# Lattice Weyl Fermion on a single spherical domain-wall

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August 8, 2024 @場の理論と弦理論 2024
arXiv:2402.09774

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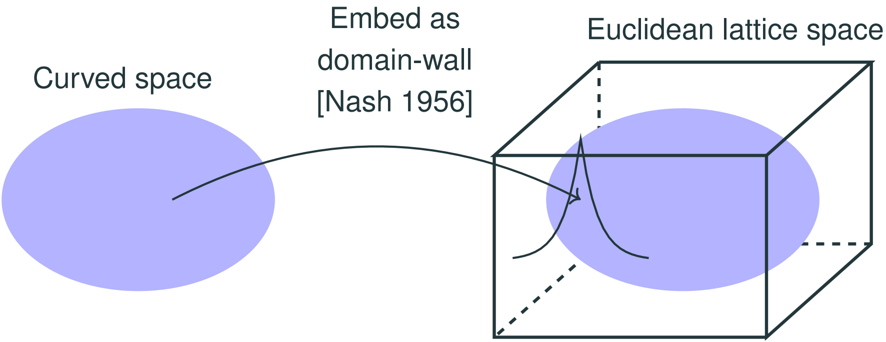
# Motivation

Every curved manifold can be isometrically embedded into some higher-dimensional Euclidean spaces.




Edge modes are localized at the curved domain-wall.

= They feel "gravity" by the equivalence principle.

$\longrightarrow$ Are they chiral?

# Embedding a Curved Space [Nash, 1956]

For any $n$-dim. Riemann space $(Y, g)$, there is an embedding $f : Y \to \mathbb{R}^m$ $(m \gg n)$ such that $Y$ is identified as

$$x^\mu = x^\mu(y^1, \cdots, y^n) \ (\mu = 1, \cdots, m)$$

$$\left( \begin{array}{ll} x^\mu & : \text{Cartesian coordinates of } \mathbb{R}^m \\ y^i & : \text{Coordinates of } Y \end{array} \right.$$

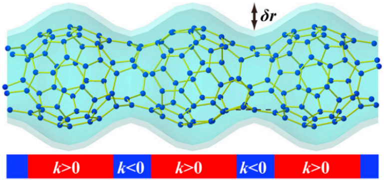
and the metric is written as

$$g_{ij} = \sum_{\mu\nu} \delta_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j}.$$

$\longrightarrow$ Vielbein and spin connection are also induced!

Any Riemannian manifold can be identified as a submanifold of a flat Euclidean space!

# “Gravity” in Condensed Matter Physics




[Onoe et al., 2012] observed a gravitational effect on $1D$ uneven peanut-shaped $C_{60}$ polymer.

The Hamiltonian on a curved surface is

[Jensen and Koppe, 1971; da Costa, 1981]

$$H = -\frac{\hbar^2}{2m_*}\left[\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) + h^2 - k\right], \quad \begin{cases} h\text{: Extrinsic curvature} \\ k\text{: Gaussian curvature} \end{cases}$$

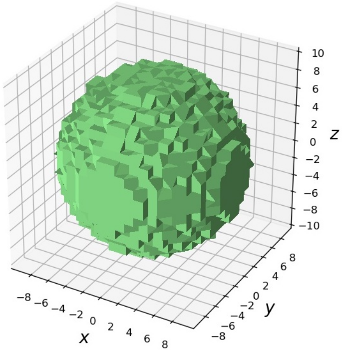
$\longrightarrow$ Density of states depends on the curvatures.

# Weyl Fermion on Single Spherical Domain-wall

We investigate a $S^2$ domain-wall fermion system with monopole.

$$D = \sum_{i=1}^{3} \sigma^i \left( \frac{\partial}{\partial x^i} - iA_i \right) - m$$

$$\to i \not{D}^{S^2} \frac{1}{2}(1 + \sigma^3)$$




- Spectrum
- Edge modes
- Continuum Limit
- Restoration of Symmetry

There is an obstacle in formulating lattice chiral gauge theory.

Cf. [Kaplan and Sen, 2024]: $S^1$ domain-wall.

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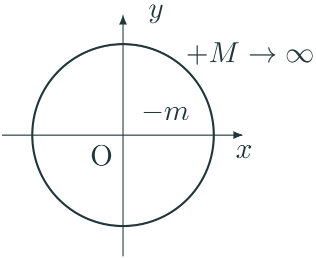
# $S^2$ domain-wall

Domain-wall:

$$m(r) = \begin{cases} -m & (r < r_0) \\ +M \to +\infty & (r \geq r_0) \end{cases}$$




Dirac operator:

$$D = \sum_{i=1}^{3} \sigma_i \frac{\partial}{\partial x^i} + m(r) = \sigma_r \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) + m(r) - \sigma_r \frac{D^{S^2}}{r},$$

$$D^{S^2} = \sum_{i=1}^{3} \sigma^i L_i + 1, \ \sigma_r = \frac{x}{r}\sigma_1 + \frac{y}{r}\sigma_2 + \frac{z}{r}\sigma_3.$$

Boundary condition: $\sigma_r \psi(x) = \pm \psi(x)$ at $r = r_0$

# Effective Dirac op and Dirac op. of $S^2$

A local Lorentz transformation

$$R = \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}} \sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}} \sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}} \cos\left(\frac{\theta}{2}\right) \end{pmatrix} e^{i\frac{\phi}{2}}$$

changes $\psi \to R^{-1}\psi$ and

$$D^{S^2} \to i\left(\sigma_1 \frac{\partial}{\partial\theta} + \frac{\sigma_2}{\sin\theta}\left(\frac{\partial}{\partial\phi} + \underbrace{\frac{i}{2} - \frac{\cos\theta}{2}\sigma_1\sigma_2}_{\text{Spin}^c \text{ connection on } S^2}\right)\right),$$

$$\sigma_r \to \sigma_3$$

Edge states feel gravity through the induced connection!

[Takane and Imura, 2013].

# Eigenstate of $D^\dagger D$ and $DD^\dagger$

Let $\chi_\pm$ satisfy

$$D^{S^2}\chi_\pm = \lambda\chi_\mp,\ (\lambda = 1, 2, \cdots)$$

$$\sigma_r\chi_\pm = \pm\chi_\pm.$$

In the large $m$ limit, we assume $\psi_\pm = \frac{1}{r}e^{-m|r-r_0|}\chi_\pm$, then we get

$$D\psi_+ = \left(\sigma_r\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + m\epsilon - \sigma_r\frac{D^{S^2}}{r}\right)\psi_+ \simeq \frac{\lambda}{r_0}\psi_-.$$

$$D^\dagger\psi_- = \left(-\sigma_r\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + m\epsilon + \sigma_r\frac{D^{S^2}}{r}\right)\psi_- \simeq \frac{\lambda}{r_0}\psi_+.$$

$\longrightarrow D^\dagger D\psi_+ = \left(\frac{\lambda}{r_0}\right)^2\psi_+$ and $DD^\dagger\psi_- = \left(\frac{\lambda}{r_0}\right)^2\psi_-$

Chiral fermion appears at the wall!

# $S^2$ Domain-wall Fermion on Lattice

We consider a lattice on $B^3$ with the radius $r_0$.

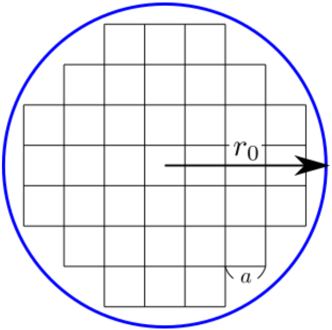
The (Wilson) Dirac op is

$$D = \frac{1}{a}\left(\sum_{i=1}^{3}\left[\sigma^i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i\nabla_i^\dagger\right] - m\right).$$

$$(\nabla_i\psi)_x = \psi_{x+\hat{i}} - \psi_x,\ (\nabla_i^\dagger\psi)_x = \psi_{x-\hat{i}} - \psi_x$$

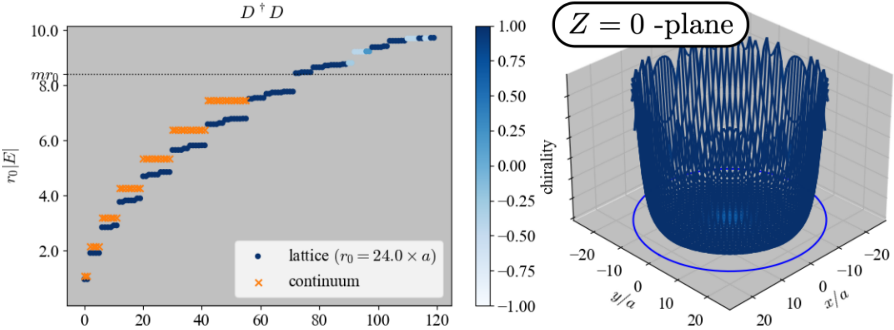
+OBC

We analyze $D^\dagger D$ and $DD^\dagger$.



## Spectrum and Edge modes of $D^\dagger D$

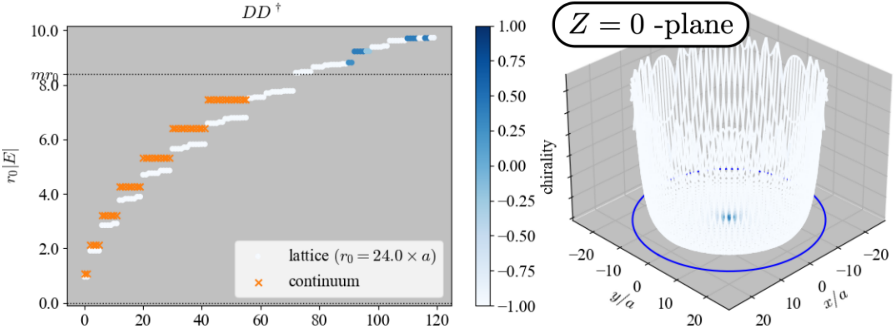
We solve $D^\dagger D\psi = E^2\psi$ when $r_0 = 24a, ma = 0.35$.




- Localized at the boundary
- $\sigma_r\psi = +\psi$
- There is a gap from zero (as a gravitational effect )
- Agrees well with the continuum prediction

## Spectrum and Edge modes $DD^\dagger$

We also compute $DD^\dagger\psi = E^2\psi$.




The result is almost the same as $D^\dagger D$, but edge modes are negative chiral modes:

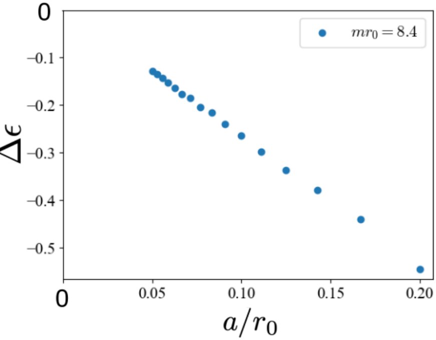
$$\sigma_r\psi = -\psi$$

It seems that a chiral theory is possible...

# Continuum Limit and Finite-volume Effect

Continuum limit $a \to 0$

($mr_0 = 8.4$ is fixed)

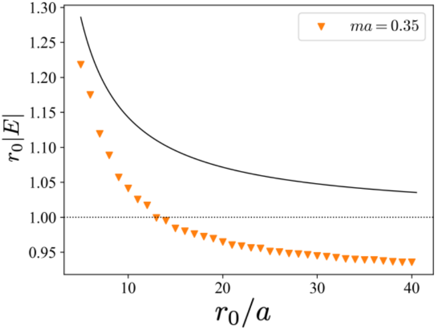



Agree well with

the conti. prediction!

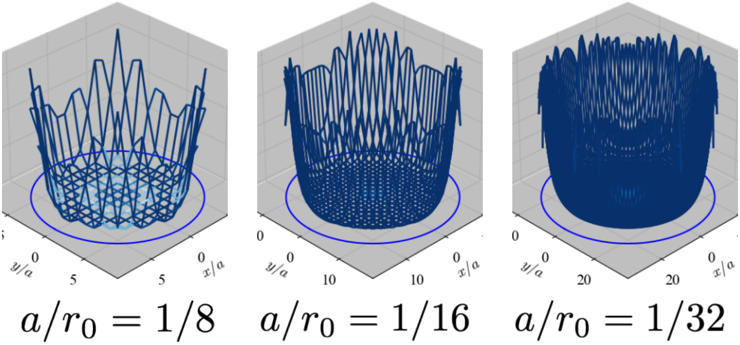
Large volume limit $r_0 \to \infty$

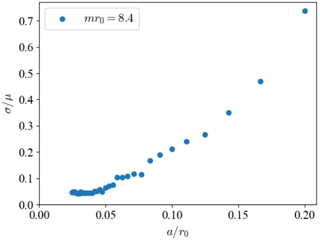
($ma = 0.35$ is fixed)




Saturates in the large $r_0$ limit!

# Restoration of Rotational Symmetry




The rotational symmetry automatically recovers in the continuum limit!

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# Chiral Anomaly

$S = \int \bar{\psi} \not{D} \psi$ has a chiral symmetry.
$\longrightarrow j^\mu_A = \bar{\psi}\gamma^5\gamma^\mu\psi$ is conserved.

$$\partial_\mu j^\mu_A = 0$$

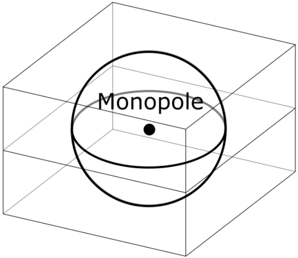
However, a quantum loop correction breaks this conservation law:

$$\int dx^2 \partial_\mu j^\mu_A = \int \frac{1}{2\pi} F = n_+ - n_- \neq 0$$

$$n_\pm = \#\left\{ \psi \;\middle|\; \not{D}\psi = 0,\ \gamma^5\psi = \pm\psi \right\}$$

This anomaly is related to the zero-mode!

# $S^2$ domain-wall with Monopole




$$A = \frac{1-\cos\theta}{2} d\phi$$

$$\text{Chiral Anomaly} = \underbrace{\frac{1}{2\pi}\int_{S^2} dA = 1}_{\text{Dirac index}}$$

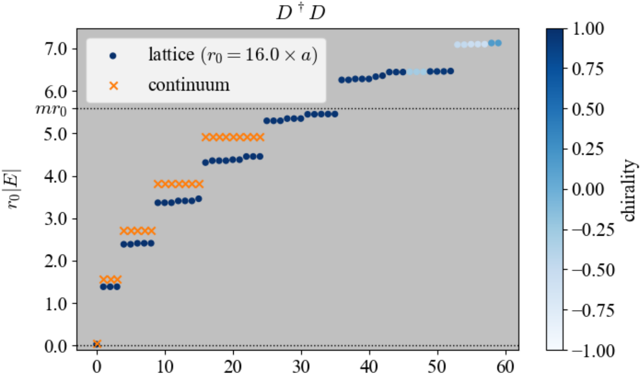
Covariant Derivative: $(\nabla_i \psi)_x = \exp\left[-i\int_{x+\hat{i}}^{x} A_i dx^i\right]\psi_{x+\hat{i}} - \psi_x$

Dirac Operator:

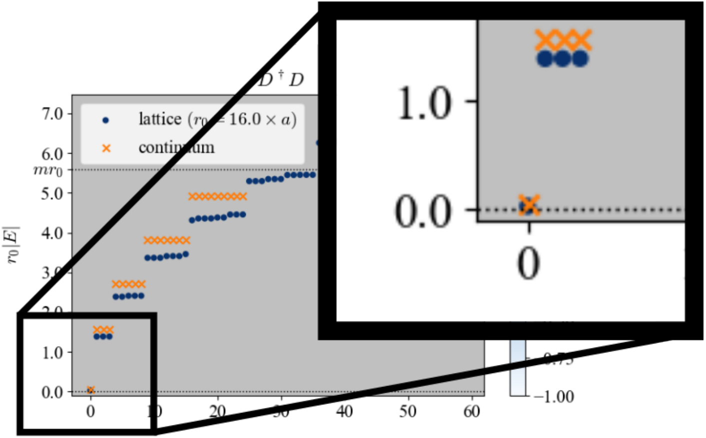
$$D = \frac{1}{a}\left(\sum_{i=1}^{3}\left[\sigma^i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i \nabla_i^\dagger\right] - ma\right).$$

$\longrightarrow n_+ - n_- = 1 \neq 0$ on the lattice?.

# Spectrum ($D^\dagger D\psi = E^2\psi$) at $r_0 = 16a, ma = 0.35$

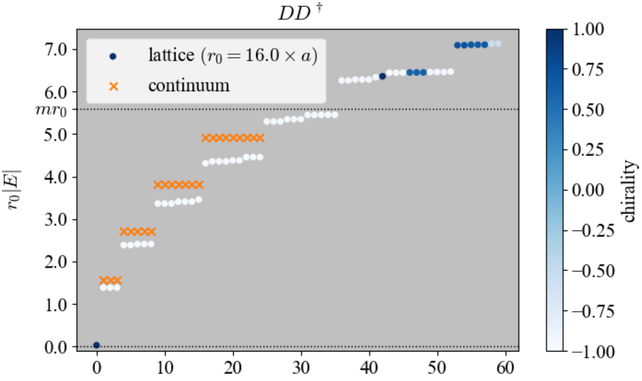


## Spectrum ($D^\dagger D\psi = E^2\psi$) at $r_0 = 16, ma = 0.35$

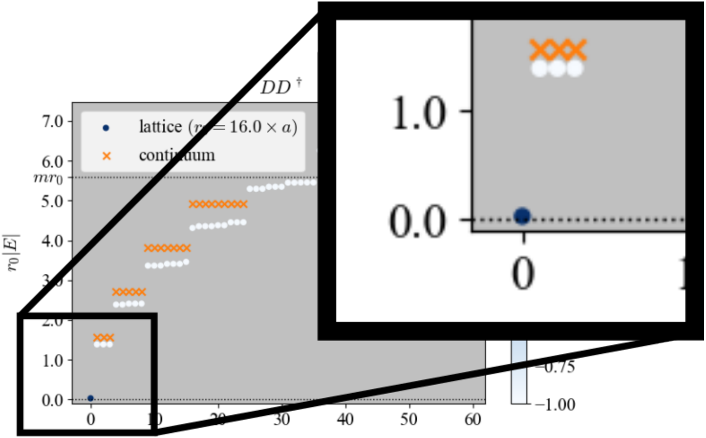



$0$ mode appears as well as the continuum theory.

# Spectrum ($DD^\dagger\psi = E^2\psi$) at $r_0 = 16, ma = 0.35$

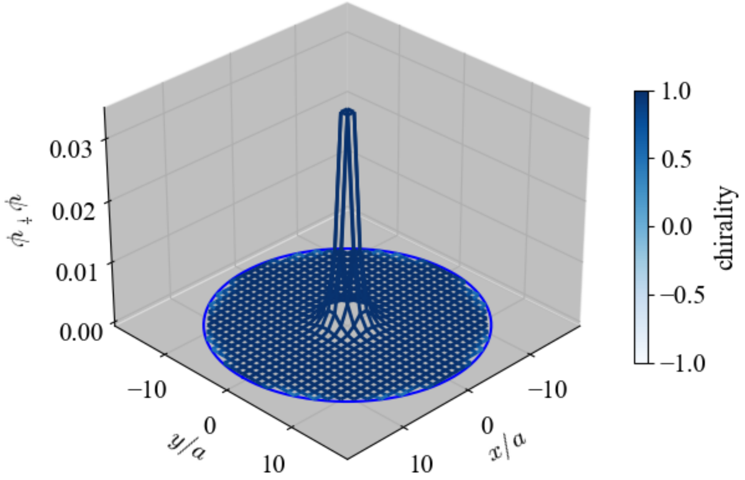


## Spectrum ($DD^\dagger\psi = E^2\psi$) at $r_0 = 16, ma = 0.35$




- An extra $0$ mode appears.
- The chirality is opposite to the localized modes.

# Distribution of Extra Zero Mode of $DD^\dagger$




The extra $0$ mode is localized around the monopole.

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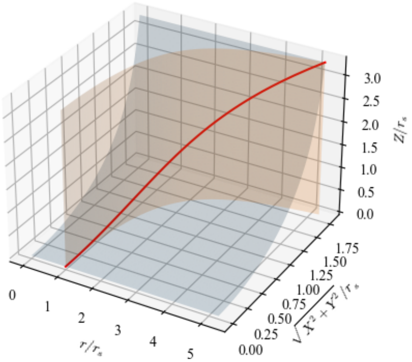
# Conclusion

We investigate a fermion system with a $S^2$ domain-wall.

- Weyl fermions appear at the wall.
- Continuum limit is good!
- There is an obstacle in formulating lattice chiral gauge theory.

# Outlook

- Symmetric mass generation [Wang and You, 2022]
- Embedding $4D$ Schwarzschild space into $6D$ flat space [Kasner, 1921].




$$
\begin{aligned}
ds^2 =& dx^2 + dy^2 + dz^2 + dX(r,\tau)^2 + dY(r,\tau)^2 + dZ(r,\tau)^2 \\
=& \left(1 - \frac{\beta}{4\pi r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{\beta}{4\pi r}} + r^2 d\Omega^2
\end{aligned}
$$

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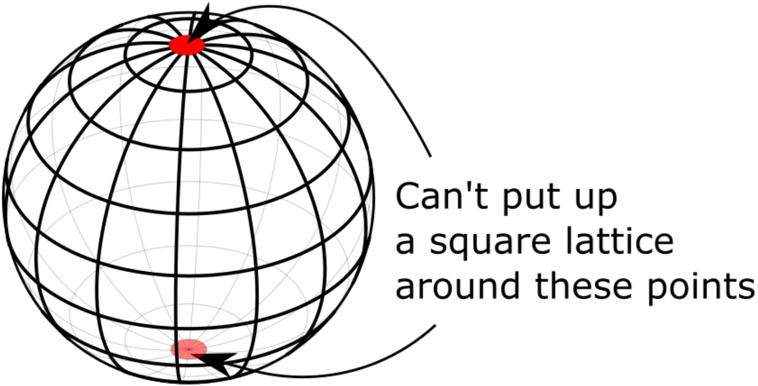
## Lattice gauge theory on a curved space

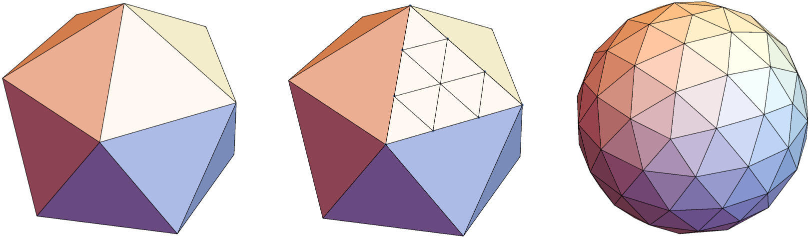



We can not approximate $d$-dim space with $d$-dim square lattice.

$\longrightarrow$ Can’t handle gravity!

# Triangular Lattice

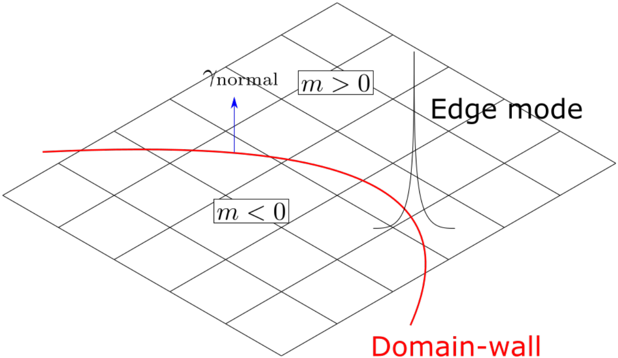
Arbitrary space can be discretezed by a triangular lattice. Their length and angle represent the gravity. [Regge [1961]; Ambjørn et al. [2001]; Brower et al. [2017]]



Triangular lattice on $2$-dim sphere[Brower et al. [2017]]

However, continuum limit is not unique and symmetry restoration is non-trivial.

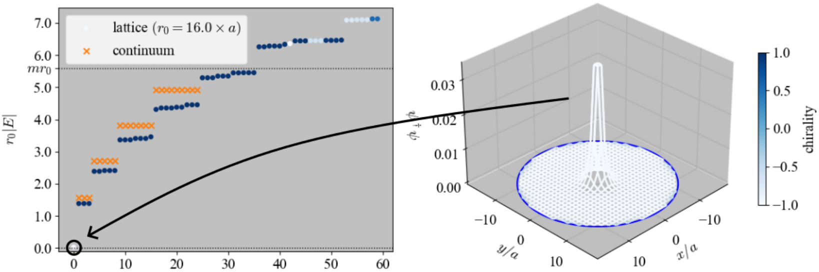
# Curved Kaplan Domain-Wall [SA and H. Fukaya, 2022]




- Edge modes ($\gamma_{\text{normal}} = +1$) appear at the wall.
- They feel gravity through the induced spin connection.
- The continuum limit is unique (lattice spacing $\to 0$).
- Rotational symmetry is automatically recovered.

cf. flat case [Kaplan, 1992], spherical TI [Takane and Imura, 2013]

# With nontrivial $U(1)$ link variables




When the monopole exists in the ball, another $0$-mode appears around the monopole.

$\longrightarrow$ An obstacle in formulating lattice chiral gauge theory.