

HOLOGRAPHIC RENORMALIZED ENTANGLEMENT AND ENTROPIC C-FUNCTION

Mitsutoshi Fujita (University of South China)

Collaborators: Song He, Yuan Sun, Jun Zhang

Based on MF-He-Sun, Phys. Rev. D102, 126019(2020), MF-He-Sun-Zhang
JHEP01(2024)079, Chen-Chen-MF-He, work in progress

Motivations of this paper

❖ QFT with **negative energy**

✧ The IR behavior of $D=4$ $N=4$ SYM on S^1 is $d=3$ pure YM theory

✧ **The negative Casimir energy** of a gauge theory

~ **The mass of an AdS soliton**

Witten '98, Horowitz-Myers '98

❖ **DOF** from EE (e.g. the coefficient of the A-type anomaly for a spherical entangling surface in CFT, *Solodkhin '08*)

✧ Renormalized EE with a spherical entangling

Liu-Mezei '12

✧ **The entropic C function** from the EE with the striped subsystem

Nishioka-Takayanagi '06

Motivations of this paper

- ❖ What is difference between two entropy? → **trace anomaly part**
 - ✧ Renormalized EE only contains either A-type or **B-type anomaly**
 - ✧ Entropic c function contains **both A-type and B-type anomalies**
-

The entanglement entropy (an extension of the thermal entropy)

- ❖ System whose total Hilbert space is a direct product:

$$H = H_A \otimes H_B$$

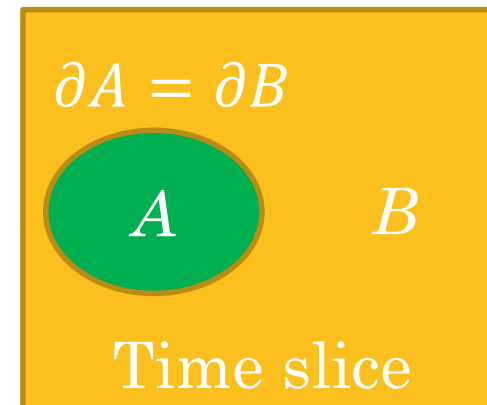
- ❖ Definition of the reduced density matrix $\rho_A = \text{Tr}_B(\rho)$ **taking the trace over H_B**

- ❖ Entanglement Entropy (EE) defined using the density matrix ρ_A as

$$S_A = -\text{Tr}_A(\rho_A \log \rho_A)$$

Von Neumann entropy of ρ_A

- ❖ In QFT, A and B : often a spatial bipartition of a system



The UV structure of entanglement entropy

❖ S_{UV} with the entangling surface $S^1 \times S^{d-3}$ vs $S_{UV,0}$ with the surface S^{d-2}

❖ The UV structure of entanglement entropy is $S_{UV} = \frac{L_\phi}{R} S_{UV,0}$ $S^A \sim \gamma \frac{\text{Area}(\partial A)}{a^{d-1}}$
+subleading terms

$$S_{ren} = \frac{1}{R} f_d(R\partial_R) R S_{EE} = L_d(R\partial_R) S_{EE}$$

$$= \begin{cases} \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2) \dots (R\partial_R - (d-3)) S_{EE}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} (R\partial_R + 1)(R\partial_R - 1) \dots (R\partial_R - (d-3)) S_{EE}, & d = \text{even}, \end{cases}$$

$$f_d(R\partial_R) S_{EE}^{(0)} = \begin{cases} \frac{1}{(d-2)!!} (R\partial_R - 1)(R\partial_R - 3) \dots (R\partial_R - (d-2)) S_{EE}^{(0)}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2) \dots (R\partial_R - (d-2)) S_{EE}^{(0)}, & d = \text{even}. \end{cases}$$

The UV structure of entanglement entropy

❖ S_{UV} with the entangling surface $S^1 \times S^{d-3}$ vs $S_{UV,0}$ with the surface S^{d-2}

❖ The UV structure of entanglement entropy is $S_{UV} = \frac{L_\phi}{R} S_{UV,0}$ $S^A \sim \gamma \frac{\text{Area}(\partial A)}{a^{d-1}}$
+subleading terms

$$S_{ren} = \frac{1}{R} f_d(R\partial_R) R S_{EE} = L_d(R\partial_R) S_{EE}$$

$$= \begin{cases} \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2) \dots (R\partial_R - (d-3)) S_{EE}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} (R\partial_R + 1)(R\partial_R - 1) \dots (R\partial_R - (d-3)) S_{EE}, & d = \text{even}, \end{cases}$$

$$f_d(R\partial_R) S_{EE}^{(0)} = \begin{cases} \frac{1}{(d-2)!!} (R\partial_R - 1)(R\partial_R - 3) \dots (R\partial_R - (d-2)) S_{EE}^{(0)}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2) \dots (R\partial_R - (d-2)) S_{EE}^{(0)}, & d = \text{even}. \end{cases}$$

Equivalent to D-1 dimensional renormalized entanglement entropy on $R^{1,d-2}$

Odd dimensional results

$$S_{ren} = R\partial_R S_{EE} \quad \text{for } d = 3,$$

$$S_{ren} = \frac{1}{3} R\partial_R (R\partial_R - 2) S_{EE} \quad \text{for } d = 5.$$

- ❖ Through Kaluza-Klein reduction along S^1 , the renormalized entanglement entropy effectively embodies $d-1$ dimensional one in the low-energy limit
 - ❖ In systems respecting Lorentz symmetry, the 2-dimensional entropic c-function is **both non-negative** and **monotonically decreasing**
 - ❖ The behavior of the 4d renormalized entanglement entropy displays non-monotonic tendencies
-

Renormalized entanglement entropy of 4d QFT (even dimensional results)

- ❖ The length scale R_1 of the subregion related to rescaling of the metric

$$R_1 \partial_{R_1} S_A = \lim_{n \rightarrow 1} (-2 \partial_n \int d^{d+1}x g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)} [w - n w|_{n=1}])$$

$$\frac{1}{2\pi} \lim_{n \rightarrow 1} \partial_n \left(\langle \int d^{d+1}x \sqrt{g} T_\mu{}^\mu(x) \rangle_{M_n} - n \langle \int d^{d+1}x \sqrt{g} T_\mu{}^\mu(x) \rangle_{M_1} \right)$$

- ❖ The log term of the entanglement entropy $S_{EE} = s \log\left(\frac{\epsilon}{R_1}\right) + \dots$

$$s = \frac{a}{180} \int_{\partial\sigma} d^2x \sqrt{h} E_2 + \frac{c}{120} \int_{\partial\sigma} d^2x \sqrt{h} I_2$$

- ❖ For cylinder type topology: $S_{ren} = \frac{1}{2}(l\partial_l + 1)(l\partial_l - 1)S_{EE} = s = \frac{cL_\phi}{240l}$. Only B-Type anomaly
- ❖ For a spherical surface S^2 : $S_{ren} = \frac{1}{2}R\partial_R(R\partial_R - 2)S_{EE} = s = \frac{a}{90}$ Only A-Type anomaly

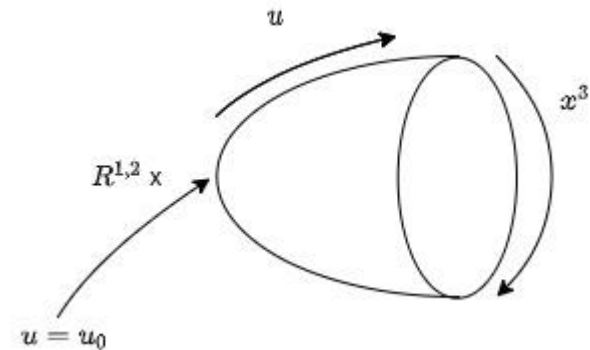
The AdS soliton dual to **confining theory**

- ❖ The **double Wick rotation** of the AdS black hole
- ❖ It corresponds to the ground state of **QFT with the anti-periodic boundary condition on fermions**

$$ds^2 = \frac{L^2}{u^2} (-dt^2 + \frac{du^2}{f(u)} + f(u)d\varphi^2 + \sum dx^i dx^i),$$

where $f(u) = 1 - \left(\frac{u}{u_0}\right)^4$

- ❖ The mass of the AdS soliton = **negative energy**



The AdS soliton with a background gauge field

- ❖ The metric of the AdS soliton **with a background gauge field**

$$ds_{d+1}^2 = \frac{L^2}{z^2} \left(\frac{dz^2}{f_d(z)} + f_d(z) d\phi^2 - dt^2 + dR^2 + R^2 d\Omega_{d-3} \right) \quad f_d(z) = 1 - \left(1 + \frac{\epsilon_1 z_+^2 a_\phi^2}{\gamma^2} \right) \left(\frac{z}{z_+} \right)^d + \frac{\epsilon_1 z_+^2 a_\phi^2}{\gamma^2} \left(\frac{z}{z_+} \right)^{2(d-1)},$$

a_ϕ **a constant gauge field** and $\gamma^2 = \frac{(d-1)g_e^2 L^2}{(d-2)\kappa^2}$

$$A_\phi = a_\phi \left(1 - \left(\frac{z}{z_+} \right)^{d-2} \right),$$

- ❖ The Kaluza-Klein mass

$$M_0 = \frac{1}{4\pi z_+} \left(d - \frac{\epsilon_1 (d-2) z_+^2 a_\phi^2}{\gamma^2} \right) > 0.$$

- ❖ The solution exists if $a_\phi \leq \frac{2\pi M_0 \gamma}{\sqrt{d(d-2)}} = a_c$

The boundary energy

❖ $M = \langle T_{00} \rangle V_{d-2} / M_0$

$$M = - \frac{V_{d-2}}{M_0} \frac{L^{d-1} \bar{a}_\phi}{2\kappa^2 z_+^d} \quad \bar{a}_\phi = 1 - \left(z_+ a_\phi \right)^2.$$

❖ The boundary energy changes the sign when we change Wilson lines a_ϕ

$$\begin{cases} M < 0 & z_+ a_\phi < 1 \\ M > 0 & z_+ a_\phi > 1. \end{cases}$$

For $a_\phi = 0$, it realizes

Casimir energy of 4d SYM theory.

Casimir energy is different among periodic and anti-periodic b.c.

The holographic entanglement entropy

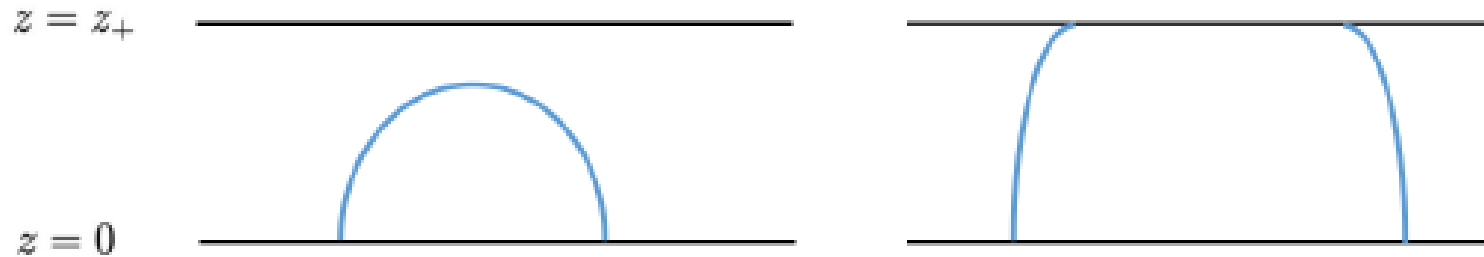
❖ The boundary region $S^1 \times S^{d-3}$: $R = l$ and $0 \leq \phi \leq L_\phi$

❖ The surface action:
$$A = \int d^{d-1}x \mathcal{L} = \Omega_{d-3} L_\phi L^{d-1} \int dz \frac{R^{d-3}}{z^{d-1}} \sqrt{1 + f \dot{R}^2}$$

❖ Boundary conditions:

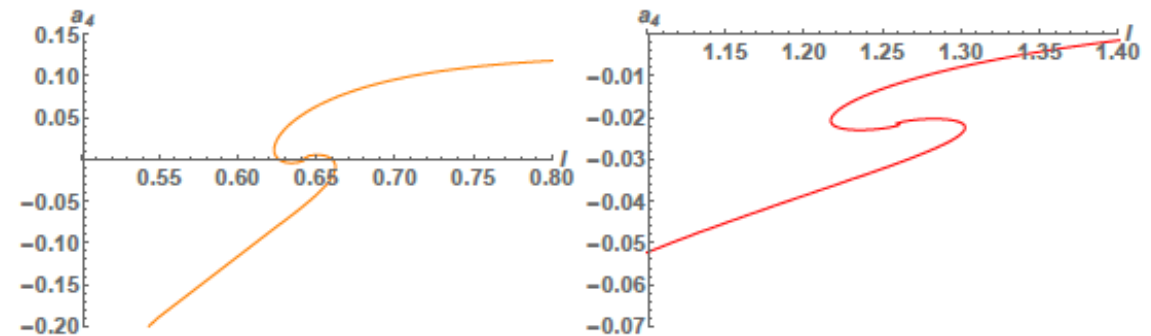
Disk type: $R(z_t) = 0, R'(z_t) = \infty$

A cylinder: $z_t = z_+$



The coefficient a_4 in $d=4$

- ❖ The AdS boundary expansion: $R(z) = l - \frac{z^2}{4l} + a_4(l)z^4 + \frac{z^4}{32l^3} \log \frac{z}{l} + \dots$
- ❖ $a_4(l)$ must be determined by the IR boundary condition



Left: $a_\varphi = i/2$

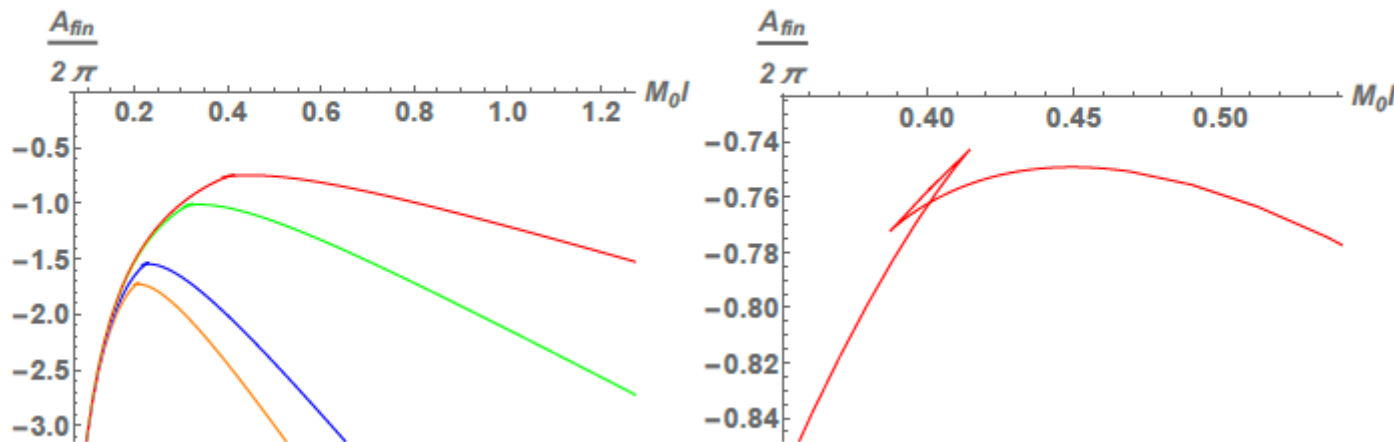
Right: $a_\varphi = 1/\sqrt{2}$

For small l , disk type surface is favored

For large l , cylinder type surface is favored

HEE in $4d$

- ❖ Finite part of HEE shows HEE with the Wilson line increases (vs confinement)



- ❖ Left: from $a_\phi = \frac{i}{2}, 0, \frac{2}{3}, 1/\sqrt{2}$ from the left to the right
- ❖ Swallow tail phase transition

An interpretation of HREE

- ❖ The renormalized EE can capture effective DOF on S^1 :
 - ✧ EE **can** detect effective DOF if $E \sim 1/l > M_0$
 - ✧ EE **can not** detect effective DOF if $E \sim 1/l < M_0$

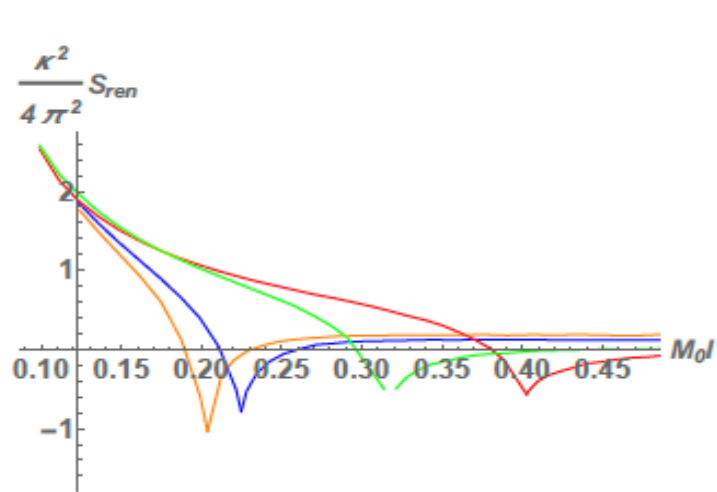
$$\frac{\kappa^2 S_{ren}}{4\pi^2 L_\phi} = -4l^2 a_4(l) - 2l^3 a'_4(l) + \frac{7}{64l} - \frac{A_{fin}}{2}$$

$$= -4l^2 a_4(l) - 2l^3 a'_4(l) + 2 \int^l l' a_4(l') dl' + \frac{1}{8l} + c_r,$$

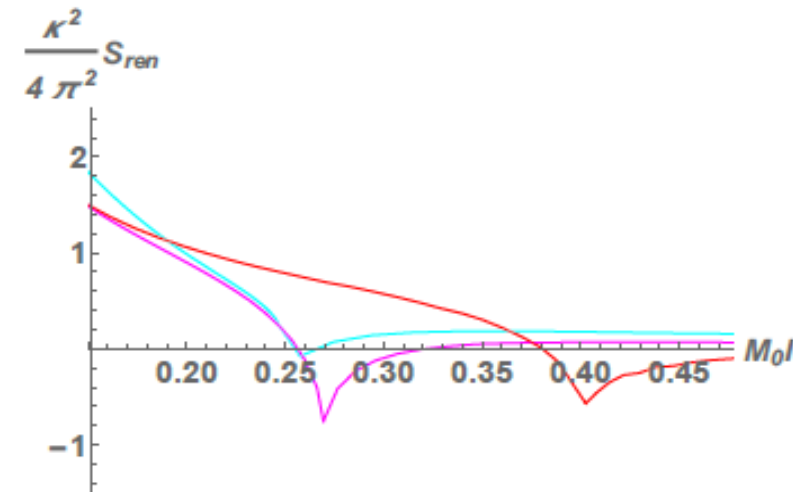
- ❖ Breaking conformal invariance
- ❖ Terms brought from the Weyl anomaly (B-type anomaly)

Renormalized EE in $4d$

- ❖ REE corresponds to effective DOF from EE with spherical surface
- ❖ REE non-monotonically behaves near critical lengths



$a_\phi = \frac{i}{2}, 0, \frac{2}{3}, \frac{1}{\sqrt{2}}$ from the left to the right

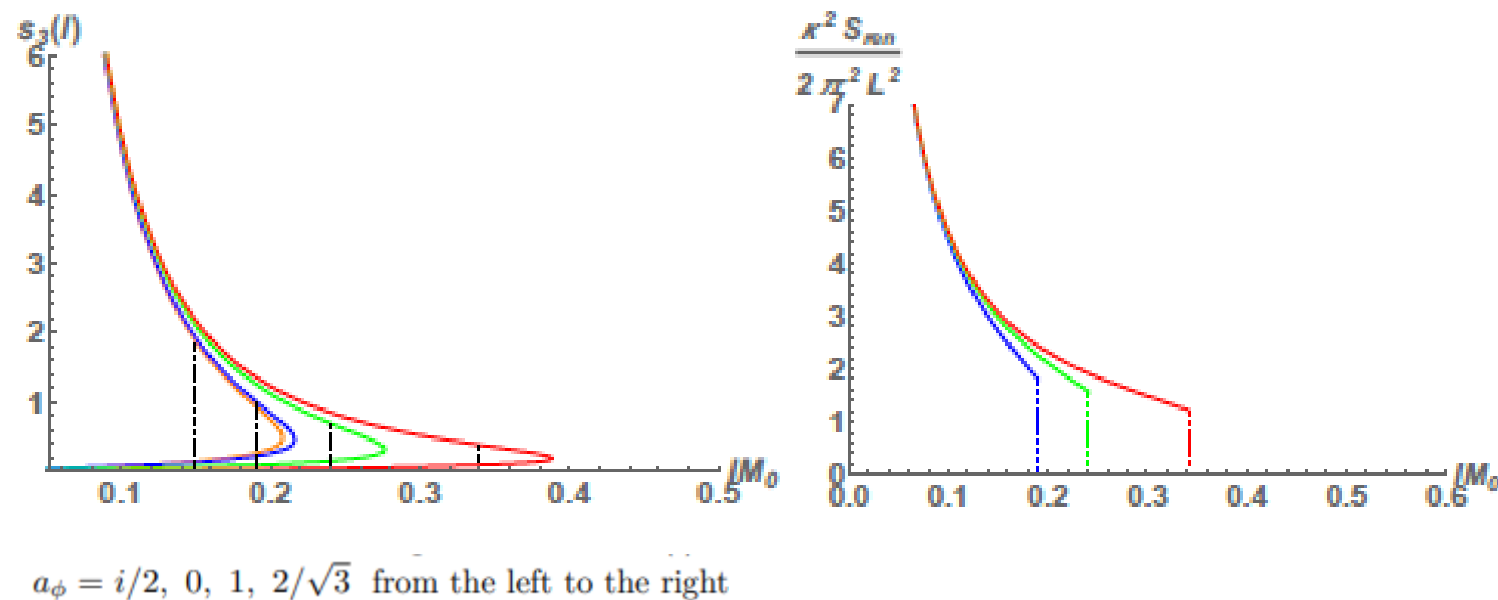


$M_0 = 1/\pi, 2/5, 3/5$ from the right to the left

- ❖ It implies that Wilson lines make particles light
- ❖ Massive modes decouple others soon

Renormalized EE in 3d

- ❖ 3d: REE is positive and monotonically decreasing.
- ✧ consistent with the entropic c function in $R^{1,1}$



- ✧ $s_3 \sim dS/dl$. The minimal surface is concave: $S'' < 0$

Discussion

Renormalized EE	Entropic C function
Increase at a region However, $S_{ren}(l \rightarrow 0) \geq S_{ren}(l \rightarrow \infty)$ (Decrease and positive for d=3)	Increase at a region and positive However, $S_{ren}(l \rightarrow 0) \geq S_{ren}(l \rightarrow \infty)$
Counting the effective DOF of Wilson lines along the S^1 direction	Counting the effective DOF of Wilson lines along the S^1 direction
Quantum phase transition	Quantum phase transition

Thank you!
