Quantum equivalence between the Polyakov, Schild and Nambu-Goto-type formulations in superstring theory

Yuhma Asano (University of Tsukuba)

August 9, 2024 @YITP, Kyoto

Based on arXiv: 2408.04000

Introduction

Perturbative string theory

(Euclidean)
$S_{\rm P}^{\rm (E)}$: Polyakov-type action

An S-matrix is described as

$$A_{j_1,\cdots,j_n}(k_1,\cdots,k_n) = \sum_{\chi=2,0,-2,\cdots} g_s^{-\chi} \int DX\, D\theta\, Dg\, V_{j_1}(k_1)\cdots V_{j_n}(k_n)\ \exp[-S_{\rm P}^{\rm (E)}]$$

[Polyakov '81]

Questions:

- Is it equivalent to the Minkowskian theory?

  ➡ Yes!

- For a Minkowskian theory, how do we define the Nambu-Goto-type theory in the path-integral formalism?

  $$\exp\left[-i\int d^2\sigma \sqrt{-\det h_{ab}}\right]$$

  ➡ $i\epsilon$ terms select a branch

- Does it have the same features as standard QFT? What about the causality?

  ➡ $\det h_{ab} > 0$ does not contribute

Introduction

Euclidean v. Minkowskian

We start with the Minkowski signature but at some point, Wick-rotate the theory to the Euclidean signature.

… because Euclidean theory is usually well-defined

But we should NOT naively Wick-rotate it; otherwise, we might study a different theory.

$$X^0 = e^{-i\theta} X^D, \qquad e_0{}^a = e^{i\theta} e_2{}^a \qquad \left(\theta : 0 \to \frac{\pi}{2}\right)$$

$$\int DX\, Dg\, \exp\left[-i \int d^2\sigma \sqrt{-g}\, g^{ab} \partial_a X^\mu \partial_b X_\mu\right] \qquad g^{ab} = e_\alpha{}^a \eta^{\alpha\beta} e_\beta{}^b$$

$$= \int DX\, Dg\, \exp\Big[-i \int d^2\sigma \sqrt{e^2}\, \big(e^{-i\theta} e_1{}^a e_1{}^b \partial_a X^i \partial_b X^i - e^{i\theta} e_2{}^a e_2{}^b \partial_a X^i \partial_b X^i$$
$$\underline{-e^{-3i\theta}} e_1{}^a e_1{}^b \partial_a X^D \partial_b X^D + e^{-i\theta} e_2{}^a e_2{}^b \partial_a X^D \partial_b X^D\big)\Big]$$

Cauchy's thm. doesn't equate the Wick-rot. theory to the original.

Introduction

Nambu-Goto type

$$S_{\mathrm{NG}} = -\int d^2\sigma \sqrt{-h}$$

$$h_{ab} = \Pi_a^\mu \Pi_{b\mu}, \quad h = \det h_{ab}$$

Schild type

$$S_{\mathrm{Schild}} = -\frac{1}{2}\int d^2\sigma \left(\frac{-h}{e_g} + e_g \right)$$

Quantum mechanically
Equivalent?

Polyakov type

$$S_{\mathrm{P}} = -\frac{1}{2}\int d^2\sigma \sqrt{-g}\, g^{ab} h_{ab}$$

$$g^{ab} = e^{-\phi} \begin{pmatrix} -\frac{1}{\Lambda_0} & \frac{\Lambda_1}{\Lambda_0} \\ \frac{\Lambda_1}{\Lambda_0} & \frac{-\Lambda_1^2 + \Lambda_0^2}{\Lambda_0} \end{pmatrix}$$

Λ_a are just Lagrange multipliers for the constraints in the NG-type theory.

Path integral—Euclidean to Minkowskian

Let's start with the Polyakov's Euclidean path integral
in the case of critical bosonic string theory.

$$Z = \int DX\, Dg\, \exp\left[-\frac{1}{2}\int d^2\sigma \sqrt{g}\, g^{ab} h_{ab}\right]$$ Polyakov-type

$$Dg = D\phi \prod_\sigma \frac{2e^\phi\, d\Lambda_1 d\Lambda_2}{(\Lambda_2)^2} \qquad g^{ab} = e^{-\phi}\begin{pmatrix} \frac{\Lambda_1^2+\Lambda_2^2}{\Lambda_2} & -\frac{\Lambda_1}{\Lambda_2} \\ -\frac{\Lambda_1}{\Lambda_2} & \frac{1}{\Lambda_2} \end{pmatrix}$$

$$\| \delta g \|^2 = \frac{1}{2}\int d^2\sigma \sqrt{g}\, g^{ab}\delta g_{bc} g^{cd} \delta g_{da} = \int d^2\sigma\, e^\phi \left(\frac{\delta\Lambda_1^2 + \delta\Lambda_2^2}{(\Lambda_2)^2} + \delta\phi^2\right)$$

$$= \int DX\, Dg\, \exp\left[-\frac{1}{2}\int d^2\sigma \left\{\frac{h_{11}}{\Lambda_2}\left(\Lambda_1 - \frac{h_{12}}{h_{11}}\right)^2 + \frac{h_{11}h_{22} - h_{12}^2}{\Lambda_2 h_{11}} + \Lambda_2 h_{11}\right\}\right]$$

$$= \int DX \left[\prod_\sigma \int_0^\infty \frac{2d\Lambda_2}{\sqrt{\Lambda_2^3 h_{11}}}\right] \exp\left[-\frac{1}{2}\int d^2\sigma \left\{\frac{h_{11}h_{22} - h_{12}^2}{\Lambda_2 h_{11}} + \Lambda_2 h_{11}\right\}\right]$$

Schild-type

Path integral — Euclidean to Minkowskian

Cauchy's integral thm. equates the path integral to its Minkowskian version by the following deformation of the contour:

$$X^D = e^{i\theta}X^0, \qquad \Lambda_2 h_{11} =: e_g^{(\mathrm{E})} = e^{i\theta}e_g, \qquad \sigma^2 \to \sigma^0 \qquad \left(\theta: 0 \to \frac{\pi}{2}, \quad 0 \to -\frac{\pi}{2}\right)$$

$$Z = \int DX \left[\prod_\sigma \int_0^\infty \frac{2de_g^{(\mathrm{E})}}{e_g^{(\mathrm{E})\,3/2}}\right] \exp\left[-\frac{1}{2}\int d^2\sigma\left\{\frac{h_{11}h_{22}-h_{12}^2}{e_g^{(\mathrm{E})}} + e_g^{(\mathrm{E})}\right\}\right]$$

$$h = \frac{1}{2}\{(\varepsilon^{ab}\partial_a X^i \partial_b X^j)^2 + 2(\varepsilon^{ab}\partial_a X^D \partial_b X^i)^2\} > 0$$

$$-\frac{1}{2}\int d^2\sigma\left\{\frac{e^{-i\theta}(\varepsilon^{ab}\partial_a X^i \partial_b X^j)^2 + 2e^{i\theta}(\varepsilon^{ab}\partial_a X^0 \partial_b X^i)^2}{2e_g} + e^{i\theta}e_g\right\}$$

$$\overset{\theta=\pi/2}{=} -\frac{i}{2}\int d^2\sigma\left\{\frac{-(\varepsilon^{ab}\partial_a X^i \partial_b X^j)^2 + 2(\varepsilon^{ab}\partial_a X^0 \partial_b X^i)^2}{2e_g} + e_g\right\}$$

$$= \int DX \left[\prod_\sigma \int_{-\infty}^\infty \frac{-de_g}{(ie_g)^{3/2}}\right] \exp\left[-\frac{i}{2}\int d^2\sigma\left\{\frac{-(h_{11}h_{00}-h_{10}^2)}{e_g} + e_g\right\}\right]$$

[Y.A. '24]

Path integral—Euclidean to Minkowskian

$$Z = \int DX \left[\prod_\sigma \int_{-\infty}^{\infty} \frac{-de_g}{(ie_g)^{3/2}}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \left\{ \frac{-(h_{11}h_{00} - h_{10}^2)}{e_g} + e_g \right\}\right]$$

$$1 = \left[\prod_\sigma \int_{-\infty}^{\infty} \frac{d\Lambda_1}{(-i\Lambda_0/h_{11})^{1/2}}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \frac{h_{11}}{\Lambda_0}\left(\Lambda_1 - \frac{h_{10}}{h_{11}}\right)^2\right]$$

$$e_g = \Lambda_0 h_{11}$$

$$g^{ab} = e^{-\phi} \begin{pmatrix} -\frac{1}{\Lambda_0} & \frac{\Lambda_1}{\Lambda_0} \\ \frac{\Lambda_1}{\Lambda_0} & \frac{-\Lambda_1^2 + \Lambda_0^2}{\Lambda_0} \end{pmatrix}$$

$$= \int DX \left[\prod_\sigma \frac{d\Lambda_0 d\Lambda_1}{(\Lambda_0)^2}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \sqrt{-g}\, g^{ab} h_{ab}\right]$$

Polyakov-type

Polyakov's Euclidean path int. is equivalent to its Minkowskian ver.

[Y.A. '24]

Path integral—Polyakov to Nambu-Goto

We then start with the Minkowskian path integral w/ $S_{\rm P}$ (Polyakov) in the case of critical Green-Schwarz superstring theory.

$$Z = \int DX\, D\theta\, Z_{\rm b}[X,\theta]\, e^{i\Delta S_{\rm fermion}}$$

$$Z_{\rm b} = \int \left[\prod_\sigma \frac{d\Lambda_0 d\Lambda_1}{(\Lambda_0)^2}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \sqrt{-g}\, g^{ab} h_{ab} \;\; -i\epsilon\,|\Lambda_0| - i\frac{\tilde\epsilon(\Lambda_1 - c)^2}{|\Lambda_0|} \right]$$

Regulators for the convergence of the path integral:

- $i\epsilon$ terms coming from the ground state wave function
- Λ_a corresponds to constraints: $\delta(\chi) = \int_{-\infty}^{\infty} \frac{d\Lambda}{2\pi}\, e^{i\Lambda\chi - \epsilon|\Lambda|}$
- gauge invariant

$\rightarrow \Lambda_a \in (-\infty, \infty)$

$$Z_{\rm b} = \left[\prod_\sigma \int_{-\infty}^{\infty} \frac{-de_g}{(ie_g)^{3/2}}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \left\{ \frac{-h}{e_g} + e_g - i\epsilon\,|e_g| - i\frac{\tilde\epsilon}{|e_g|} \right\}\right]$$

$e_g = \Lambda_0 h_{11}$

[Y.A. '24]

Path integral—Polyakov to Nambu-Goto

$$Z_{\rm b} = \left[\prod_\sigma \int_{-\infty}^{\infty} \frac{-de_g}{(ie_g)^{3/2}}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \left\{ \frac{-h}{e_g} + e_g - i\epsilon |e_g| - i\frac{\tilde{\epsilon}}{|e_g|} \right\}\right]$$

$$= \prod_\sigma \left(\frac{\sqrt{2\pi}}{\sqrt{-h-i\epsilon}} e^{-i\Delta\Sigma\sqrt{-h-i\epsilon}} + \frac{\sqrt{2\pi}}{\sqrt{-h+i\epsilon}} e^{i\Delta\Sigma\sqrt{-h+i\epsilon}} \right)$$

cancel if $h > 0$

$$= \left[\prod_\sigma \sum_{s(\sigma)=\pm 1} \frac{\sqrt{2\pi}}{\sqrt{-h-i\epsilon s}}\right] \exp\left[-i\int d^2\sigma\, s\sqrt{-h-i\epsilon s}\right] \quad \begin{cases} s=1\text{: F1} \\ s=-1\text{: anti-F1} \end{cases}$$

The Polyakov, Schild and Nambu-Goto types are quantum mechanically equivalent.

The causality is realised by an anti-F1. [Y.A. '24]

Toward the non-pert. definition

The path integral of perturbative string theory:

$$A(k_1,\cdots,k_n) = \sum_{\chi=2,0,-2,\cdots} g_s^{-\chi}\int DX\, D\theta\, De_g\, \exp\left[iS_{\rm Schild}\right] V(k_1)\cdots V(k_n)$$

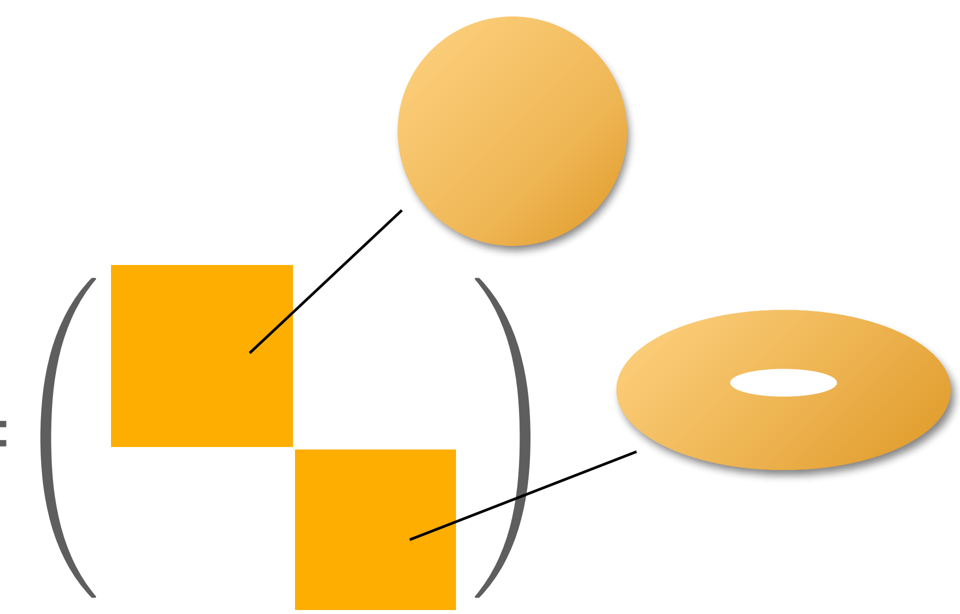
$$S_{\rm Schild} = -\frac{1}{2\pi}\int d^2\sigma\left\{-\frac{1}{2}\left(\frac{h}{e_g}-e_g\right) - i\varepsilon^{ab}\partial_a X^\mu(\theta^{1T}\Gamma_\mu\partial_b\theta^1 - \theta^{2T}\Gamma_\mu\partial_b\theta^2) + \varepsilon^{ab}\theta^{1T}\Gamma^\mu\partial_a\theta^1\,\theta^{2T}\Gamma_\mu\partial_b\theta^2\right\}$$

… This is merely perturbation theory around the 10D flat spacetime.

Matrix regularisation ⬇

$$A(k_1,\cdots,k_n) = \int d\mu\, \exp\left[iS_{\rm MM}\right] V(k_1)\cdots V(k_n)$$

$X^\mu =$



We expect the matrices describe multi-body systems of superstrings.

Matrix regularisation

A map of functions on a compact space to matrices

$$f(\sigma) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_{lm}(\sigma) \longmapsto \sum_{l=0}^{N-1} \sum_{m=-l}^{l} f_{lm} (Y_{lm})_{ij} = f_{ij}$$

fn. on S^2 spherical harmonics matrix

Matrix regularisation of the Schild-type theory

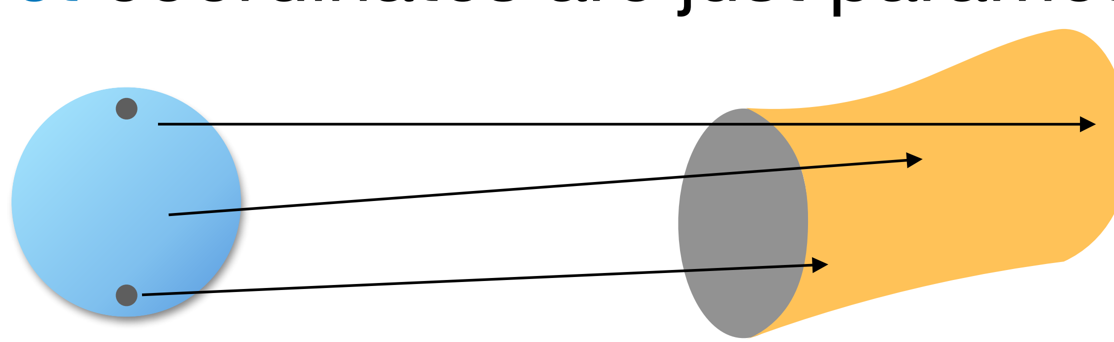
1. Matrix Regularisation after the Wick rotation

   It is manifestly well-defined because
   the worldsheet and the target space are Riemannian.

   Cauchy's thm. equates the Euclidean theory to the original.

2. Matrix Regularisation w/o Wick rotation

   Though the target space is Lorentzian,
   the worldsheet coordinates are just parameters.



Toward the non-pert. definition

We fix the gauge of the Schild-type theory by

$$\varphi = \theta^1 + i\theta^2 = 0$$

Ghosts are decoupled

Then we obtain

$$S_{\text{Schild}} = \frac{1}{2\pi}\int d^2\sigma\left[\frac{1}{4e_g}\{X^\mu, X^\nu\}^2_{\hat{\text{P}}} + 2i\psi^T\Gamma_\mu\{X^\mu, \psi\}_{\hat{\text{P}}} - \frac{e_g}{2}\right]$$

[Ishibashi, Kawai, Kitazawa, Tsuchiya '96]

By matrix regularisation, $\{\,\cdot\,,\,\cdot\,\}_{\hat{\text{P}}} \mapsto \frac{N}{i}[\,\cdot\,,\,\cdot\,]$, $\quad \frac{1}{\pi}\int d^2\sigma \mapsto \frac{1}{N}\,\text{tr}$,

$$\int DX\, D\theta\, De_g\, e^{iS_{\text{Schild}}}$$

1. MR after Wick rot. w/ $e_g^2 = 1$ $\mapsto$ $\displaystyle\int DX\, D\psi\, e^{-S^{(\text{E})}_{\text{IKKT}}}$

2. MR w/o Wick rot. $\mapsto$ $\displaystyle\int DX\, D\psi\, DY\, e^{iS_{\text{NBI}}}$

$$S^{(\text{E})}_{\text{IKKT}} = -N\,\text{tr}\left(\frac{1}{4}[X^m, X^n]^2 + \frac{1}{2}\psi^T\Gamma_m[X^m, \psi]\right)$$

$$S_{\text{NBI}} = N\,\text{tr}\left(\frac{1}{4}Y^{-1}[X^\mu, X^\nu]^2 + \frac{1}{2}\psi^T\Gamma_m[X^m, \psi] + Y + \frac{i}{N}(N+\tfrac{1}{2})\ln(-iY)\right)\Bigg]$$

[Fayyazuddin, Makeenko, Olesen, Smith, Zarembo '96]

Summary

- The Minkowskian superstring theory is quantum mechanically equivalent to its Euclidean version in terms of path integration.
- The Polyakov, Schild and Nambu-Goto-type formulations are quantum mechanically equivalent in the case of critical string theory (bosonic & type II).
- Full integration over the worldsheet metric provides the causality. Since configs. with $\det h_{ab} > 0$ don't contribute to the path integral, string propagation between points at space-like separation is prohibited.
- We obtained two matrix models as matrix regularisation of IIB string: the Euclidean IKKT model and the Minkowskian NBI-type IKKT model.
- I plan to find out the exact relationship between perturbative superstring d.o.f. and matrix d.o.f., starting with the identification for the vertex operator.

[cf. Kitazawa '02; Iso, Terachi, Umetsu '04; Kitazawa, Mizoguchi, Saito '07]