{2234}, c_{2334}, c_{2434}, c_{3334}\} \\
&= \{\infty\}
\end{aligned} \tag{62}$$

9.1.2 State-space Stability

135. Entropy corresponding to single center specification takes to a constant value of $S(\Gamma = \Lambda(0,6,0,-12)) = \pi\sqrt{10368}\Lambda^2$.

136. Possible stability of internal state-space configurations reduce to the positivity of

$$p_1 = \sqrt{2}\pi,\ p_2 = -3\pi^2,\ p_3 = 9\sqrt{2}\pi^3,\ p_4 \ = \ 9\pi^4 \tag{63}$$

137. The scalar curvature remains non zero, positive and take the value of

$$R(\Gamma = \Lambda(0,6,0,-12)) = \frac{\sqrt{2}}{54\pi\Lambda^2} \tag{64}$$

138. Thus, the state-space correlation volume vary as an inverse function of the single center brane entropy.

9.2 Double Center D_6-D_4-D_2-D_0 Configurations

9.2.1 State-space Correlations at the First Center

139. The brane charges $p0 := 1$; $p := 3\Lambda$; $q := 6\Lambda^2$; and $q0 := -6\Lambda$ defining first center of the two center D_6-D_4-D_2-D_0 configurations, we have the following components of the state-space metric tensor

$$\begin{aligned}
g_{11} &= 108\pi\Lambda^3\frac{6\Lambda^2+12\Lambda^4+8\Lambda^6+1}{(3\Lambda^4-1)^{3/2}},\ g_{12} = -54\pi\Lambda^2\frac{7\Lambda^2+16\Lambda^4+12\Lambda^6+1}{(3\Lambda^4-1)^{3/2}} \\
g_{13} &= 54\pi\Lambda^3\frac{4\Lambda^2+4\Lambda^4+1}{(3\Lambda^4-1)^{3/2}},\ g_{14} = -\pi\frac{18\Lambda^4+27\Lambda^6-1}{(3\Lambda^4-1)^{3/2}} \\
g_{22} &= 18\pi\Lambda\frac{13\Lambda^2+30\Lambda^4+24\Lambda^6+2}{(3\Lambda^4-1)^{3/2}},\ g_{23} = -3\pi\frac{42\Lambda^4+12\Lambda^2+45\Lambda^6+1}{(3\Lambda^4-1)^{3/2}} \\
g_{24} &= 9\pi\Lambda^3\frac{1+2\Lambda^2}{(3\Lambda^4-1)^{3/2}},\ g_{33} = 3\pi\Lambda\frac{2+9\Lambda^2+12\Lambda^4}{(3\Lambda^4-1)^{3/2}} \\
g_{34} &= -\frac{3}{2}\pi\Lambda^2\frac{1+3\Lambda^2}{(3\Lambda^4-1)^{3/2}},\ g_{44} = \frac{1}{2}\pi\Lambda^3\frac{1}{(3\Lambda^4-1)^{3/2}}
\end{aligned} \tag{65}$$

140. Subsequent notations of the relative state-space correlations are prescribed by defining $c_{ijkl} := g_{ij}/g_{kl}$.

9.2.2 State-space Correlations at the Second Center

141. The charges $p0 := -1$; $p := 3\Lambda$; $q := -6\Lambda^2$; and $q0 := -6\Lambda$ define the second center of the two center configurations.

142. Following limiting values are achieved for the state-space pair correlation functions

$$\begin{aligned}
g_{11} &= 108\pi\Lambda^3\frac{6\Lambda^2+12\Lambda^4+8\Lambda^6+1}{(3\Lambda^4-1)^{3/2}},\ g_{12} = 54\pi\Lambda^2\frac{7\Lambda^2+16\Lambda^4+12\Lambda^6+1}{(3\Lambda^4-1)^{3/2}} \\
g_{13} &= 54\pi\Lambda^3\frac{4\Lambda^2+4\Lambda^4+1}{(3\Lambda^4-1)^{3/2}},\ g_{14} = \pi\frac{18\Lambda^4+27\Lambda^6-1}{(3\Lambda^4-1)^{3/2}} \\
g_{22} &= 18\pi\Lambda\frac{13\Lambda^2+30\Lambda^4+24\Lambda^6+2}{(3\Lambda^4-1)^{3/2}},\ g_{23} = 3\pi\frac{42\Lambda^4+12\Lambda^2+45\Lambda^6+1}{(3\Lambda^4-1)^{3/2}} \\
g_{24} &= 9\pi\Lambda^3\frac{1+2\Lambda^2}{(3\Lambda^4-1)^{3/2}},\ g_{33} = 3\pi\Lambda\frac{2+9\Lambda^2+12\Lambda^4}{(3\Lambda^4-1)^{3/2}} \\
g_{34} &= \frac{3}{2}\pi\Lambda^2\frac{1+3\Lambda^2}{(3\Lambda^4-1)^{3/2}},\ g_{44} = \frac{1}{2}\pi\Lambda^3\frac{1}{(3\Lambda^4-1)^{3/2}}
\end{aligned} \tag{66}$$

143. The relative correlations of the state-space configuration concerning second center of the D_6-D_4-D_2-D_0 system are similarly analyzed.

9.2.3 State-space Stability of Double Center D_6-D_4-D_2-D_0 Configurations

144. [*Denef and Moore: arXiv:0705.2564v1, hep-th/ 0702146v2*] have shown the two centered bound state configurations arise with charge centers $\Gamma_1 = (1, 3\Lambda, 6\Lambda^2, -6\Lambda)$ and $\Gamma_2 = (-1, 3\Lambda, -6\Lambda^2, -6\Lambda)$.

145. The entropies of both the two charge centers Γ_1, Γ_2 match, and in particular we have
$$S(\Gamma_1) = S(\Gamma_2) = \pi\sqrt{108\Lambda^6 - 36\Lambda^2} \sim \Lambda^3 \tag{67}$$

146. Apart from definite scaling in Λ, the above two center D_6-D_4-D_2-D_0 configurations form two type of state-space pair correlation functions
$$\begin{aligned} C^{(1)}_{ij}(\Gamma) &:= \{g_{ij}(\Gamma_1) = g_{ij}(\Gamma_2); (i,j) \in \{(1,1),(1,3),(2,2),(2,4),(3,3),(3,4),(4,4)\}\} \\ C^{(2)}_{ij}(\Gamma) &:= \{g_{ij}(\Gamma_1) = -g_{ij}(\Gamma_2); (i,j) \in \{(1,2),(1,4),(2,3)\}\} \end{aligned} \tag{68}$$

147. For both the Γ_1 and Γ_2, the respective state-space metric constraints satisfy
$$g_{ii}(X_a) \;>\; 0 \;\forall\, i \in \{1,2,3,4\} \tag{69}$$

148. Both the centers have the same principle minors
$$\begin{aligned} p_1 &= 108\pi|\Lambda|^3\left(\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{(3\Lambda^4 - 1)^{3/2}}\right) \\ p_2 &= -972\pi^2|\Lambda|^4\left(\frac{1 + 8\Lambda^2 + 24\Lambda^4 + 32\Lambda^6 + 16\Lambda^8}{(3\Lambda^4 - 1)^{-2}}\right) \\ p_3 &= 972\pi^3|\Lambda|^3\left(\frac{6\Lambda^2 + 12\Lambda^4 + 8\Lambda^6 + 1}{(3\Lambda^4 - 1)^{-3/2}}\right) \end{aligned} \tag{70}$$

149. The general expression of determinant of the metric tensor implies well-defined state-space manifold (M_4, g)

150. The state-space scalar curvature again remains non-zero, positive quantity and takes the same values for both of the two charge centers
$$R(\Gamma_1) = R(\Gamma_2) = \frac{4}{9\pi}\frac{1}{\sqrt{3\Lambda^6 - \Lambda^2}} \sim \frac{1}{\Lambda^3}, \quad for\ large\ \Lambda \tag{71}$$

151. Thus, the global statistical correlations are identical for both the centers of two charge centered D_6-D_4-D_2-D_0 black brane configurations.

152. In a chosen basin of D_6-D_4-D_2-D_0 brane charges, the correlation volume, as the scalar curvature of the (M_4, g), both the single center and double center solutions modulate as an inverse function of the entropy
$$\begin{aligned} R^{single}(\Gamma) &\sim \frac{1}{\Lambda^2} \\ R^{double}(\Gamma) &\sim \frac{1}{\Lambda^3} \end{aligned} \tag{72}$$

10 Exact Fluctuating 1/2-BPS Configurations

[*S. Bellucci and BNT: JHEP (2010)*].

153. We shall now consider the role of the statistical fluctuations, in the two parameter giant and superstar configurations, characterized by an ensemble of arbitrary liquid droplets or irregular shaped fuzzballs.

154. Covariant thermodynamic geometries are analyzed for the giant solutions in terms of the chemical configuration parameters and arbitrarily excited boxes of random Young tableaux.

155. Underlying moduli configurations appear horizonless and smooth, but one acquires an entropy associated with average horizon area of the black hole in the classical limit.

156. We shal work in the limit defined as (i) Planck length $l_P \to 0$ and (ii) AdS throat scale $L \to 0$ such that the ratio $l_P/L \to \infty$

157. To compare with the corresponding quantum picture, one may use AdS/CFT correspondence

$$AdS/CFT: \ \{\mathcal{N} = 4 \ SYM\} \ \leftrightarrow \ \{Type \ IIB \ String \ Theory \ on \ AdS_5 \times S^5\} \qquad (73)$$

For further details see [*H. Lin, O. Lunin, J. Maldacena (LLM): Bubbling AdS space and 1/2 BPS geometries, JHEP* **0410**, *025 (2004), arXiv:hep-th/0409174v2*].

158. As per the Ref. [*Balasubramanian, De Boer, Jejjala and Simon: arXiv:hep-th/0508023v2*], the supergravity description emerges in the strong coupling limit $g_{YM}^2 N >> 1$, whereas, the dual CFT is in the weakly coupled regime $g_{YM}^2 N << 1$

159. Four the purpose of chemical geometry, the canonical energy is a function of two distinct parameters (T, λ), where T is an effective canonical temperature and λ is the chemical potential dual to the R symmetry of the theory.

160. For the purpose of the state-space geometry, the box counting entropy is a function of two distinct large integers (n, M), where n corresponds to number of excited boxes and M^2 corresponds to the total number of possible boxes in Young tableaux.

161. Expressions for the energy and counting entropy, as the type IIB string theory black hole, are known from the viewpoint of dual $\mathcal{N} = 4$ super Yang Mills theory and $AdS_5 \times S^5$.

10.1 Chemical Description

162. To analyze the chemical correlation of large number of excited free fermion states, we now consider Weinhold geometry.

163. Typical correlation of the statistical states are characterized by arbitrary Young diagrams.

164. The average canonical energy defined in terms of the effective canonical temperature T, and R- chemical potential λ is

$$< E(T,\lambda) >= \sum_{j=0}^{\infty} \frac{j\ exp(-(\lambda + j/T))}{1 - exp(-(\lambda + j/T))} \tag{74}$$

165. To investigate the chemical fluctuations, we consider two neighboring statistical states characterized by (T,λ) and $(T+\delta T, \lambda + \delta\lambda)$.

166. Chemical pair correlation functions are defined as

$$g_{ij}^{(E)} = \partial_i \partial_j < E(T,\lambda) >,\ i,j = T,\lambda \tag{75}$$

167. The components of Weinhold metric tensor find following series expansions

$$\begin{aligned} g_{TT}(T,\lambda) &= \sum_{j=0}^{\infty} (-2\frac{j^2}{T^3}\frac{exp(-\lambda - j/T)}{(1-exp(-\lambda - j/T))} + \frac{j^3}{T^4}\frac{exp(-\lambda - j/T)}{(1-exp(-\lambda - j/T))} \\ &+3\frac{j^3}{T^4}\frac{exp(-\lambda - j/T)^2}{(1-exp(-\lambda - j/T))^2} + 2\frac{j^3}{T^4}\frac{exp(-\lambda - j/T)^3}{(1-exp(-\lambda - j/T))^3} \\ &-2\frac{j^2}{T^3}\frac{exp(-\lambda - j/T)^2}{(1-exp(-\lambda - j/T))^2}) \\ g_{T\lambda}(T,\lambda) &= \sum_{j=0}^{\infty} (-\frac{j^2}{T^2}\frac{exp(-\lambda - j/T)}{(1-exp(-\lambda - j/T))} - 3\frac{j^2}{T^2}\frac{exp(-\lambda - j/T)^2}{(1-exp(-\lambda - j/T))^2} \\ &-2\frac{j^2}{T^2}\frac{exp(-\lambda - j/T)^3}{(1-exp(-\lambda - j/T))^3}) \\ g_{\lambda\lambda}(T,\lambda) &= \sum_{j=0}^{\infty} (j\frac{exp(-\lambda - j/T)}{(1-exp(-\lambda - j/T)} + 3j\frac{exp(-\lambda - j/T)^2}{(1-exp(-\lambda - j/T))^2} \\ &+2j\frac{exp(-\lambda - j/T)^3}{(1-exp(-\lambda - j/T))^3}) \end{aligned} \tag{76}$$

168. Define a level function

$$b_j(T,\lambda) := exp(-\lambda - j/T) \tag{77}$$

169. The stability of arbitrary chemical configurations is thence determined by the determinant of thermodynamic Weinhold metric tensor

$$\begin{aligned} ||g(T,\lambda)|| &= \sum_{j=0}^{\infty}(-j^2\frac{b_j}{T^4}\frac{(-2T + 2Tb_j + j + jb_j)}{(b_j - 1)^3)}) \times \\ & \sum_{j=0}^{\infty}(-jb_j\frac{(1+b_j)}{(b_j-1)^3}) - (\sum_{j=0}^{\infty}(j^2\frac{b_j}{T^2}\frac{(1+b_j)}{(b_j-1)^3)})^2 \end{aligned} \tag{78}$$

170. Conclusive nature of the global chemical correlations are analyzed by scalar curvature invariant

$$R(T,\lambda) = \frac{R_{T\lambda T\lambda}(T,\lambda)}{||g(T,\lambda)||} \tag{79}$$

where the covariant Riemann tensor $R_{T\lambda T\lambda}(T,\lambda)$ turns out to be

$$\begin{aligned}
R_{T\lambda T\lambda}(T,\lambda) &= -\frac{1}{4}\{(\sum_{j=0}^{\infty}(-j^2\frac{b_j}{T^4}\frac{(-2T+2Tb_j+j+jb_j)}{(b_j-1)^3})) \times (\sum_{j=0}^{\infty}(-jb_j\frac{(1+b_j)}{(b_j-1)^3}) \\
&-(\sum_{j=0}^{\infty}(j^2\frac{b_j}{T^2}\frac{(1+b_j)}{(b_j-1)^3}))^2\}^{-1}\{-(\sum_{j=0}^{\infty}(-jb_j\frac{(1+b_j)}{(b_j-1)^3})) \times \\
&(\sum_{j=0}^{\infty}(-j^2\frac{b_j}{T^4}\frac{(-2T+2Tb_j^2+j+4jb_j+jb_j^2)}{(b_j-1)^4}))^2 + (\sum_{j=0}^{\infty}(-jb_j\frac{(1+b_j)}{(b_j-1)^3})) \times \\
&(\sum_{j=0}^{\infty}(j^2\frac{b_j}{T^6}\frac{(j^2+4j^2b_j+j^2b_j^2-6jT+6T^2+6jTb_j^2-12T^2b_j+6T^2b_j^2)}{(b_j-1)^4})) \\
&\times(\sum_{j=0}^{\infty}(j^2\frac{b_j}{T^2}\frac{(1+4b_j+b_j^2)}{(b_j-1)^4})) + (\sum_{j=0}^{\infty}(j^2\frac{b_j}{T^2}\frac{(1+b_j)}{(b_j-1)^3})) \times \\
&(\sum_{j=0}^{\infty}(-j^2\frac{b_j}{T^4}\frac{(-2T+2Tb_j^2+j+4jb_j+jb_j^2)}{(b_j-1)^4})) \times \\
&(\sum_{j=0}^{\infty}(j^2\frac{b_j}{T^2}\frac{(1+4b_j+b_j^2)}{(b_j-1)^4})) - (\sum_{j=0}^{\infty}(j^2\frac{b_j}{T^2}\frac{(1+b_j)}{(b_j-1)^3})) \times \\
&(\sum_{j=0}^{\infty}(j^2\frac{b_j}{T^6}\frac{(j^2+4j^2b_j+j^2b_j^2-6jT+6T^2+6jTb_j^2-12T^2b_j+6T^2b_j^2)}{(b_j-1)^4})) \\
&\times(\sum_{j=0}^{\infty}(-jb_j\frac{(1+4b_j+b_j^2)}{(b_j-1)^4})) - (\sum_{j=0}^{\infty}(-j^2\frac{b_j}{T^4}\frac{(-2T+2Tb_j+j+jb_j)}{(b_j-1)^3})) \times \\
&(\sum_{j=0}^{\infty}(j^2\frac{b_j}{T^2}\frac{(1+4b_j+b_j^2)}{(b_j-1)^4}))^2 + (\sum_{j=0}^{\infty}(-j^2\frac{b_j}{T^4}\frac{(-2T+2Tb_j+j+jb_j)}{(b_j-1)^3})) \times \\
&(\sum_{j=0}^{\infty}(-j^2\frac{b_j}{T^4}\frac{(-2T+2Tb_j^2+j+4jb_j+jb_j^2)}{(b_j-1)^4})) \times (\sum_{j=0}^{\infty}(-jb_j\frac{(1+4b_j+b_j^2)}{(b_j-1)^4}))\}
\end{aligned} \tag{80}$$

171. The Weinhold geometry allows dual entropy representation for the statistical correlations between the states characterizing arbitrary Young diagrams. The dual state-space geometry is defined by Legendre transform

$$\begin{aligned} q &= e^{-\beta} \\ \xi &= e^{-\lambda}, \end{aligned} \tag{81}$$

where the canonical temperature is defined by $T = 1/\beta$.

10.2 The Fluctuating Young Tableaux

172. Typical statistical fluctuations are divulged over an ensemble of states with large charge $\Delta = J = N^2$ in the limit $N \to \infty$, $\hbar \to 0$ such that $N\hbar$ remains fixed.

173. State-space geometry arises from the coarse graining of microscopic 1/2 BPS supergravity.

174. An ensemble of degenerate microstates are described by Young tableaux characterizing arbitrary phase-space configurations having M^2 cells with at most n random excited cells.

175. The pictorial view of a typical Young diagram may be given as

$$\begin{aligned} Y(N, N_c) \;&=\; \begin{array}{l} \square\ \maltese\ \square\ \ldots \\ \maltese\ \square\ \maltese \\ \square\ \square\ \square \\ \qquad\quad \ddots \\ \ \vdots \\ \downarrow_{(RG\ trans.)} \\ \downarrow \\ \square\ \square\ \square\ \ldots \\ \square\ \square\ \square \\ \square\ \square\ \square \\ \qquad\quad \ddots \\ \ \vdots \end{array} \\ &=\; Phase\ Space\ (N, N_c) \end{aligned} \tag{82}$$

176. In an arbitrary Young diagram $Y(N, N_c)$, there are n filled boxes with maltese and rest $(M^2 - n)$ of them are empty boxes.

177. Therefore, the degeneracy in choosing random n maltese (excited boxes) out of the total M^2 boxes is $M^2C_n = \frac{(M^2)!}{(n)!(M^2-n)!}$.

178. From the first principle of statistical mechanics, the canonical counting entropy is

$$S(n, M) = \ln(M^2)! - \ln(n)! - \ln(M^2 - n)! \tag{83}$$

179. The subsequent analysis do not exploit any approximation, such as Stirling's approximation or thermodynamic limit.

180. The present statistical fluctuations over the canonical ensemble offer exact expressions of the state-space pair correlations and global correlation length.

181. To demonstrate so, let n excited droplets are arbitrarily chosen among M^2 fundamental cells which form an ensemble of states.

182. Then, the state-space geometry describes correlations between two neighbouring statistical states (n, M) and $(n + \delta n, M + \delta M)$ in random Young tableaux $Y(N, N_c)$.

183. The statistical fluctuations (in the droplets or fuzzballs picture having a pair (n, M)) are defined via the state-space metric tensor

$$g_{ij}^{(S)} = -\partial_i \partial_j S(n, M), \; i, j = n, M \tag{84}$$

184. The components of covariant state-space metric tensor thus defined are

$$\begin{aligned} g_{nn}(n, M) &= \Psi(1, n+1) + \Psi(1, M^2 - n + 1) \\ g_{nM}(n, M) &= -2M\Psi(1, M^2 - n + 1) \\ g_{MM}(n, M) &= 4M^2\Psi(1, M^2 - n + 1) - 2\Psi(M^2 + 1) \\ &\quad -4M^2\Psi(1, M^2 + 1) + 2\Psi(M^2 - n + 1), \end{aligned} \tag{85}$$

where $\Psi(n, x)$ is the n^{th} polygamma function, defined as the n^{th} derivative of the digamma function.

185. The digamma function $\Psi(x)$ is defined as

$$\Psi(x) = \frac{\partial}{\partial x} \ln(\Gamma(x)) \tag{86}$$

186. State-space stability holds locally, if the metric satisfies

$$\begin{aligned} g_{nn} &> 0, \; \forall \; (n, M) \mid \Psi(1, n+1) + \Psi(1, M^2 - n + 1) > 0 \\ g_{MM} &> 0, \; \forall \; (n, M) \mid \Psi(1, M^2 - n + 1) - \Psi(1, M^2 + 1) > \\ & \frac{1}{2M^2}(\Psi(M^2 + 1) - \Psi(M^2 - n + 1)) \end{aligned} \tag{87}$$

187. Modulus of the ratio of excited-excited and excited-unexcited statistical pair correlation functions determines selection parameter

$$a := \frac{1}{2M} \left| \frac{\Psi(1, n+1)}{\Psi(1, M^2 - n + 1)} \right| \tag{88}$$

188. For $n > 1$, state-space correlations involve ordinary rational number $\Psi(n, x) = \Psi(n) + \gamma$, where γ is the standard Euler's constant.

189. For small n, the $\Psi(n)$ is computed as a sum of gamma, which is again a rational number.

190. To perform this computation for a larger value of n, we have used

$$\Psi(n, x) = \frac{\partial^n \Psi(x)}{\partial x^n}, \tag{89}$$

for given initial condition $\Psi(0, x) = \Psi(x)$.

191. Stability of underlying statistical configurations is analyzed by computing the determinant of the state-space metric tensor

$$\begin{aligned} g(n, M) &= -4M^2\Psi(1, n+1)\Psi(1, M^2+1) - 2\Psi(1, n+1)\Psi(M^2+1) \\ &+4M^2\Psi(1, n+1)\Psi(1, M^2-n+1) + 2\Psi(1, n+1)\Psi(M^2-n+1) \\ &-4M^2\Psi(1, M^2-n+1)\Psi(1, M^2+1) - 2\Psi(1, M^2-n+1)\Psi(M^2+1) \\ &+2\Psi(1, M^2-n+1)\Psi(M^2-n+1) \end{aligned} \tag{90}$$

192. For a family of boxes and their excitations, this shows that there exists positive definite volume form on the (M_2, g).

193. Generic global properties of 1/2-BPS black holes state-space configurations are examined by the scalar curvature

$$\begin{aligned} R(n, M) &= \frac{1}{2}\{-2\Psi(1, n+1)\Psi(1, M^2+1)M^2 - \Psi(1, n+1)\Psi(M^2+1) \\ &+2M^2\Psi(1, n+1)\Psi(1, M^2-n+1) + \Psi(1, n+1)\Psi(M^2-n+1) \\ &-2M^2\Psi(1, M^2-n+1)\Psi(1, M^2+1) - \Psi(1, M^2-n+1)\Psi(M^2+1) \\ &+\Psi(1, M^2-n+1)\Psi(M^2-n+1)\}^{-2} \\ &[\Psi(1, M^2-n+1)^3 + \Psi(1, n+1)\Psi(1, M^2-n+1)^2 \\ &-\Psi(1, M^2-n+1)\Psi(2, n+1)\Psi(M^2+1) \\ &+\Psi(1, M^2-n+1)\Psi(2, n+1)\Psi(M^2-n+1) \\ &+\Psi(2, M^2-n+1)\Psi(1, M^2-n+1)\Psi(M^2+1) \\ &-\Psi(2, M^2-n+1)\Psi(1, M^2-n+1)\Psi(M^2-n+1) \\ &+2M^2\{\Psi(2, M^2-n+1)\Psi(1, M^2-n+1)\Psi(1, M^2+1) \\ &-\Psi(2, M^2-n+1)\Psi(1, n+1)\Psi(1, M^2-n+1) \\ &-\Psi(2, M^2-n+1)\Psi(2, n+1)\Psi(M^2+1) \\ &+\Psi(2, M^2-n+1)\Psi(2, n+1)\Psi(M^2-n+1) \\ &-2\Psi(1, M^2-n+1)^2\Psi(2, n+1) \\ &+2\Psi(1, M^2-n+1)\Psi(2, n+1)\Psi(1, M^2+1) \\ &+3\Psi(2, M^2-n+1)\Psi(1, n+1)\Psi(1, M^2+1)\} \\ &+4M^4\{\Psi(2, M^2-n+1)\Psi(1, n+1)\Psi(2, M^2+1) \\ &+\Psi(1, M^2-n+1)\Psi(2, n+1)\Psi(2, M^2+1) \\ &-\Psi(2, M^2-n+1)\Psi(2, n+1)\Psi(1, M^2+1)\}] \end{aligned} \tag{91}$$

194. The chemical and state-space geometric descriptions exhibit an intriguing set of exact pair correction functions and the global correlation length.

195. Chemical configuration shows, non-trivially curved determinant and the scalar curvature, and surprisingly the results remain valid even for single component $j = 1$ configuration.

196. The Gaussian fluctuations over an equilibrium chemical and state-space configurations accomplish well-defined, non-degenerate, curved and regular intrinsic Riemannian manifolds for all physically admissible domains of parameters.

11 The Fuzzball Solutions

[*S. Bellucci and B.N.T.: Phys. Rev. D (2010)*] and [*Entropy (2010)*].

197. We have analyzed the state-space geometry for the two charge extremal rotating black brane solutions in the viewpoint of the Mathur's Fuzzball, [*S. D. Mathur: arXiv:hep-th/0502050v1*].

198. The throat geometry of black hole space-time ends in a very quantum Fuzzball, have been introduced in [*S. D. Mathur: arXiv:0706.3884v1; O. Lunin, S. D. Mathur: hep-th/ 0109154v1; O. Lunin, S. D. Mathur: hep-th/ 0202072v2*].

11.1 State-space Geometry: Fuzzy Rings

199. Bekenstein-Hawking entropy obtained from the area of stretched horizon or coarse graining statistical entropy is

$$S(Q, P, J) = C\sqrt{QP - J} \tag{92}$$

where the electric-magnetic charges, (Q, P) and angular momentum J form co-ordinate charts on the intrinsic state-space manifold (M_3, g).

200. Explicitly, the components of the metric tensor are

$$\begin{aligned} g_{PP} &= \frac{1}{4}CQ^2(PQ-J)^{-3/2},\ g_{PQ} = -\frac{1}{4}C(PQ-2J)(PQ-J)^{-3/2} \\ g_{PJ} &= -\frac{1}{4}CQ(PQ-J)^{-3/2},\ g_{QQ} = \frac{1}{4}CP^2(PQ-J)^{-3/2} \\ g_{QJ} &= -\frac{1}{4}CP(PQ-J)^{-3/2},\ g_{JJ} = \frac{1}{4}C(PQ-J)^{-3/2} \end{aligned} \tag{93}$$

201. $\forall i \neq j \in \{P, Q\}$ and J, the relative pair correlation functions scale as

$$\begin{aligned} \frac{g_{ii}}{g_{jj}} &= (\frac{j}{i})^2,\ \frac{g_{ii}}{g_{JJ}} = j^2,\ \frac{g_{ij}}{g_{ii}} = -\frac{1}{j^2}(PQ-2J) \\ \frac{g_{ii}}{g_{iJ}} &= -j,\ \frac{g_{iJ}}{g_{jJ}} = \frac{j}{i},\ \frac{g_{ii}}{g_{jJ}} = -\frac{j^2}{i} \\ \frac{g_{iJ}}{g_{JJ}} &= -j,\ \frac{g_{ij}}{g_{iJ}} = \frac{1}{j}(PQ-2J),\ \frac{g_{ij}}{g_{JJ}} = -(PQ-2J) \end{aligned} \tag{94}$$

202. For all admissible parameters, the three parameter Fuzzball solutions stable if the following state-space minors are positive

$$p_1 = \frac{1}{4}CQ^2(PQ-J)^{-3/2},\ p_2 = \frac{1}{4}C^2J(PQ-J)^{-2} \tag{95}$$

203. For non-zero brane charges and angular momentum, the determinant of the metric tensor is non-zero

$$\|g\| = -\frac{1}{16}C^3(PQ - J)^{-5/2} \tag{96}$$

204. Thus, the Fuzzball black rings do not correspond to an intrinsic stable statistical basis, when all the configuration parameters fluctuate.

205. Important state-space global properties of the fuzzy black rings configurations are determined by the nature of state-space scalar curvature invariant

$$R(P, Q, J) = -\frac{5}{2C}(PQ - J)^{-1/2} \tag{97}$$

206. The state-space scalar curvature can be expressed as an inverse function of the entropy with a negative constant of proportionality, and thus Mathur's fuzzy ring is a regular and an attractive statistical configuration.

207. For all non-zero rotation, both the constant entropy and constant state-space scalar curvature curves are just some hyperbolic paraboloid

$$PQ - J = k, \tag{98}$$

on which the state-space geometry turns out to be well-defined, and in interacting statistical system.

208. In present case, the constants $k := (k_S, k_R)$ are respectively defined as

$$\begin{aligned} k_S &:= \frac{S_0^2}{C^2} \; for\ constant\ entropy \\ k_R &:= \frac{25}{4C^2R_0^2} \; for\ constant\ scalar\ curvature \end{aligned} \tag{99}$$

209. The vanishing angular momentum limit $J \to 0$ makes an ill-defined state-space geometry, which in turn is the same case as that of the two charge small black holes.

11.2 Subensemble Theory

210. Mathur's Fuzzball proposal: the microstates of an extremal hole can not have singularity. For a detailed introduction see, [*S. D. Mathur: arXiv:0706.3884v1; O. Lunin, S. D. Mathur: hep-th/ 0109154v1; O. Lunin, S. D. Mathur: hep-th/ 0202072v2*].

211. State-space geometry and Mathur's subensemble theory are married each other by considering large number of subsets of the states which are characterized by conserved quantities of the black brane solution.

212. For D_1-D_5-J solutions having total ring entropy $S(n_1, n_5, J)$, if there are M number of subensembles with entropy $\tilde{S}(n_1', n_5', J)$, then Mathur has shown that the entropy in each subensemble is given by

$$\tilde{S}(n_1', n_5', J) = \frac{1}{M} S(n_1, n_5, J) \tag{100}$$

213. We have shown that the non-vanishing state-space scalar curvature indicates that the extremal $D_1 D_5 J$ system corresponds to an interacting statistical basis.

214. In particular, the infinite subensemble limit $M \to \infty$ implies that the each subensemble with given number of microstates has $R \to 0$, and therefore large subensemble limit corresponds to a non-interacting statistical system.

215. In conclusion, we find that the state-space geometry defining statistical correlations among an ensemble of equilibrium microstates in the chosen subensemble of extremal holes remains non-singular.

12 Bubbling Black Brane Foams

[*S. Bellucci and B.N.T.: Phys. Rev. D (2010)*] and [*Entropy (2010)*].

216. Bubbling black brane solutions are considered as the black foams and axi-symmetric merger solutions [*I. Bena, C. W. Wang, N. P. Warner: hep-th/0604110v2; arXiv:0706.3786v2 [hep-th]*].

217. State-space geometry of charge foamed black brane configurations in M-theory characterizes statistical correlations over an ensemble of equilibrium microstates.

218. The most general bubbling supergravity solutions possessing three brane charges corresponding to each GH center of the bubbled black brane foam configuration.

219. Considering all possible partitioning of the flux parameters $\{k_i^1, k_i^2, k_i^3\}$, the leading order topological entropy is given by

$$S(Q_1, Q_2, Q_3) := \frac{2\pi}{\sqrt{6}}\{(\frac{Q_2Q_3}{Q_1})^{1/4} + (\frac{Q_1Q_2}{Q_3})^{1/4} + (\frac{Q_1Q_3}{Q_2})^{1/4}\} \tag{101}$$

220. Characterized the coordinate chart of the state-space manifold in terms of the charges $\{Q_i\}$ of the equilibrium foam solution, we find that the components of covariant state-space metric tensor are

$$\begin{aligned}
g_{Q_1Q_1} &= -\pi\{\frac{5\sqrt{6}}{48Q_1^2}(\frac{Q_2Q_3}{Q_1})^{1/4} - \frac{\sqrt{6}}{16Q_1}(\frac{Q_2}{Q_3Q_1})^{1/4} - \frac{\sqrt{6}}{16Q_1}(\frac{Q_3}{Q_2Q_1})^{1/4}\} \\
g_{Q_1Q_2} &= -\pi\{-\frac{\sqrt{6}Q_3}{48Q_1^2}(\frac{Q_1}{Q_2Q_3})^{3/4} + \frac{\sqrt{6}}{48Q_3}(\frac{Q_3}{Q_1Q_2})^{3/4} - \frac{\sqrt{6}Q_3}{48Q_2^2}(\frac{Q_2}{Q_1Q_3})^{3/4}\} \\
g_{Q_1Q_3} &= -\pi\{-\frac{\sqrt{6}Q_2}{48Q_1^2}(\frac{Q_1}{Q_2Q_3})^{3/4} - \frac{\sqrt{6}Q_2}{48Q_3^2}(\frac{Q_3}{Q_1Q_2})^{3/4} + \frac{\sqrt{6}}{48Q_2}(\frac{Q_2}{Q_1Q_3})^{3/4}\} \\
g_{Q_2Q_2} &= -\pi\{\frac{5\sqrt{6}}{48Q_2^2}(\frac{Q_1Q_3}{Q_2})^{1/4} - \frac{\sqrt{6}}{16Q_2}(\frac{Q_3}{Q_1Q_2})^{1/4} - \frac{\sqrt{6}}{16Q_2}(\frac{Q_1}{Q_3Q_2})^{1/4}\} \\
g_{Q_2Q_3} &= -\pi\{\frac{\sqrt{6}}{48Q_1}(\frac{Q_1}{Q_2Q_3})^{3/4} - \frac{\sqrt{6}Q_1}{48Q_3^2}(\frac{Q_3}{Q_1Q_2})^{3/4} - \frac{\sqrt{6}Q_1}{48Q_2^2}(\frac{Q_2}{Q_1Q_3})^{3/4}\} \\
g_{Q_3Q_3} &= -\pi\{\frac{5\sqrt{6}}{48Q_3^2}(\frac{Q_1Q_2}{Q_3})^{1/4} - \frac{\sqrt{6}}{16Q_3}(\frac{Q_2}{Q_1Q_3})^{1/4} - \frac{\sqrt{6}}{16Q_3}(\frac{Q_1}{Q_2Q_3})^{1/4}\}
\end{aligned} \tag{102}$$

221. State-space metric constraints over the diagonal pair correlation functions are

$$g_{Q_iQ_i}(Q_1, Q_2, Q_3) \;\; > \;\; 0 \;\forall\; i \in \{1, 2, 3\} \mid f_{ii} < 0, \tag{103}$$

where

$$\begin{aligned} f_{11}(Q_1, Q_2, Q_3) &:= \frac{5\sqrt{6}}{48Q_1^2}(\frac{Q_2Q_3}{Q_1})^{1/4} - \frac{\sqrt{6}}{16Q_1}(\frac{Q_2}{Q_3Q_1})^{1/4} - \frac{\sqrt{6}}{16Q_1}(\frac{Q_3}{Q_2Q_1})^{1/4} \\ f_{22}(Q_1, Q_2, Q_3) &:= \frac{5\sqrt{6}}{48Q_2^2}(\frac{Q_1Q_3}{Q_2})^{1/4} - \frac{\sqrt{6}}{16Q_2}(\frac{Q_3}{Q_1Q_2})^{1/4} - \frac{\sqrt{6}}{16Q_2}(\frac{Q_1}{Q_3Q_2})^{1/4} \\ f_{33}(Q_1, Q_2, Q_3) &:= \frac{5\sqrt{6}}{48Q_3^2}(\frac{Q_1Q_2}{Q_3})^{1/4} - \frac{\sqrt{6}}{16Q_3}(\frac{Q_2}{Q_1Q_3})^{1/4} - \frac{\sqrt{6}}{16Q_3}(\frac{Q_1}{Q_2Q_3})^{1/4} \end{aligned} \tag{104}$$

222. Precise scaling properties of possible ratios consisting of the components of metric tensor are visualized by considering C_{BB}, as in the three charge toy model bubbling black branes.

223. To accomplish state-space stability, all the principle minors should be positive definite.

224. The local stability conditions on the one dimensional line, two dimensional surfaces and three dimensional hyper-surfaces of the state-space manifold are respectively measured by

$$\begin{aligned} p_1(Q_1, Q_2, Q_3) &= -\frac{\sqrt{6}\pi}{48}Q_1^{-15/4}Q_2^{-7/4}Q_3^{-7/4}(5Q_1^{3/2}Q_2^2Q_3^2 - 3Q_1^2Q_2^{3/2}Q_3^2 - 3Q_1^2Q_2^2Q_3^{3/2}) \\ p_2(Q_1, Q_2, Q_3) &= -\frac{\pi}{96}Q_1^{-7/2}Q_2^{-7/2}Q_3^{-3/2}(4Q_1^{3/2}Q_2^2Q_3^2 + Q_1^{3/2}Q_2^2Q_3^{3/2} - 8Q_1^{3/2}Q_2^{3/2}Q_3^2 \\ &\quad -2Q_1^2Q_2^2Q_3 + Q_1^2Q_2^{3/2}Q_3^{3/2} + 4Q_1^2Q_2Q_3^2) \end{aligned} \tag{105}$$

225. The global stability on the full state-space configuration is achieved by demanding positivity of the determinant of the state-space metric tensor

$$\|g\| = -\frac{\pi^3\sqrt{6}}{384}(Q_1Q_2Q_3)^{-13/4}f_1(Q1, Q2, Q3), \tag{106}$$

where the factor $f_1(Q_1, Q_2, Q_3)$ is defined by

$$\begin{aligned} f_1(Q_1, Q_2, Q_3) &:= -Q_1^{3/2}Q_2Q_3^2 - Q_1Q_2^{3/2}Q_3^2 + 3Q_1^{3/2}Q_2^{3/2}Q_3^{3/2} - Q_1^{3/2}Q_2^2Q_3 \\ &\quad -Q_1Q_2^2Q_3^{3/2} - Q_1^2Q_2^{3/2}Q_3 + Q_1^2Q_2^{1/2}Q_3^2 + Q_1^2Q_2^2Q_3^{1/2} \\ &\quad -Q_1^2Q_2Q_3^{3/2} + Q_1^{1/2}Q_2^2Q_3^2 \end{aligned} \tag{107}$$

226. Information about the global correlation volume of underlying statistical system is read-off form the intrinsic state-space scalar curvature

$$R = -\frac{\sqrt{6}}{12\pi}(Q_1Q_2Q_3)^{7/4}f_2(Q_1, Q_2, Q_3)f_1(Q_1, Q_2, Q_3)^{-3}, \tag{108}$$

where f_2 is defined by

$$
\begin{aligned}
f_2(Q_1,Q_2,Q_3) \ := \ & -10Q_1^4Q_2^2Q_3^2 - 10Q_1^2Q_2^2Q_3^4 - 10Q_1^2Q_2^4Q_3^2 + Q_1^4Q_2^4 + Q_1^4Q_3^4 + Q_2^4Q_3^4 \\
& +4Q_1^4Q_2^{5/2}Q_3^{3/2} - 27Q_1^3Q_2^2Q_3^3 + 4Q_1^{5/2}Q_2^{5/2}Q_3^4 + 4Q_1^{3/2}Q_2^{5/2}Q_3^4 \\
& +4Q_1^{5/2}Q_2^4Q_3^{3/2} - 4Q_1^4Q_2^{7/2}Q_3^{1/2} + 4Q_1^3Q_2^4Q_3 - 4Q_1^{7/2}Q_2^4Q_3^{1/2} \\
& -27Q_1^3Q_2^3Q_3^2 - 4Q_1^{1/2}Q_2^{7/2}Q_3^4 - 4Q_1^2Q_2^{7/2}Q_3^{5/2} + 6Q_1^3Q_2^{3/2}Q_3^{7/2} \\
& -4Q_1^{7/2}Q_2^2Q_3^{5/2} - 4Q_1^{7/2}Q_2^{5/2}Q_3^2 + 22Q_1^{5/2}Q_2^{5/2}Q_3^3 - 4Q_1^{5/2}Q_2^{7/2}Q_3^2 \\
& -4Q_1^{5/2}Q_2^2Q_3^{7/2} + 2Q_1Q_2^{7/2}Q_3^{7/2} + 2Q_1^{7/2}Q_2Q_3^{7/2} + 2Q_1^{7/2}Q_2^{7/2}Q_3 \\
& +22Q_1^{5/2}Q_2^3Q_3^{5/2} + 6Q_1^{7/2}Q_2^{3/2}Q_3^3 + 4Q_1^4Q_2Q_3^3 - 4Q_1^{7/2}Q_2^{1/2}Q_3^4 \\
& +4Q_1^3Q_2Q_3^4 + 4Q_1Q_2^4Q_3^3 + 4Q_1^{3/2}Q_2^4Q_3^{5/2} + 4Q_1^4Q_2^{3/2}Q_3^{5/2} \\
& +4Q_1Q_2^3Q_3^4 + 4Q_1^4Q_2^3Q_3 - 4Q_1^{1/2}Q_2^4Q_3^{7/2} - 27Q_1^2Q_2^3Q_3^3 \\
& +6Q_1^{7/2}Q_2^3Q_3^{3/2} + 6Q_1^3Q_2^{7/2}Q_3^{3/2} + 22Q_1^3Q_2^{5/2}Q_3^{5/2} - 4Q_1^2Q_2^{5/2}Q_3^{7/2} \\
& +6Q_1^{3/2}Q_2^3Q_3^{7/2} + 6Q_1^{3/2}Q_2^{7/2}Q_3^3 - 4Q_1^4Q_2^{1/2}Q_3^{7/2} \qquad (109)
\end{aligned}
$$

227. Underlying state-space geometry thus remains well-defined only as an intrinsic Riemannian manifold, $M := M_3 \setminus B$, where the set of charges define a degenerate set

$$B := \{(Q_1,Q_2,Q_3) | f_1(Q_1,Q_2,Q_3) = 0\} \qquad (110)$$

228. For some given entropy S_0, the constant entropy curve is defined by

$$(\frac{Q_2Q_3}{Q_1})^{1/4} + (\frac{Q_1Q_2}{Q_3})^{1/4} + (\frac{Q_1Q_3}{Q_2})^{1/4} = c, \qquad (111)$$

where the real constant c has the value of $c := \sqrt{6}S_0/2\pi$.

229. The curve of constant curvature scalar is

$$f_1(Q_1,Q_2,Q_3)^3 = K(Q_1Q_2Q_3)^{7/4} f_2(Q_1,Q_2,Q_3) \qquad (112)$$

230. For the equal values of brane charges $Q_i := Q$, the principle minors and determinant of metric tensor reduce to

$$p_1(Q) = \frac{\sqrt{6}\pi}{48} Q^{-7/4}, \ p_2(Q) = 0, \ g(Q) = 0 \qquad (113)$$

231. Thus, the equal brane charge foam system is not stable over planes and the hyper-planes of the state-space configurations and the state-space scalar curvature $R(Q)$ indexes out of the range in division procedure.

232. As, the foam solution of [*I. Bena and P. Kraus: arXiv:hep-th/0408186v2*] can be dualized to the frame of D_1D_5P charges which asymptotically reduces to the $AdS_3 \times S^3 \times T^4$ configurations.

233. Thus, state-space co-ordinate transformations give an interesting clue of classical string dualities and offer statistical correlation properties for the parameters of an ensemble of dual D_1-D_5-P CFT states.

13 Concluding Remarks

234. For a pair of distinct state-space variable $\{X_i, X_j\}$, the state-space pair correlations of an extremal configurations scale as

$$\frac{g_{ii}}{g_{jj}} = (\frac{X_j}{X_i})^2, \ \frac{g_{ij}}{g_{ii}} = -\frac{X_i}{X_j} \tag{114}$$

235. In general, the black brane configurations in string theory are categorized as

    (a) The underlying sub-configurations turn out to be well-defined over possible domains, whenever there exist respective set of non-zero state-space principle minors.
    (b) The underlying full configuration turns out to be everywhere well-defined, whenever there exist a non-zero state-space determinant.
    (c) The underlying configuration corresponds to an interacting statistical system, whenever there exist a non-zero state-space scalar curvature.

236. Intrinsic state-space manifold of extremal/ non-extremal and supersymmetric/ nonsupersymmetric string theory black holes may intrinsically be described by an embedding

$$(M_{(n)}, g) \hookrightarrow (M_{(n+1)}, \tilde{g}) \tag{115}$$

237. The extremal state-space configuration may be examined as a restriction to the full counting entropy with an intrinsic state-space metric tensor $g \mapsto \tilde{g}|_{r_+=r_-}$.

238. For supersymmetric black holes, the restriction $g \mapsto \tilde{g}|_{M=M_0(P_i,Q_i)}$ should be applied to an assigned nonsupersymmetric black brane configuration.

239. For large N_c gauge theories, we have shown that vacuum fluctuations yield an interacting ensemble that can physically be realized as a collection of fluctuating metastable states (in the light of the parity odd bubbles).

14 Future Directions and Open Issues

240. State-space Instabilities and dual CFTs:

     (i) Multi-center Gibbons-Hawking solutions with generalized base space manifolds having mixing of positive and negative residues.

     (ii) Dual CFTs and microscopic string duality symmetries

     (iii) Stabilization against local and/ or global perturbations: such as GL modes, chemical potential fluctuations, electric-magnetic charges and dipole charges, rotational fluctuations and the thermodynamic temperature fluctuations for the near-extemal and non-extremal black brane solutions

241. D Dimensional Black Brane Configurations: various black rings with horizon topology $S^1 \times S^{D-3}$ for $D > 5$, higher horizon topologies $S^1 \times S^1 \times S^2$, $S^3 \times S^3$, etc?

242. Bubbling Black Brane Solutions: Lin, Lunin and Maldacena (LLM) geometries, Liquid droplets, and Mathur's Fuzzball conjecture(s).

243. Generalized Hyper-Kähler Manifolds: Mathur's conjecture reduces to classifying and counting asymptotically flat four dimensional hyper Kähler manifolds which have moduli regions of uniform signature $(+,+,+,+)$ and $(-,-,-,-)$.

244. Physics at the Planck Scale: The thermodynamic state-space geometry may be explored with foam geometries, and empty space virtual black holes whose statistical correlations among the microstates would involve foam of two-spheres.

245. The present exploration thus opens an avenue to give new insight into the promising vacuum structures of black brane space-time at very small scales.

The present talk is largely[1] based on the following papers

246. "State-space Geometry, Statistical Fluctuations and Black Holes in String Theory", S. Bellucci, B. N. Tiwari, arXiv:1103.2064[hep-th].

247. "State-space geometry, non-extremal black holes and Kaluza-Klein monopoles", S. Bellucci, B. N. Tiwari, arXiv:1102.2391[hep-th].

248. "State-space Correlations and Stabilities", S. Bellucci, B. N. Tiwari, [Phys. Rev. D (2010)], arXiv:0910.5309v1 [hep-th].

249. "On the Microscopic Perspective of Black Branes Thermodynamic Geometry", S. Bellucci, B. N. Tiwari, [Entropy (2010)], arXiv:0808.3921v1 [hep-th].

250. "State-space Manifold and Rotating Black Holes", S. Bellucci, B. N. Tiwari, [To Appear], arXiv:1010.1427v1 [hep-th].

251. "Black Strings, Black Rings and State-space Manifold", S. Bellucci, B. N. Tiwari, [Communicated], arXiv:1010.3832v1 [hep-th].

252. "An Exact Fluctuating 1/2-BPS Configuration", S. Bellucci, B. N. Tiwari, J. High Energy Phys. 05 (2010) 023, arXiv:0910.5314v2 [hep-th].

253. "Thermodynamic Geometry and Hawking Radiation", S. Bellucci, B. N. Tiwari, J. High Energy Physics 030 (2010) 1011, arXiv:1009.0633v1 [hep-th].

254. "Thermodynamic Geometry and Extremal Black Holes in String Theory", T. Sarkar, G. Sengupta, B. N. Tiwari, J. High Energy Phys., 0810, 076, 2008, arXiv:0806.3513v1 [hep-th].

255. "Sur les corrections de la géométrie thermodynamique des trous noirs", B. N. Tiwari, arXiv:0801.4087v1 [hep-th]. New Paths Towards Quantum Gravity, Sominestationen in Holbaek, Denmark (May 12-18, 2008).

256. "On the Thermodynamic Geometry of BTZ Black Holes", T. Sarkar, G. Sengupta, B. N. Tiwari, J. High Energy Phys. 0611 (2006) 015, arXiv:hep-th/0606084v1 [hep-th].

257. "On Generalized Uncertainty Principle", B. N. Tiwari, arXiv:0801.3402v1 [hep-th]. New Paths Towards Quantum Gravity, Sominestationen in Holbaek, Denmark (May 12 18, 2008).

258. "Thermodynamic Geometry and Free Energy of Hot QCD", S. Bellucci, B. N. Tiwari, V. Chandra, [Int. J. Mod. Phys. A], arXiv:0812.3792v1 [hep-th].

259. "Thermodynamic Stability of Quarkonium Bound States", S. Bellucci, B. N. Tiwari, V. Chandra, arXiv:1010.4225v1 [hep-th].

[1]A set of closely associated references are mentioned in the sequel.

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Acknowledgements

260. I would like to thank my doctoral thesis supervisors, Prof. V. Ravishankar and Prof. S. Bellucci, and Prof. Jagdish Rai Luthra and Prof. Padmakali Banerjee for their valuable support and encouragements towards this research.

261. I would like to thank the organizers of the **"YITP Workshop Strings and Fields 2018"** for their support towards the presentation of this work.

262. I further express my thanks to the organizers for their support towards my participation in **"New Frontiers in String Theory 2018"**, July 02- August 03, 2018, Yukawa Institute for Theoretical Physics, Kyoto University, wherefore to offer an opportunity to learn new aspects of the modern string theory.

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Thank You !