Grassmann Tensor Renormalization Group for $N_f = 2$ Schwinger model with a θ term

菅野 颯人 (Hayato Kanno)
RIKEN Nishina Center, RIKEN BNL Research Center

Based on the work with
秋山 進一郎 (筑波大),
村上 耕太郎 (東工大),
武田 真滋 (金沢大)
(in preparation)

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アメリカ所属ですが、
未だ日本(和光)にいます!
9月の学会も参加します!

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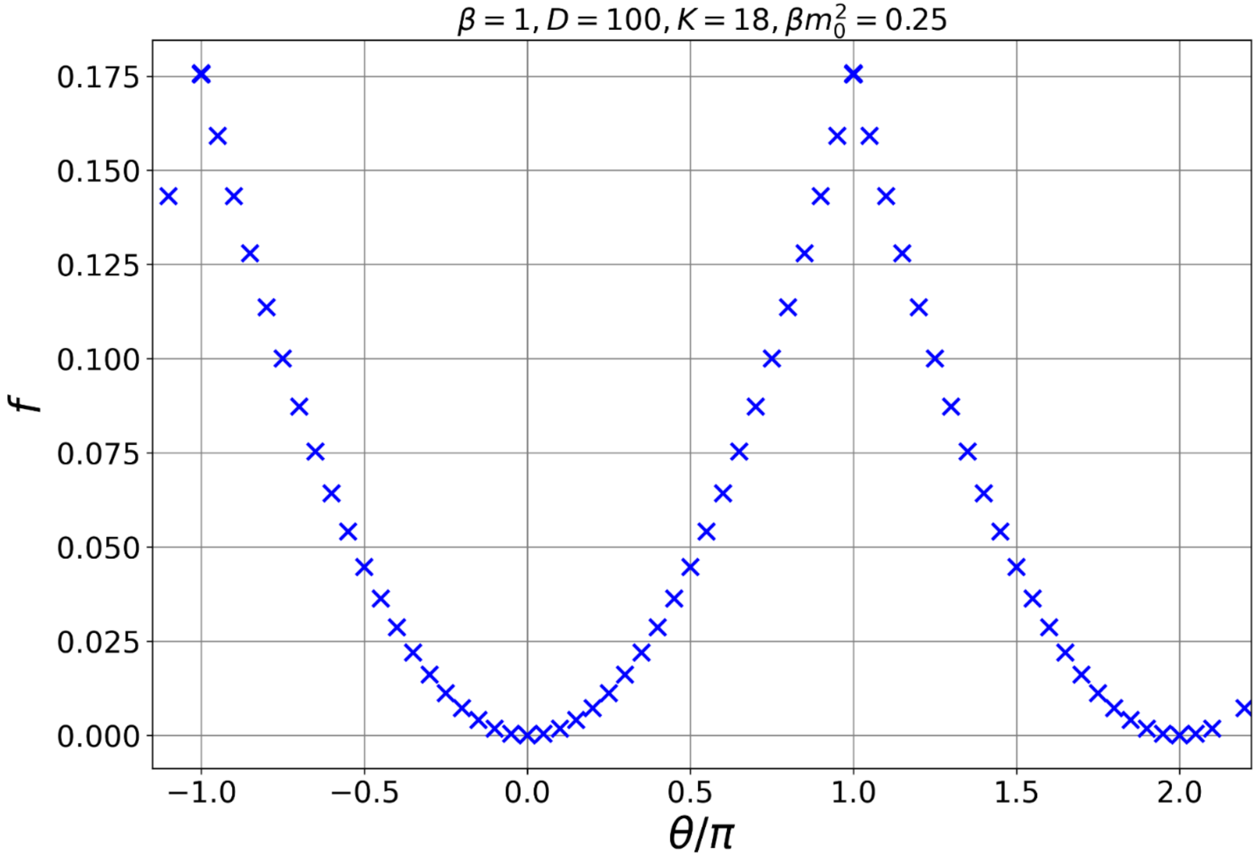
Short summary

We numerically calculate the θ dependence of the free energy (θ vacuum structure) in $N_f = 2$ Schwinger model.



Why is a θ term important?

What is a θ term?

$$Z_{YM} = \int \mathcal{D}A \, \mathrm{e}^{-S_{YM}} \supset \sum_n \mathrm{e}^{in\theta}$$

- A topological term in 4d QCD or Yang-Mills theory.
- Related to the instanton number.

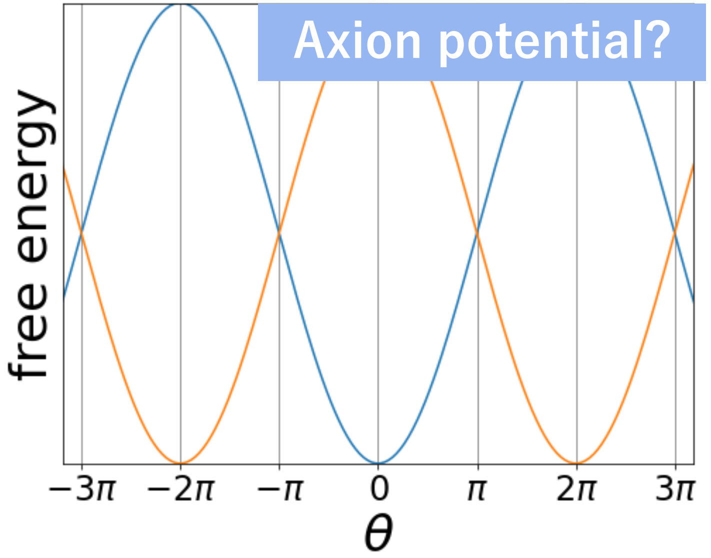
Strong CP problem

- QCD in our world, $\theta < 10^{-10}$. (from neutron EDM experiments)
- Why is it too small? (**Strong CP problem**)

Axion [Peccei, Quinn 1977]

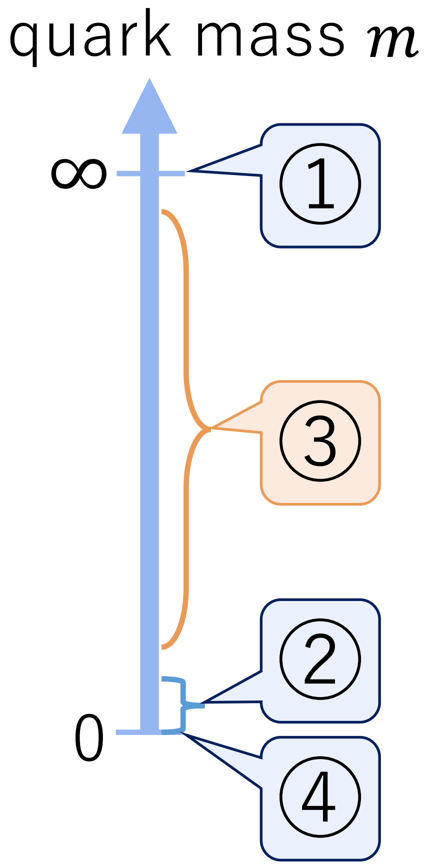
- Axion $\simeq$ scalar field which couples to the QCD like θ.
- Candidate for a dark matter
- Application to inflation; Natural inflation [Freese et al. 1990]
- Axion potential $=$ **θ dependence of the free energy**

  ($\theta = 0 \pmod{2\pi}$) is the stable point)



Known facts for the free energy

$SU(N_c)$ QCD with $N_f \geq 2$ flavors



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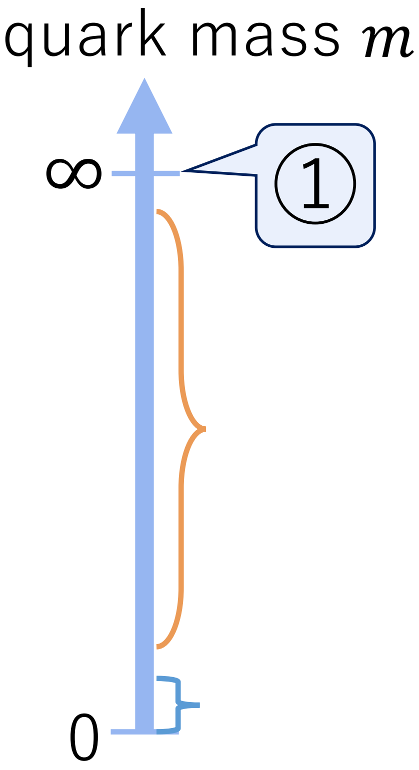
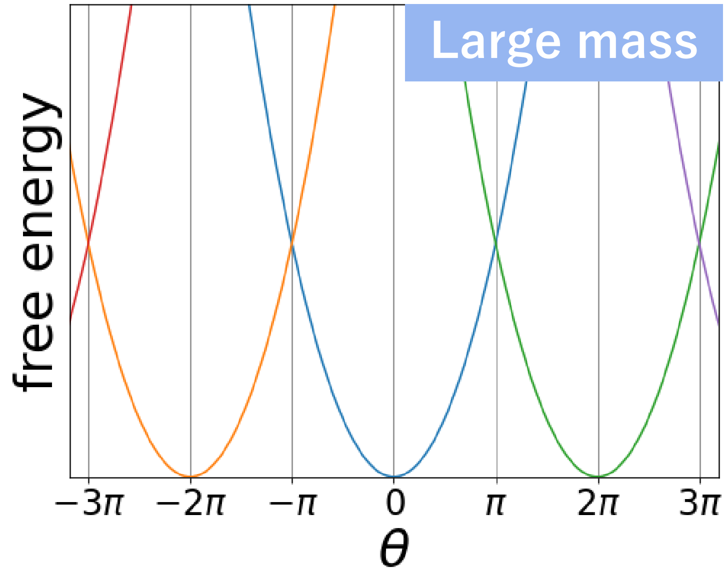


① $SU(N_c)$ Yang-Mills theory
θ dependence: known for large N_c
Free energy $\propto \theta^2$



cf.) Yokokura-kun's talk (am, today)

Free energy $\simeq$ vacuum energy
In my talk, I will focus on **true vacua**.

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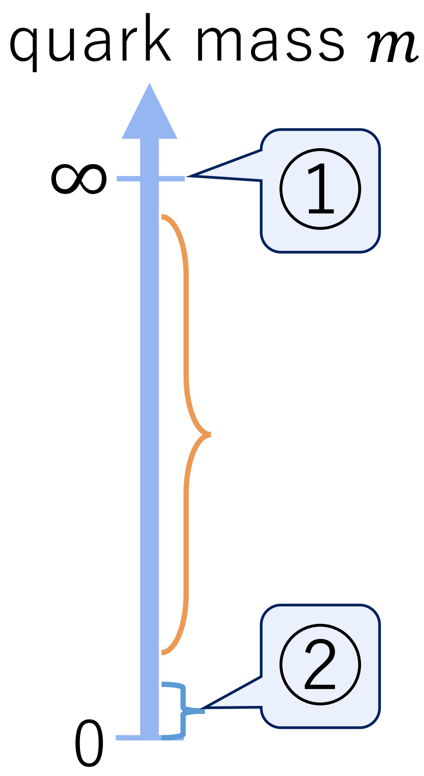

① $SU(N_c)$ Yang-Mills theory
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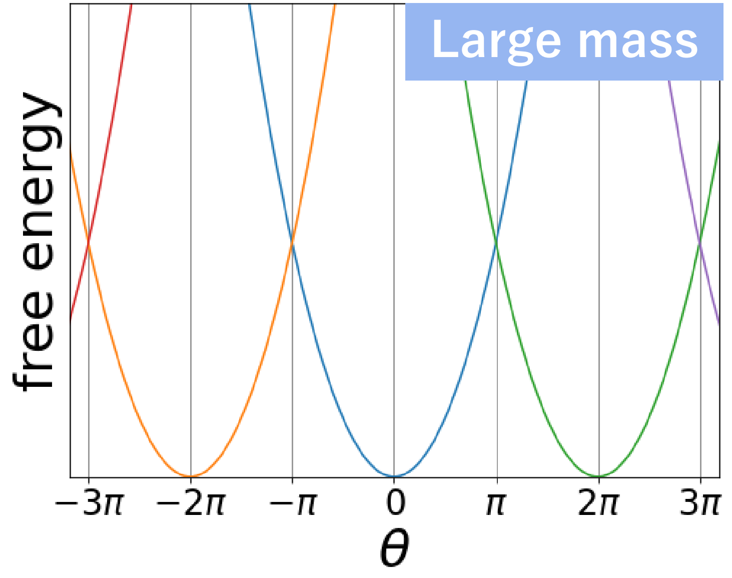
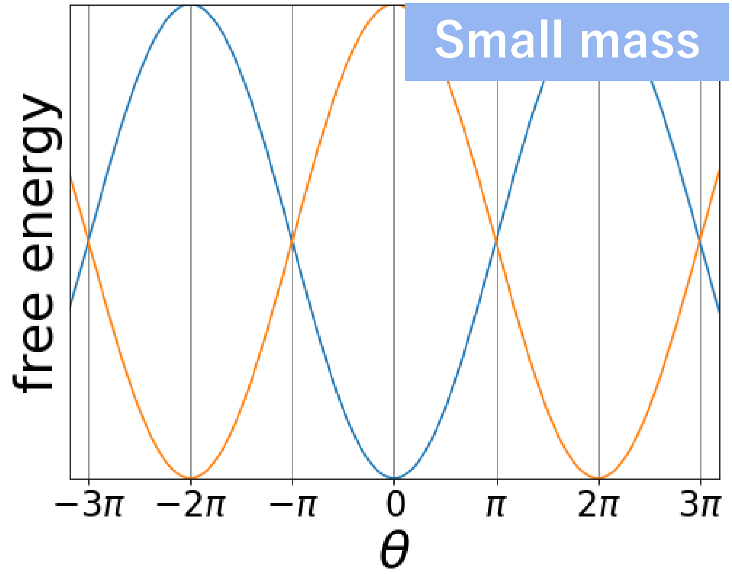
② QCD with mass perturbation
θ dependence: known for small m
Free energy $\propto \cos(\theta/N_f)$





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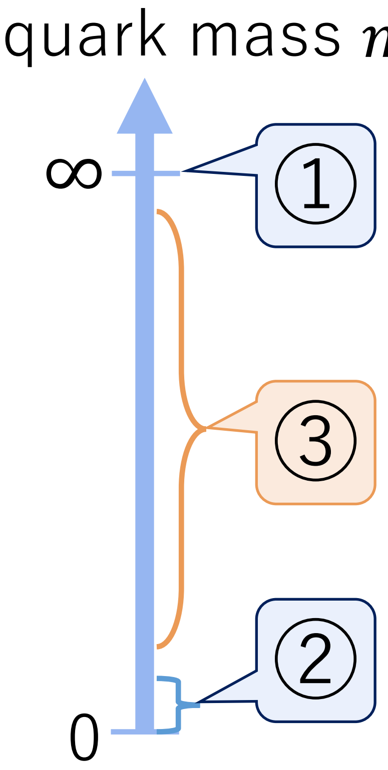

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③ Intermediate mass
θ dependence: unknown
Numerical calculation is needed

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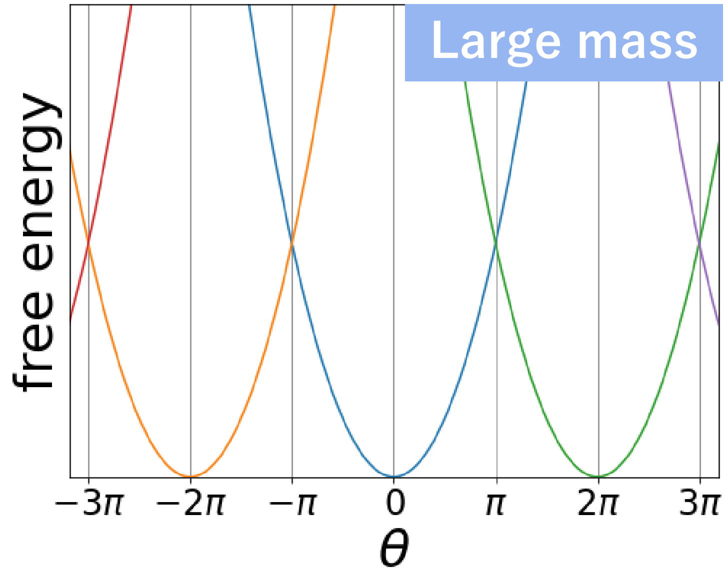

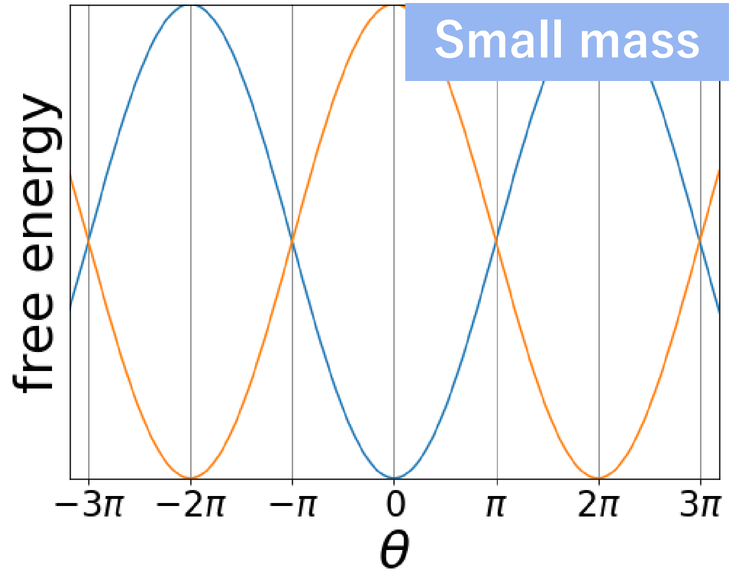

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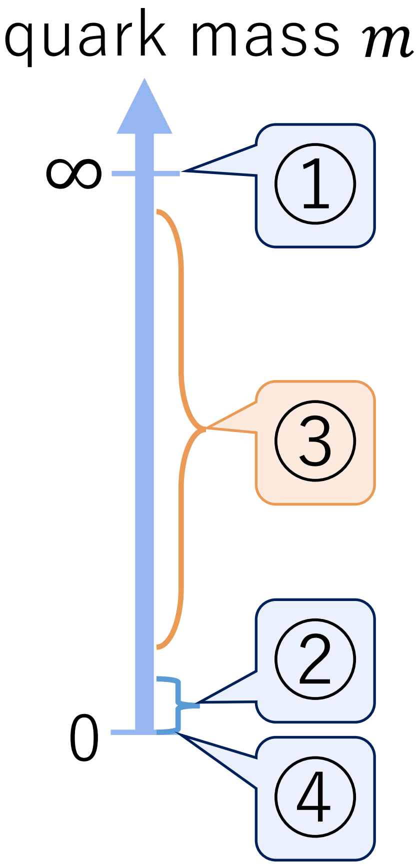

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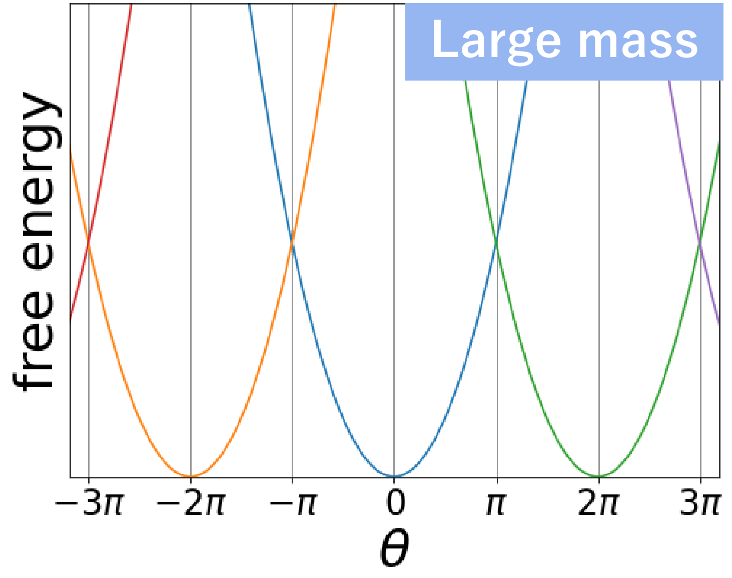
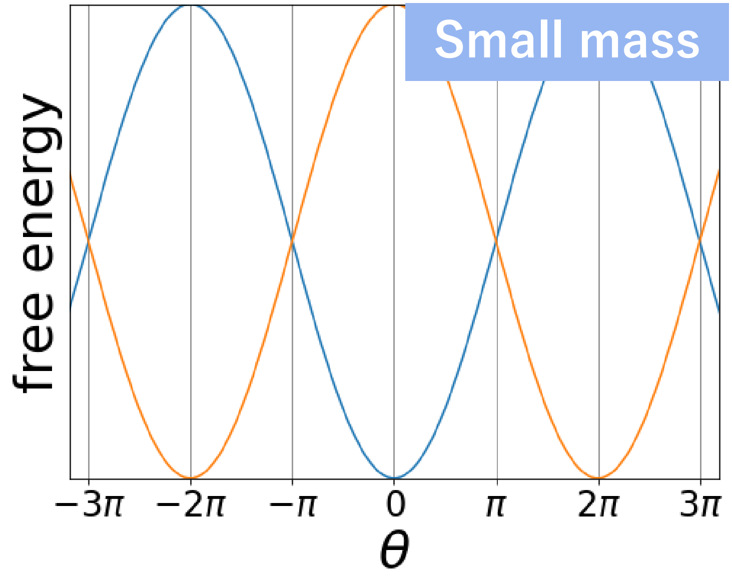
④ Massless QCD, chiral symmetry
No θ dependence ($U(1)_A$ anomaly)





How to calculate QCD with the θ term

Monte Carlo method: the sign problem

$$Z_{YM} = \int \mathcal{D}A \, \mathrm{e}^{-S_{YM}} \supset \sum_n \mathrm{e}^{in\theta}$$

- With finite θ, the partition function includes imaginary part.

→ The Monte Carlo simulation does not work well.

- There are some studies by the Monte Carlo.
  e.g.) 4d $SU(2)$ YM theory [Kitano et al. 2102.08784]
- But, $\theta = \pi$ point is tough…

Tensor network methods do not have the sign problem!

- It is hard to use tensor network methods for 4d QCD.
- However, tensor networks work well for 2d theories.

→ We calculate the Schwinger model (2d toy model of the QCD) by tensor renormalization group (TRG). [Levin, Nave 2007]

Plan

1. Introduction (4)
   - Short summary
   - Why is the θ term important?
   - How to calculate QCD with θ

2. Schwinger model (2)
   - What is the Schwinger model?
   - What we want to calculate

3. TRG (2)
   - TRG
   - Lattice action

4. Results (6)
   - 2π periodicity
   - Large mass limit
   - Small mass limit
   - Intermediate mass

5. Conclusion (1)

What is the Schwinger model? [Schwinger 1962, Coleman 1976, …]

Schwinger model = 2d QED

cf.) Matsumoto-san's talk

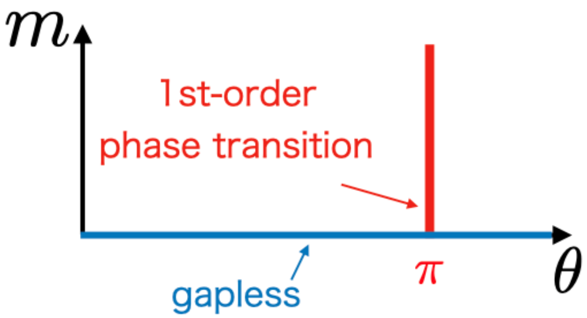
$$S=\int \mathrm{d}^2x\left\{\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}+\frac{\mathrm{i}\theta}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}+\bar{\psi}\mathrm{i}\gamma^{\mu}(\partial_\mu+\mathrm{i}A_\mu)\psi+m\bar{\psi}\psi\right\}$$

- $U(1)$ gauge theory + (fundamental) fermions (2dim $U(1)$ gauge theory has a strong coupling.)

For $N_f \geq 2$ case,

- Bosonization → pion theory
- In the massless point, $SU(N_f)_1$ WZW model in the IR limit
- First order phase transition @$\theta=\pi$

For $\mathbb{C}P^1$ model,
Shimizu-kun's poster

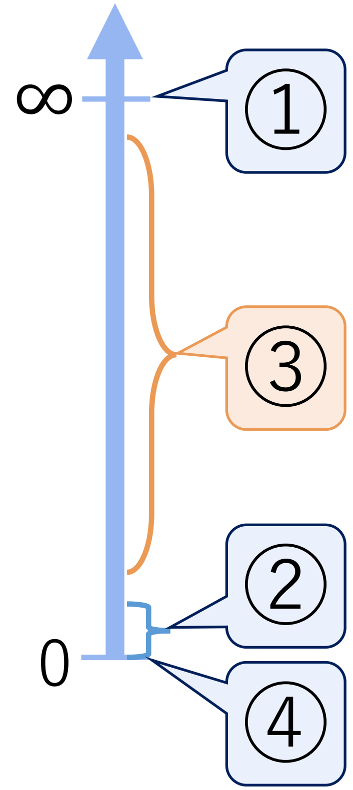


→ How much does the vacuum structure similar?

What we want to calculate

$N_f = 2$ Schwinger model

fermion mass m



What we want to calculate

$N_f = 2$ Schwinger model

$V = \int d^2x$ (volume)

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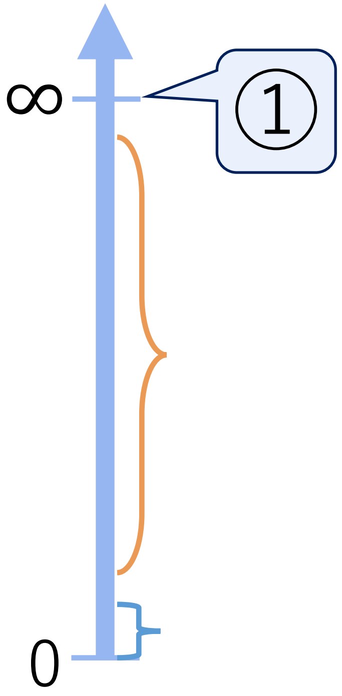
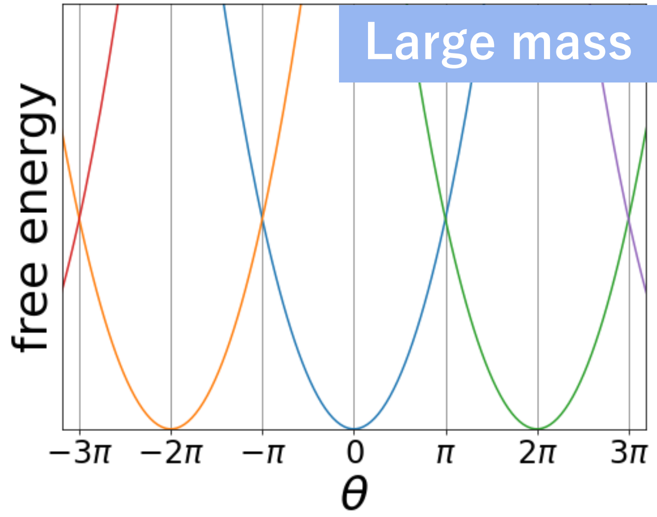


① $U(1)$ Maxwell theory

$$-\frac{\log Z(\theta)}{g^2 V} = \min_n \frac{1}{8\pi^2}(\theta - 2\pi n)^2$$

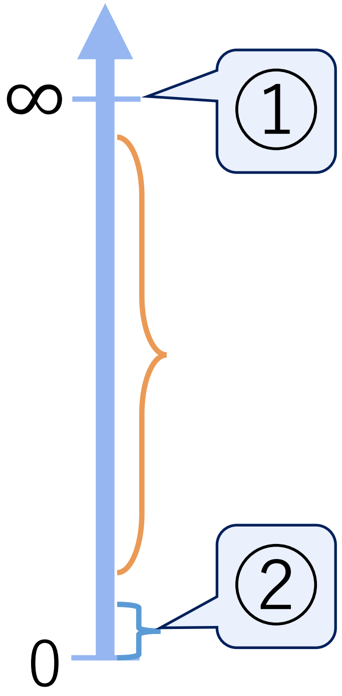


What we want to calculate

$N_f = 2$ Schwinger model

$V = \int d^2x$ (volume), $\gamma = 0.577\ldots$ (Euler constant)

fermion mass m



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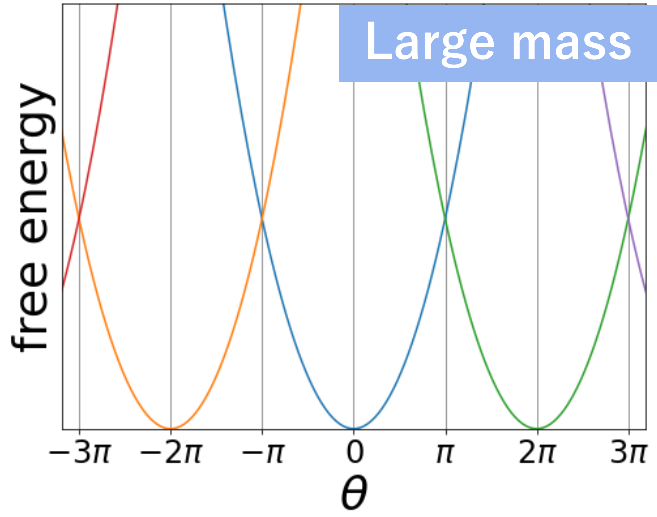
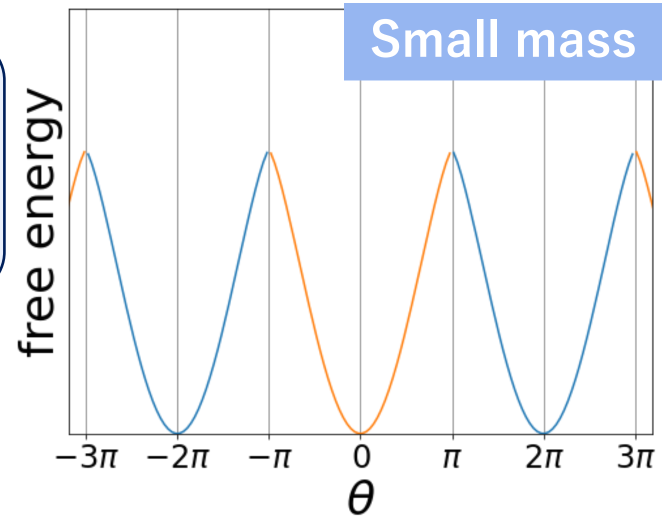


② Mass perturbation [Coleman 1976]

$$-\frac{\log Z(\theta)}{g^2 V} = \min_n \left\{ (\mathrm{e}^{\gamma})^{\frac{4}{3}} \pi^{-\frac{5}{3}} 2^{\frac{1}{3}} \left(\frac{m^2}{g^2}\right)^{\frac{2}{3}} \cos^{\frac{4}{3}}\left(\frac{\theta - 2\pi n}{2}\right) \right\}$$



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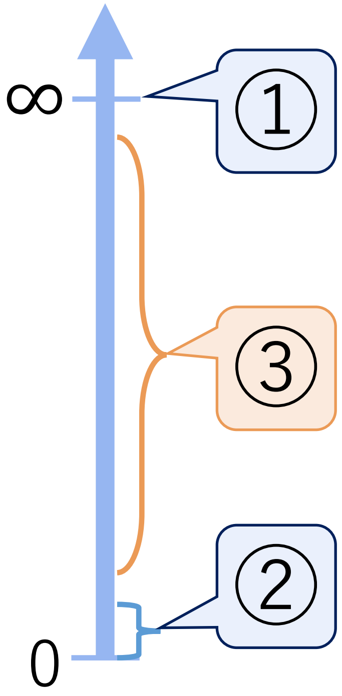

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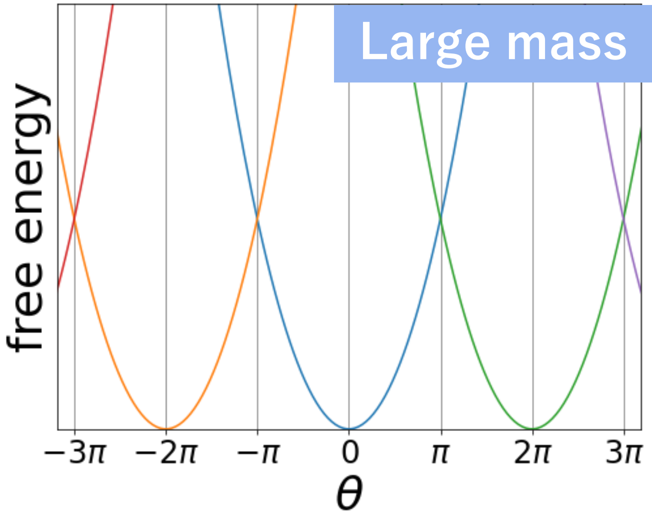

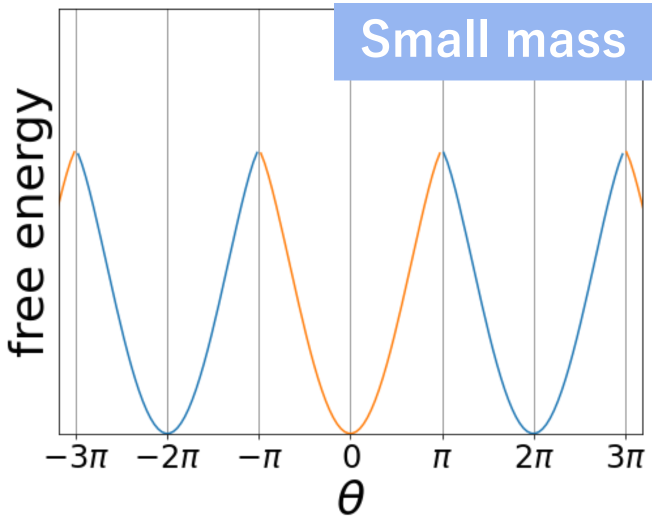

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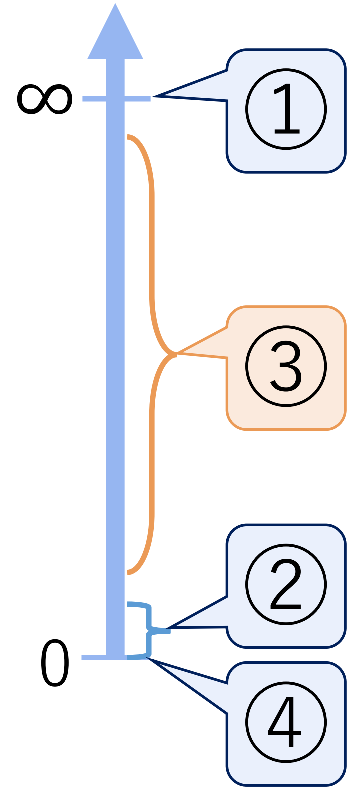

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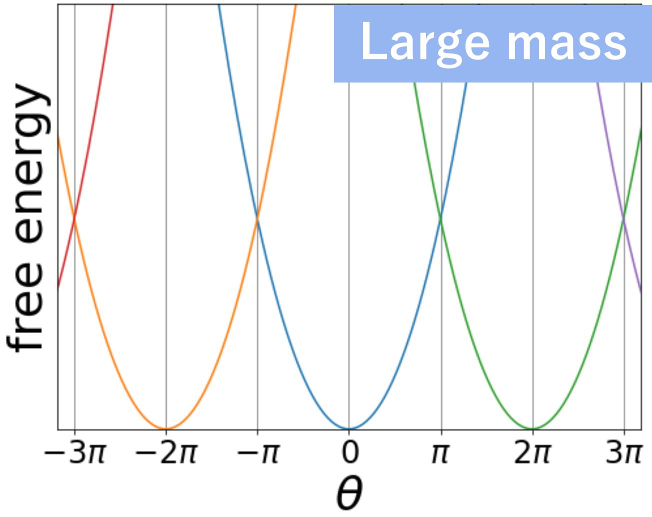
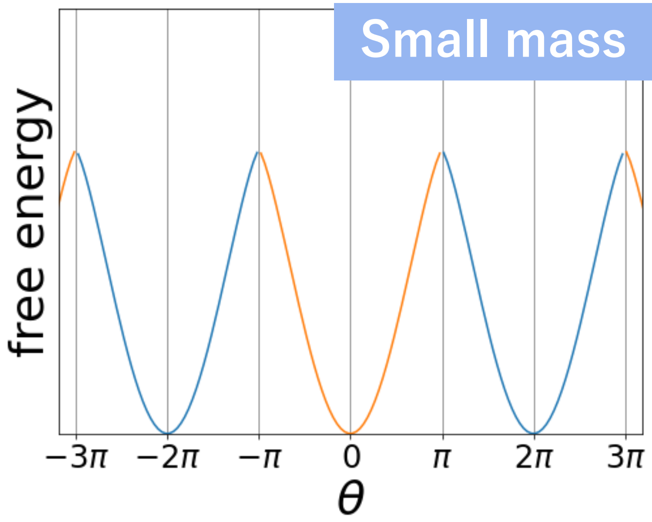
④ Massless, chiral symmetry ($SU(2)_1$ WZW model)
No θ dependence ($U(1)_A$ anomaly)





TRG

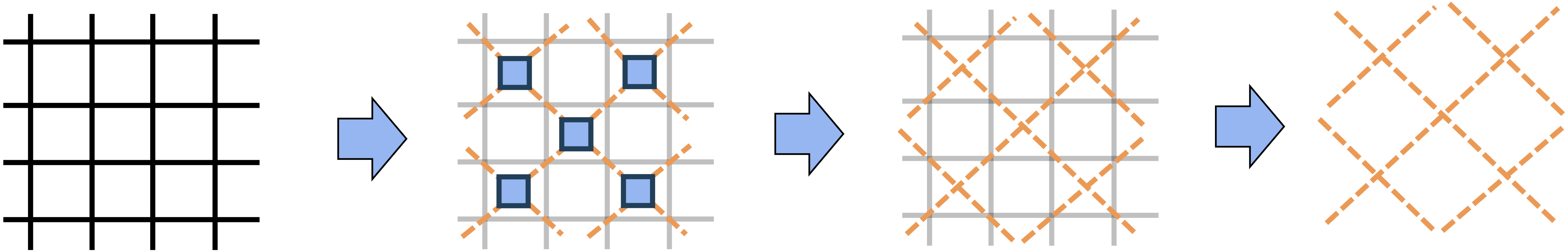
Tensor Renormalization Group

TRG

[Levin, Nave 2007]

Real space renormalization for one initial tensor

- From the translation invariance, we just focus on single tensor.
- Singular value decomposition (SVD)
  - Finite cut off for singular values : bond dimension
  - Approximation for TRG.
- Grassmann-TRG [Gu, Verstraete, Wen 1004.2563]
  - Fermion has less d.o.f. by Grassmann path integral.



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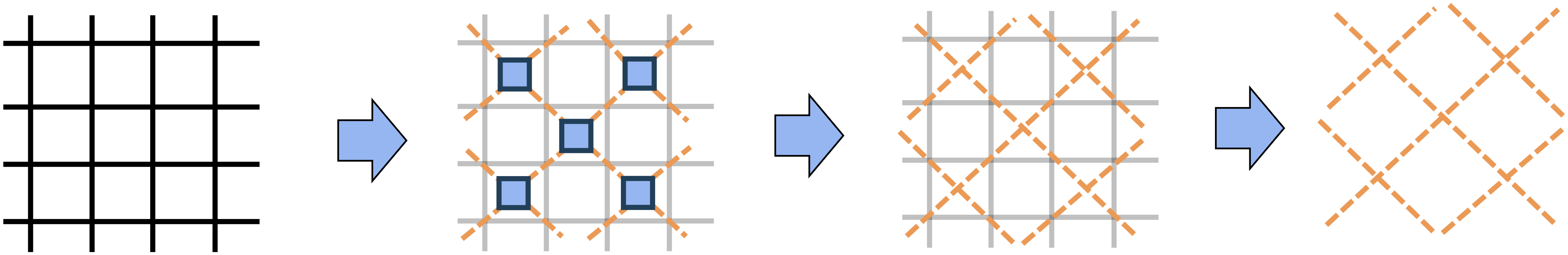

Lattice action for Schwinger model

$$S = \sum_{n,\mu} \left[-\beta \cos(A_p(n)) - \frac{i\theta}{2\pi}\tilde{A}_p(n) + \frac{1}{2}\left[\eta_\mu(n)\{\bar{\chi}(n)U_\mu(n)\chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu})U_\mu^\dagger(n)\chi(n)\} + m_0\bar{\chi}(n)\chi(n)\right]\right]$$

$U_\mu = \mathrm{e}^{iA_\mu}$: link variable

$\eta_1 = 1,\ \eta_2 = (-1)^{n_1}$: staggered phase

Staggered fermion (χ)

- 2d one staggered fermion ↔ 2-flavor Dirac fermion

Gauge field (A_μ)

- $\log U_p$ type θ term (2π periodicity of θ is realized.)

$$\tilde{A}_p(n) = -i \log U_p(n)$$

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- We use Gauss-Legendre quadrature to discretize gauge field.

  [Kuramashi, Yoshimura 1911.06480] (TRG for 2dim Maxwell with the θ term) $A_\mu \in U(1) \to K$ points

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Parameters of this study

Continuum limit: $\beta \to \infty$, w/ βm_0 and β/L^2 fixed
$D \to \infty$, $K \to \infty$

- $\beta = 4$ ($\beta = 1/a^2g^2$), (θ, m_0)($m_0 = ma$, we search for these parameters.)
- $D = 120$ (bond dimension), $K = 25$, $L^2 = 2^{32}$ ($L^2 = V/a^2$, we take large volume limit)

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Lattice action for Schwinger model

$$S = \sum_{n,\mu} \left[-\beta\cos(A_p(n)) - \frac{i\theta}{2\pi}\tilde{A}_p(n) + \frac{1}{2}\left[\eta_\mu(n)\{\bar{\chi}(n)U_\mu(n)\chi(n+\hat{\mu}) - \bar{\chi}(n+\hat{\mu})U_\mu^\dagger(n)\chi(n)\} + m_0\bar{\chi}(n)\chi(n)\right]\right]$$

$U_\mu = e^{iA_\mu}$: link variable

$\eta_1 = 1,\ \eta_2 = (-1)^{n_1}$: staggered phase

Staggered fermion (χ)

- 2d one staggered fermion ↔ 2-flavor Dirac fermion

Gauge field (A_μ)

- $\log U_p$ type θ term (2π periodicity of θ is realized.) $\tilde{A}_p(n) = -i\log U_p(n)$
- We use Gauss-Legendre quadrature to discretize gauge field.

  [Kuramashi, Yoshimura 1911.06480] (TRG for 2dim Maxwell with the θ term) $A_\mu \in U(1) \to K$ points

Parameters of this study

Continuum limit: $\beta \to \infty$, w/ βm_0 and β/L^2 fixed

$D \to \infty$, $K \to \infty$

- $\beta = 4$ ($\beta = 1/a^2g^2$), (θ, m_0)($m_0 = ma$, we search for these parameters.)
- $D = 120$ (bond dimension), $K = 25$, $L^2 = 2^{32}$ ($L^2 = V/a^2$, we take large volume limit)

Results

2π periodicity

What we calculate

→ free energy

$$f = -\left\{\frac{\log Z(\theta)}{g^2 V} - \frac{\log Z(\theta = 0)}{g^2 V}\right\}$$

$$= -\left\{\beta \frac{\log Z(\theta)}{L^2} - \beta \frac{\log Z(\theta = 0)}{L^2}\right\}$$

(Dimension-less free energy density normalized at $\theta = 0$)

- We also check $\partial f / \partial \theta$

2π periodicity

What we calculate

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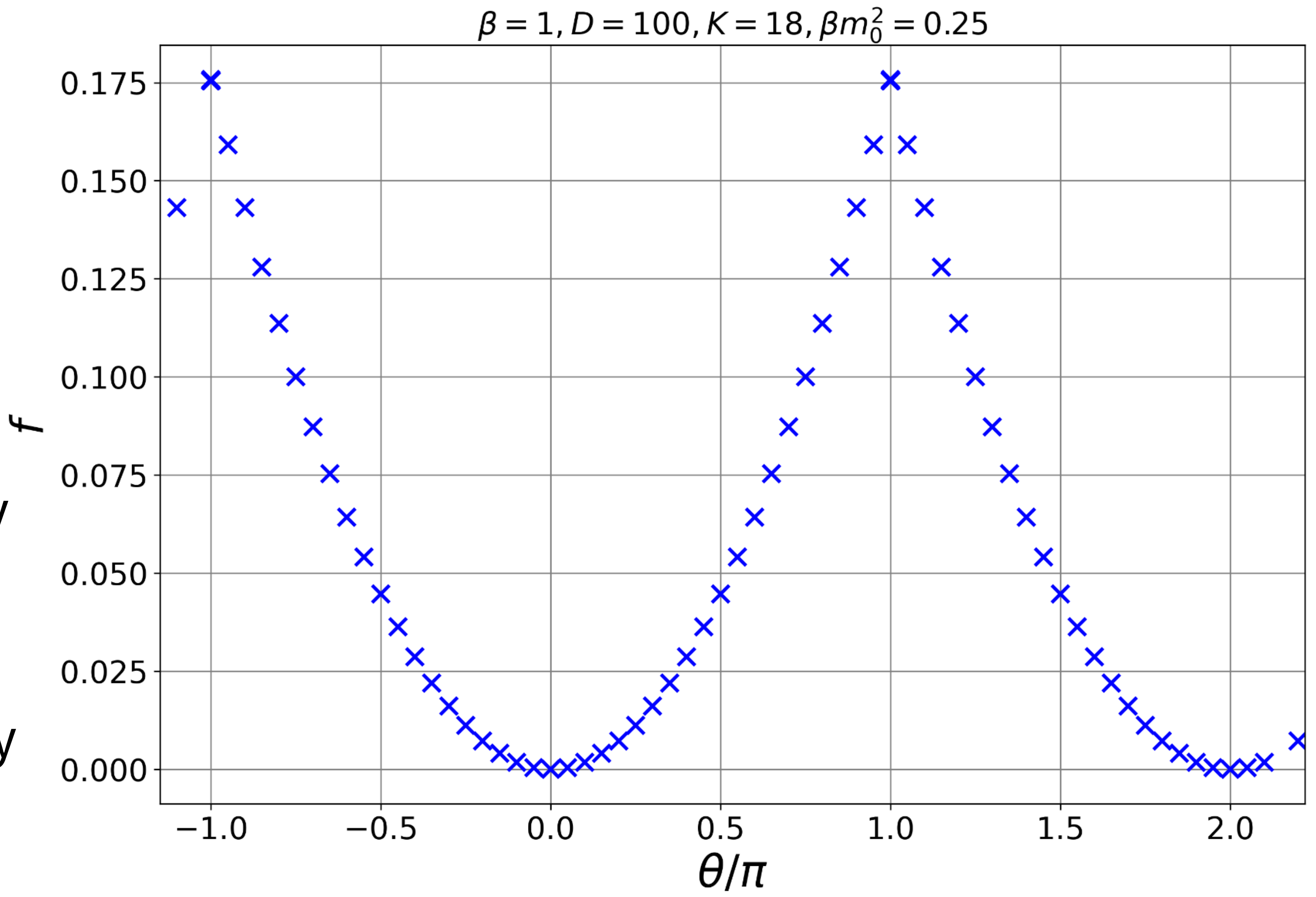
(Dimension-less free energy density normalized at $\theta = 0$)

- We also check $\partial f/\partial\theta$
- Plot for free energy density vs θ

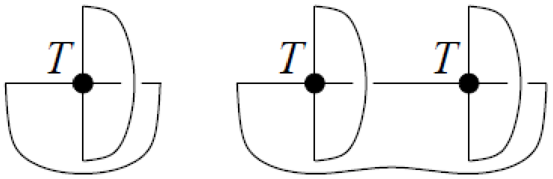


Degeneracy

Ground state degeneracy in TRG

- We can calculate ground state (or vacuum) degeneracy in TRG. [Gu, Wen 0903.1069]

$$X_1 = \frac{\left(\sum_{ru} T_{ruru}\right)^2}{\sum_{ruld} T_{rulu} T_{ldrd}} = \frac{(a)^2}{(b)}$$

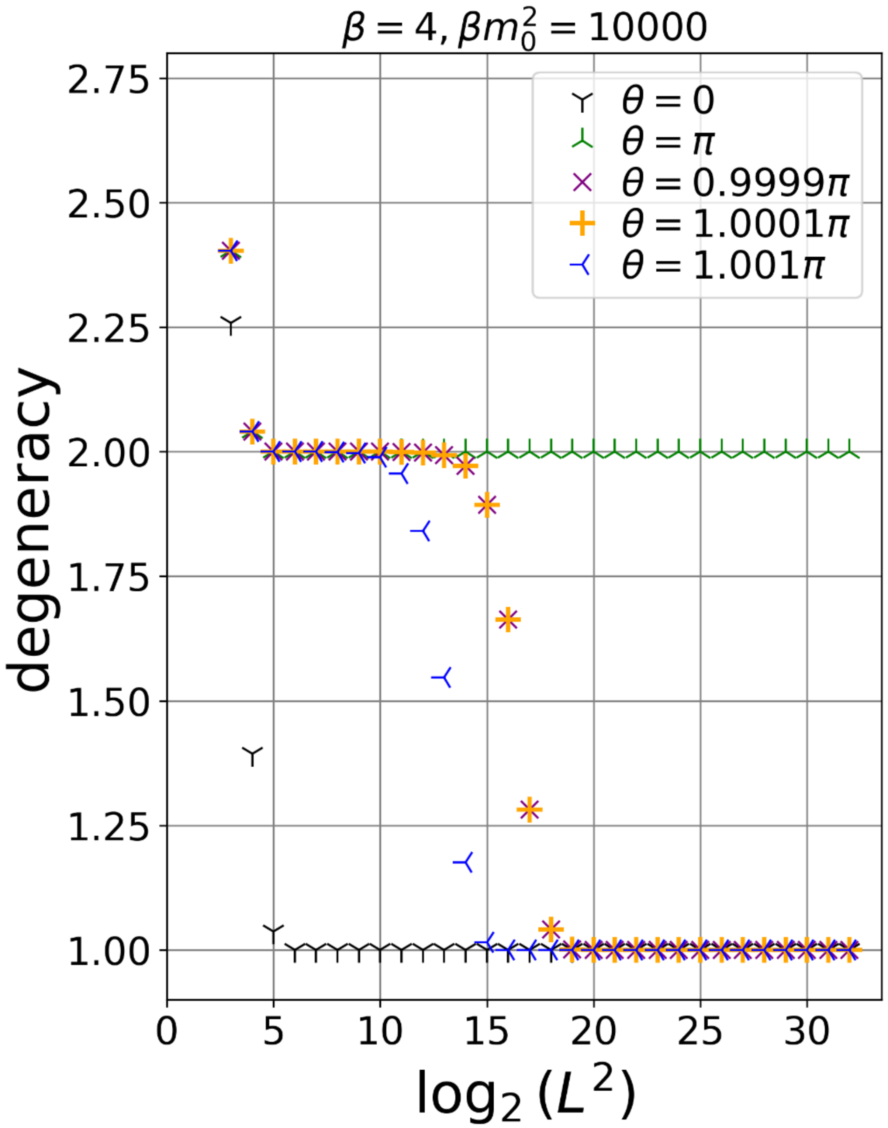


(a) (b)

- We checked 2-vacua degeneracy at $\theta = \pi$ for large mass parameters.
- $\theta = \pi \pm 0.0001\pi$ shows a single vacuum!

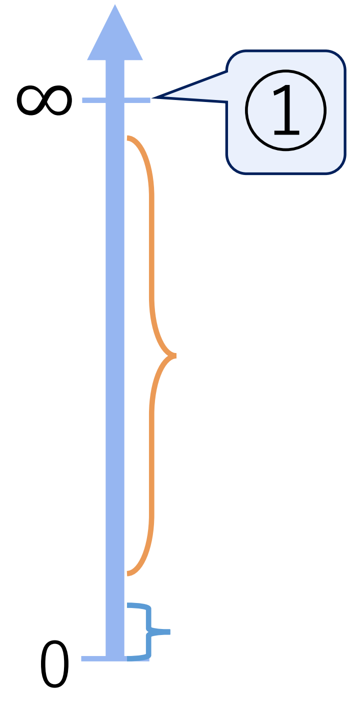
$\rightarrow$ 2π periodicity is obvious!

- In the following parts, we just focus on $\theta \in [0, \pi]$

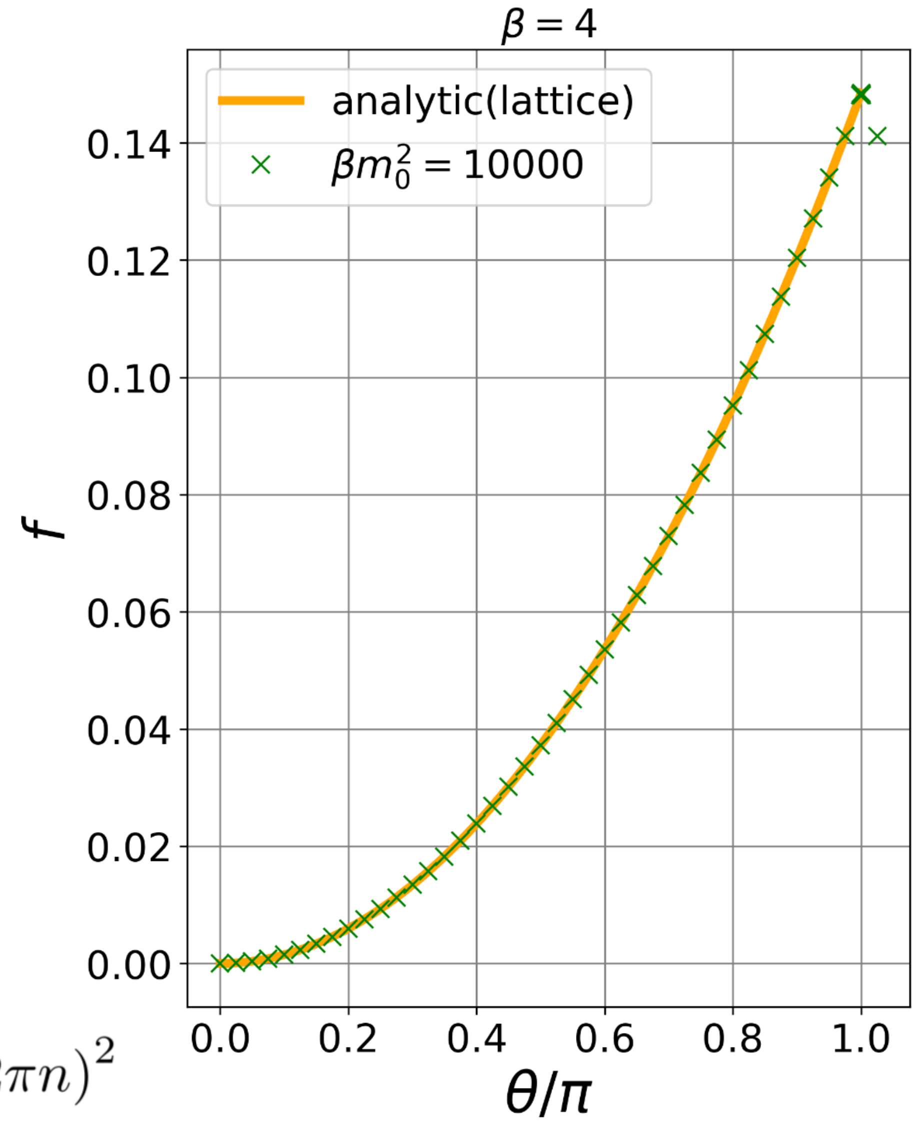


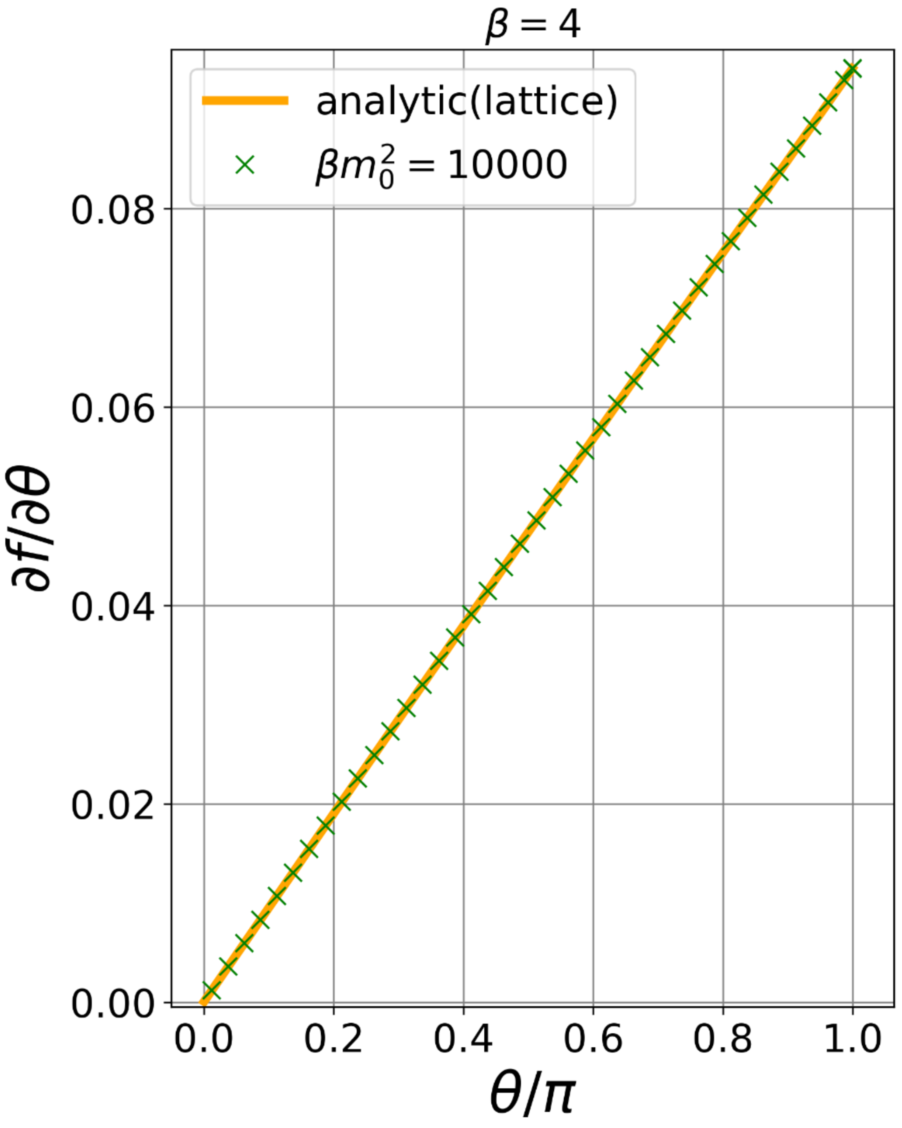
Large mass limit

fermion mass m

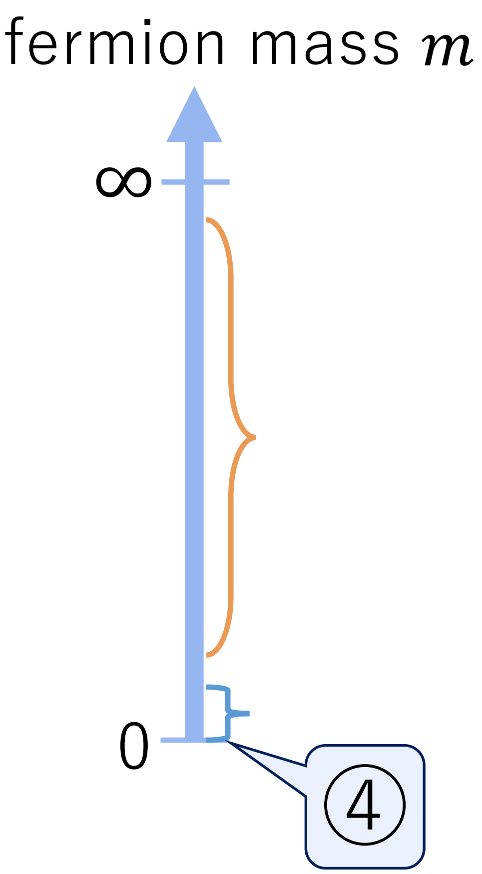


$$-\frac{\log Z(\theta)}{g^2V} = \min_n \frac{1}{8\pi^2}(\theta - 2\pi n)^2$$



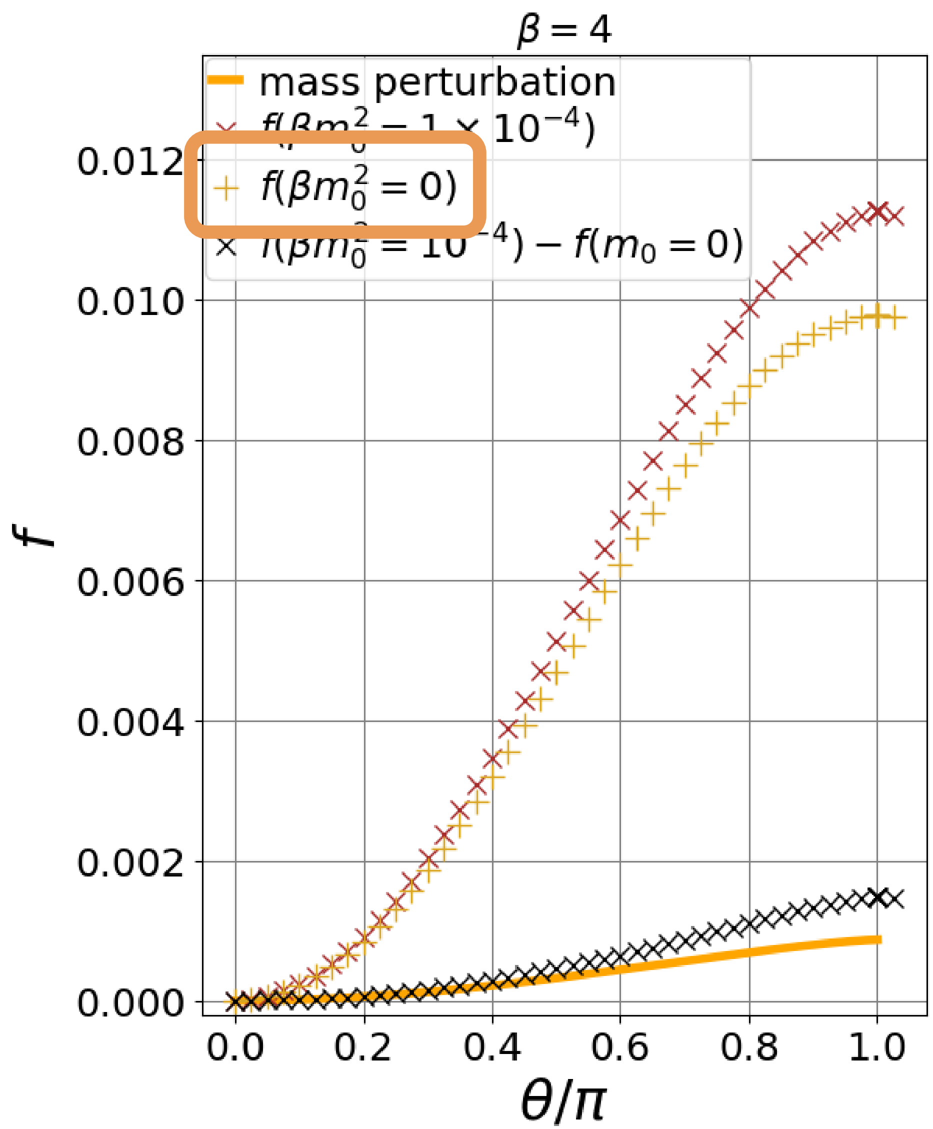


Small mass limit (1)

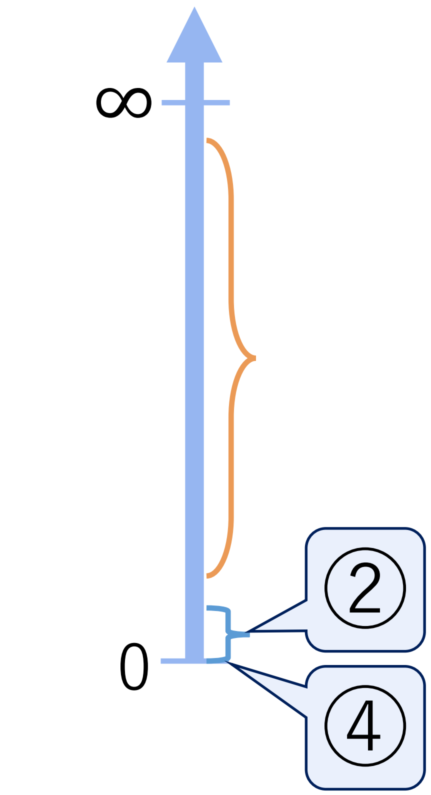


④: massless point

- There is a lattice artifact @ $\beta m_0^2 = 0$. (Gold plots)
- It disappears in $\beta \to \infty$ limit.



Small mass limit (1)



④: massless point

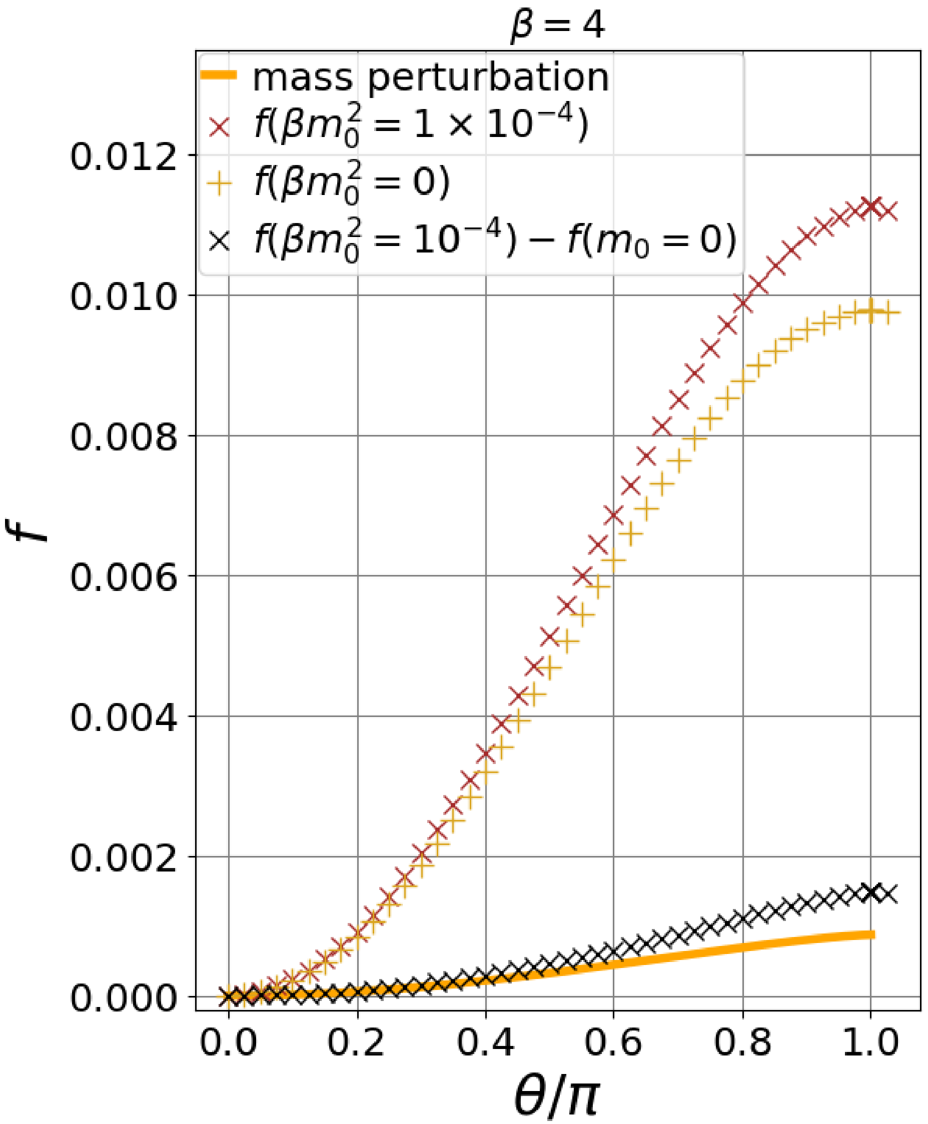
- There is a lattice artifact @ $\beta m_0^2 = 0$. (Gold plots)
- It disappears in $\beta \to \infty$ limit.

②: small mass

- We calculate $\beta m_0^2 = 10^{-4}$. (Brown plots)
- We subtract the lattice artifact from small mass results.

$$f(m_0) - f(m_0 = 0)$$

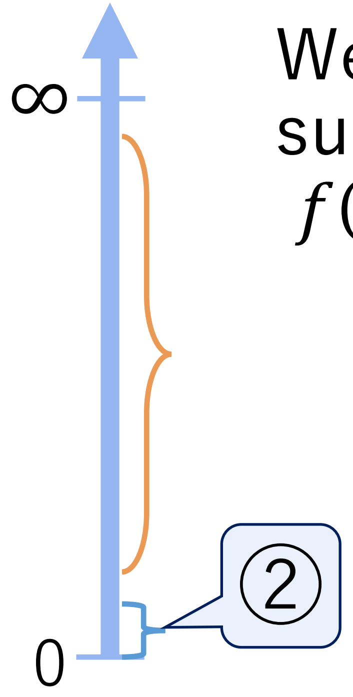
(**Black plots**)



Small mass limit (2)

Check of the finite β effect

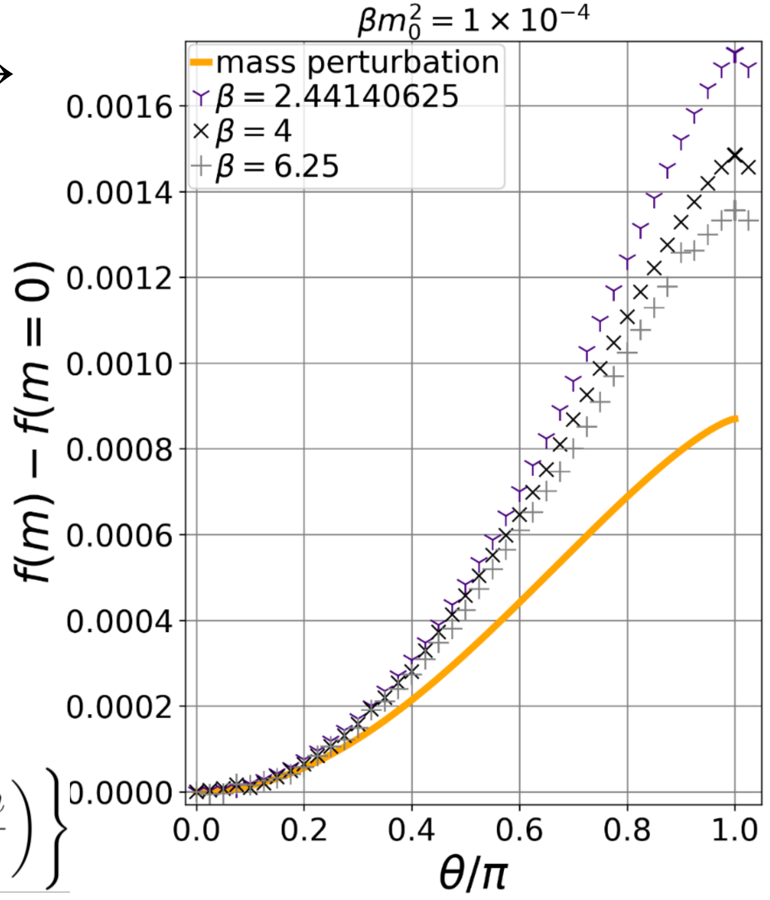
fermion mass m

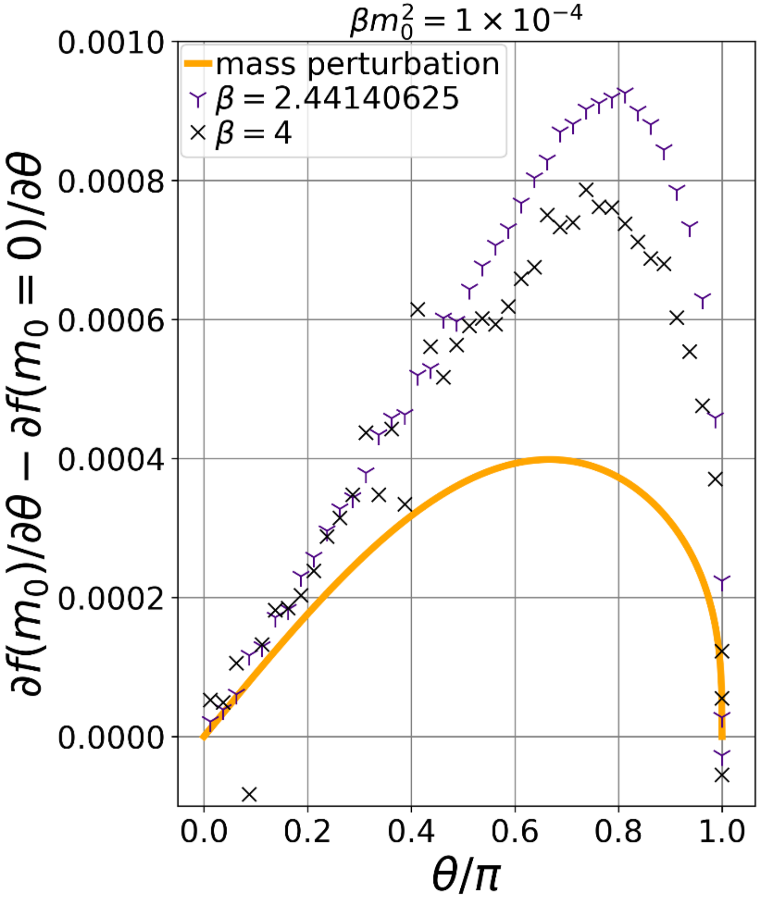


We analyze the subtracted plots →
$f(m_0) - f(m_0 = 0)$

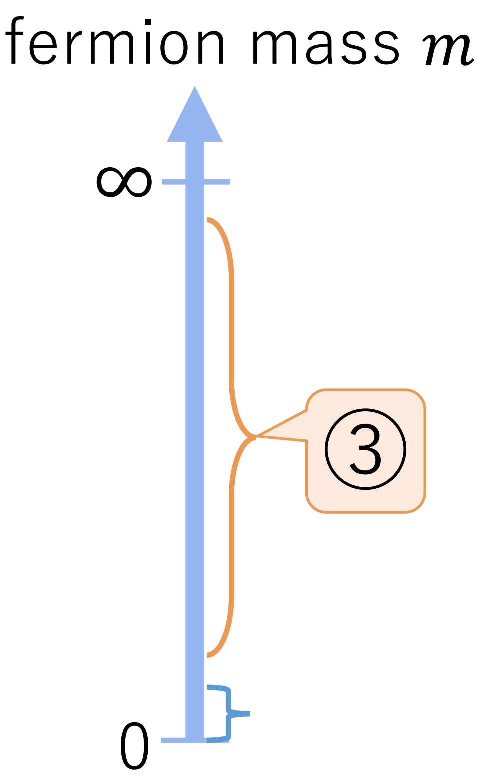
$$-\frac{\log Z(\theta)}{g^2 V} = \min_n \left\{ (\mathrm{e}^{\gamma})^{\frac{4}{3}} \pi^{-\frac{5}{3}} 2^{\frac{1}{3}} \left(\frac{m^2}{g^2}\right)^{\frac{2}{3}} \cos^{\frac{4}{3}} \left(\frac{\theta - 2\pi n}{2}\right) \right\}$$

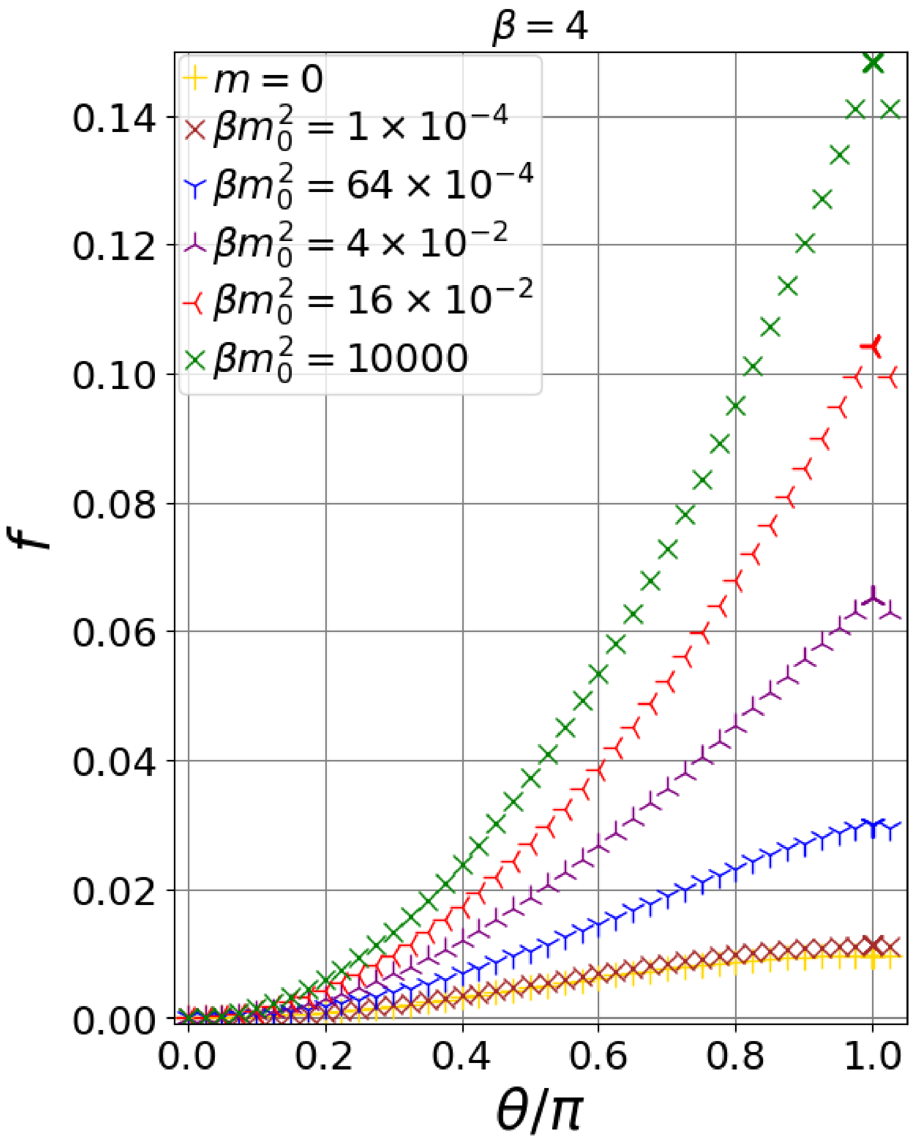
- In $\beta = 4$, we found discrepancy from the mass perturbation.
- Larger β calculations are required.

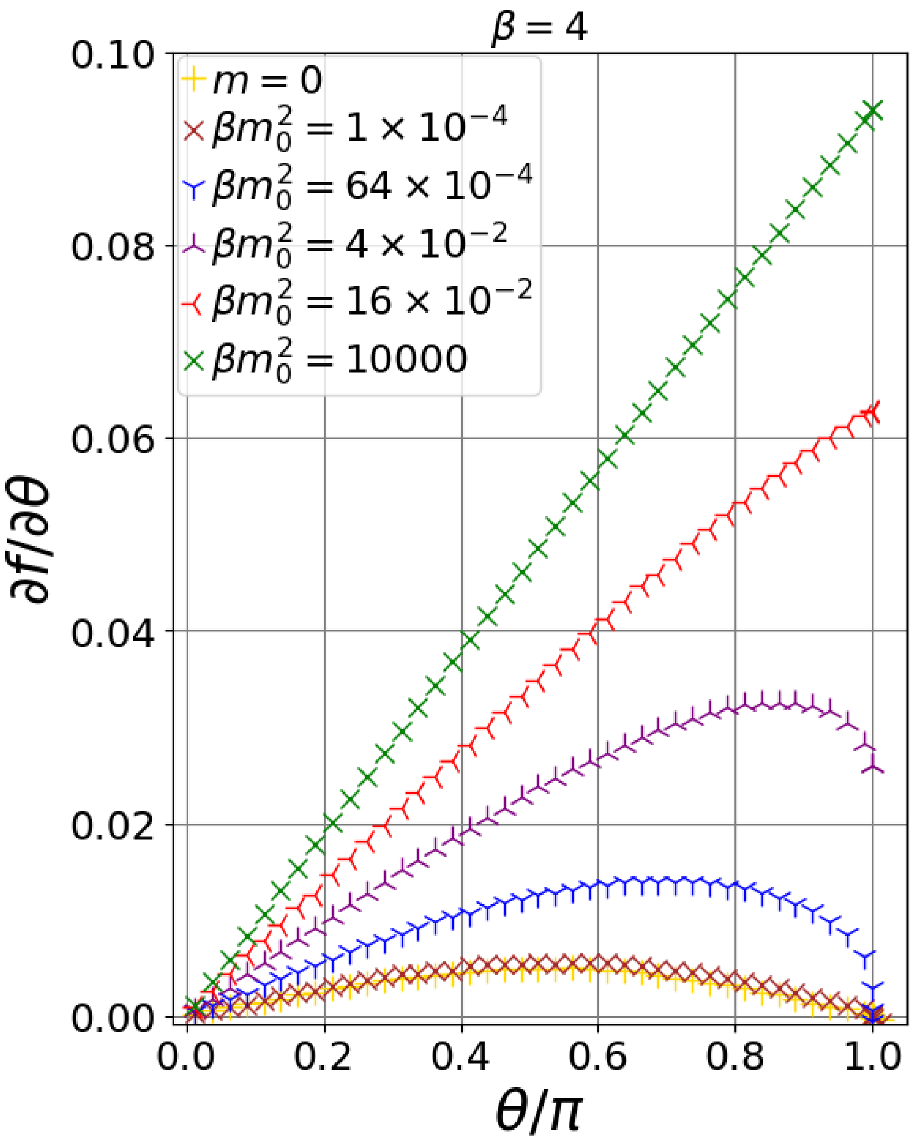




Intermediate mass







Conclusion

$N_f = 2$ Schwinger model in TRG

- Schwinger model : 2dim QED
  - 4dim QCD-like theory (chiral sym, vacuum structure, ...)
  - **θ dependence of free energy** is also similar.
  - Good to calculate by TRG (Smaller d.o.f. than 4dim theory)
- We calculated θ dependence of the free energy by Grassmann-TRG.
  - **2π periodicity** of θ is obvious.
  - Large mass region is consistent.
  - Small mass region is not consistent enough. (finite β effect)
  - **Finite mass effects** for the intermediate mass regime.
- Future directions
  - Larger β calculations for small mass parameters (To check the consistency with the mass perturbation.) → Larger D calculation is required!