

The refined quantum extremal surface prescription from the asymptotic equipartition property

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INTRODUCTION

- Information theoretic ideas have provided numerous insights in the progress of fundamental physics, especially in our pursuit of quantum gravity.
- In particular, the [holographic entanglement entropy](#) is a very useful tool in understanding AdS/CFT, and its efficacy is manifested in the recent black hole page curve calculation.
- On the other hand, the one-shot information theoretic entropies, such as the [smooth min/max-entropies](#), are less discussed in AdS/CFT. They are however more fundamental entropy measures from the quantum information perspective and should also play pivotal roles in holography.
- We study the [quantum extremal surface](#) (QES) prescription that was recently revised to highlight the significance of one-shot entropies in characterizing the QES phase transition.
- Motivated by the [asymptotic equipartition property](#) (AEP), we derive the [refined quantum extremal surface prescription](#) for fixed-area states via a novel AEP replica trick, demonstrating the synergy between quantum information and quantum gravity.
- The refinement also applies beyond AdS/CFT to the black hole page curve.

OUTLINE

- Holographic entanglement entropy and quantum extremal surfaces
- One-shot entropies and the asymptotic equipartition property
- The refined quantum extremal surface prescription
- The refined Page curve

HOLOGRAPHIC ENTANGLEMENT ENTROPY

- In AdS/CFT, the Ryu-Takayanagi formula claims that the **entanglement entropy** (EE) of a boundary subregion B is given by $S(B) = \frac{A[\gamma]}{4G_N}$, where the area term is attributed to some bulk minimal surface γ .

Ryu, Takayanagi

- It evolves into the well-established **quantum extremal surface prescription (QES)**

$$S(B)_\rho = \min_\gamma \text{ext}_\gamma \left[\frac{A[\gamma]}{4G_N} + S(\Sigma_{\gamma,B})_\rho \right], \text{ where } \Sigma_{\gamma,B} \text{ denotes the entanglement wedge bounded by } B \text{ and } \gamma.$$

Hubeny, Rangamani, Takayanagi; Wall Faulkner, Lewkowycz, Maldacena

Engelhardt, Wall

- We shall focus on a time-symmetric slice in this talk $S(B)_\rho = \min_\gamma \left[\frac{A[\gamma]}{4G_N} + S(\Sigma_{\gamma,B})_\rho \right].$

- The QES prescription plays a pivotal role in obtaining the Page curve characterising the entropy of an evaporation black hole. The “Island Formula” is a particular instance of it.

Penington, Almheiri, Engelhardt, Marolf, Maxfield, Maldacena, Hartman, Shaghoulian, Tajdini, Shenker, Stanford, Yang ...

QUANTUM EXTREMAL SURFACE PRESCRIPTION

- How can one make sense of the QES prescription $S(B)_\rho = \min_\gamma \left[\frac{A[\gamma]}{4G_N} + S(\Sigma_{\gamma,B})_\rho \right]$?
- There is a Euclidean gravitational path integral derivation due to Lewkowycz-Maldacena. They gravitationally compute $S = \lim_{n \rightarrow 1} S_n = \lim_{n \rightarrow 1} \frac{1}{1-n} \log \text{Tr } \rho^n$ by preparing n replicas of the original system. By assuming a $\mathbb{Z}_n$ -**symmetry** of the dominant saddle geometry, one can take the $\mathbb{Z}_n$ quotient and go back to the original geometry up to a conical singularity at the $\mathbb{Z}_n$ fixed point, γ . It contributes $e^{-(n-1)A[\gamma]/4G_N}$ to the partition function. One can do analytic continuation and obtain the RT formula.
Lewkowycz, Maldacena
- When there are several candidate surfaces γ_i , we pick the minimal one that dominates the partition function. Bulk quantum contributions can be also accounted.

Faulkner, Lewkowycz, Maldacena

QUANTUM EXTREMAL SURFACE PRESCRIPTION

- However, there are examples in which the QES prescription fails badly near the transition from one surface to another.
- One cannot pick out the $\mathbb{Z}_n$ -symmetric saddle to be the dominant one after analytic continuation. There is a **replica-symmetry breaking**.
- Including all the saddle geometries gives a correction to the QES prescription. If so, we shall conclude that the QES prescription gives only the **upper bounds**

$$S(B)_\rho \leq \min_\gamma \left[\frac{A[\gamma]}{4G_N} + S(\Sigma_{\gamma,B})_\rho \right].$$

Penington, Shenker, Stanford, Yang
Akers, Penington

- Can we systematically characterize the corrections?

ONE-SHOT ENTROPIES

- Entropies characterize the operational tasks in information theory: data compression, randomness extraction, teleportation/state merging.
- The entanglement/von Neumann/Shannon entropy shows up only in the **asymptotic regime**, whereas one-shot entropies show up in the **one-shot regime**. The latter is more realistic and mathematically fundamental in information theory.
- Most notably, the min/max-entropies are particularly relevant. They sit at the ends of the spectrum of Rényi entropies.
Renner, Dupuis, Tomamichel, Berta, Colbeck, König...
- $S = \lim_{n \rightarrow \infty} S_n$, $H_{\min} := S_\infty$, $H_{\max} = S_0$, $H_{\min} \leq S \leq H_{\max}$
- Roughly speaking, one can think of the min-entropy $H_{\min}(\rho)$ as the $-\log(\text{the largest eigenvalue})$, whereas the max-entropy $H_{\max}(\rho)$ as the $\log(\text{rank})$.

ASYMPTOTIC EQUIPARTITION PROPERTY (AEP)

- The connection between the one-shot regime and the asymptotic regime is formally captured by the AEP, which is a consequence of the law of large number.

Tomamichel, Colbeck, Renner

- $$S(\rho) = \lim_{m \rightarrow \infty} \frac{1}{m} H_{\min}^{\varepsilon}(\rho^{\otimes m}) := \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{m(1-n)} \max_{\tilde{\rho}_m \approx_{\varepsilon} \rho^{\otimes m}} \log \text{Tr} [\tilde{\rho}_m]^n .$$

- It involves **no analytic continuation**.

- This formula is operationally more meaningful than the usual formula used in the replica trick:

$$S(\rho) = \lim_{n \rightarrow 1} \frac{1}{1-n} \log \text{Tr} \rho^n .$$

- When finding the maximiser is difficult, one can always resort to a feasible state $\tilde{\rho}_m$ to obtain a lower bound on the EE. Similarly, the AEP works for the max-entropy where one can obtains upper bounds.
- We shall use this AEP formula to devise a replica trick that can be used in refining the QES prescription.

THE REFINED QUANTUM EXTREMAL SURFACE PRESCRIPTION

- Recently Akers and Penington proposed that the QES prescription should be revised. In particular, when there are two competing QES', we have the **refined QES prescription**

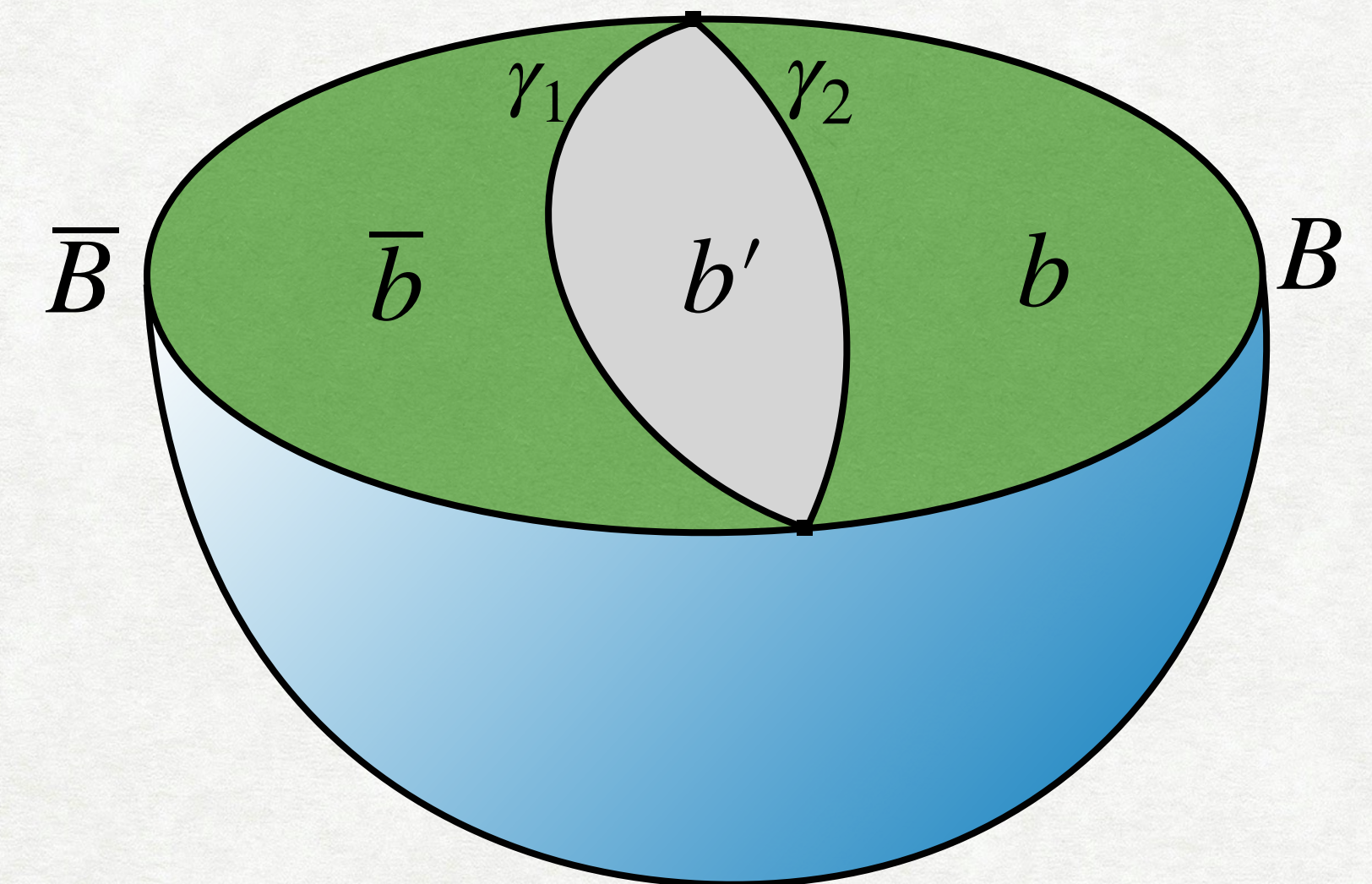
Akers, Penington

$$S(B)_\rho =^\epsilon \begin{cases} \frac{A_1}{4G_N} + S(bb')_\rho, & H_{\max}^\epsilon(b'|b)_\rho \leq \frac{A_2 - A_1}{4G_N} & \text{(regime 1)} \\ \text{(indefinite)}, & H_{\min}^\epsilon(b'|b)_\rho \leq \frac{A_2 - A_1}{4G_N} \leq H_{\max}^\epsilon(b'|b)_\rho & \text{(regime 2)} \\ \frac{A_2}{4G_N} + S(b)_\rho, & H_{\min}^\epsilon(b'|b)_\rho \geq \frac{A_2 - A_1}{4G_N} & \text{(regime 3)} \end{cases}$$

where $\epsilon \sim \text{Poly}(G_N)$, and the standard QES prescription says

$$S(B)_\rho = \begin{cases} \frac{A_1}{4G_N} + S(bb')_\rho, & S(b'|b)_\rho \leq \frac{A_2 - A_1}{4G_N} \\ \frac{A_2}{4G_N} + S(b)_\rho, & S(b'|b)_\rho \geq \frac{A_2 - A_1}{4G_N}, \end{cases} \quad (S(b'|b)_\rho := S(bb') - S(b))$$

- Recall that $H_{\min}^\epsilon(b'|b)_\rho \leq S(b'|b)_\rho \leq H_{\max}^\epsilon(b'|b)_\rho$, i.e. the refined QES identifies an **indefinite regime** where the EE cannot be described by the generalized entropy. The refinement can give large corrections near the transition at the leading order.
- AP supported their claim by translating the problem to the **random tensor network model**, where they can invoke QI (decoupling) theorems. Their argument is convincing but rather indirect. Most importantly, it was not obvious whether it holds beyond AdS/CFT.



DERIVATION

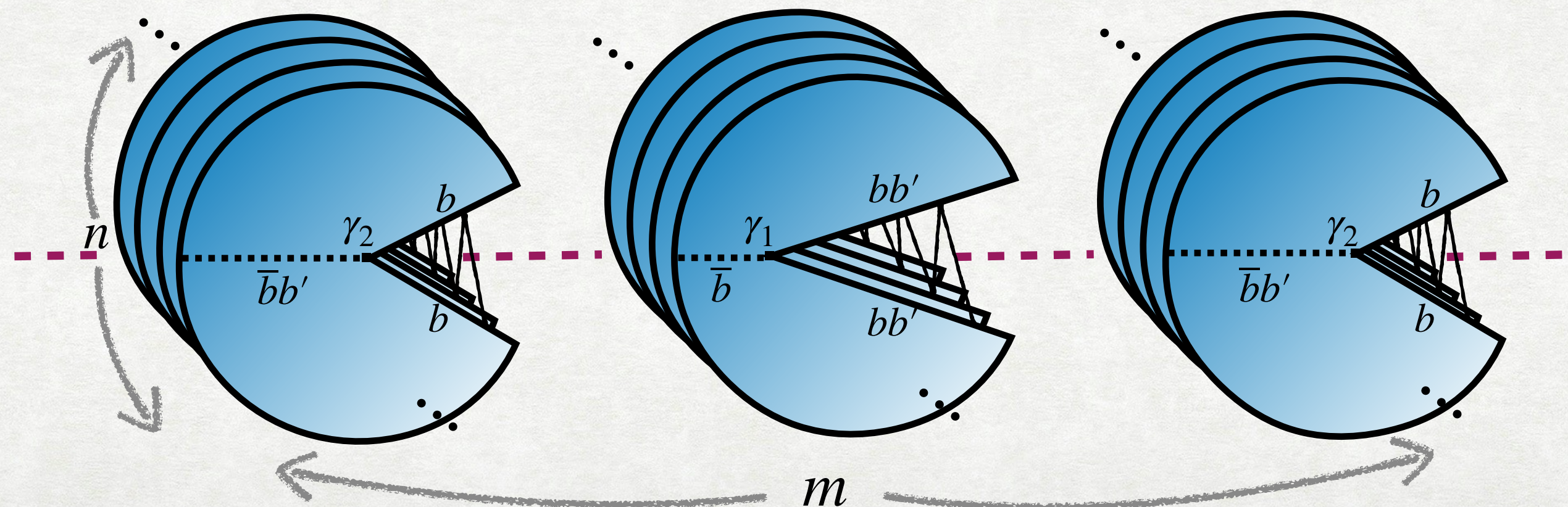
- Evaluate $\text{Tr} [\tilde{\rho}_m]^n$ gravitationally for **fixed-area states** ρ , $\tilde{\rho}_m$ is a carefully chosen feasible state.

- Then apply the AEP formula $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{m(1-n)} \max_{\tilde{\rho}_m \approx_\epsilon \rho^{\otimes m}} \log \text{Tr} [\tilde{\rho}_m]^{\otimes n} \tau_n$ to obtain lower bounds.

- Remark: area-fixing is a technical convenience so that we can work with the same geometry in each replica even for the higher Renyi entropy. The leading order refinement also extends to general states with an additional subleading correction near the transitions due to area fluctuation.

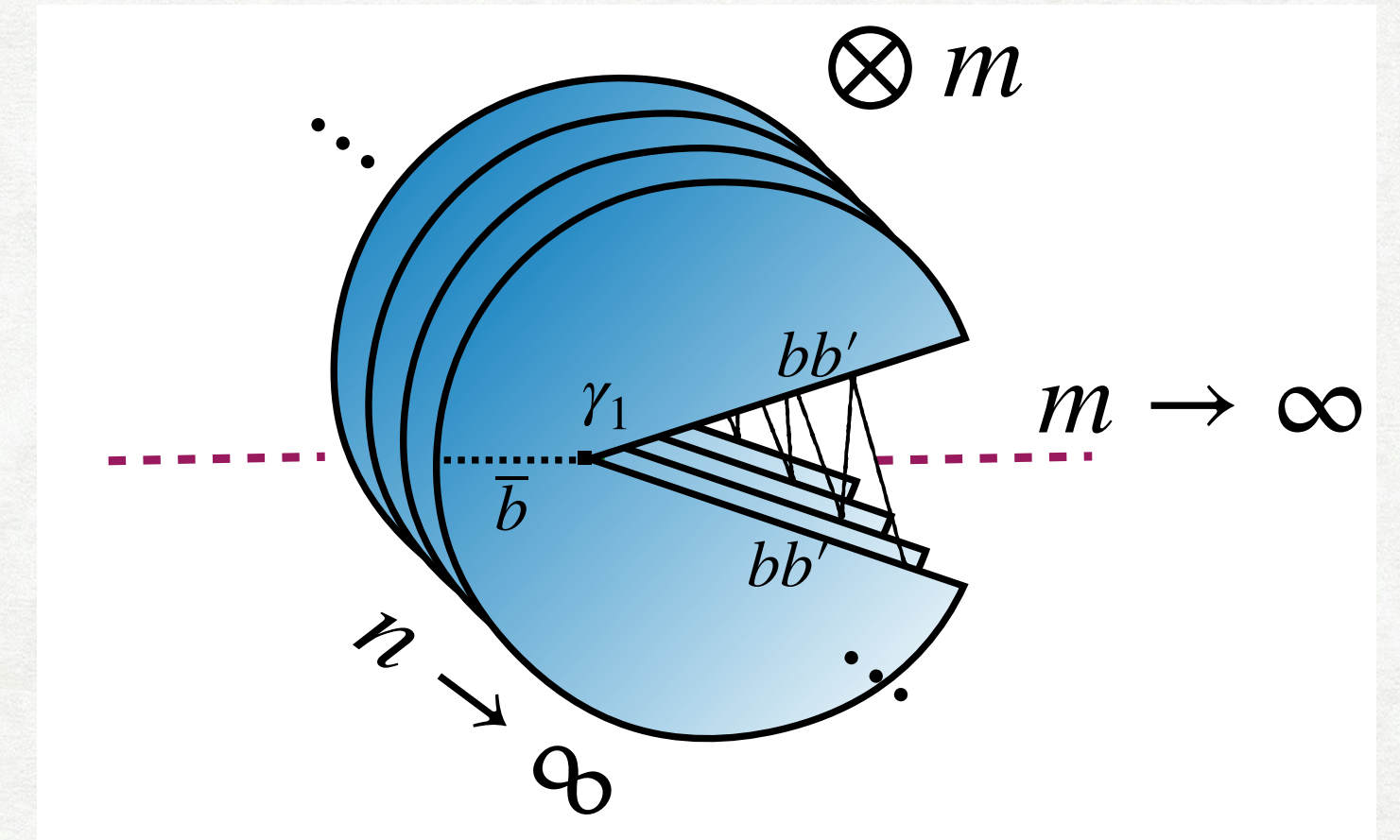
Dong, Wang; Marolf, Wang, Wang

- Instead of considering n replicas and take the $\mathbb{Z}_n$ quotient back to $n = 1$, we consider $n \times m$ replicas and take the limit of $n, m \rightarrow \infty$. Assuming $\mathbb{Z}_n$ -replica symmetry at large n , there are 2^m **saddles**.



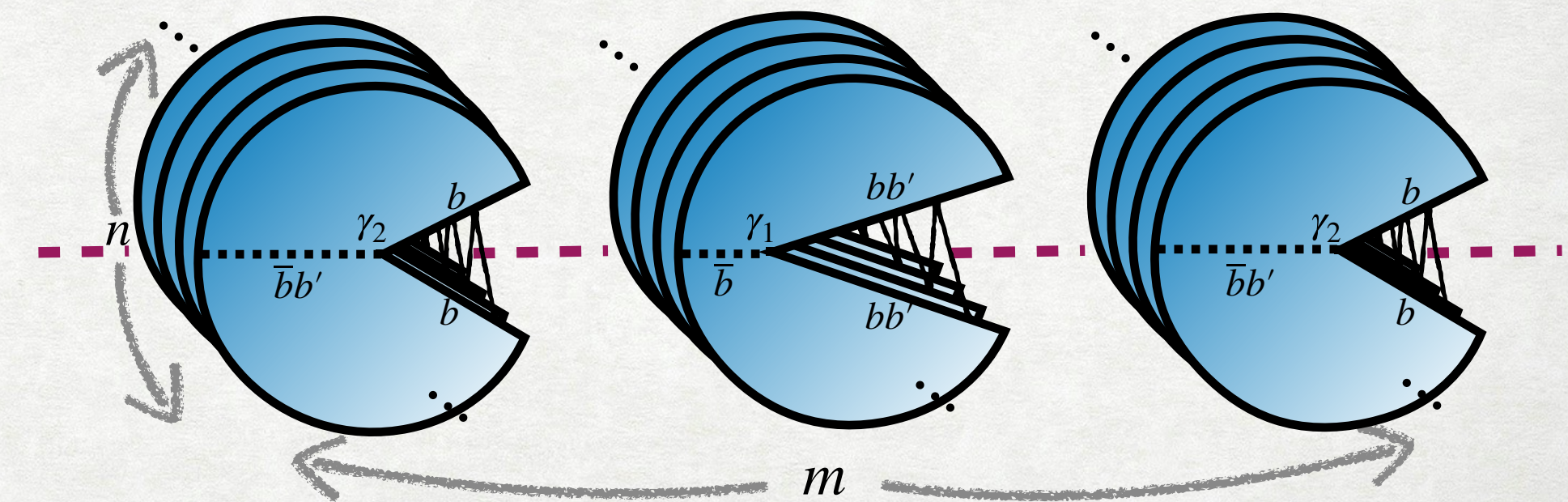
DERIVATION

- By carefully choosing a **feasible smoothing state** $\tilde{\rho}_m$, we can pick out the dominant saddle out of 2^m saddles under the min/max-entropy conditions in regimes 1&3.
- To argue the dominance, we use the chain rules relating the min-entropies.
- In regime 1&3, We obtain **lower bounds** of the EE in terms of the RT formula, $S(B)_\rho \geq \min_i \left[\frac{A[\gamma_i]}{4G_N} + S(\Sigma_{\gamma_i, B})_\rho \right]$ that matches the upper bounds from the standard replica trick.



$$H_{\max}(b'|b)_\rho \leq \frac{A_2 - A_1}{4G_N}$$

$$\gg$$



BEYOND ADS/CFT

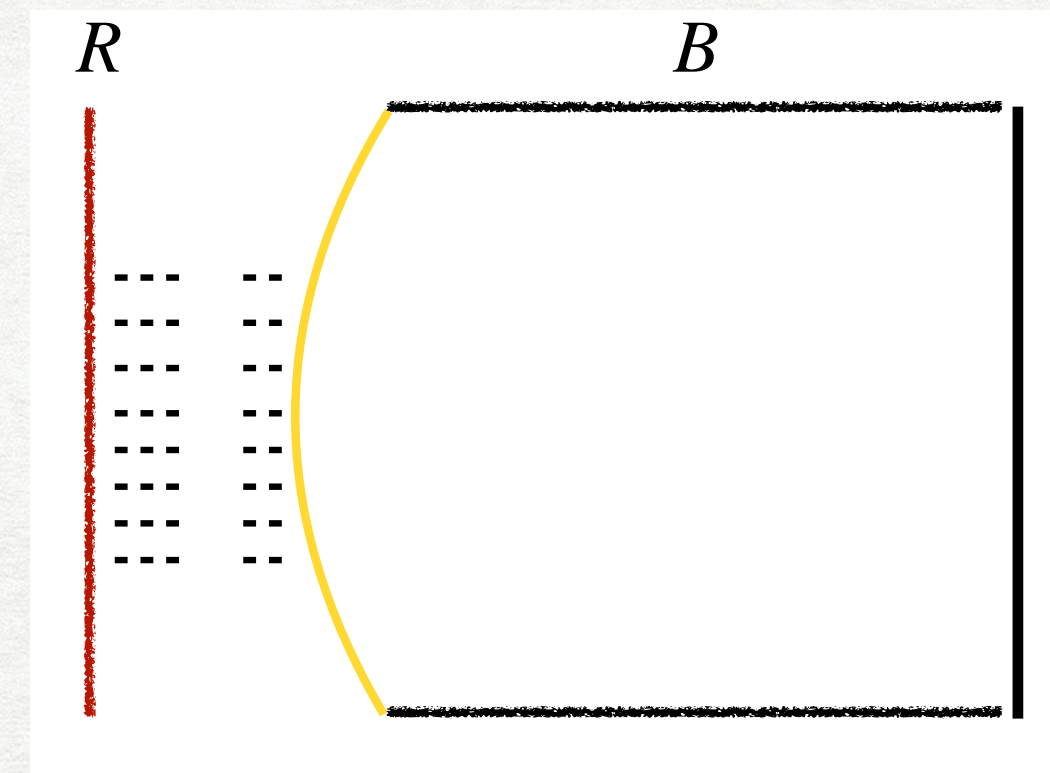
- The Lewkowycz-Maldacena derivation implies that the QES prescription is applicable beyond holographic theories in semiclassical gravity. Our the path-integral derivation indicates the refinement should also apply beyond AdS/CFT .

THE REFINED PAGE CURVE

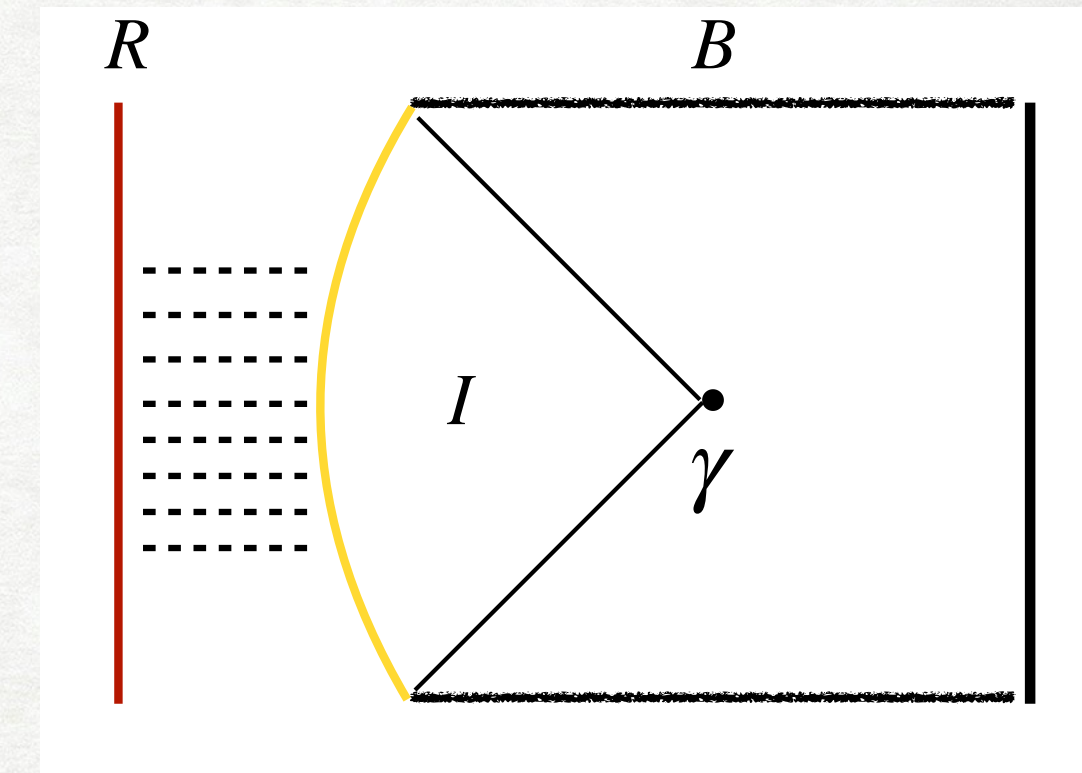
- Let's verify it in the JT+End of World brane model of PSSY, which is a toy model involving a 2D black hole and its Hawking radiation correlated by hand. Penington, Shenker, Stanford, Yang

$$\bullet \quad |\Psi\rangle_{RB} = \frac{1}{\sqrt{k}} \sum_i^k |i\rangle_R |\phi_i\rangle_B$$

- The model can be analysed in full detail and the entire entanglement spectrum can be computed.
- The resulting EE behaviour gives the Page curve.



$$S_{k < e^{S_{BH}}} = 0 + \log k$$



$$S_{k > e^{S_{BH}}} = e^{S_{BH}} + 0$$

$$S(B)_\rho = \min_\gamma \left[\frac{A[\gamma]}{4G_N} + S(\Sigma_{\gamma,B})_\rho \right]$$

THE REFINED PAGE CURVE

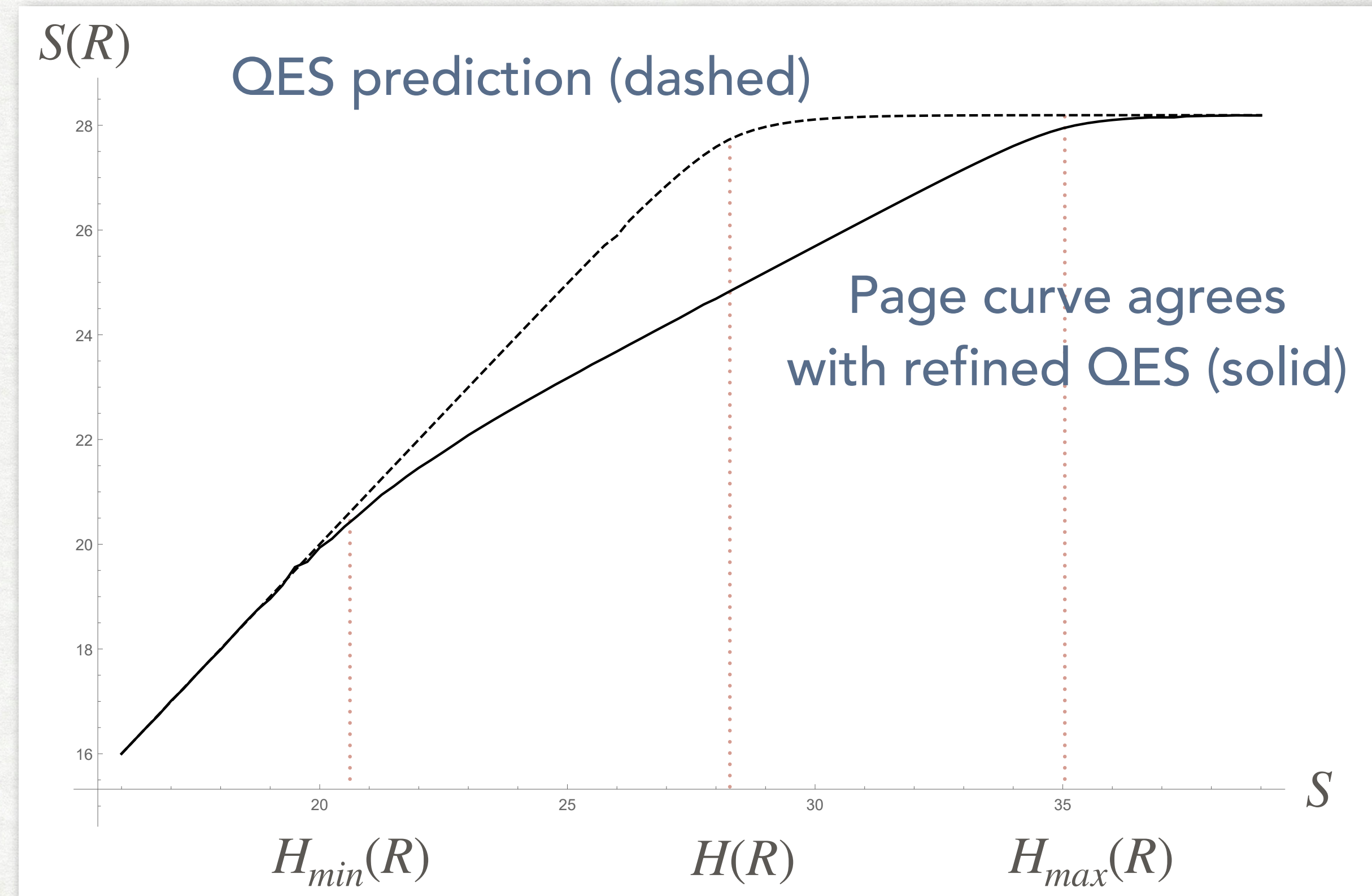
- The Schmidt coefficients are uniform so $H_{min}(R) = H_{max}(R) = H(R)$, so the refinement is simply not manifested in the original model.

- Let's modify the model by changing the BH microstate so that $H_{min}(R) < H(R) < H_{max}(R)$.

- $$|\Psi\rangle_{RB} = \sum_i^k c_i |i\rangle_R |\phi_i\rangle_B$$

- Then the resulting EE behaviour is different (solid), consistent with the **refined** QES prescription.
- This is to be contrasted with the subleading correction due to energy/area fluctuations, which has also been analysed by people.

Akers, Penington
Marolf, Wang, Wang
Dong, Wang



TAKE-HOME MESSAGE

- Operationally, the entanglement/von Neumann/Shannon entropy is an “average/asymptotic” quantity.
- One-shot entropies are more fundamental in information theory.
- The AEP replica trick can be useful in situations when the analytic continuation (as in the standard replica trick) is difficult.
- This leading order QES refinement goes beyond holography and applies to the Black Hole Page curve.

OUTLOOK

- Other applications of the AEP replica trick.
- Can one hope to characterize the indefinite regime?
- What about Entanglement Wedge Reconstruction in light of this refinement and Harlow’s error correction theorem?
Harlow
- What about a realistic evaporating black hole? What are the interesting scenarios that produce a large gap between min and max entropies?

Thank you!

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