Towards Hodge Theoretic Characterizations of 2d Rational SCFTs

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based on a joint work with
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2d Rational CFT

At some special points in a moduli space of CFT, the symmetry of CFT is enhanced.

(the chiral algebra $\mathcal{A}$ is larger than the Virasoro algebra
$\rightarrow$ # (primary fields (irrep.) of $\mathcal{A}$) $< \infty$)

$\rightarrow$ Rational CFT

known examples of 2d RCFT

- It is determined when a torus-target CFT is rational.
  e.g. S^1 target with radius R such that $\frac{R^2}{\alpha'} \in \mathbb{Q}$
- Gepner model $\rightarrow$ K3 target

How many more RCFTs? characterization by target geometry?

(Buckup) RCFT: definition and example

- CFT

$$\mathcal{H}_{\text{tot}} = \bigoplus_{i\in\mathcal{I}} \mathcal{H}_{(h_i,c_i)} \otimes \tilde{\mathcal{H}}_{(\tilde{h}_i,\tilde{c}_i)}$$

$\mathcal{H}_{(h_i,c_i)}$: an irrep. of Vir

- chiral algebra

$$\mathcal{A} = \bigoplus_{i\in\mathcal{I},\tilde{h}_i=0} \mathcal{H}_{(h_i,c_i)}$$

$$\left(\begin{array}{l} \text{Under state-op. corresp.,} \\ \mathcal{A} = \{\text{hol. fields}\} \ni \mathbf{1}, T(z) \\ \therefore \mathcal{A} \supset \text{Vir} \end{array} \right)$$

- irreducible decomposition

$$\mathcal{H}_{\text{tot}} = \bigoplus_{(\alpha,\tilde{\beta})\in\text{Irr}(\mathcal{A})\times\text{Irr}(\tilde{\mathcal{A}})} (\mathcal{M}_\alpha \otimes \widetilde{\mathcal{M}}_{\tilde{\beta}})^{d_{\alpha,\tilde{\beta}}}$$

- CFT is rational

$$\Leftrightarrow |\text{Irr}(\mathcal{A})|, |\text{Irr}(\tilde{\mathcal{A}})|, \sum_{(\alpha,\tilde{\beta})} d_{\alpha,\tilde{\beta}} < \infty$$

S^1_R target free boson CFT

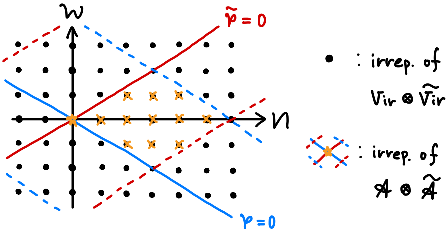
$$\begin{cases} p|n,w\rangle = (\frac{n}{R} + \frac{wR}{\alpha'})|n,w\rangle \\ \tilde{p}|n,w\rangle = (\frac{n}{R} - \frac{wR}{\alpha'})|n,w\rangle \end{cases}$$

$$\mathcal{H}_{\text{tot}} = \bigoplus_{(n,w)\in\mathbb{Z}^2} H_{(n,w)}$$

$$H_{(n,w)} = \text{Span}_{\mathbb{C}}\{\alpha_{-m_1}\alpha_{-m_2}\cdots|n,w\rangle\}$$

$$\mathcal{A} = \bigoplus_{(n,w),\frac{1}{2}\tilde{p}^2=0} H_{(n,w)}$$

$$= \begin{cases} \bigoplus_{(n,w)\in\mathbb{Z}(k,l)} H_{(n,w)} & (\frac{R^2}{\alpha'} = \frac{k}{l} \in \mathbb{Q}) \\ H_{(0,0)} & (\text{otherwise}) \end{cases}$$



Gukov-Vafa conjecture

Conjecture [GV02]

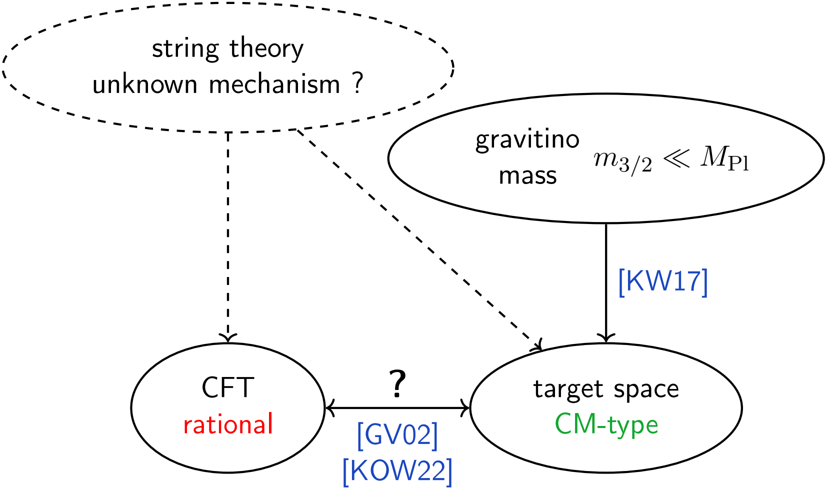
For a Ricci-flat Käler target $(M; G, B)$,

the SCFT is rational $\overset{?}{\Longleftrightarrow}$ M and its mirror W are of CM-type (complex multiplication) with the same CM field

In [KOW22],

- we pointed out some need for refinement of this statement.
- we examined this statement for $M = T^4$.
  $\rightarrow$ "RCFT $\Rightarrow$ CM" but the converse needs more.

(Backup) Motivation from string phenomenology

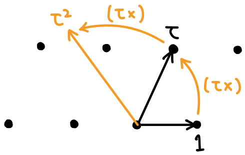


Complex Multiplication

CM-type : a generalized notion of

an elliptic curve with complex multiplication.

$T^2_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$



If $^\exists a, b \in \mathbb{Z}$ such that $\tau^2 = a\tau + b$,

$(\tau\times)$ is a linear map $\mathbb{Z} \oplus \tau\mathbb{Z} \to \mathbb{Z} \oplus \tau\mathbb{Z}$.

$(\quad T^2_\tau \quad \to \quad T^2_\tau \quad)$

Otherwise,

$(\tau\times)$ is not such a linear map.

$\rightsquigarrow$

If $^\exists D \in \mathbb{Z}_{<0}$ such that $\tau \in \mathbb{Q}(\sqrt{D})$,

$$\mathrm{End}(H^1(T^2_\tau;\mathbb{Q}))^{\mathrm{Hdg}} \cong \mathbb{Q}(\sqrt{D}).$$

"T^2_τ is an elliptic curve with complex multiplication."

"The Hodge structure on $H^1(T^2_\tau;\mathbb{Q})$ is of CM-type."

(Otherwise, $\mathrm{End}(H^1(T^2_\tau;\mathbb{Q}))^{\mathrm{Hdg}} \cong \mathbb{Q}$.)

Supporting evidence of GV conjecture [GV02]

- elliptic curves $M = T^2$

  $M \cong T^2_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$

  $\rho = \int_M \frac{B + i\omega}{(2\pi)^2\alpha'}$

  RCFT $\Leftrightarrow$ ${}^\exists D \in \mathbb{Z}_{<0}$ such that $\tau, \rho \in \mathbb{Q}(\sqrt{D})$

  $M \cong T^2_\tau$ and $W \cong T^2_\rho$ are of CM-type with $\mathbb{Q}(\sqrt{D})$.

- the Fermat quintic $M = \{z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0\} \subset \mathbb{P}^4$

  $\cdots$ target space of the Gepner model $(\mathrm{Min}^{\mathcal{N}=2}_{c=9/5})^{\otimes 5}/\mathbb{Z}_5$

  $\cdots$

$$
\underline{M}\;
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\Big\} =
\begin{array}{cccc}
\boxed{1\;\;1\;\;1\;\;1} \\
+ \\
0\;\;\boxed{2\;\;2}\;\;0 \times 50
\end{array}
\qquad
\underline{W}\;
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

The Hodge structures are of CM-type with $\mathbb{Q}(\zeta_5)$.

GV conjecture revisited

There is some need for refinement:

- Choice of complex structure
  - $\mathcal{N} = (1,1)$ SCFT $\leftrightsquigarrow$ non-linear sigma model $(M; G, B)$
  - the Hodge structure on $H^*(M;\mathbb{Q}) \leftrightsquigarrow (M; I)$

  There exists a continuous choice of I compatible with G on T^{2n} $(n \geq 2)$, hyperkähler manifolds.
- Existence and choice of mirror

  In general, for $\mathcal{N} = (2,2)$ SCFT $(M; G, B; I)$,

  - the mirror $(W; G_\circ, B_\circ; I_\circ)$ does not always exist.
  - even if exists, not necessarily unique.
- Which Hodge structure should be of CM-type?

T^4

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

CY 3-fold

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

CY 4-fold

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{3,3} & & 0 & & \\ & 0 & & h^{3,2} & & h^{2,3} & & 0 & \\ 1 & & h^{3,1} & & h^{2,2} & & h^{1,3} & & 1 \\ & 0 & & h^{2,1} & & h^{1,2} & & 0 & \\ & & 0 & & h^{1,1} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Classification of CM-type T^4

(based on the classificatioin of degree-4 CM field [Shi98])

(A') $T^4 \overset{\text{isog.}}{\simeq} E_1 \times E_2$
$E_1 \not\simeq E_2$: non-isogenous CM-type elliptic curves

$$\mathrm{End}(H^1(T^4;\mathbb{Q}))^{\mathrm{Hdg}} \cong \mathbb{Q}(\sqrt{p_1}) \oplus \mathbb{Q}(\sqrt{p_1}) \quad \left(\begin{smallmatrix} p_1, p_2 \in \mathbb{Q}_{<0} \\ p_1/p_2 \notin \mathbb{Q} \end{smallmatrix} \right)$$

(A) $T^4 \overset{\text{isog.}}{\simeq} E \times E$
E : a CM-type elliptic curve

$$\mathrm{End}(H^1(T^4;\mathbb{Q}))^{\mathrm{Hdg}} \cong M_2(\mathbb{Q}(\sqrt{p})) \quad (p \in \mathbb{Q}_{<0})$$

(B,C) T^4 : a simple Abelian surface

$$\mathrm{End}(H^1(T^4;\mathbb{Q}))^{\mathrm{Hdg}} \cong K$$

K : a CM field, $[K:\mathbb{Q}] = 4$, (B) Galois/$\mathbb{Q}$, (C) non-Galois/$\mathbb{Q}$.

Results 1 (digest) [KOW22], partially owed to [Wen00] and [Che05]

- $(T^4; G, B)$ gives RCFT

  $\Rightarrow {}^{\exists}$complex structure I such that

  $(T^4; I)$ and its mirror $(T^4_\circ; I_\circ)$ are of CM-type, and

  the Hodge structures on $H^*(T^4; \mathbb{Q})$ and

  the Hodge structures on $H^*(T^4_\circ; \mathbb{Q})$ are Hodge isomorphic.

Results 1 (detailed statement)

Theorem [KOW22, Thm. 5.8]

If $(T^4; G, B)$ gives RCFT, then

1. $^{\exists}$complex structure I, with which G is compatible, ...

Under such a complex structure,

2. the horizontal and vertical level-n Hodge structures are of CM-type and Hodge isomorphic,
   the Kähler form is algebraic $\omega \in H^2(T^4; \mathbb{Q}) \cap H^{1,1}(T^4)$, ...
3. all other hrz. and vrt. Hodge structures are of CM-type, ...
4. there exists a mirror $(T^4_\circ; G_\circ, B_\circ; I_\circ)$ ...
5. the vrt. Hodge structure on T^4 can be interpreted as the hrz. Hodge structure on $T^4_\circ$.
6. the hrz. and vrt. Hodge structures are Hodge isomorphic.
7. the Hodge isomorphism in 6. can be interpreted as ...

- showing some directions for refinement [work in progress].
- If ω is not algebraic, $(T^4; G, B)$ can be non-RCFT [Che05].

Results 2 [KOW22]

- The converse is not true for Case (A', B, C):

  If we impose 1, 2, 3 on $(T^4; G, B)$, a non-RCFT family remains.

  e.g. Case (A')

	$T^4 \simeq E_1 \times E_2$		$T^4_\circ \simeq E^\circ_1 \times E^\circ_2$	
	E_1	E_2	E°_1	E°_2
cpx. struc.	$\tau_1 \in \mathbb{Q}(\sqrt{p_1})$	$\tau_2 \in \mathbb{Q}(\sqrt{p_2})$	$\tau^\circ_1 \in \mathbb{Q}(\sqrt{p_i})$	$\tau^\circ_2 \in \mathbb{Q}(\sqrt{p_j})$
cpx. Kähler prm.	$\rho_1 \in \mathbb{Q}(\sqrt{p_i})$	$\rho_2 \in \mathbb{Q}(\sqrt{p_j})$	$\rho^\circ_1 \in \mathbb{Q}(\sqrt{p_1})$	$\rho^\circ_2 \in \mathbb{Q}(\sqrt{p_2})$

$$(i, j) = \begin{cases} (1,2) & \rightarrow \quad \text{RCFT} \\ (2,1) & \rightarrow \quad \text{non-RCFT} \end{cases}$$

Work in progress

It is expexted that ...

- Even if we impose all 1-7, the non-RCFT family remains.
- The additional condition
  8. There exists an isogeny $\phi : T^4_\circ \to T^4$ such that $\phi^* : H^1(T^4; \mathbb{Z}) \to H^1(T^4_\circ; \mathbb{Z})$ satisfies
  $$\phi^*|_{\Gamma_b^\vee} = \mathrm{id}|_{\Gamma_b^\vee}$$
     where $\Gamma_b \subset H_1(T^4; \mathbb{Z})$ denotes the "non-T-dualized directions".

  eliminates the non-RCFT family.
  $\to$ a necessary and sufficient condition for RCFT of $M = T^4$.

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