von Neumann Algebra and Quantum Gravity

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This is basically an ultra compact version of a rather comprehensive and audacious series of intensive lectures given last year at KEK and at Komaba, using 850 slides,

with some further thoughts added.

What we wish to convey in this talk is the novel possibility and perspective of the classic mathematical structure of the theory of von Neumann algebras as a rather universal and comprehensive framework for understanding

WHAT IS "QUANTUM GRAVITY"

This point of view was emphasized in particular by E. Witten a few years ago and since has been being pursued by him and his collaborators, as well as others, yielding remarkable results and provoking deep questions.

Plan of the Talk

1. Introduction

- Where does this approach stand ?

2. Minimum basics of von Neumann algebras

- What is a von Neumann algebra ?
- Classification and characteristics of von Neumann algebra (factors): type I, II, III
- Concrete examples

3. von Neumann algebra for QFT in space-time

- Net of algebra
- von Neumann algebra for QFT in flat Minkowski space

4. Modular theory of von Neumann algebra (Tomita-Takesaki theory)

- Tomita-Takesaki theory and modular operators
- Case of a finite dimensional quantum system
- Relative entropy and its basic properties

- Modular operators for QFT in flat spacetime: Importance of Reeh-Schlieder theorem.
- Reeh Schlieder theorem and entanglement
- Modular automorphism and its analyticity properties

5. Crossed product and understanding of type III factor

- Concept of “crossed product”
- Connes-Takesaki duality theorem and the structure of type III factor

6. Application to black hole physics

- Eternal black hole in the strict large N limit
- Witten's observation:

Inclusion of $1/N^2$ corrections and transitions from type $\mathbf{III_1}$ to type $\mathbf{II_\infty}$ via a crossed product

- Classical-quantum states and the definitions of trace and entropy

7. Large N algebra and generalized entropy

- Formula for the generalized entropy and the quantum extremal surface (QES)
- Microcanonical version of TFD and von Neumann algebra at large N (skipped)
- Entropy for the semiclassical states and the genralzed second law
- Effect of perturbations of a black hole (skipped)

8. Application to physics in de Sitter space

- Questions and brief summary
- Algebra of the static patch and the type $\mathbf{II_1}$ factor
- Renormalized entropy for $\mathbf{II_1}$
- Thermal nature of the de Sitter
- Algebra of observables for an observer in the static patch
- Bulk formula for the semiclassical entropy
- Comparison with the case of the black hole

9. What is the essence of “quantum gravity” ?

- Conspicuous characteristics of quantum gravity
- Split property and notion of subregions
- Important problems for investigating the essence of quantum gravity in the framework of von Neumann algebra

Due to lack of time:

Only the essence will be discussed.

♦ Details will not be elaborated.

♦ Proofs will not be given.

♦ Where appropriate, pedagogy is given priority over rigor.

For more detailed descriptions, we refer to the basic references listed below (and further references cited therein.)

Basic references

1. E. Witten, "Notes on some entanglement properties of quantum field theory, RMP **90** (2018) 045003, arXiv:1803.04993
2. E. Witten, "Why does quantum field theory in curved spacetime make sense ? And what happens to the algebra of observables in the thermodynamic limit ?", arXiv:2112.11614
3. S. Leutheusser and H. Liu, "Emergent times in holographic duality", arXiv: 2112.12156
4. E. Witten, "Gravity and the crossed product", arXiv: 2112.12828 (28 pages)
5. V. Chandrasekharan, R. Longo, G. Penington and E. Witten, "An algebra of observables for de Sitter space", arXiv: 2206.10780

6. V. Chandrasekharan, G. Penington and E. Witten, “ Large $\boldsymbol{N}$ algebra and generalized entropy, arXiv: 2209.10454 (70 pages)

7. A. Strohmaier and E. Witten, “ Analytic states in quantum field theory on curved spacetimes”, arXiv: 2302.02709

8. E. Witten, “Algebras, regions and observers”, arXiv: 2303.02837

9. A. Strohmaier and E. Witten, “The timelike tube theorem in curved spacetime”, arXiv:2303.16380

10. E. Witten, “Background independence of algebra in quantum gravity”, arXiv:2308.03663
(Physicist version of the mathematically rigorous 1st paper with Strohmaier.)

Books

11. R. Haag, "Local Quantum Physics", (Springer-Verlag, 1992)
12. H. Araki, "Mathematical Theory of Quantum Fields", Oxford Univ. Press, 1999.
13. M. Takesaki, "Theory of Operator Algebras I, II, III", (Springer-Verlag, 1979) (especially volume II)
14. A. Connes, "Noncommutative Geometry", (Academic Press, 1994) (especially chapter V)

1 Introduction

- Various frameworks for attempting to understand "quantum gravity"

[1] "**QUANTIZATION**" of a **SPECIFIC LOCAL ACTION**

- ♦ (Semiclassical) **quantization of GR** and supergravity
- ♦ Quantization of GR using Ashteker variables
  $\Rightarrow$ **Loop Quantum Gravity**
- ♦ Idea of **asymptotic safety** by causal dynamical triangulation and renormalization group

[2] **EMERGENT GRAVITY** **together with a host of other fields**: **SUPERSTRING THEORY** (worldsheet, SFT)

[3] Definition of quantum gravity by **NON-GRAVITATIONAL QFT** In particular, **AdS/CFT CORRESPONDENCE**

[4] Framework of **ALGEBRAIC QFT**, based on the theory of **VON NEUMANN ALGEBRAS**. (Quantized from the outset) **NO SPECIFIC ACTION, except for some constraints.**

[4'] **Axiomatic path integral approach** for **UV complete theory** to define quantum gravity. (Marolf et al[1]). **No specific action assumed**.

[1] arXiv: 2310.02189, 2402.09691

♦ We will focus on [4] as a universal framework and see how much we can capture the essence of quantum gravity.

2 Minimum basics of von Neumann algebras

2.1 Fundamental notions and definitions needed to discuss von Neumann algebras

♦ $\boxed{\text{C}^*\text{-algebra}}$ = a complex algebra $\mathcal{A}$ of continuous linear operators acting on a complex **Hilbert space** $\mathcal{H}$, with two additional properties: (**C** stands for "closed".)

(1) $\mathcal{A}$ is closed under the $*$ involution ($\sim$ taking the adjoint).

(2) $\mathcal{A}$ is a topologically closed set in the **strong topology**.

- **strong convergence**: $\lim_{n\to\infty} ||T_n x - Tx|| = 0)$
- **weak convergence**: $\lim_{n\to\infty}(x, T_n y) = (x, Ty)$

♦ $\boxed{\textbf{Commutant } \mathcal{A}'}$ of an algebra $\mathcal{A}$

- $\mathcal{A}' \equiv \{\{\mathbf{a}'\} \big| \; [\mathbf{a}', \mathbf{a}] = 0, \; ^{\forall}\mathbf{a} \in \mathcal{A}\}$

This notion, although looking rather trivial, **plays crucial roles in many situations.**

□ **Two ways to define a von Neumann algebra $\mathcal{A}$**:

[1] Most common definition:

It is a **weakly closed** $*$-algebra of **bounded operators** on a Hilbert space **containing the identity**.

∴ von Neumann algebra is a special type of C^*-algebra.

Bounded operator (Product is also a bounded operator)

T = a linear operator acting on $\mathcal{H}$

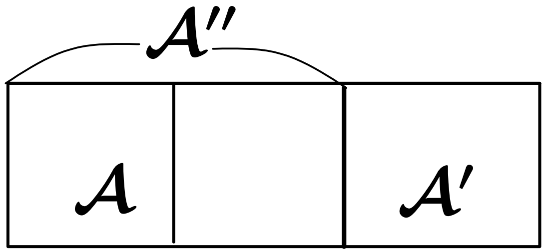
Then **if $^\exists$ real non-negative number λ** such that

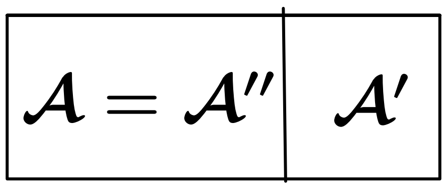
$||T\Psi|| \leq \lambda||\Psi||$ holds for any $\Psi \in \mathcal{H}$,

then T is said to be a bounded operator on $\mathcal{H}$.

$[2]$ It is a subalgebra of the bounded operators closed under $*$-involution and **equal to its double commutant, $i.e.$ $\mathcal{A} = \mathcal{A}''$**.

In general, $\mathcal{A} \subset \mathcal{A}''$. But for von Neumann algebra $\mathcal{A} = \mathcal{A}''$





(This characterization is powerful in proving various theorems.)

• The fundamental **von Neumann's bicommutant theorem** (1930) states that the definitions **[1]** and **[2]** are equivalent.

$\square$ **Some more notions and definitions**:

state: A state ω is **a linear functional on the algebra $\mathcal{A}$**

$\boxed{\omega : \mathbf{a} \in \mathcal{A} \longrightarrow \omega(\mathbf{a}) \in \mathbb{C}}$ with the following properties

(1) linearity

(2) positivity: $\omega(\mathbf{a}^*\mathbf{a}) \geq 0\,, \quad {}^\forall\mathbf{a} \in \mathcal{A}$

(3) normalization $||\omega|| = \sup\{\omega(\mathbf{a}) \;:\; ||\mathbf{a}|| = 1, \mathbf{a} \in \mathcal{A}\} = 1$

positive operator:

- $\mathbf{a} \in \mathcal{A}$ is a **positive operator** $\Leftrightarrow \langle \mathbf{a}v, v\rangle \geq 0, {}^\forall v \in \mathcal{H}$

This is equivalent to $\mathrm{sp}\,\mathbf{a} \subseteq [0, \infty)$. ($\mathrm{sp}$ stands for spectrum)

weight:

Let $\mathcal{A}_+ = \{$ positive operators $\}$
A **weight ϕ on $\mathcal{A}$** is a mapping $\boxed{\phi: \mathcal{A}_+ \longrightarrow [0, \infty]}$ such that $\phi(\lambda \mathbf{a} + \mathbf{b}) = \lambda\phi(\mathbf{a}) + \phi(\mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathcal{A}_+$ and $\lambda \in [0, \infty)$.

- A weight ϕ is said to be

(i) faithful if $\mathbf{a} \in \mathcal{A}_+$ and $\mathbf{a} \neq 0$ imply $\phi(\mathbf{a}) > 0$.
(ii) normal if $\phi(\mathbf{a}) = \sup \phi(\mathbf{a}_i)$ whenever $\mathbf{a}$ is the strong limit of a monotonically increasing sequence $\{\mathbf{a}_i\}$ in $\mathcal{A}_+$.
(iii) a **trace** if $\phi(\mathbf{a}^*\mathbf{a}) = \phi(\mathbf{a}\mathbf{a}^*)$ for all $\mathbf{a} \in \mathcal{A}$.

trace (a more general definition)

trace = linear map $\mathcal{A} :\longrightarrow \mathbb{C}$ satisfiying $\mathbf{Tr}(\mathbf{ab}) = \mathbf{Tr}(\mathbf{ba})$ for all $\mathbf{a}, \mathbf{b}$.

trace class $= \{\mathbf{a} \in \mathcal{A} | \mathrm{Tr}(\mathbf{a}) =$ **finite**$\}$

Cyclic and separating vectors for an algebra $\mathcal{A}$

The notion of cyclic and separating vector is extremely important and ubiquitous in the theory of von Neumann algebras.

- $\Psi \in \mathcal{H}$ is **cyclic for $\mathcal{A}$** $\Longleftrightarrow \{\mathbf{a}\Psi\}, (\mathbf{a} \in \mathcal{A})$ is **dense** in $\mathcal{H}$

$\Leftrightarrow$ **Any element of $\mathcal{H}$ is generated from Ψ by the action of $\mathcal{A}$ with arbitrary precision**

- $\Psi \in \mathcal{H}$ is **separating for $\mathcal{A}$** $\Longleftrightarrow$ $\mathbf{a}\Psi = 0$ implies $\mathbf{a} = 0$ for $\mathbf{a} \in \mathcal{A}$.

$\Leftrightarrow$ **No operator, except 0, annihilates Ψ**

♦ These notions are intimately related to the "**Reeh-Schlieder theorem**" and **entanglement**, to be described later.

2.2 Classification of von Neumann algebra factors and their characteristics

Hereafter, we will focus on von Neumann algebras

(1) which are **hyperfinite** $\sim$ constructable as **a limit of matrix algebra** and

(2) acting on a **separable Hilbert space**, *i.e.* with **a countable basis**

□ Concept of a "factor" and its importance:

$\boxed{\textbf{factor}}$ = a von Neumann algebra whose **center consists only of multiples of the identity operator**.

> **Theorem** (von Neumann (1949)): **Classification** of von Neumann algebra on a separable Hilbert space is reduced to that of **isomorphism classes of factors**.

□ Classification of hyperfinite von Neumann factors[2]:

$\boxed{\textbf{Type I}}$ Relevant to **ordinary QM**

- ♦ The algebra $\mathcal{A}$ consists of **all the bounded operators** acting on a Hilbert space $\mathcal{H}$.

[2] Initiated by E.J. Murray and J. von Neumann (1936) and completed by von Neumann (1940).

♦ Type I_n for $\dim \mathcal{H} = n$ =finite. $\exists$ trace for all the elements. **IRREP is unique**.

♦ Type I_∞ for $\dim \mathcal{H} =$ countably ∞. $\exists$ trace only for **trace-class** elements. **IRREP is highly non-unique.**

Type II Relevant to **quantum gravity**

♦ There are two hyperfinite type II factors

– $\boxed{\textbf{Type II}_1}$: It has **a trace for all the elements** of $\mathcal{A}$. **It has no IRREP.**

– $\boxed{\textbf{Type II}_\infty}$: It has **a trace only for traceclass elements** of $\mathcal{A}$. It has only **one IRREP**.

$\boxed{\text{Type III}}$ **Relevant to QFT. Most difficult to handle:**

Needed the works of Tomita, Takesaki, Connes, Araki and others to understand.

- **Its representations are all isomorphic.**
- $\boxed{\text{\textbf{It has no trace} } \Rightarrow \text{\textbf{ No (reduced) density matrix, no entropy}}}$
- There are following **classes of hyperfinite type III factors**
  - $\boxed{\text{Type III}_1}$: Relevant to **two typical systems**

    (1) **QFT defined in some local region**.

    (2) **Thermodynamic limit of a generic many body system**

$-$ $\boxed{\textbf{Type III}_\lambda\ (0 \le \lambda < 1)}$: More general Type **III** factor[3]

2.3 Concrete examples of various types of von Neumann algebras

2.3.1 Type I

□ **System of spins (qubits)**:

A qubit $\{|\uparrow\rangle, |\downarrow\rangle\}$

- **Algebra of operators for n-qubit system:**

$$\sigma_{a,k}\sigma_{b,k} = \delta_{ab} + i\epsilon_{abc}\sigma_{c,k} \tag{2.1}$$

$$[\sigma_{a,k}, \sigma_{b,k'}] = 0\,, \qquad k \neq k'\,, \qquad k = 1, 2, \ldots, n \tag{2.2}$$

[3] Araki and Woods (1968) clarified many important relations among Type **III** factors, using further generalization of $\mathbf{III}_\lambda$ factor

- **If n is finite,** $^{\exists}$ unique IRREP of dimension $\mathbf{2^n}$.

□ **Case of the system of countably infinite set of qubits**:

$\mathcal{H}^*$ = **full Hilbert space**

This has **uncountable dimensions**

(since the set of all possible configurations are uncountable)

- Let $\mathcal{H} \equiv n \longrightarrow \infty$ limit of n-dimensional Hilbert space, which

is of **countably infinite** dimensions=**separable Hilbert space**

- **Such $\mathcal{H}$ is much smaller than $\mathcal{H}^*$**

- **Choice of $\mathcal{H}$ out of $\mathcal{H}^*$ depends on our interest in**

  (1) what class of observables ?

  (2) what class of states ?

Take $\mathcal{A}_0 = \{$ **finite** linear combinations of products

$$\boldsymbol{\sigma}_{a_1,k_1}\boldsymbol{\sigma}_{a_2,k_2}\cdots\boldsymbol{\sigma}_{a_r,k_r}\}$$

Need some **completion** to get the algebra $\mathcal{A}$ we want.

Completion depends on what states to care about

$\Leftrightarrow$ what measurements to consider

Example of states of interest

$(\star 1)$ Suppose we are interested in states where **almost all the spins are up**:

$$\cdots\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\cdots\uparrow\downarrow\uparrow\uparrow\uparrow\cdots\in\mathcal{H}^*$$

- We want to construct a suitable separable Hilbert space $\mathcal{H}_\uparrow \subset \mathcal{H}^*$:

$\mathcal{H}_\uparrow$ should at least contain the state with all the spins up.

$$\Omega_\uparrow = \otimes_n |\uparrow\rangle_n \quad n \text{ runs over infinite range of integers} \tag{2.3}$$

Also, $\mathcal{H}_\uparrow \ni \mathcal{A}_0 \Omega_\uparrow = \mathcal{H}_{\uparrow,0} =$ **pre-Hilbert space**.

Allow $\mathcal{A}_0$ **to act on a finite set of qubits in** $\Omega_\uparrow$

$\Rightarrow$ We get all the states of the form $(\star 1)$.

- We then need to **complete** $\mathcal{A}_0$ by adding **suitable** convergent limits of sequence of operators in $\mathcal{A}_0$.

Hint for the suitable limit: Any given experiment measures **finitely many matrix elements** to finite precision

$$\Downarrow$$

weak limit: $a_1, a_2, \ldots, \in \mathcal{A}_0$ converges weakly if $\lim_{n\to\infty} \langle \psi | a_n | \chi \rangle$ exists for all $\psi, \chi \in \mathcal{H}_\uparrow$.

(In this sense one writes $\lim_{n\to\infty} a_n = a$)

Completion using this limit

$\Rightarrow \mathcal{A}_{\uparrow}$ = algebra of **all** bounded operators on $\mathcal{H}_{\uparrow}$

= **von Neumann algebra of type** I

$(\star 2)$ Similarly, we may be interested in states where **almost all the spins are down**

$$\cdots \downarrow\downarrow\downarrow\uparrow\downarrow\downarrow \cdots \downarrow\uparrow\downarrow\downarrow\downarrow \cdots \in \mathcal{H}^*$$

Start from

$$\Omega_{\downarrow} = \otimes_n |\downarrow\rangle_n \quad n \text{ runs over infinite integers} \tag{2.4}$$

Go through the same procedure $\Rightarrow \{\mathcal{A}_{\downarrow}, \mathcal{H}_{\downarrow}\}$

• $\mathcal{A}_\uparrow \neq \mathcal{A}_\downarrow$ **since** $\mathcal{H}_\uparrow \neq \mathcal{H}_\downarrow$ so that the weak limits are defined differently.

Thus, for ∞ degrees of freedom, IRREP is not unique.

2.3.2 Conditions for the irreducibility of the representation and the concept of a "pure state"

□ **An example where the Hilbert space $\mathcal{H}$ built on a vector $\Psi \in \mathcal{H}^*$ is reducible**:

Consider a state of the form

$$\Psi = \frac{1}{\sqrt{2}}\Omega_\uparrow + \frac{1}{\sqrt{2}}\Omega_\downarrow \tag{2.5}$$

By acting arbitrary elements of $\boldsymbol{\mathcal{A}}$ on $\boldsymbol{\Psi}$, we get the sum of Hilbert spaces built on $\boldsymbol{\Omega}_\uparrow$ and $\boldsymbol{\Omega}_\downarrow$. So

$$\boldsymbol{\mathcal{H}}_{\boldsymbol{\Psi}} = \boldsymbol{\mathcal{H}}_\uparrow \oplus \boldsymbol{\mathcal{H}}_\downarrow \tag{2.6}$$

Then $\boldsymbol{\mathcal{A}}$ **does not act irreducibly** on $\boldsymbol{\mathcal{H}}_{\boldsymbol{\Psi}}$ **since (2.6) is an** $\boldsymbol{\mathcal{A}}$**-invariant decomposition of** $\boldsymbol{\mathcal{H}}_{\boldsymbol{\Psi}}$.

Thus, the choice of $\boldsymbol{\Psi}$ in (2.5) is an example which does not give an IRREP of the operator algebra.

□ **Characterization of the base vector Ψ which gives an IRREP**:

• The notion of $\boxed{\textbf{pure state}}$, as defined below, completely characterizes the base vector that gives an IRREP.

Definition of a pure state

A state $\boldsymbol{F} : \boldsymbol{\mathcal{A}} \longrightarrow \mathbb{C}$ on the algebra is called **"pure"** if it is **not possible** to write it in terms of two states $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ as

$$(\star) \quad \boldsymbol{F}(\mathbf{a}) = \boldsymbol{p_1 F_1}(\mathbf{a}) + \boldsymbol{p_2 F_2}(\mathbf{a})\,, \qquad \boldsymbol{p_1, p_2 > 0}$$
$$\boldsymbol{p_1 + p_2 = 1} \tag{2.7}$$

(• This condition is indispensable for **Type II and III** algebras.)

Irreducibility theorem:

Let $F_\Psi(\mathbf{a}) \equiv \langle\Psi|\mathbf{a}|\Psi\rangle$ be a state associated with the vector Ψ. Then $\mathcal{H}_\Psi$ is an IRREP for the algebra $\mathcal{A}$ **if and only if $F_\Psi(\mathrm{a})$ is a pure state.**

An example of the use of this theorem:

As for Ψ given in (2.5), *i.e.* $\Psi = \frac{1}{\sqrt{2}}\Omega_\uparrow + \frac{1}{\sqrt{2}}\Omega_\downarrow$, we have

$$F_\Psi(\mathbf{a}) = \langle\Psi|\mathbf{a}|\Psi\rangle = \frac{1}{2}\underbrace{\langle\Omega_\uparrow|\mathbf{a}|\Omega_\uparrow\rangle}_{F_1(\mathbf{a})} + \frac{1}{2}\underbrace{\langle\Omega_\downarrow|\mathbf{a}|\Omega_\downarrow\rangle}_{F_2(\mathbf{a})} \qquad (2.8)$$

Thus it is **not a pure state** and hence $\mathcal{H}_\Psi$ does not provide an IRREP.

2.3.3 Type II

■ **Construction of hyperfinite Type II_1 factor**

□ **Relevant Hilbert space**:

• It can be constructed from **a countably infinite set of maximally entangled** qubits.

(1) Case of one qubit system

• **It is useful to represent states of one qubit system by a 2×2 matrix.**

$\mathcal{V}$ = a vector space consisting of 2×2 **complex matrices**, with the inner product $\langle v, w \rangle = \mathrm{Tr}\, v^\dagger w$.

M_2 and M_2' = two copies of A_2 $\equiv$ **the algebra of 2×2 complex matrices** **acting on a vector $V \in \mathcal{V}$ as $M_2 V M_2'$**

$\mathcal{V}$ can be viewed as a space of tensor product $W \otimes W'$

W = a space of two-component **column vectors** (**qubits**) acted on by M_2

W' = a space of two-component **row vectors** (**qubits**) acted on by M_2'.

Let I_2 be the 2×2 identity matrix.

- In this representation, **a normalized maximally entangled vector $V \in \mathcal{V}$ is given by the "normalized" unit matrix $I_2' \equiv \frac{1}{\sqrt{2}} I_2$.**

$\mathbf{2 \times 2}$ matrix can be expressed in terms of the $\mathbf{4}$ **basis** of the tensor product states with the following correspondence:

$$v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \longleftrightarrow \sum_{i,j=1,2} v_{\bar{i}j} |\bar{i}\rangle_l \otimes |j\rangle_r$$

For the qubit system,

$$|1\rangle = |\uparrow\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = |\downarrow\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The **basic quantity** in this notation is

$$v = I_2' = \frac{1}{\sqrt{2}} I_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longleftrightarrow V = \frac{1}{\sqrt{2}} (|1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle) = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$$

(2) Use infinite set of qubits to construct II_1

Generalize the construction above:

$V^{[k]}$ = a space of 2×2 matrices acted on as $M_2^{[k]} V {M'}_2^{[k]}$ for $k \geq 1$.

We wish to consider the space of infinite tensor product

$$\mathcal{V} = \{V^{[1]} \otimes V^{[2]} \otimes \cdots \otimes V^{[k]} \otimes \cdots\}.$$

But, the dimension of $\mathcal{V}$ is **uncountably infinite** ($i.e.$ cannot be ordered.).

To extract **countably infinite subspace**, consider

$$\mathcal{H}_0 = \{v_1 \otimes v_2 \otimes \cdots \otimes v_k \otimes \cdots\} \in V^{[1]} \otimes V^{[2]} \otimes \cdots \otimes V^{[k]} \otimes \cdots$$

such that all but finitely many of the v_k's are I'_2.

Since the number of non-trivial v_k's ($\neq I'_2$) is **finite**, this gives a nice **practically finite dimensional** subspace of $\mathcal{V}$.

To make it into a Hilbert space $\mathcal{H}$, we need to

(i) introduce an **inner product** and

(ii) make a **completion**

(i) Inner product

Let $$v \equiv v_1 \otimes v_2 \otimes \cdots \otimes \cdots \in \mathcal{H}_0$$
$$w \equiv w_1 \otimes w_2 \otimes \cdots \otimes \cdots \in \mathcal{H}_0$$

Then $^\exists n$ such that $v_k = w_k = I'_2$ for all $k > n$.

Truncate v and w at such n as

$$v_{(n)} = v_1 \otimes v_2 \otimes \cdots \otimes v_n, \qquad w_{(n)} = w_1 \otimes w_2 \otimes \cdots \otimes w_n$$

and define the inner product as[4]

$$\langle v, w \rangle \equiv \mathrm{Tr}(v_{(n)}^{\dagger} w_{(n)}) \tag{2.9}$$

(ii) Completion to $\mathcal{H}$: restricted tensor product of $V^{[k]}$

For $v_1 \otimes v_2 \otimes \cdots \otimes \cdots$ to be in $\mathcal{H}$, v_n must tend rapidly to I'_2 as $n \to \infty$.

[4] This is independent of the choice of n since (infinitely many) common factors of I'_2 give $\mathbf{1}$.

□ Construction of the algebra $\mathcal{A}$:

- The procedure is very similar to the one for the Hilbert space.

(1) First define the algebra $\mathcal{A}_0$.

Wish to define the algebra $\boldsymbol{\mathcal{A}}$ as an infinite tensor product

$$\boldsymbol{M_2^{[1]} \otimes M_2^{[2]} \otimes \cdots \otimes M_2^{[n]} \otimes \cdots}$$

Define an algebra $\mathcal{A}_0 = \{\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \cdots \otimes \mathbf{a}_n \otimes \cdots\}$

such that all but finitely many of the $\mathbf{a}_i$'s are equal to I_2

This can act on $\boldsymbol{\mathcal{H}}$ but is not yet closed.

(2) Close the algebra by adding limits

A sequence $\mathbf{a}_{(k)} \in \mathcal{A}_0$ is defined to converge if $\lim_{n\to\infty} \mathbf{a}_{(n)}\chi$ exists for all $\chi \in \mathcal{H}$.

$\Rightarrow$ Define an operator $\mathbf{a} : \mathcal{H} \longrightarrow \mathcal{H}$ by $\mathbf{a}\chi = \lim_{n\to\infty} \mathbf{a}_{(n)}\chi$.
$\mathcal{A}$ is defined to include all such limits.

□ **A natural linear function on the algebra $\mathcal{A}$ and the trace :**

Define

$$\boxed{F(\mathbf{a}) \equiv \langle\Psi|\mathbf{a}|\Psi\rangle, \text{ where } |\Psi\rangle \text{ is separating for } \mathcal{A}.}$$

Then, any $\mathbf{a} \neq 0 \in \mathcal{A}$ satisfies $\mathbf{a}\Psi \neq 0$ and hence $F(\mathbf{a}^\dagger\mathbf{a}) > 0$.

♦ ***F* has the defining property of a trace:**
$F(\mathbf{ab}) = F(\mathbf{ba})$: (proof omitted)

♦ We may write $\boxed{F(\mathbf{a}) \text{ as } \mathrm{Tr}\, \mathbf{a}}$.

Remarks on the trace

- In the case of a **Type $\mathbf{I}_\infty$** algebra, one can define a trace only for the **traceclass** elements. (The trace of the identity element is infinite.)
- **By contrast, a hyperfinite Type II_1 has a trace that is defined on the whole algebra, and we can normalize it as $\mathrm{Tr}\,1 = 1$.**
- **Fact:** **Type II_1 algebra $\mathcal{A}$ has no IRREP**

  (This is due to its ∞-dim nature. Proof is interesting but omitted)

2.3.4 Algebra of Type $\mathbf{II}_\infty$

- Type $\mathbf{II}_\infty$ factor $= \mathbf{II}_1 \otimes \mathcal{B}(\mathcal{H})$,

where $\dim \mathcal{H} = \infty$ and $\mathcal{B}(\mathcal{H}) =$ bounded operators on $\mathcal{H}$.

It is very similar to $\mathbf{II}_1$

- A conspicuous difference:

— Trace for $\mathbf{II}_1$ is **finite for all operators** including the identity.

—Trace for $\mathbf{II}_\infty$ is finite only for (a dense set of) operators called the **trace-class operators.**

2.3.5 Type III

The type $\mathbf{III}$ factors can be constructed **just like the case of $\mathbf{II_1}$**, but using the qubits **with more general entanglement**.

For $\mathbf{0 < \lambda < 1}$, consider a matrix

$$\boldsymbol{K_{2,\lambda}} = \frac{\mathbf{1}}{(\mathbf{1} + \boldsymbol{\lambda})^{\mathbf{1/2}}} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda}^{\mathbf{1/2}} \end{pmatrix}. \tag{2.10}$$

This matrix describes a pair of qubits that are entangled in a more general way with the parameter $\boldsymbol{\lambda}$.

$$\boldsymbol{K_{2,\lambda}} \Longleftrightarrow \frac{\mathbf{1}}{\sqrt{\mathbf{1} + \boldsymbol{\lambda}}} \Big(| \uparrow\uparrow \rangle + \sqrt{\boldsymbol{\lambda}} | \downarrow\downarrow \rangle \Big) \tag{2.11}$$

(1) In the procedure of constructing the Hilbert space $\mathcal{H}$ of type II, **replace $\boldsymbol{I'_2}$ everywhere by $\boldsymbol{K_{2,\lambda}}$.**

That is, consider the space $\mathcal{H}_{\lambda,0}$ spanned by vectors

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes \cdots \in V^{[1]} \otimes V^{[2]} \otimes \cdots \otimes V^{[n]} \otimes \cdots$$

such that all but finitely many of the v_n are equal to $K_{2,\lambda}$.

(2) Denote by $\mathcal{H}_\lambda$ the **Hilbert space closure of** $\mathcal{H}_{\lambda,0}$.

(3) Similarly, define the **algebra** $\mathcal{A}_\lambda$ in the following way:
Define $\mathcal{A}_0$ with I_2 for II_1 case replaced by $K_{2,\lambda}$, and **take its closure in the space of bounded operators acting on** $\mathcal{H}_\lambda$

$\Rightarrow$ **von Neumann algebra** $\mathcal{A}_\lambda$.

(4) The **commutant** $\mathcal{A}'_\lambda$ is defined similarly and is isomorphic to $\mathcal{A}_\lambda$.

(5) The vector $\Psi = K_{2,\lambda} \otimes K_{2,\lambda} \otimes \cdots \otimes K_{2,\lambda} \otimes \cdots$ can be shown to be **cyclic and separating for $\mathcal{A}_\lambda$ and for $\mathcal{A}'_\lambda$**.

(6) No trace exists:

For example, **the natural linear function defined by $F(\mathsf{a}) = \langle\Psi|\mathsf{a}|\Psi\rangle$ does not satisfy $F(\mathsf{ab}) = F(\mathsf{ba})$.**

- There is a **general proof for the non-existence of a trace**[5], **independent of such an example**.

[5]For example, Prop.8.5.4 in "Fundamentals of the Theory of Operator Algebras", Vol 2 (Kadison and Ringrose).

♦ **The entanglement entropy for $\mathcal{A}_\lambda$ (and $\mathcal{A}'_\lambda$) in the state Ψ is divergent**
because Ψ describes an infinite collection of qubit pairs each with the same entanglement.

This divergence is universal; any state in $\mathcal{H}_\lambda$ has the same leading divergence in the entanglement entropy.

♦ Just as for II_1, **the action of $\mathcal{A}_\lambda$ on $\mathcal{H}_\lambda$ is far from irreducible.**

But it can be shown that **all the reduced representations are isomorphic and in that sense there is only <u>one IRREP</u>**.

3 von Neumann algebra for QFT in space-time

3.1 Net of algebra

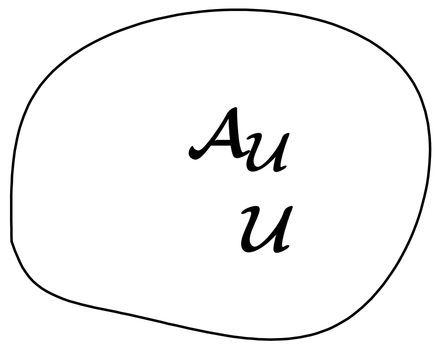
In applying the theory of von Neumann algebra to QFT in **space-time**, the basic notion is the **"net of algebra"**.

Let

$\phi(x)$ = local field operator in spacetime

$\mathcal{U}$ = an open region in spacetime

$\phi_f \equiv \int dx f(x)\phi(x)$ = field smeared with test function f with their support in $\mathcal{U}$.



To each $\mathcal{U}$, associate

$\mathcal{A}_{\mathcal{U}}$ = algebra generated by "all local operators" ϕ_f.

$\boxed{\textbf{Net of algebra} = \textbf{the correspondence } \mathcal{U} \longrightarrow \mathcal{A}_{\mathcal{U}}}$

♦ **What should we include in "all local operators" ?**

(1) Operators of the form $F(\phi_{f_1}, \phi_{f_2}, \ldots, \phi_{f_n})$ where
F = a **bounded function** of n complex variables
Bounded operators can be multiplied without trouble and naturally form an algebra.

(2) **Weak limits of sequence of such operators**

(3) **Hermitian ($*$) conjugate** of all such operators.

Such set of operators form a von Neumann algebra

3.2 Important property of von Neumann algebra in flat Minkowski spacetime

- **von Neumann algebra for a local region $\mathcal{U}$ in the case of QFT in the flat Minkowski spacetime is Type III_1**

- **This was first shown by Araki for a free scalar field**[6]

Now there are many refined arguments for this fact for many different types of regions $\mathcal{U}$. (Details omitted).

[6]H. Araki in "Type of von Neumann algebra associated with free field", PTP 32 (1964) 956. There he called it type $\mathbf{III}_\infty$.

4 Modular theory of von Neumann Algebra: (Tomita-Takesaki theory)

The so called **modular theory** for von Neumann algebra will be **extremely important in many respects**. Among them,

- It allows one to define and study the properties of the $\boxed{\textbf{(relative) quantum entropy.}}$
- **(Connes) spectrum of the modular operator** characterizes the **classification** of the von Neumann algebra factors.
- $\boxed{\textbf{Modular automorphism}}$
  $\Rightarrow$ concept of "**crossed product**"
  $\Rightarrow$ **structure for type III factor and its duality property**

$\Rightarrow$ **Physics of blackhole and quantum fields in de Sitter**

4.1 Tomita-Takesaki theory and modular operator

$\mathcal{A}$ =von Neumann algebra acting on a Hilbert space $\mathcal{H}$
$\mathcal{A}' =$ commutant of $\mathcal{A}$

Tomita operator S_Ψ

Let Ψ be a vector in $\mathcal{H}$ which is **cyclic and separating for $\mathcal{A}$**.
Define the **antilinear** operator S_Ψ, called **Tomita operator**[7]

$$\boxed{S_\Psi(\mathbf{a}|\Psi\rangle) = \mathbf{a}^\dagger|\Psi\rangle\,, \quad {}^\forall\mathbf{a} \in \mathcal{A} \quad (4.1)}$$

[7] M.Tomita (1967), as "deciphered " and systematized by M. Takesaki.

Need of the separating and cyclic properties for Ψ:

- Ψ must be **separating** for $\mathcal{A}$ to avoid the possible inconsistency where $\mathbf{a}|\Psi\rangle = 0$ while $\mathbf{a}^\dagger|\Psi\rangle \neq 0$.
- Ψ must be **cyclic** for $\mathcal{A}$ in order to define the action of S_Ψ on a dense subspace of $\mathcal{H} = \{\mathbf{a}|\Psi\rangle\}$.

Important properties of S_Ψ

(1) From the definition, we have, $\forall \mathbf{a}$

$$S_\Psi S_\Psi(\mathbf{a}|\Psi\rangle) = S_\Psi(\mathbf{a}^\dagger|\Psi\rangle) = \mathbf{a}|\Psi\rangle$$

$$\therefore \qquad \boxed{S_\Psi S_\Psi = 1} \quad \Rightarrow (S_\Psi)^{-1} = S_\Psi \tag{4.2}$$

(2) Taking $\mathbf{a}=1$, we see that $|\Psi\rangle$ is invariant under S_Ψ:

$$\boxed{\boldsymbol{S_\Psi|\Psi\rangle = |\Psi\rangle} \qquad (4.3)}$$

(3) **Tomita operator S'_Ψ for the commutant $\mathcal{A}'$** is given by

$$\boxed{\boldsymbol{S'_\Psi = S^\dagger_\Psi} \qquad (4.4)}$$

- The same properties hold for S'_Ψ since one can easily show

$$\boxed{\boldsymbol{S'_\Psi(\mathsf{a}'|\Psi\rangle) = \mathsf{a}'^\dagger|\Psi\rangle\,, \quad \mathsf{a}' \in \mathcal{A}'} \qquad (4.5)}$$

□ **Polar decomposition of the Tomita operator, the modular operator, and the modular conjugation :**

♦ An **invertible** matrix $\boldsymbol{A}$ can be uniquely polar decomposed as

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{P}\,, \quad \boldsymbol{U} = \text{unitary matrix}\,, \; \boldsymbol{P} = \sqrt{\boldsymbol{A}^{\dagger}\boldsymbol{A}} \tag{4.6}$$

$$\det \boldsymbol{A} = \det \boldsymbol{P} \det \boldsymbol{U} = \boldsymbol{r}e^{i\boldsymbol{\theta}} \tag{4.7}$$

♦ Likewise, similar decomposition holds for the Tomita operator[8]

$$\boxed{\begin{aligned} S_{\Psi} &= J_{\Psi}\Delta_{\Psi}^{1/2} = \Delta_{\Psi}^{-1/2}J_{\Psi} && (4.8)\\ \Delta_{\Psi} &= \text{modular operator} = S_{\Psi}^{\dagger}S_{\Psi} && (4.9)\\ J_{\Psi} &= \text{modular conjugation} && (4.10) \end{aligned}}$$

[8] Δ_{Ψ} had been known. The discovery that a more premitive operator S_{Ψ} contains Δ_{Ψ} is the Tomita's most important result.

Some important properties of Δ_Ψ and J_Ψ

- Since $S_\Psi|\Psi\rangle = S_\Psi^\dagger|\Psi\rangle = |\Psi\rangle$, we have

$$\boxed{\Delta_\Psi|\Psi\rangle = |\Psi\rangle \qquad (4.11)}$$

Thus, for any function f, $f(\Delta_\Psi)|\Psi\rangle = f(1)|\Psi\rangle$.

- The above property can be reexpressed as

$$\boxed{\Delta_\Psi = e^{-H_\Psi}, \qquad H_\Psi|\Psi\rangle = 0 \qquad (4.12)}$$

H_Ψ = modular Hamiltonian and $|\Psi\rangle$ is like a vacuum.

- Other useful relations that can be easily shown are

$$\boxed{\begin{aligned} J_\Psi \Delta_\Psi^{1/2} J_\Psi &= \Delta_\Psi^{-1/2} \qquad (4.13) \\ J_\Psi^2 &= 1 \qquad (4.14) \end{aligned}}$$

- Squaring (4.13) and using $J_\Psi^2 = 1$, we get
$J_\Psi \Delta_\Psi J_\Psi = \Delta_\Psi^{-1}$.

From the antiunitarity of J_Ψ, we then get

$\boxed{J_\Psi f(\Delta_\Psi) J_\Psi = \overline{f}(\Delta_\Psi^{-1})}$ for any function $f(x)$.

- In particular, we can choose $f(x) = x^{is}$ for real s. Then,

$$\boxed{J_\Psi \Delta_\Psi^{is} J_\Psi = \Delta_\Psi^{is}\,, \quad s \in \mathbb{R} \qquad (4.15)}$$

The operator Δ_Ψ^{is} will generate an automorphism of $\mathcal{A}$ and $\mathcal{A}'$, called modular automorphism,

which will be **extremely important**.

- The modular operators for the commutant $\mathcal{A}'$ are related to those for $\mathcal{A}$ as

$$\boxed{J'_\Psi = J_\Psi\,, \qquad \Delta'_\Psi = \Delta_\Psi^{-1} \quad (4.16)}$$

4.2 Relative modular operator

H. Araki (1975)[9] introduced an **important generalization** of the Tomita operator **referring to two vectors Ψ and Φ**, called **relative Tomita operator $S_{\Psi|\Phi}$** for the algebra $\mathcal{A}$ [10].
It is defined by the action

$$\boxed{S_{\Psi|\Phi}\mathbf{a}|\Psi\rangle = \mathbf{a}^{\dagger}|\Phi\rangle \qquad (4.17)}$$

where we assume $|\Psi\rangle$ and $|\Phi\rangle$ are normalized to unity.

[9] H. Araki, “Inequalities in von Neumann algebras”, RCP25 **22** (75)1-25,section 2.

[10] In general $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ may belong to different Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ on which the same algebra $\mathcal{A}$ acts.

- For $S_{\Psi|\Phi}$ to make sense as a densely defined operator,

  — Ψ must be cyclic and separating for $\mathcal{A}$.

  — Φ can be any state.

- <u>If Φ is cyclic and separating for $\mathcal{A}$</u>, one can switch the roles of

Ψ and Φ and define

$$\boxed{S_{\Phi|\Psi}\mathbf{a}|\Phi\rangle \equiv \mathbf{a}^{\dagger}|\Psi\rangle \qquad (4.18)}$$

- <u>If both Ψ and Φ are cyclic and separating</u>,

then $S_{\Phi|\Psi}S_{\Psi|\Phi} = 1 \quad \Rightarrow S_{\Psi|\Phi}$ is invertible.

Relative modular operator $\Delta_{\Psi|\Phi}$

$$\boxed{\Delta_{\Psi|\Phi} \equiv S^{\dagger}_{\Psi|\Phi} S_{\Psi|\Phi} \qquad (4.19)}$$

- Clearly, $\Delta_{\Psi|\Phi}$ is positive semidefinite.

  It is positive definite iff $S_{\Psi|\Phi}$ is invertible.
- Case $\Phi = \Psi$: $\Delta_{\Psi|\Psi} = \Delta_{\Psi}$.

Polar decomposition

$$\boxed{S_{\Psi|\Phi} = J_{\Psi|\Phi} \Delta^{1/2}_{\Psi|\Phi} \qquad (4.20)}$$

- $J_{\Psi|\Phi}$ = **relative modular conjugation**

4.3 Forms of S_Ψ etc. for a finite dimensional quantum system

Most typical example:

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, **with** $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = n$.

Let $\{\psi_k = |k\rangle\}$ and $\{\psi'_k = |k\rangle'\}$ be the orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

Write the expansion of a general vector $\Psi \in \mathcal{H}$ as

$$\boxed{\Psi = \sum_{k=1}^{n} c_k |k,k\rangle = \sum_{k=1}^{n} c_k |k\rangle \otimes |k\rangle'}$$

From the explicit action of the Tomita operator on this state, one can easily obtain

$$\boxed{S_\Psi|j,i\rangle = \frac{c_j}{\overline{c}_i}|i,j\rangle. \qquad (4.21)}$$

$$\boxed{(\star 1) \quad \Delta_\Psi|j,i\rangle = \frac{|c_j|^2}{|c_i|^2}|j,i\rangle. \qquad (4.22)}$$

$$(\star 2) \quad J_\Psi\,|j,i\rangle = \sqrt{\frac{c_j c_i}{\overline{c}_j \overline{c}_i}}\;|i,j\rangle$$

$$= |i,j\rangle \quad \text{when } c_i \text{ are taken to be real} \qquad (4.23)$$

Explicit form for the **relative modular operator**

Introduce another vector $\Phi \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ as

$$\Phi = \sum_{\alpha=1}^{n} d_\alpha |\alpha, \alpha\rangle = \sum_{k=1}^{n} d_\alpha |\alpha\rangle \otimes |\alpha\rangle' \quad (4.24)$$

Then, from the definition of $S_{\Psi|\Phi}$, it is easy to find

$$(\star 3) \quad \Delta_{\Psi|\Phi} |\alpha, i\rangle = \frac{|d_\alpha|^2}{|c_i|^2} |\alpha, i\rangle. \quad (4.25)$$

□ **Expressions in terms of the density operators**:

Normalize $\mathbf{\Psi}$ and $\mathbf{\Phi}$ to be unit vectors.

The density operators (and their reductions) for $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are

$$\boldsymbol{\rho}_{12} = |\mathbf{\Psi}\rangle\langle\mathbf{\Psi}|\,, \quad \boldsymbol{\sigma}_{12} = |\mathbf{\Phi}\rangle\langle\mathbf{\Phi}| \tag{4.26}$$

$$\boldsymbol{\rho}_1 = \mathbf{Tr}_2\,\boldsymbol{\rho}_{12}, \qquad \boldsymbol{\rho}_2 = \mathbf{Tr}_1\,\boldsymbol{\rho}_{12} \tag{4.27}$$

$$\boldsymbol{\sigma}_1 = \mathbf{Tr}_2\,\boldsymbol{\sigma}_{12}, \qquad \boldsymbol{\sigma}_2 = \mathbf{Tr}_1\,\boldsymbol{\sigma}_{12} \tag{4.28}$$

Then by using the previous expressions in terms of the expansion coefficients, it is easy to find **a convenient and important expressions**

$$\boxed{\mathbf{\Delta}_{\mathbf{\Psi}} = \boldsymbol{\rho}_1 \otimes \boldsymbol{\rho}_2^{-1}. \quad \mathbf{\Delta}_{\mathbf{\Psi}|\mathbf{\Phi}} = \boldsymbol{\sigma}_1 \otimes \boldsymbol{\rho}_2^{-1}. \qquad (4.29)}$$

4.4 Relative entropy and its properties

Relative modular operator allows one to define and study **relative entropy**, from which **useful inequalities** are derived.

4.4.1 Relative entropy: general definition

The relative entropy $S_{\Psi|\Phi}(\mathcal{U})$ between two states $\mathbf{\Psi}$ and $\mathbf{\Phi}$ for measurements in the **region $\mathcal{U}$** is **defined** by [11]

$$\boxed{S_{\Psi|\Phi}(\mathcal{U}) \equiv -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle \quad (4.30)}$$

[11] H.Araki, Publ.RIMS.Kyoto.U **11** (1976) 809

• **This definition does not make use of a trace** so that **it can be defined for type III factor as well.**

♦ The most important properties of the relative entropy are

$$\boxed{\textbf{positivity}} \quad \text{and} \quad \boxed{\textbf{monotonicity}}\ .$$

4.4.2 Positivity property of $S_{\Psi|\Phi}(\mathcal{U})$

Theorem (proof omitted)

(1) $S_{\Psi|\Phi}(\mathcal{U}) \geq 0$
(2) Equality holds iff $\mathbf{\Phi} = \mathbf{a}'\mathbf{\Psi}$, where $\mathbf{a}'$ is a **unitary** element of the commutant algebra $\mathcal{A}'_{\mathcal{U}}$.

- **The built-in positivity of the relative entropy has many implications,** such as the **subadditivity of quantum entropy**.

4.4.3 Monotonicity of relative entropy

Meaning of monotonicity

In QFT, the algebra and the modular operator are associated with an open set $\mathcal{U}$, indicated as $\mathcal{A}_{\mathcal{U}}$ and $\Delta_{\Psi,\mathcal{U}}$.

Monotonicity refers to the behavior when $\mathcal{U}$ is replaced by a smaller open set $\boxed{\widetilde{\mathcal{U}} \subset \mathcal{U}}$.

■ **Monotonicity theorem for relative entropy:**

$$\boxed{S_{\Psi|\Phi}(\mathcal{U}) \geq S_{\Psi|\Phi}(\widetilde{\mathcal{U}})\,, \quad \text{for} \quad \mathcal{U} \supset \widetilde{\mathcal{U}} \qquad (4.31)}$$

- For QFT, the proof was given by Araki (1976)

4.4.4 Consequences of positivity and monotonicity of relative entropy

♦ **Positivity and monotonicity of relative entropy** lead to several **important consequences in quantum information theory**, which in particular play roles **in black hole physics**.

□ **Example: Case of type I and II, where Tr is defined** :

In this case, **Araki's definition reduces** to the **standard definition** of the von Neumann entropy

$$S(\rho) = -\mathrm{Tr}\rho \log \rho \tag{4.32}$$

Let

$AB =$ a bipartite system with a factorized Hilbert space

$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, with the density matrix ρ_{AB}.

$\rho_A = \mathrm{Tr}_B\, \rho_{AB}$ $\qquad\qquad$ $\rho_B = \mathrm{Tr}_A\, \rho_{AB}$.

$S_{AB} = S(\rho_{AB})$ $\qquad\qquad$ $S_A = S(\rho_A)$

etc..

Def. of mutual information $I(A, B)$ between A and B

$$\boxed{I(A, B) \equiv S_A + S_B - S_{AB}}$$

- **Positivity of the relative entropy**

  $\Longrightarrow$ **non-negativity of the mutual information**

$\Longrightarrow$ **subadditivity of quantum entropy**

$$I(A, B) \geq 0 \quad \text{for all } \rho_{AB} \tag{4.33}$$

$$i.e. \quad S_A + S_B \geq S_{AB} \tag{4.34}$$

Strong subadditivity of quantum entropy[12]

Let ABC = a **tripartite system** with the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and the density matrix ρ_{ABC}.
One can define various reduced density matrices such as
$\rho_{AB} = \mathrm{Tr}_C\, \rho_{ABC}$, etc.

Strong subadditivity $\Leftrightarrow$ mutual information is monotonic

in the sense

[12] E.H.Lieb and M.B.Ruskai, JMP **14** (1973) 1938.

$$\boxed{\boldsymbol{I(A,B) \leq I(A,BC)} \quad (4.35)}$$

Substituting the definition of $\boldsymbol{I(A,B)}$, this is equivalent to

$$\boxed{\boldsymbol{S_B + S_{ABC} \leq S_{AB} + S_{BC}} \qquad (4.36)}$$

This is a consequence of the **monotonicity** of the relative entropy.

(Proof is straightforward but is omitted.)

4.5 Modular operators for QFT in flat Minkowski space-time

$\Sigma \equiv$ the initial value surface $t = 0$

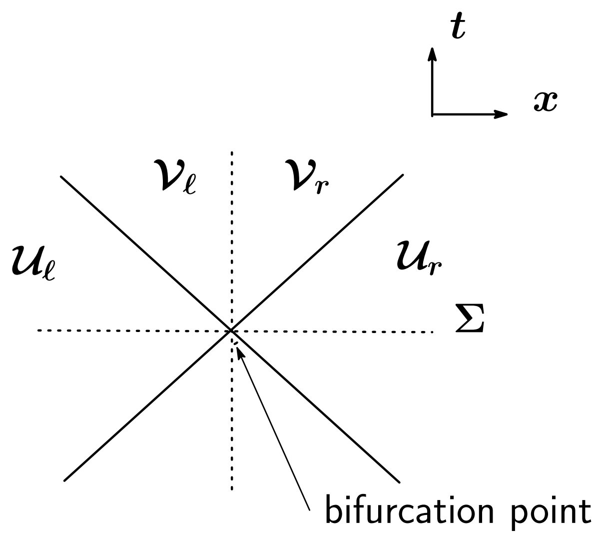
$\mathcal{V}_r$ = the open right half-space in Σ, defined by $x > 0$.

$\mathcal{U}_r$ = the right wedge, defined by $x > |t|$.

$\mathcal{H}_r$ = Hilbert space of DoF located at $x < 0$.

$\mathcal{A}_r$ = the algebras of observables in $\mathcal{U}_r$ acting on $\mathcal{H}_r$.

Similarly for $\mathcal{V}_\ell, \mathcal{U}_\ell, \mathcal{H}_\ell, \mathcal{A}_\ell$.



For rigorous treatment, the bifurcation point should be excluded and $\mathcal{H} \neq \mathcal{H}_r \otimes \mathcal{H}_\ell$.

Two approaches for determining J_Ψ and Δ_Ψ

(1) **Path integral approach** (intuitive but not quite rigorous)

Useful in studying the thermal nature of

— Unruh vacuum, blackhole physics, correlation functions in de Sitter space

(2) **Use of the analyticity of the correlation function**

Bisognano-Wichmann analysis[13] (**rigorous treatment**)

- Use of analyticity property $\Leftarrow$ **Reeh-Schlieder theorem**.

[13] J. Bisognano and E.H. Wichmann, "On the duality condition for quantum fields", JMP **17** (76) 303.

Although the results for J_Ψ and Δ_Ψ are important, we only give a very rough sketch.

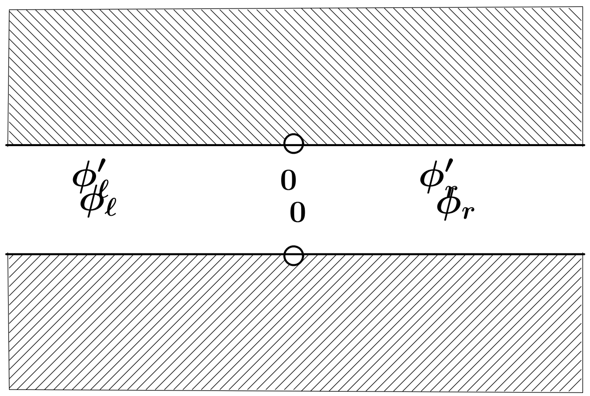
- First, forget rigor and assume the factorization $\mathcal{H} = \mathcal{H}_r \otimes \mathcal{H}_\ell$.
- Try to make use of the expression $\Delta_\Psi = \rho_r \otimes \rho_\ell^{-1}$ and first consider

$$\boxed{\rho_r(\phi'_r, \phi_r) = \int D\phi_\ell |\Omega(\phi_\ell, \phi'_r)\rangle\langle\Omega(\phi_\ell, \phi_r)|. \qquad (4.37)}$$

- This has the following simple path integral interpretation:

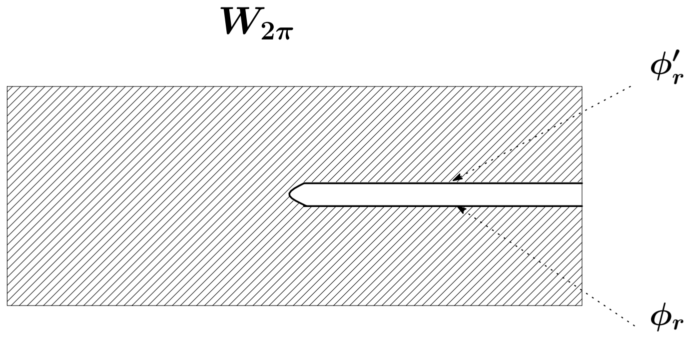
$|\Omega(\phi'_\ell, \phi'_r)\rangle =$

$\langle\Omega(\phi_\ell, \phi_r)| =$



Identify ϕ'_ℓ and ϕ_ℓ and integrate as above

$\Rightarrow$ glue over the portion $x < 0$



This gives the spacetime $W_{2\pi}$ $\equiv$ a copy of Euclidean space with a "cut" along the half-hyperplane $\tau = 0$, $x > 0$.

- 2π rotation in Euclidean space corresponds in Minkowski space to a real Lorentz boost by 2π by the generator

$$\boxed{K = K_r - K_\ell \quad (4.38)}$$

where $\boldsymbol{K_r}$ and $\boldsymbol{K_\ell}$ are partial Lorentz boost generators

$$\boldsymbol{K_r = \int_{t=0, x\geq 0} \mathrm{d}x\, \mathrm{d}\vec{y}\ xT_{00}}$$
$$\boldsymbol{K_\ell = - \int_{t=0, x\leq 0} \mathrm{d}x\, \mathrm{d}\vec{y}\ xT_{00}}$$

$\boldsymbol{K_r}$ and $\boldsymbol{K_\ell}$ separately are not well-defined but $\boldsymbol{K}$ is.

- With factorization assumed, this gives

$$\boxed{\begin{aligned}\boldsymbol{\Delta_\Omega} &= \boldsymbol{\rho_r \otimes \rho_\ell^{-1} = \exp(-2\pi K_r)\exp(2\pi K_\ell)} \\ &= \boldsymbol{\exp(-2\pi K).}\end{aligned}}$$

The method above is not rigorous but the result is correct.

♦ J_Ω can be obtained from the formula $S_\Omega = J_\Omega \Delta_\Omega^{1/2}$. $\Delta_\Omega^{1/2}$ is a rotation by π, *i.e.*

$$\boxed{\Delta_\Omega^{1/2} \mathbf{a}|\Omega\rangle = e^{-\pi K} \mathbf{a}|\Omega\rangle = \tilde{\mathbf{a}}|\Omega\rangle}$$

$\mathbf{a}$ = a general product of local operators in $\mathcal{A}_r$

$\tilde{\mathbf{a}} \in \mathcal{A}_\ell$ = obtained from $\mathbf{a}$ by $(t, x, \vec{y}) \to (-t, -x, \vec{y})$.

$$\boxed{J_\Omega = \mathbf{CRT} : \ (t, x, \vec{y}) \to (-t, -x, \vec{y})}$$

Rigorous derivation by Bisognano-Wichmann

♦ Action of the "Lorentz boost" $e^{-2\pi i \eta K}$ for real η is unitary. From **Reeh-Schlieder theorem** $\Rightarrow$ η **can be analytically continued to the range** $0 \leq \mathrm{Im}\, \eta \leq 1/2$.

Taking $\eta = i/2$ rigorously defines the operator $\Delta_\Omega^{1/2}$ as $e^{-\pi K}$.

4.6 Reeh-Schlieder theorem and entanglement

Reeh-Schlieder theorem[14]

- RS theorem is a **truly remarkable theorem**, which in its essence states that the **"vacuum state " is fully entangled at all scales** and at the same time, implies **important analyticity property** in various situations.

Set up

Consider a QFT in D-dimentional Minkowski spacetime M_D.

[14] H. Reeh and S. Schlieder, "Bemerkungen zur Unitaäräquivalenz von Lorentzinvarienten Ferdern", Nuovo Cimento 22(1961) 1051.

$|\Omega\rangle$ = the vacuum state, $H|\Omega\rangle = 0$.

$\mathcal{H}_0$ = the vacuum sector of the Hilbert space

generated by all the states of the form

$$|\Psi_{\vec{f}}\rangle \equiv \phi_{f_1}\phi_{f_2}\cdots\phi_{f_n}|\Omega\rangle \tag{4.39}$$

ϕ_f = smeared field $\int d^D x f(x)\phi(x)$ for any smooth function f

RS theorem

$\mathcal{V}(\subset \Sigma)$ = **an arbitrary small** open set. $\mathcal{U}_{\mathcal{V}}$ = small neighborhood of $\mathcal{V}$ in spacetime. Even if we restrict the functions f_i to be supported in $\mathcal{U}_{\mathcal{V}}$, the states $\{\Psi_{\vec{f}}\}$ still suffice to generate **the entire $\mathcal{H}_0$**.

Proof is omitted[15].

[15] For a proof, see "PCT, spin and statistics and all that", Streater and

A compact expression for the RS theorem

Let $|\Omega\rangle$ be the vacuum state in Minkowski space and let $\mathcal{A}_{\mathcal{U}}$ be the field algebra corresponding to **any** open set $\mathcal{U}$. Then, $|\Omega\rangle$ is a **cyclic vector for $\mathcal{A}_{\mathcal{U}}$**.

Remarks: There are several important and interesting remarks on RS theorem. However, we will only concentrate on a few and skip the rest.

[1] Relation to **cluster property** SKIP

[2] Relation to **causality** SKIP

Wightman (Benjamin, 1964). More pedagogical proof, E. Witten, "Notes on some entanglement properties of quantum field theory, RMP **90** (2018) 045003, arXiv:1803.04993.

[3] Relation to **entanglement**

To appreciate the deep relation to entanglement, consider the RS theorem using the orthonormal bases.

$\mathcal{U}$ ≡ **an open set,** $\mathcal{U}'$ ≡ **its complement**.

For pedagogy assume[16] that we can write

$$\mathcal{H} = \mathcal{H}_{\mathcal{U}} \otimes \mathcal{H}_{\mathcal{U}'}$$

$\{\psi^i_{\mathcal{U}}\}$ and $\{\psi^i_{\mathcal{U}'}\}$ be orthonormal basis of $\mathcal{H}_{\mathcal{U}}$ and $\mathcal{H}_{\mathcal{U}'}$.

- First we decompose the vacuum state $\mathbf{\Omega}$ in these bases as

$$\mathbf{\Omega} = \sum_i \sqrt{p_i}\,\psi^i_{\mathcal{U}} \otimes \psi^i_{\mathcal{U}'}\,, \qquad p_i > 0 \ {}^{\forall} i \tag{4.40}$$

[16] In the case of QFT, the vacuum state $\mathbf{\Omega}$ is fully entangled but with divergent entropy between the regions $\mathcal{U}$ and $\mathcal{U}'$ and cannot stricly be written as a tensor product. Still, RS-theorem has been rigorously shown to hold.

Note that we have assumed that **Ω is fully entangled** *i.e.* $p_i > 0$ for all i.

♦ Next we expand a **general state $\Psi \in \mathcal{H}$** in the basis $\psi^i_{\mathcal{U}'}$ and write it as

$$\Psi = \sum_i \lambda^i_{\mathcal{U}} \otimes \psi^i_{\mathcal{U}'}, \qquad \lambda^i_{\mathcal{U}} \in \mathcal{H}_{\mathcal{U}} \tag{4.41}$$

Now define an **operator X acting only on $\mathcal{H}_{\mathcal{U}}$**, by requiring the property

$$X(\sqrt{p_i}\,\psi^i_{\mathcal{U}}) = \lambda^i_{\mathcal{U}} \tag{4.42}$$

Then, acting X on the vacuum, we obtain

$$X\Omega = \sum_i X(\sqrt{p_i}\psi^i_{\mathcal{U}}) \otimes \psi^i_{\mathcal{U}'} = \sum_i \lambda^i_{\mathcal{U}} \otimes \psi^i_{\mathcal{U}'} = \Psi$$

This is an analogue of the RS theorem since it says that $X \in \mathcal{A}_{\mathcal{U}}$ acting on the vacuum produces **a general state Ψ**.

It is clear that such an operator X exists **if and only if $p_i > 0$ for all i**, *i.e.* **when Ω is fully entangled at all scales.**

[4] Characterization of cyclic and separating properties in terms of basis expansion

Expand $\Psi \in \mathcal{H}$ as

$$\Psi = \sum_i \sqrt{p_i} \psi^i_{\mathcal{U}} \otimes \psi^i_{\mathcal{U}'} \tag{4.43}$$

and let **a** $\in \mathcal{A}_{\mathcal{U}}$ be an operator which acts **only on $\mathcal{H}_{\mathcal{U}}$**.

Then

$$\boxed{\begin{array}{l}\text{(i)}\quad \boldsymbol{\Psi} \text{ is } \textbf{cyclic} \text{ for } \mathcal{A}_{\mathcal{U}} \text{ if } \{\psi^i_{\mathcal{U}'}\} \textbf{ forms a basis of } \mathcal{H}_{\mathcal{U}'}. \\ \text{(ii)}\quad \boldsymbol{\Psi} \text{ is } \textbf{separating} \text{ for } \mathcal{A}_{\mathcal{U}} \text{ if } \{\psi^i_{\mathcal{U}}\} \textbf{ forms a basis of } \mathcal{H}_{\mathcal{U}}.\end{array}}$$

Proof is a simple exercise.

♦ Thus, **intrinsic description of the RS theorem naturally introduces the notions of both "cyclic" and "separating" vectors.**

4.7 Modular automorphism and its Analyticity properties

4.7.1 Modular automorphism group and Tomita-Takesaki theorems

By introducing a parameter s, one can define a **modular translation** [17]

$$\boxed{\Delta_{\Psi}^{is} = e^{is \log \Delta_{\Psi}} \qquad (4.44)}$$

with

$\log \Delta_{\Psi} \equiv$ **modular Hamiltonian.**

Using this, one can define the **modular automorphism group**

[17] Recall that the definition of the modular operator Δ_{Ψ} depends on a cyclic and separating vector $|\Psi\rangle$.

$\sigma_s(\mathcal{A})$, which is a **unitary transformation** for real s.

$$\boxed{\sigma_s(\mathcal{A}) \equiv \Delta_\Psi^{is} \mathcal{A} \Delta_\Psi^{-is} \qquad (4.45)}$$

Tomita and Takesaki showed the following **important theorems**

Central theorem of Tomita and Takesaki

[1]

$$\sigma_s(\mathcal{A}) = \mathcal{A}\,, \quad \sigma_s(\mathcal{A}') = \mathcal{A}' \qquad (4.46)$$
$$s \in \mathbb{R}$$
$$J_\Psi \mathcal{A} J_\Psi = \mathcal{A}'\,, \quad J_\Psi \mathcal{A}' J_\Psi = \mathcal{A} \qquad (4.47)$$

This theorem says that **$\sigma_s(\mathcal{A})$ is indeed an automorphism.**

Further, one can define the **relative modular group** generated by $\Delta^{is}_{\Psi|\Phi}$ and it satisfies the property

[2]

$$\boxed{\Delta^{is}_{\Psi|\Phi}(\mathbf{a}\otimes 1)\Delta^{-is}_{\Psi|\Phi} = \Delta^{is}_{\Psi'|\Phi}(\mathbf{a}\otimes 1)\Delta^{-is}_{\Psi'|\Phi}, \quad \mathbf{a}\in\mathcal{A} \quad (4.48)}$$

where $\underline{\Psi \text{ and } \Psi'}$ **are two arbitrary cyclic separating vectors**.

- Thus, **the relative modular transformation depends only on Φ**.

♦ For **general** ∞-dimensional von Neumann algebras, the Tomita-Takesaki theorems are **not so easy to prove**[18].

[18]However, in the case of hyperfinite algebras, there is a relatively simple proof: R. Longo, "A simple proof of the existence of modular automorphisms

□ **Important difference of the property of Δ_Ψ between the ∞-dimensional and the finite dimensional case**:

- **Essential difference** exists between them when the **parameter s** of the modular automorphism is **complex** $\longrightarrow \Delta_\Psi^{iz}$.

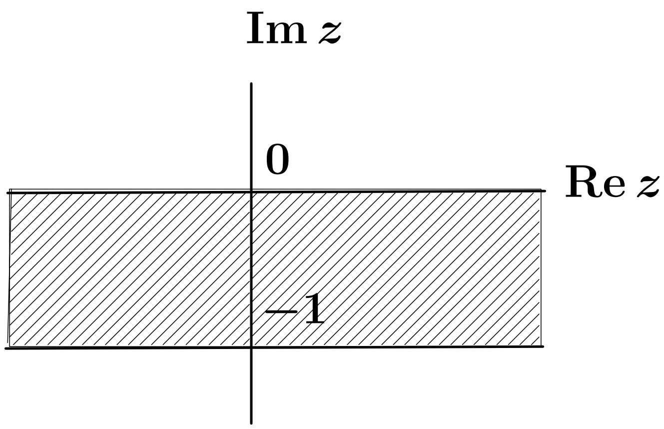
  - ♦ **Finite dimensional case: Δ_Ψ^{iz} is an entire function of z.**
  - ♦ **∞-dimensional case: Analyticity property of $\Delta_\Psi^{iz}\chi$ depends very much on the state χ.**

---

in approximately finite dimensional von Neumann algebras" Pacific Journal of Mathematics 75 (1978) 199.

[1] **Theorem** (proof omitted)

Let $\boldsymbol{\Psi}$ be a vector **cyclic for $\mathcal{A}$**.
Then, $\Delta_{\Psi}^{iz}\mathsf{a}\Psi$ is continuous in the strip $0 \geq \mathrm{Im}\, z \geq -1$ and holomorphic in the interior of the strip.



[2] Act by an additional operator $\mathsf{b} \in \mathcal{A}$ and look at the analytic property of the function

$$\boldsymbol{F}(z) \equiv \langle \Psi | \mathsf{b}\Delta_{\Psi}^{iz}\mathsf{a} | \Psi \rangle\,, \quad z = s - ir \qquad (4.49)$$

- What will be important hereafter is that we have **two operators a and b acting in a certain order.**

It is easy to show that $F(z)$ **has the same analyticity property** as the quantity in the previous Theorem.

The value at the lower boundary of the strip $z = s - i$

$F(s-i)$ can be manipulated **without the use of trace operation**[19] and can be rewritten as

$$F(s-i) = \langle \Psi | \mathbf{a} \Delta_{\Psi}^{-is} \mathbf{b} | \Psi \rangle \tag{4.50}$$

The order of a and b are interchaged compared with the

[19]Thus this type of calculation is valid even for Type III von Neumann algebra, where trace does not exist ! A great advantage of Tomita-Takesaki theory.

original $F(z) = \langle\Psi|\mathbf{b}\Delta_\Psi^{iz}\mathbf{a}|\Psi\rangle$.

Note that **different operator ordering can be obtained by analytic continuation.**

♦ **This is important in the derivation of a general bound on quantum chaos**[20] **and in many other applications.**

■ **More explicit form** in the case of the **fininite dimentional** system with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

In this case, $\rho_{12} = |\Psi\rangle\langle\Psi|$, $\rho_1 = \mathrm{Tr}_2\,\rho_{12}$, and $\rho_2 = \mathrm{Tr}_1\,\rho_{12}$ and $\Delta_\Psi = \rho_1 \otimes \rho_2^{-1}$.

[20] J. Maldacena, S.H. Shenker, and D. Stanford, "A bound on chaos", JHEP 1608 (2016) 106; arXiv:1503.01409.

If we define the **Modular Hamiltonian** H by $\rho_1 = e^{-H}$ explicit computation of $F(z)$ gives

$$F(z) = \langle \Psi | \mathbf{b} \Delta_\Psi^{iz} \mathbf{a} | \Psi \rangle = \mathrm{Tr}_1\, e^{-H} \mathbf{b} e^{-izH} \mathbf{a} e^{izH} \tag{4.51}$$

From this, and the previously obtained expression for $F(s-i)$, we have

$$\boxed{\begin{aligned} F(s) &= \mathrm{Tr}_1\, e^{-H} \mathbf{b} e^{-isH} \mathbf{a} e^{isH} && (4.52) \\ F(s-i) &= \mathrm{Tr}_1\, e^{-H} e^{-isH} \mathbf{a} e^{isH} \mathbf{b} && (4.53) \end{aligned}}$$

4.7.2 Conspicuous similarity to the KMS condition

• The analyticity property above coincides with what is known in statistical mechanics as the **KMS condition**, which **characterizes the condition of thermal equilibrium.**

Below we recap the analysis above suitable in that context.

[1] KMS (Kubo-Martin-Schwinger)[21] condition for the modular automorphism

$\mathbf{a}, \mathbf{b}, \ldots \in \mathcal{A}$
$\alpha_s(\mathbf{a}) \equiv \Delta_\Psi^{is} \mathbf{a} \Delta_\Psi^{-is}$ = **automorphism group of** $\mathcal{A}$

[21] R. Kubo, "Statistical mechanical theory of irreversible processes I", J. Math. Soc. Japan 12, 570 (1957); P.C. Martin and J. Schwinger, "Theory of many particle systems: I", Phys. Rev. 115, 1342 (1959)

A State ϕ on $\mathcal{A}$ is said to satisfy the KMS condition at β **$(0 < \beta < \infty)$**[22] if for any $\mathbf{a}, \mathbf{b} \in \mathcal{A}$ there exists a complex function $F_{\mathbf{a},\mathbf{b}}(z)$ **which is analytic on the strip**

$$\{z \in \mathbb{C} \,|\, 0 < \operatorname{Im} z < \beta\}$$

and continuous on the closure of this strip such that

$$\boxed{F_{\mathbf{a},\mathbf{b}}(s) = \phi(\alpha_s(\mathbf{a})\mathbf{b}) \quad \text{and} \quad F_{\mathbf{a},\mathbf{b}}(s + i\beta) = \phi(\mathbf{b}\alpha_s(\mathbf{a}))}$$

- Such a state $\phi(\mathbf{a}) \in \mathbb{C}$ is called a **KMS state**.

- Among a number of nice properties about the KMS states, which we skip, one property, however, should be mentioned:

[22] As usual, in statmech, β is the inverse temperature.

Let ϕ be a **faithful normal state** on $\mathcal{A}$ induced by a cyclic separating vector Ψ. It is of the form

$$\phi(\mathbf{a}) = \frac{1}{||\Psi||^2}\langle\Psi|\mathbf{a}|\Psi\rangle\,, \quad \mathbf{a} \in \mathcal{A} \tag{4.54}$$

Then, **the state ϕ is a KMS state** at $\boxed{\beta = 1}$.

For such value of β, the KMS condition is called the **modular condition**.

■ Important to **clarify in general context the precise relation between the modular automorphism in von Neumann algebra and the KMS condition in quantum stat. mech.**.

◆ In a simple case, we have already seen that in flat Minkowski space, Δ_Ω is generated by the Lorentz boost, and **the Unruh effect** is nothing but the phenomenon expressing such a relation. .

5 "Crossed product" and its physical significance

5.1 Concept of "crossed product"

- The study of von Neumann algebra of type $\mathbf{III}$ was intractable for a long time.
- The breakthrough was achieved by A. Connes[23] in his thesis and by a paper of M. Takesaki[24] in 1973, who obtained the **"structure theorem for III factor"** and the **"duality theorem"**, using the notion of the **crossed product**.

[23] A. Connes, Ann. Sci. de L.E.N.S. 6(73) 133.
[24] M. Takesaki, Acta Math. 131 (73) 249

• **The crossed product is similar to a semi-direct product in group theory** (introduced already in the seminal paper of Murray and von Neumann).

□ **Definition of the crossed product**:

$\mathcal{A}$ = a von Neumann algebra , acting on a Hilbert space $\mathcal{H}$, with an **automorphism group** $\mathrm{Aut}(\mathcal{A})$, associated with a continuous action α of a locally compact group G.

$$\alpha : G \ni g \rightarrow \alpha_g \in \mathrm{Aut}(\mathcal{A})$$

$$\alpha_g : \mathcal{A} \rightarrow \mathcal{A}$$

$L^2(\mathcal{H}, G) = \{\mathcal{H}\text{-valued } L^2 \text{ functions on } G\}$

Π_α= A representation of $\mathcal{A}$ on $L^2(\mathcal{H}, G)$ associated with α

λ= A representation of G on $L^2(\mathcal{H}, G)$

Crossed product of $\mathcal{A}$ by G

• crossed product of $\mathcal{A}$ by the action α of G $\mathcal{A} \rtimes \alpha =$ von Neumann algebra on $L^2(\mathcal{H}, G)$ generated by $\pi_\alpha(\mathcal{A})$ and $\lambda(G)$

More intuitive description

Let $T =$ self-adjoint operator that generates $\mathrm{Aut}(\mathcal{A})$

$$e^{isT}\, \mathbf{a}\, e^{-isT} \in \mathcal{A}, \quad \mathbf{a} \in \mathcal{A},\ s \in \mathbb{R} \tag{5.1}$$

$L^2(\mathbb{R}) = L^2$ functions of $X \in \mathbb{R}$.

- Then, the crossed product $\mathcal{A} \rtimes \mathbb{R}$ acts on $\mathcal{H} \otimes L^2(\mathbb{R})$ by adjoining bounded functions of $T + X$ to $\mathcal{A}$.

That is, it is the set of elements of the form

$$\boxed{\mathcal{A} \rtimes \mathbb{R} = \{\mathsf{a}\, e^{isT} \otimes e^{isX}\} \qquad (5.2)}$$

which forms an algebra.

- If $e^{isT} \in \mathcal{A}$, this automorphism is said to be **inner**.

In this case, X is the only new variable added and nothing essentially interesting happens by forming a crossed product.

<u>**Fact:**</u> **For type I factor, all $\mathrm{Aut}(\mathcal{A})$ is inner.**

Type II and III can contain outer automorphisms.

Thus, **the crossed product construction is interesting for** <u>**Type II and III.**</u>

5.2 Structure of type III factor and Connes-Takesaki duality theorem

- **For the study of type III factor, the crossed product plays a crucial role.**

Below we restrict to $G =$ locally compact **abelian** group

$\widehat{G} =$ (Pontrjagin) dual group of $G \equiv \mathbf{Hom}\,(G, T)$

where $T =$ circle group

$\widehat{\widehat{G}} \simeq G$: Pontrjagin duality theorem

$\widehat{\alpha} =$ dual action of $\widehat{G}$ on $\mathcal{A} \rtimes \alpha$

In particular, $G = \mathbb{R} = \widehat{\mathbb{R}} = \widehat{G}$ will be relevant for us.

Duality theorem of Connes and Takesaki

$$(\mathcal{A} \rtimes \alpha) \rtimes \widehat{\alpha} = \mathcal{A} \otimes \mathcal{L}(L^2(G))$$

where $\mathcal{L}(L^2(G))$ = type $\mathbf{I}$ factor of all bounded operators on $L^2(G)$

- Under certain technical conditions, this implies $\boxed{(\mathcal{A} \rtimes \alpha) \rtimes \widehat{\alpha} \text{ is isomorphic to } \mathcal{A} \text{ itself}}$.

Theorem (Takesaki)

If $\mathcal{A}$ is of type $\mathbf{III}$, then $\mathcal{A} \rtimes \mathbb{R}$ must be of type $\mathbf{II}_\infty$,
i.e. $\boxed{\mathbf{III} \rtimes \mathbb{R} = \mathbf{II}_\infty}$

- **So by taking the crossed product with $G = \mathbb{R}$, type III factor reduces to type II_∞ factor.**

- Further, **using this result in the duality theorem, with $\mathcal{A} = \mathrm{III}$**, we obtain

■ **structure theorem for type III factor**:

$$\boxed{\mathrm{III} \simeq (\mathrm{III} \rtimes \mathbb{R}) \rtimes \mathbb{R} = \mathrm{II}_\infty \rtimes \mathbb{R}}$$

- This theorem **reduces the difficult study of III von Neumann algebra to that of the crossed product of simpler factor II_∞.**

- **Witten found the importance of the notion of the crossed product and its use in the theorems of Connes and Takesaki in the study of black hole physics using the von Neumann algebra .**

This will be sketched in the next section

6 Application to black hole physics

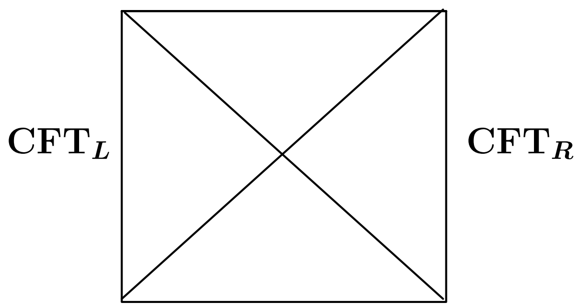
6.1 Eternal black hole in AdS in the strict large N limit

In this subsection, we **assume AdS/CFT**.

- The eternal black hole in AdS is dual to 2 copies of boundary CFT's[25]:

$\mathbf{CFT}_L, \mathbf{CFT}_R$: $\mathcal{N} = 4\ SU(N)$ SYM

Above Hawking-Page temperature T_{HP}, it is stable and is expressed by the **thermofield double state Ψ_{TFD}**.



[25] J. Maldacena, "Eternal black holes in anti-de Sitter", JHEP 04 (2003) 021, hep-th/0106112

□ **Relevant operators in the strict large N limit**:

Unsubtracted single trace operator $\langle \mathrm{Tr} W \rangle \sim N \to \infty$

Subtracted single-trace operator

$$\mathcal{W} \equiv \mathrm{Tr} W - \langle \mathrm{Tr} W \rangle \sim \mathcal{O}(1)$$

$\langle \mathcal{W}_1 \mathcal{W}_2 \rangle \sim \mathcal{O}(1)\,,\ \langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \rangle \sim 1/N \to 0$.

- Only 1- and 2-point functions survive and

$$[\mathcal{W}_1, \mathcal{W}_2] = c\# \tag{6.1}$$

$\Rightarrow$ **Subtracted single trace operators describe "generalized free field (GFF)**

$\boxed{\textbf{Bounded functions of GFF form a von Neumann algebra}}$

- $\mathcal{A}_{R,0}, \mathcal{A}_{L,0}$= right and left **boundary** von Neumann algebra

The subscript $\mathbf{0}$ signifies that they **do not contain the central elements**

- $\mathcal{A}_{r,0}, \mathcal{A}_{l,0}$ = corresponding **bulk quantities**

denoted with **lower case letter subscripts** .
They are the operator algebras of the bulk field theory
for the region outside the horizon and hence are of Type $\mathbf{III_1}$.

By duality, $\mathcal{A}_{R,0}$ and $\mathcal{A}_{L,0}$ are also of type $\mathbf{III_1}$.

- We will mention a more intrinsic logic without the use of duality

later.

Importance of the BH mass (energy) M

Conserved charges possessed by a BH
$=$ M (or energy), angular momentum, gauge charges (like $\boldsymbol{R}$-charges of $\mathcal{N} = 4$ SYM)
They can be considered as **collective coordinates, shared by Right and Left**.

- Of particular importance is M.

M is not a generator of the bulk algebras $\mathcal{A}_{r,0}$ and $\mathcal{A}_{l,0}$.

$$\xrightarrow{\text{duality}} \boxed{\boldsymbol{H_L}, \boldsymbol{H_R} \text{ do not belong to } \boldsymbol{\mathcal{A}_{L,0}}, \boldsymbol{\mathcal{A}_{R,0}}}$$

For a while focus on $\mathcal{A}_{R,0}$.

Above $T_{\rm HP}$, $\langle H_R \rangle = \mathcal{O}(N^2)$.

$H'_R \equiv H_R - \langle H_R \rangle$ is not defined as $N \to \infty$.

What is meaningful is the important **time translation operator**

$\boxed{U \equiv \frac{1}{N} H'_R}$ **Its bulk dual is the black hole energy M.**

$U = \mathcal{O}(1)$ at large N and has a large N limit.

- But U is **central** and is **not a part of $\mathcal{A}_{R,0}$**.

Indeed for $^\forall \mathbf{a} \in \mathcal{A}_{R,0}$,

$$[U, \mathbf{a}] = \frac{1}{N}[H_R, \mathbf{a}] = -\frac{i}{N}\frac{\partial \mathbf{a}}{\partial t} \xrightarrow[N \to \infty]{} 0 \tag{6.2}$$

$\Rightarrow$ $\boxed{\text{Time-translation is an outer automorphism of } \mathcal{A}_{R,0}}$

Modular Hamiltonian on the boundary and in the bulk

- On the boundary, H_R and H_L do not have a large N limit.

Meaningful operator is their difference

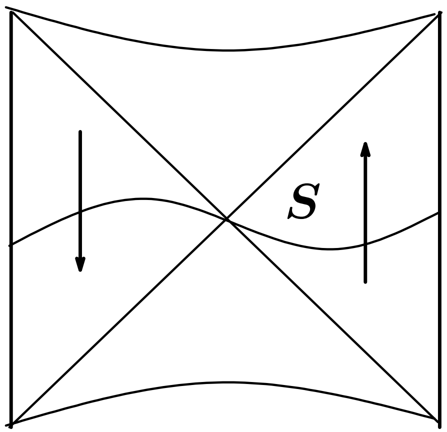
$$\boxed{\widehat{H} = H_R - H_L = H'_R - H'_L}$$

- Bulk dual of $\widehat{H}$: It is given by the operator

$$\widehat{h} = \int_S d\Sigma^\mu V^\nu T_{\mu\nu} = h_r - h_l$$

where S = Cauchy surface and V is the time-like Killing vector as dipicted.

Killing vector V



The boundary dual of $\widehat{\boldsymbol{h}}$ is the **modular Hamiltonian**

$$\boxed{\widehat{\boldsymbol{h}} \Leftrightarrow \widehat{\boldsymbol{H}}_{\mathbf{mod}} = \boldsymbol{\beta}\widehat{\boldsymbol{H}} \qquad (6.3)}$$

where $\widehat{\boldsymbol{H}}$ is the Hamiltonian corresponding to physical energy.

Modular operators on the boundary and in the bulk

$$\boldsymbol{\Delta}_{\boldsymbol{\Psi}_{\mathbf{TFD}}} = \begin{cases} \boldsymbol{e}^{-\boldsymbol{\beta}\widehat{\boldsymbol{H}}} & \text{for the boundary} \\ \boldsymbol{e}^{-\widehat{\boldsymbol{h}}} & \text{for the bulk} \end{cases} \tag{6.4}$$

A comment on the pioneering work of Leutheusser and Liu [LL] [26]

- [LL] are the ones who first attempted to make use of the von Neumann algebra seriously to study **how one can go into the interior or a black hole by defining emergent time there**.
- They made use of the property called **half-sided modular translation**, which exists only for type $\mathbf{III_1}$. (Description omitted due to its high technicality.)
- **It is their work which ignited interest** of many people, in particular Witten, to focus due light on von Neumann algebra concerning gravity.

[26] S. Leutheusser and H. Liu, "Emergent times in holographic duality", arXiv: 2112.12156

$\square$ **Inclusion of the central variable U and extended TFD states**:

- Since we have an extra central variable $\boldsymbol{U}$, we must add its bounded functions to complete the algebra.

For the right sector the full large $\boldsymbol{N}$ algebra becomes

$$\mathcal{A}_R = \mathcal{A}_{R,0} \otimes \mathcal{A}_U$$

$\mathcal{A}_U$ = abelian algebra of bounded functions of $\boldsymbol{U}$

$\mathcal{A}_R$ **acts on** $\mathcal{H}_{\mathrm{TFD}} \otimes L^2(\mathbb{R})$

$\therefore$ **Element $\widehat{a}$ of $\mathcal{A}_R = U$-dependent element of $\mathcal{A}_{R,0}$**

General form of $\widehat{a} \in \mathcal{A}_R$

$$\widehat{a} = \int_{-\infty}^{\infty} du\, a(u) e^{iuU}\,, \quad a(u) \in \mathcal{A}_{R,0} \tag{6.5}$$

Extended TFD state $\widehat{\Psi}_{\mathrm{TFD}}$

We should now define the **extended TFD state in the space including the new variable U:**

$$\boxed{\widehat{\Psi}_{\mathrm{TFD}} \equiv \sqrt{g(U)}\Psi_{\mathrm{TFD}} \quad (6.6)}$$

$g(U)$ is a suitable Gaussian.

Then the thermal expectation value of $\widehat{a} \in \mathcal{A}_R$ can be expressed as

$$\begin{aligned}\langle \widehat{a} \rangle &= \langle \widehat{\Psi}_{\mathrm{TFD}} | \widehat{a} | \widehat{\Psi}_{\mathrm{TFD}} \rangle \\ &= \int_{-\infty}^{\infty} dU g(U) \int_{-\infty}^{\infty} du e^{iuU} \langle \Psi_{\mathrm{TFD}} | a(u) | \Psi_{\mathrm{TFD}} \rangle\end{aligned} \tag{6.7}$$

Classical-Quantum (CQ) state[27]

- $\widehat{\Psi}_{\mathbf{TFD}}$ introduced above is a typical example of the so-called **"classical quantum states"**. **Such states will be important when we discuss the modular operator**

Definition of CQ state:

Let $\mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$
$\mathcal{H}_A =$ **"classical"**, $\mathcal{H}_B =$ **"quantum"**

Examples of CQ state

- U is central in the algebra $\mathcal{A}_R = \mathcal{A}_{R,0} \otimes \mathcal{A}_U$
  $\Rightarrow \widehat{\Psi}_{\mathbf{TFD}}$ is a classical-quantum state.

[27] CQ is not the same as a semi-classical state.

6.2 Inclusion of $\mathcal{O}(G_N) \sim 1/N^2$ corrections and the transition from type III_1 to type II_∞

Witten's observation

- In the strict large N limit, $\boldsymbol{\mathcal{A}}$ of left and right sides of the horizon contains **central operator** dual to the bulk collective coordinate M_{BH} (or A_{horizon}).

> - With $\mathcal{O}(G_N) = 1/N^2$ corrections, **this is no longer the case** and the **von Neumann algebra changes from III_1 to II_∞**, through the mechanism of the **crossed product**.

Below we will sketch his logic.

6.2.1 $1/N$ expansion and the crossed product

□ **Change of the character of $(1/N)H'_R$ when including $1/N$ correction**:

- $(1/N)H'_R$ is **no longer a central element**

$\because$ For any $\mathbf{a} \in \mathcal{A}_R$, we have $[H'_R/N, \mathbf{a}] = -i(1/N)\partial_t \mathbf{a} \neq 0$

- **Correction** to the **bulk dual** of H'_R/N ?

$(1/\beta)\widehat{h}$ **has the same commutation relation** and it coincides with ∂_t on the right boundary.

This suggests the modification (recall $(1/\beta)\widehat{h} \Leftrightarrow \widehat{H} = H'_R - H'_L$)

$$\boxed{\frac{1}{N}H'_R = U + \frac{1}{N}\widehat{H} \qquad (6.8)}$$

The left and the right side have the desired commutation relations with any element of $\mathcal{A}_R$.

6.2.2 Crossed product and II_∞ nature for the eternal black hole

Consideration above shows:

- Order by order in $1/N$, one can interpret the boundary algebra $\mathcal{A}_R$ as generated by $\mathcal{A}_{R,0}$ and $U + \widehat{H}/N$.

This is expressed as the **crossed product**

$$\boxed{\mathcal{A}_R = \mathcal{A}_{R,0} \rtimes \mathcal{A}_{U+\widehat{H}/N} \qquad (6.9)}$$

□ von Neumann algebra for the eternal black hole is type II_∞:

- von Neumann algebra of the boundary and the bulk are considered to be **isomorphic**.

Hereafter we will treat them with **common notations**.

- Recall the **important result of the Tomita-Takesaki theory**

$$\boxed{\Delta_\Psi^{-is} = e^{i\widehat{h}_\Psi s} = \text{automorphism of } \mathcal{A} \qquad (6.10)}$$

where

$\Psi(\in \mathcal{H})$ = a cyclic and separating vector for $\mathcal{A}$.

$\Delta_\Psi = e^{-\widehat{h}_\Psi}$ = corresponding modular operator.

I.e. one can take the automorphism generator T to be $\widehat{h}_\Psi$.

Define

$$\boxed{\begin{aligned} \mathbb{R}_{\Psi} &= \text{automorphism group generated by } \widehat{h}_{\Psi}. \\ &= \textbf{modular} \text{ automorphism group.} \end{aligned}}$$

- Then, in the case of the eternal black hole in AdS,

with the $\mathcal{O}(G_N)$ correction taken into account, the **theorem of Takesaki** dictates

$$\boxed{\begin{gathered} \mathcal{A} = \mathcal{A}_0 \rtimes \mathbb{R}_{\Psi_{\mathrm{TFD}}} \\ \mathcal{A} = \mathrm{II}_{\infty}, \quad \mathcal{A}_0 = \mathrm{III}_1 \end{gathered}}$$

Important theorem of Connes

> The crossed product $\mathcal{A} \rtimes \mathbb{R}_{\Psi}$ is **independent of Ψ as long as Ψ is cyclic and separating for $\mathcal{A}$**.

$\Rightarrow$ study of $\mathcal{A} \rtimes \mathbb{R}$ is that of $\mathcal{A}$, **not** of the pair $(\mathcal{A}, \Psi)$.
This is referred to as "**state independence**.

- The proof of the state independence is accomplished by making use of the operator called the **Connes cocycle**

$$u_{\Psi|\Omega}(t) \equiv \Delta_{\Psi|\Omega}^{it} \Delta_{\Omega}^{-it} = \Delta_{\Psi}^{it} \Delta_{\Omega|\Psi}^{-it} \qquad (6.11)$$

(Detail is omitted.)

6.3 "Classical-quantum (CQ) states" and the definition of the trace and the entropy

6.3.1 Usefulness of a special type of CQ states

Consider a CQ state of the form

$$\boxed{\widehat{\Psi} = \sqrt{g(X)}\,\Psi\,, \quad X \in \mathbb{R}}$$

where $\sqrt{g(X)}$ is assumed to be everywhere positive so that $\widehat{\Psi}$ is cyclic separating.

- As we shall see **the modular operator $\widehat{\Delta}_{\widehat{\Psi}}$ for this state**, to be defined below, **has a very useful factorization property.**

6.3.2 Definition of $\widehat{\Delta}_{\widehat{\Psi}}$ and its factorization property

- To define the modular operator $\widehat{\Delta}_{\widehat{\Psi}}$, **we demand that it satisfy the same properties which characterize Δ_Ψ.**

(1) The first important property of Δ_Ψ:

$$\boxed{\langle\Psi|\mathbf{ab}|\Psi\rangle = \langle\Psi|\mathbf{b}\Delta_\Psi\mathbf{a}|\Psi\rangle\,, \quad {}^\forall\mathbf{a},\mathbf{b}\in\mathcal{A} \qquad (6.12)}$$

The order of **b** and **a** are reversed with an insertion of Δ_Ψ.

(2) The second property is the **KMS condition**.

Write

$$\widehat{h}_\Psi \equiv -\log\Delta_\Psi \quad \Leftrightarrow \quad \Delta_\Psi = e^{-\widehat{h}_\Psi} \tag{6.13}$$

$$\mathbf{a}_u \equiv e^{i\widehat{h}_\Psi u}\mathbf{a}e^{-i\widehat{h}_\Psi u} \tag{6.14}$$

KMS condition is obtained by replacing **a** by $\mathbf{a}_u$ in (6.12) characterizing $\mathbf{\Delta}_\Psi$. After a simple calculation this leads to

$$\boxed{\textbf{KMS condition} \quad \langle\Psi|\mathbf{a}_u\mathbf{b}|\Psi\rangle = \langle\Psi|\mathbf{b}\mathbf{a}_{u+i}|\Psi\rangle \qquad (6.15)}$$

- Further, with no change of the order of **a** and **b**, one can show

$$\boxed{\langle\Psi|\mathbf{a}_u\mathbf{b}_v|\Psi\rangle = \langle\Psi|\mathbf{a}_{u+s}\mathbf{b}_{v+s}|\Psi\rangle\,, \quad {}^{\forall}s \qquad (6.16)}$$

In the context of thermofield double this is the **"time" translation symmetry for $s \in \mathbb{R}$**.

Explicit solution for $\widehat{\boldsymbol{\Delta}}_{\widehat{\boldsymbol{\Psi}}}$

The first demand that $\widehat{\boldsymbol{\Delta}}_{\widehat{\boldsymbol{\Psi}}}$ **must satisfy**[28]

$$\boxed{(\star) \quad \langle \widehat{\boldsymbol{\Psi}} | \widehat{\mathbf{a}}\widehat{\mathbf{b}} | \widehat{\boldsymbol{\Psi}} \rangle = \langle \widehat{\boldsymbol{\Psi}} | \widehat{\mathbf{b}} \widehat{\boldsymbol{\Delta}}_{\widehat{\boldsymbol{\Psi}}} \widehat{\mathbf{a}} | \widehat{\boldsymbol{\Psi}} \rangle \qquad (6.17)}$$

Solution:

$$\boxed{\widehat{\boldsymbol{\Delta}}_{\widehat{\boldsymbol{\Psi}}} = \boldsymbol{\Delta}_{\boldsymbol{\Psi}} \frac{g(\widehat{\boldsymbol{h}}_{\boldsymbol{\Psi}} + \boldsymbol{X})}{g(\boldsymbol{X})} = \widetilde{\boldsymbol{K}} \boldsymbol{K} \qquad (6.18)}$$

[28] Recall: $\widehat{\mathbf{a}}$ was defined in (6.5).

where

$$\boxed{\begin{aligned} \boldsymbol{K} &\equiv \boldsymbol{\Delta_\Psi e^{-X} g(\widehat{h}_\Psi + X) = e^{-(\widehat{h}_\Psi + X)} g(\widehat{h}_\Psi + X)} & (6.19) \\ \widetilde{\boldsymbol{K}} &\equiv \boldsymbol{\frac{e^X}{g(X)}} & (6.20) \end{aligned}}$$

- Checking that this indeed satisfies $(\star)$ **requires the KMS condition and the translation symmetry.**
- The fact that $\widehat{\boldsymbol{\Delta}}_{\widehat{\boldsymbol{\Psi}}}$ can be **factorized as** $\widetilde{\boldsymbol{K}}\boldsymbol{K}$ will be important.

□ **Implication of the factorization $\widehat{\Delta}_{\widehat{\Psi}} = \widetilde{K}K$:**

(1) K part is only a function of $\widehat{h}_{\Psi}+X$ and hence $K \in \mathcal{A}\rtimes\mathbb{R}_{\Psi}$.

(2) **Fact:** **Existence of such a factorization**
$\Rightarrow$ **modular automorphism group of $\widehat{\Psi}$ is inner.**
(Proof omitted)

(3) **Fact:** **Any modular automorphism of Type III algebra is always outer.** Hence, the crossed product algebra $\mathcal{A} \rtimes \mathbb{R}$ must be of Type $\mathbf{I}$ or Type $\mathbf{II}$.

(4) Obvious non-uniqueness of the factorization:

There is an outer automorphism group of the crossed product
$X \to X + c$, under which $K \to e^{-c}K, \widetilde{K} \to e^{c}\widetilde{K}$.

• There is no further indeterminacy in the factorization provided the crossed product is a "factor".

Fact: $\boxed{\text{This happens precisely when } \mathcal{A} \text{ is of Type } \mathrm{III}_1.}$

6.4 Trace

• **Factorization $\widehat{\Delta}_{\widehat{\Psi}} = \widetilde{K} K$ for a CQ state Ψ**

$\Rightarrow$ existence of a trace (in general for a subalgebra)

One can define the trace by (note the insertion of K^{-1})

$$\boxed{\mathrm{Tr}\, \widehat{\mathbf{a}} \equiv \langle \widehat{\Psi} | \widehat{\mathbf{a}} K^{-1} | \widehat{\Psi} \rangle \qquad (6.21)}$$

Then, **it satisfies $\mathrm{Tr}\, \widehat{\mathbf{a}}\widehat{\mathbf{b}} = \mathrm{Tr}\, \widehat{\mathbf{b}}\widehat{\mathbf{a}}$**

- Further, using the form of $\boldsymbol{K}$, $\mathbf{Tr}\,\widehat{\mathbf{a}}$ can be shown to take a simple form

$$\boxed{\mathbf{Tr}\,\widehat{\mathbf{a}} = \int_{-\infty}^{\infty} d\boldsymbol{X} e^{\boldsymbol{X}} \langle \Psi | \widehat{\mathbf{a}} | \Psi \rangle \qquad (6.22)}$$

Here $\widehat{\mathbf{a}}$ is taken to be a function of $\boldsymbol{X}$, like the basis element $\widehat{\mathbf{a}} = \mathbf{a} e^{is(\widehat{h}_{\Psi} + \boldsymbol{X})}$.

♦ Type $\mathbf{II}_{\infty}$ property: This formula shows: trace on $\boldsymbol{\mathcal{A}} \rtimes \mathbb{R}$ is **not defined for all the elements of this algebra** since the integral over $\boldsymbol{X}$ may not converge.

Example: For $\widehat{\mathbf{a}}(\boldsymbol{X}) = \mathbf{a}$, clearly the trace above is divergent.

♦ There is an **ambiguity due to an outer automorphism.**

Under $X \longrightarrow X + c$ the trace is rescaled:

$$\mathrm{Tr}\widehat{a} \longrightarrow e^{c}\mathrm{Tr}\widehat{a} \tag{6.23}$$

□ **Possibility of rescaling the trace for von Neumann algebra that is a factor**:

- Type $\mathbf{III}$ algebra does not admit a trace to begin with.
- Type $\mathbf{I}$ and $\mathbf{II}_1$ algebras admit a trace but it is **not** rescaled by an outer automorphism.

- **Type II_∞ algebra is the only case which admits a rescaling by an outer automorphism.**

6.5 Density matrices and entropies

□ Density matrix ρ for an algebra $\mathcal{A}$:

- $\mathcal{A}$ has a density matrix ρ if it has a nondegenerate trace that is positive in the sense $\mathbf{Tr}\,\mathbf{a}^\dagger\mathbf{a} > \mathbf{0}$ for all non-zero $\mathbf{a} \in \mathcal{A}$.

Definition of ρ of a state $|\Psi\rangle \in \mathcal{H}$

It is an element in $\mathcal{A}$ such that for $\mathbf{a} \in \mathcal{A}$,

$$(i) \quad \langle\Psi|\mathbf{a}|\Psi\rangle = \mathbf{Tr}\,\rho\mathbf{a} \tag{6.24}$$

(ii) ρ is positive in the sense

$$\mathbf{Tr}\,\rho\mathbf{a}^\dagger\mathbf{a} > \mathbf{0} \tag{6.25}$$

$$(iii) \quad \text{Normalizable to unity:} \quad \mathbf{Tr}\,\rho = \mathbf{1} \tag{6.26}$$

- **Caution:** **ρ may be non-trivial** due to the form of the trace.

□ **Definition of the von Neumann entropy and its properties**:

• When a trace and the density matrix exist, one can define the von Neumann entropy in the usual way by

$$\boxed{S(\rho) \equiv -\mathrm{Tr}\,\rho \log \rho \qquad (6.27)}$$

■ **In the case of the crossed product algebra $\mathcal{A} \rtimes \mathbb{R}_{\Psi}$**, the definition of $S(\rho)$ is **not unique** due to the existence of the outer automorphism $X \longrightarrow X + c, c \in \mathbb{R}$,
under which, $\mathrm{Tr} \longrightarrow e^{c}\mathrm{Tr}$.
To preserve $\mathrm{Tr}\rho = 1$, ρ must be rescaled by $\rho \longrightarrow e^{-c}\rho$.
This shifts the entropy as

$$\boxed{S(\rho) \longrightarrow S(\rho) + c \qquad (6.28)}$$

- The shift is **the same for all states.**
- **Only the entropy difference has physical significance for this algebra.**

♦ **The entropy defined by (6.27) is not bounded below.**

Example:

Consider the CQ state $\widehat{\Psi} = \Psi \otimes \sqrt{g(X)}$.

Recall the definition of the trace: $\quad \mathrm{Tr}\,\widehat{\mathbf{a}} = \langle\widehat{\Psi}|\widehat{\mathbf{a}}K^{-1}|\widehat{\Psi}\rangle$.

Replace $\widehat{\mathbf{a}} \longrightarrow \widehat{\mathbf{a}}K$. Then, $\mathrm{Tr}\,\widehat{\mathbf{a}}K = \langle\widehat{\Psi}|\widehat{\mathbf{a}}|\widehat{\Psi}\rangle$.

Therefore $\boxed{\rho = K}$ in this case.

Inserting the form of K into the definition and using $\widehat{h}_{\Psi}|\widehat{\Psi}\rangle = 0$, one can get

$$S(K) = \int_{-\infty}^{\infty} dX (Xg(X) - g(X)) \log g(X) \tag{6.29}$$

with the normalizaion of $\widehat{\Psi}$ $\Leftrightarrow \displaystyle\int_{-\infty}^{\infty} dX g(X) = 1$

We can check that this expression is unbounded below (and above).

Remark

- In the large N limit of the thermofield double, $g(X)$ is a Gaussian peaked near $X = 0$, not near $X = A/4G$.

$\Rightarrow$ $S(K)$ will not be $A/4G$.

- But this is no problem, since the entropy is defined only up to an arbitrary constant. Only the relative entropy is meaningful.

7 Large N algebra and generalized entropy

Based on: V. Chandrasekharan, G. Penington and E. Witten, " Large N algebra and generalized entropy", arXiv: 2209.10454

We first give the topics and summary of this section.

[1] Formula for the generalized entropy and the quantum extremal surface (QES).

QES = quantum extremal surface prescription[29]

$$\boxed{(S(B) =)\ S_{\rm gen}(b) = \frac{A(\partial b)}{4G} + S_{\rm bulk}(b) \qquad (7.1)}$$

[29] "Generalization of the Ryu-Takayanagi formula": Engelhardt and Wall, 1408.3203

$\boldsymbol{S(B)}$ =entanglement entropy of the CFT region $\boldsymbol{B}$

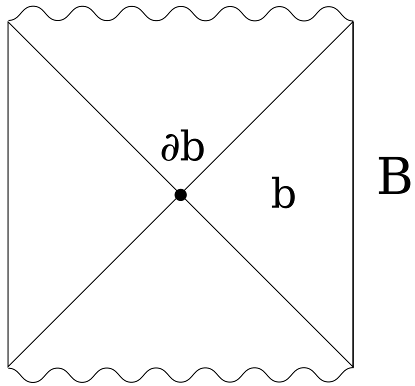
$\boldsymbol{S_{\text{gen}}(b)}$ = generalized entanglement entropy of $\boldsymbol{b}$

$\boldsymbol{B}$ = Right asymptotnic boundary of the 2-sided black hole

$\boldsymbol{b}$ = entanglement wedge of $\boldsymbol{B}$.

$\boldsymbol{\partial b}$ = bifurcation surface

$\boldsymbol{S_{\text{bulk}}}$ = quantum correction due to bulk fields



The von Neumann algebras for the CFTs on the right and the left boundaries will be denoted by $\boldsymbol{\mathcal{A}_{R,0}}$ and $\boldsymbol{\mathcal{A}_{L,0}}$.

- **Generalized entropy and the QES prescription** have been discussed and derived **so far** by
  - Relating the boundary entanglement entropy to the **bulk Euclidean gravity path integral** on a replicated manifold.
  - This method is essentially the old calculation of Gibbons and Hawking in Euclidean derivation of Bekenstein-Hawking entropy.
- **Euclidean gravity acts as a magical black box, which is not clearly justified.**
- **Wish to derive the generalized entropy formula more clearly and rigorously** as the **entanglement entropy** of a certain class of CFT in the large N limit.

[2] Use of the microcanonical version of TFD states and the von Neumann algebra at large N

- Previously, Type $\mathbf{II}_\infty$ nature of the von Neumann algebra for the black hole was derived by adding a formal **$1/N$ correction** to the strict large N algebra of type $\mathbf{III_1}$.

- In this work, using the **microcanonical** (**rather than the usual canonical**) **version of TFD states** and including the DoF of **relative time shifts** between left and right boundaries, it is shown that actually **the algebra is already of Type $\mathbf{II}_\infty$ in the strict large N limit** in this formulation.

- **Reason for the difference:**

Fluctuations under focus have **different N-dependence** between two ensembles considered.

[3] Entropy for the semicalssical states and the generalized second law

- Entropy for the **semiclassical states** can be expressed in terms of the **relative entropy**, from which **the form of the generalized entropy can be derived.**
- Then, the **generalized second law** is derived from the **inequality for the entropy**.

[4] Effect of perturbations of a black hole between a long time interval and formation of a long wormhole supported by shock waves

- A study is made of **the perturbation of a black hole at different time interval larger than the scrambling time**. In this case, the relevant von Neumann algebra is constructed as so called the **"free product"** of von Neumann algebras. It acts on states that describe **long wormholes supported by shock waves**.

7.1 Formula for the generalized entropy and the quantum extremal surface (QES)

Aim of this section

- Relate the **entropy of the semiclassical state of $\mathcal{A}_R$** and the **generalized entropy of its bulk dual**.
- $\mathcal{A}_R = \mathcal{A}_{R,0} \rtimes \mathbb{R}_h$

  $\mathbb{R}_h$ = modular group for the state $|\Phi\rangle$ below

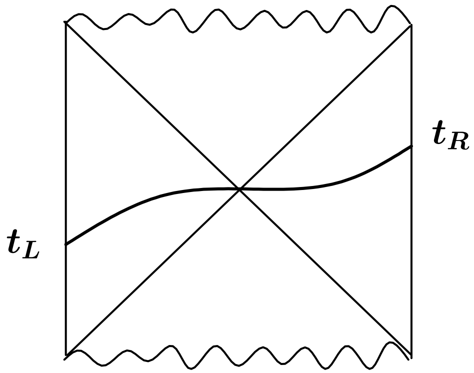
Definition of the "semiclassical state" $|\Phi\rangle \in \mathcal{H}$

Δp = fluctuations in the **timeshift** p between L and R boundaries.

$$\boxed{(\star) \quad |\Phi\rangle \text{ is semiclassical} \Leftrightarrow \Delta p = \mathcal{O}(\epsilon)\,, \epsilon \ll 1.}$$

Time shift p

A fixed Schwarzschild **time slice** extends from a boundary time t_R through the bifurcation point to reach the left boundary time t_L.



Time shift $p \equiv t_R + t_L$.

Symmetry of the SS solution $\Rightarrow$ equal and opposite shift of the time on the two sides $\Rightarrow$ p is independent of the choice of t_R.

- p can be eliminated but **only by a diffeo that acts non-trivially at the asymptotic boudaries**.

Thus **it is a physical mode**, not a gauge mode.

$$\boxed{\Delta x = \text{fluctuation in energy}}$$

Uncertainty principle, $\Delta x \Delta p \geq \frac{1}{2} \longrightarrow \Delta x \geq \mathcal{O}(1/\epsilon)$

- **General form of semiclassical state**, **denoted with** $\widehat{\ }$, is

$$|\widehat{\Phi}\rangle = \int_{-\infty}^{\infty} dx \epsilon^{1/2} g(\epsilon x) |\Phi\rangle |x\rangle \quad (7.2)$$

for arbitrary $|\Phi\rangle \in \mathcal{H}_0$ and normalized[30] $g(x) \in L^2(\mathbb{R})$ ($i.e.$ $\int_{-\infty}^{\infty} du |g(u)|^2 = 1$.)

$$\boxed{|\widehat{\Phi}\rangle \text{ is contributed from large } x.}$$

[30]The factor of $\epsilon^{1/2}$ is for this purpose.

■ Calculation of the entropy $S(\widehat{\Phi})_{\mathcal{A}_R}$

♦ Form of the density matrix $\rho_{\widehat{\Phi}}$

- To compute the entropy, we need the density matrix.

For Type $\mathbf{I}$ and $\mathbf{II}$ algebras, the density matrix is related to the **relative modular operator** by

$$(*) \quad \Delta_{\Psi|\Phi} = \rho_{\Psi} {\rho'}_{\Phi}^{-1} \tag{7.3}$$

ρ_{Ψ} = density matrix of $|\Psi\rangle$ on $\mathcal{A}$
ρ'_{Φ} = density matrix of $|\Phi\rangle$ on $\mathcal{A}'$

The original definition of $\Delta_{\Psi|\Phi}$ in terms of the relative Tomita operator $S_{\Psi|\Phi}$ is

$$(\star 1) \quad \Delta_{\Psi|\Phi} = S^{\dagger}_{\Psi|\Phi} S_{\Psi|\Phi} \tag{7.4}$$

$$(\star 2) \quad S_{\Psi|\Phi} \mathsf{a} |\Phi\rangle = \mathsf{a}^{\dagger} |\Psi\rangle \,, \quad \mathsf{a} \in \mathcal{A}_{R,0} \tag{7.5}$$

For Type $\mathbf{III}$ algebra, such as $\mathcal{A}_{R,0}$, RHS of $(*)$ is not well-defined.

But one can still define $\Delta_{\Psi|\Phi}$ by $(\star 1), (\star 2)$.

This is the power of Tomita-Takesaki theory.

Answer for the semiclassical density matrix

$$\boxed{\rho_{\widehat{\Phi}} \approx \epsilon \overline{g}(\epsilon h_R) e^{-\beta x} \Delta_{\Phi|\Psi} g(\epsilon h_R) \qquad (7.6)}$$

where $\overline{g}$ is the complex conjugate and $\approx$ means up to $\mathcal{O}(\epsilon)$.

- To prove that this is the desired density matrix, we must show[31]

$$\begin{aligned} &(\boldsymbol{a}) \quad \boldsymbol{\rho}_{\widehat{\boldsymbol{\Phi}}} > \mathbf{0} && (7.7) \\ &(\boldsymbol{b}) \quad \boldsymbol{\rho}_{\widehat{\boldsymbol{\Phi}}} \in \boldsymbol{\mathcal{A}}_{\boldsymbol{R}} && (7.8) \end{aligned}$$

(More precisely $\boldsymbol{\rho}_{\widehat{\boldsymbol{\Phi}}}$ is **"affiliated to"** $\boldsymbol{\mathcal{A}}_{\boldsymbol{R}}$
$\Leftrightarrow$ **bounded functions** of $\boldsymbol{\rho}_{\widehat{\boldsymbol{\Phi}}} \in \boldsymbol{\mathcal{A}}_{\boldsymbol{R}}$) (7.9)

$$(\boldsymbol{c}) \quad \mathbf{tr}[\boldsymbol{\rho}_{\widehat{\boldsymbol{\Phi}}} \widehat{\mathbf{a}}] \approx \langle \widehat{\boldsymbol{\Phi}} | \widehat{\mathbf{a}} | \widehat{\boldsymbol{\Phi}} \rangle \quad {}^{\forall}\widehat{\mathbf{a}} \in \boldsymbol{\mathcal{A}}_{\boldsymbol{R}} \tag{7.10}$$

- $(\boldsymbol{a})$ is evident from the form (7.6) above.

- $(\boldsymbol{b})$ is shown by using the **Connes cocycle** (skipped)

[31] For Type **II**, density matrix is in general unbounded. The form of the condition $(\boldsymbol{b})$ reflects this fact.

- The property (c) can be shown by considering a generic operator $\widehat{\mathsf{a}} \in \mathcal{A}_R$

$$\boxed{\widehat{\mathsf{a}} = \int_{-\infty}^{\infty} ds\, \mathsf{a}(s) e^{is(x+\widehat{h})}\,, \quad \mathsf{a}(s) \in \mathcal{A}_{R,0} \qquad (7.11)}$$

- By putting $\rho_{\widehat{\Phi}}$ above into the definition for the entropy $S(\widehat{\Phi})_{\mathcal{A}_R} = -\mathrm{tr}(\rho_{\widehat{\Phi}} \log \rho_{\widehat{\Phi}}) = -\langle\widehat{\Phi}| \log \rho_{\widehat{\Phi}} |\widehat{\Phi}\rangle$

we obtain the entropy of a semicalssical state as

$$\boxed{\begin{aligned} S(\widehat{\Phi})_{\mathcal{A}_R} &= \langle\widehat{\Phi}|\beta h_R|\widehat{\Phi}\rangle - \langle\widehat{\Phi}|h_{\Psi|\Phi}|\widehat{\Phi}\rangle \\ &\quad -\langle\widehat{\Phi}| \log[\epsilon|g(\epsilon h_R)|^2]|\widehat{\Phi}\rangle + \mathcal{O}(\epsilon) \qquad (7.12) \end{aligned}}$$

where $h_{\Psi|\Phi} = -\log \Delta_{\Psi|\Phi}$.

Below, we will first see that **this formula can be interpreted as**

$$\boxed{\boldsymbol{S(\widehat{\Phi})_{\mathcal{A}_R} = S_{\text{gen}}(b) + \text{const.}} \quad (7.13)}$$

7.2 Correspondence of the RHS of (7.12) to the generalized entropy in the bulk

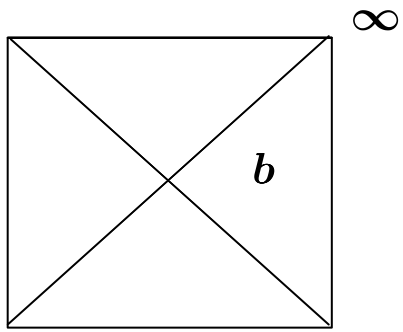
A.Wall showed[32] that **the relative entropy is the differnce of the generalized entropies**

$$\boxed{\boldsymbol{S_{\text{rel}}(\Phi||\Psi) = \Delta S_{\text{gen}} = S_{\text{gen}}(\infty) - S_{\text{gen}}(b)} \quad (7.14)}$$

[32] A.C. Wall, "A proof of the generalized second law for rapidly changing fields and arbitrary horizon slices", PRD85 (2012)104049, arXiv:1105.3445

$S_{\text{gen}}(b)$ = generalized entropy of the **entire right exterior**

$S_{\text{gen}}(\infty)$ = generalized entropy of the exterior of a cut in the black hole horizon, when the cut is taken to future infinity.



Brief description of Wall's argument

♦ In asymptotically AdS space, **any bulk quantum state** $|\Phi\rangle$ on a black hole background becomes **indistinguishable from the vacuum** $|\Psi\rangle$ **at sufficiently late time**.

Reason: Perturbations fall into the black hole across the horizon

Hence

$$\boxed{S_{\text{bulk}}(\infty)_\Phi = S_{\text{bulk}}(\infty)_\Psi = S_{\text{bulk}}(b)_\Psi \qquad (7.15)}$$

The second equality $\Leftarrow$ Ψ is independent of time.

♦ By using the **Raychaudhuri equation**, the **total "expansion"** θ around the SS black hole, which can be related to the difference of the area $A(\partial b) - A(\infty)$, can be identified with a term in $S(\widehat{\Phi})_{\mathcal{A}_R}$, and after some manipulations, one can obtain

$$\boxed{\begin{aligned} S_{\mathrm{gen}}(\infty) - S_{\mathrm{gen}}(b) &= \frac{A(\infty)}{4G} - \frac{A(\partial b)}{4G} \\ &\quad + S_{\mathrm{bulk}}(b)_\Psi - S_{\mathrm{bulk}}(b)_\Phi \\ &= -\langle\Phi|\log\rho_\Psi|\Phi\rangle - S_{\mathrm{bulk}}(b)_\Phi \\ &= S_{\mathrm{rel}}(\Phi||\Psi) \qquad (7.16) \end{aligned}}$$

This proves the Wall's formula (7.14), which is valid even for Type III algebra (where $\log \rho_\Psi$ itself is not well-defined).

This essentially justifies the QES prescription without using holography.

7.3 Comparison with the previous derivation of Bekenstein-Hawking entropy

It is worth making a comparison to previous derivations of the Bekenstein-Hawking entropy in order to clarify **the advantages of the new derivation** given above.

♦ **Hawking's argument**[33]:

His derivation, using the **"the first law of thermodynamics"** for two equilibrium black hole states of slight difference, pointed out by Bekenstein, was as follows.

For spinless chargeless black hole, the **Bekenstein relation** is

$$d(Mc^2) = \frac{\kappa c^2}{8\pi G} dA \tag{7.17}$$

$$A = \text{horizon area of the black hole} \tag{7.18}$$

$$\kappa = \text{surface gravity} \tag{7.19}$$

which is similar to $dE = TdS$.

Hawking then used the the relation between the Hawking tem-

[33] S.W. Hawking, "Black holes and thermodynamics", PRD 13 (1976) 191.

perature T_{H} and κ, discovered by himself, namely

$$T_{\mathrm{H}} = \frac{\kappa \hbar}{2\pi k_B c} \tag{7.20}$$

to eliminate κ and obtained the famous formula identifying the entropy withe the area

$$dS = \frac{k_B c^3}{4G\hbar} dA \tag{7.21}$$

Although the use of T_{H} in the intermediate stage involves quantum physics, **the entropy computed is a classical thermodynamic one, not a statisitical entropy**.

♦ Derivation by Gibbons and Hawking[34]

They rely on the interpretation of the **Euclidean** gravity partition functions as computing a trace over a canonical ensemble.
Such an interpretation is **justified only a posteriori** by the fact that it gives the desired result.

♦ Derivation using von Neumann algebra

- Only a minimal assumptions of **Lorentzian** effective quantum gravity are required.
- As in the Hawking derivation, one derives the variation formula (7.21) over a small range of energies.
- The derivation is performed in the limit where N ($=$ number

[34] G.W. Gibbons and S.W. Hawking, "Action integrals and partition functions in quantum graviaty", PRD 15 (1977) 2752.

of microstates) diverges ($i.e.$ $\boldsymbol{G_N \to 0}$).

• **The primary improvement** = **a direct connection to the statitstical von Neumann entropy of an operator algebra** rather than just a classical thermodynamic relationship between energy and temperature.

7.4 Entropy for the semiclassical states and the generalized second law

Result:

Generalized second law
= **generalized entropy of the black hole horizon increases monotonically with time**.

This is obtained by

Monotonicity of the entropy for "trace-preserving inclusions" of algebras

- This is an interesting topic for which the monotonicity of the von Neumann algebra entropy plays the crucial role.

However as it requires the notion of **trace-preserving inclusions** and technical manipulations, we will have to skip.

8 Application to physics in de Sitter space

8.1 Questions and brief summary

8.1.1 Main questions

- What is the **von Neumann algebra** of observables **for the de Sitter** space ?
- What is **the role of gravity** for determining the algebra ?
- What is **the role of the observer(s)** ?
- Can one construct **the Hilbert space behind the horizon** ?
- What is the **entropy of the static patch** of the de Sitter space ?

♦ What is the **difference from the case of the black hole**?

8.1.2 Brief summary

Results:

♦ The algebra of observables for the static patch of the de Sitter space is the von Neumann algebra of **Type $\mathbf{II_1}$**.
Type $\mathbf{II_1}$ algebra has **a state of maximal entropy**[35] **which, after subtractive renormalization, is zero.**

♦ One can compute the generalized entropy for the de Sitter of

[35] Segal, "A note on the concept of entropy", J.Math.Mech. 9(1960) 623: See also Longo and Witten, "A note on continuous entropy", ArXiv:2202.03357.

the form

$$S_{\text{gen}} = \frac{A}{4G_N} + S_{\text{out}} \tag{8.1}$$

using the **Wall's formula**[36] (already discussed), which expresses S_{gen} in terms of S_{relative}.

♦ If one **assumes the existence of an observer outside the horizon**, who is (almost) maximally entangled with a static patch observer inside, then one can **construct the global Hilbert space for the entire space** (*i.e.* inside and outside) on which the Type $\mathbf{II_1}$ algebra operates.

[36] Wall, "A proof of the generalized second law for rapidly changing fields and arbitrary horizon slaices", PRD85.104049 (hep-th/1105.3445)

Difference from the case of black hole:

- **Case of the black hole:** The algebra is of Type $\mathbf{II}_\infty$

  There is no upper bound to the entropy.

  Case of the de Sitter: The algebra is of Type $\mathbf{II}_1$ and **the entropy is finite**.

  $\mathbf{II}_\infty \Rightarrow \mathbf{II}_1$ is due to the **need to include the DoF of the observer with a positive energy and impose the Wheeler-de Witte constraint $\boldsymbol{H_{\mathrm{tot}} = H + H_{\mathrm{obs}} = 0}$.**

- For de Sitter, **there are no canonical asymptotic observers and different observers see different horizons.**

  $\Rightarrow$ The gravitational dressing, which makes the operators diffeo-invariant, should be made with respect to **each observer**.

Thus, **the observables are observer-dependent**.

8.2 Algebra of the static patch and the type II_1 factor

Static patch of dS space

Static coordinates for the n-dimensional dS space:

$$x^0 = \sqrt{\alpha^2 - r^2}\,\sinh(t/\alpha) \tag{8.2}$$

$$x_1 = \sqrt{\alpha^2 - r^2}\,\cosh(t/\alpha) \tag{8.3}$$

$$x_i = r z_i, \qquad 2 \leq i \leq n \quad z_i \text{ describes } S^{n-2} \tag{8.4}$$

The metric in this coordinate:

$$\boxed{ds^2 = -\left(1-\frac{r^2}{\alpha^2}\right)dt^2 + \frac{1}{1-\frac{r^2}{\alpha^2}}dr^2 + r^2 d\Omega_{n-2}^2 \quad (8.5)}$$

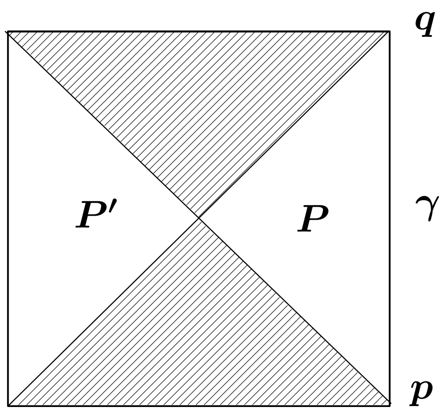
Cosmological horizon at $r = \alpha = \sqrt{3/\Lambda}$.

Comparison with the case of the **SS black hole**

$$ds_{SS}^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega_{n-2}^2 \quad (8.6)$$

Penrose diagram for the static patches

X = entire de Sitter space



γ = a time-like **world line of an observer** traveling from the past infinity p to the future infinity q.

P = **the static patch causally accessible by the observer.**

P depends on p and q but not on γ between them.

P' = **complementary static patch**.

Ordinary QFT on X without gravity

$\mathcal{H}$ = Hilbert space of physical states.

- The algebra of observables in any **local region** is a von Neumann algebra of type $\mathbf{III_1}$[37]
  - Algebra in P = Type $\mathbf{III_1}$ algebra $\mathcal{A}$ acting on $\mathcal{H}$.
  - Algebra in P' = another Type $\mathbf{III_1}$ algebra $\mathcal{A}'$ acting on $\mathcal{H}$, which is a commutant of $\mathcal{A}$. ($[\mathcal{A}, \mathcal{A}'] = \mathbf{0}$.)
- Type $\mathbf{III_1}$ algebra has **no trace**.

$\Rightarrow$ Entropy of a state $\Psi \in \mathcal{H}$ reduced to a region of spacetime has **UV divergence and cannot be defined**[38].

[37] H. Araki, "Type of von Neumann algebra associated with free field", PTP32(64)956,

[38] Sorkin (83) in a talk available at arXiv:1402.3589; Bombelli, Koul, Lee and

With the effect of gravity taken into account

- Effect of gravity in the case of a black hole:

$$\mathbf{III_1} \Longrightarrow \mathbf{II_\infty} = \mathbf{III_1} \rtimes \mathbb{R}_{\mathbf{\Psi}}.$$

- With the gravitational degrees of freedom taken into account, **the automorphism of dS space must be treated as a gauge constraint**.

To achieve this properly:

- Include the **DoF of the obeserver with** $E_{\mathrm{obs}} > 0$
- **Impose the Wheeler-de Witt eq.** $H_{\mathrm{total}} = 0$

This procedure changes the algebra

$$\mathcal{A} = \mathbf{III_1} \Longrightarrow \widehat{\mathcal{A}} = \boxed{\mathbf{II_1} = \Pi(\mathcal{A} \times \mathbb{R})\Pi}$$

Sorkin, PRD34(1986)373.

$\Pi =$ a projection operator for positive energy for the observer.

♦ **For II_1, the trace, the density matrix and the (renormalized) entropy can be defined.**

8.3 Subtractively renormalized entropy for II_1

Consider the system of N **qubits**

- The von Neumann entropy for maximally mixed case is $\boldsymbol{N}$
- **Define the renormalized entropy for a state Ψ_{TFD} reduced to system A by subtracting N before taking the limit**.

⇒ **Entropy of maximally mixed state of the infinite qubits**

$= 0.$

- Thus, **for other general states, entropy is negative.**

This can be checked directly from the definition $S(\rho) = -\mathrm{Tr}\rho \log \rho$ (demonstration omitted).

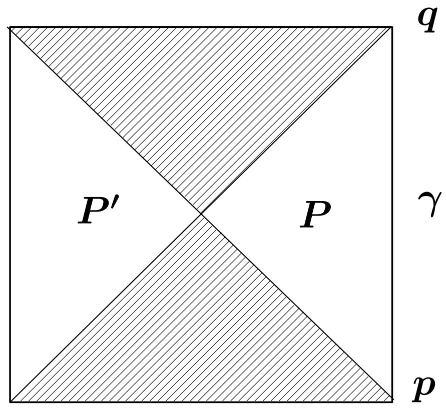
8.4 Static patch of the de Sitter and its thermal nature

What can an observer in the static patch see ?

(1) In a fixed dS background

(2) Including the effect of semi-classical quantization of gravity

Basic fact: The von Neumann algebras associated to the **local regions P and P'**, $\mathcal{A}_P$, $\mathcal{A}_{P'}$, are of Type III_1. Thus **entropies in P and P' are UV divergent and not defined.**



- **However**, it has long been known, since the work of Gibbons and Hawking[39] that the **de Sitter space has a thermal nature** and the approriately defined **correlation functions exhibit this physical feature.**

[39] Gibbons and Hawking, "Cosmological event horizon, thermodynamics and particle creation", PRD15 (77)2738.

Thermal nature of de Sitter

Notations

$\mathcal{H}_{\mathrm{dS}}$ = Hilbert space of a QFT in a fixed dS background
Ψ_{dS} = a natural "vacuum" state, called **Bunch-Davies state**[40] $\in \mathcal{H}_{\mathrm{dS}}$.

Ψ_{dS} is invariant under $SO(1, D)$ (or its double cover to include spin).

[40] Ψ_{dS} can be obtained simply by the analytic continuation of the most symmetric state in the Euclideanized dS $= S^D$.

□ **The case of dS as a background:** *i.e.* **without dynamical gravity**:

- Focus on the static patch P and its compliment P'

Automorphism group of the patch $P = G_P = \mathbb{R}_t \times SO(D-1)$ (similarly for P')

Focus on $\mathbb{R}_t$ = "time translation" generated by Killing vector V (Hamiltonian H)

V can be chosen to be

future-directed time-like in P

past-directed time-like in P'

How to see the thermal nature of de Sitter

When **continued to Euclidean space**, the Killing vector $\boldsymbol{V}$ becomes a generator of **rotation** in a direction on $\boldsymbol{S^D}$.

$\Rightarrow$ Correlation functions in the state $\boldsymbol{\Psi_{\mathrm{dS}}}$[41] become **periodic in imaginary time direction** and can be **interpreted as those in a thermal ensemble** with the following properties:

— a Hamiltonian $\boldsymbol{H_P}$ generating time translation in the patch $\boldsymbol{P}$

— with the inverse temperature $\boxed{\boldsymbol{\beta_{\mathrm{dS}} = 2\pi r_{\mathrm{dS}}}}$

($\boldsymbol{r_{\mathrm{dS}}}$ =radius of curvature of dS).

$\Leftrightarrow$ thermal density operator $\sim \exp(-\boldsymbol{\beta_{\mathrm{dS}} H_P})$

[41] To define this precisely, one need to impose a sort of "brick wall" boundary condition on the cosmologicl horizon, since otherwise, the fluctuations around the horizon hampers the definition of the operator.

Note:

- $\boldsymbol{H_P}$ is not a conserved charge associated with $\boldsymbol{V}$.

It does not act on $\boldsymbol{P'}$ and is hence **"one-sided"** in that sense.

- In algebraic QFT, this corresponds to the statement:

Modular Hamiltonian $\boldsymbol{H_{\mathrm{mod}}}$ which generates the modular automorphism group of $\boldsymbol{\mathcal{A}}$ of the static patch $\boldsymbol{P}$ for the state $\boldsymbol{\Psi_{\mathrm{dS}}}$ is

$$\boldsymbol{H_{\mathrm{mod}} = \beta_{\mathrm{dS}} H_P} \tag{8.7}$$

- **With dynamical gravity taken into account**:

- **The picture above, without gravity, must be <u>substantially modified</u>**

- **<u>Physical argument:</u>**

  - With gravitational degrees of freedom taken into account, diffeomorphisms must be treated as **gauge constraints**.
    $\Rightarrow$ Associated charge H_P **is computed as the surface term**.

  - But the static patch P of dS has no boundary at infinity and its only boundary is the cosmological horizon.
    $\Rightarrow$ H_P **is a boundary term on the horizon.**

♦ For large dS, the leading boundary term is the **area of the horizon**.

Argument of Gibbons and Hawking $\Rightarrow$ Area $\simeq$ entropy of P.

Thus, **“energy” in the static patch $=$ entropy**

This claim will be substantiated by an analysis of the operator algebra of the theory in the presence of gravity.

8.5 Algebra of observables for an observer in the static patch

♦ **What algebra of observables is accessible to an observer following a world line γ in the static patch P ?**

Answer: **Timelike Tube Theorem**[42] says:

It is the **whole algebra $\mathcal{A}$ of observables of P**, **not just the one along the trajectory γ of the observer**.
(Very similar in spirit to the Reeh-Schlieder theorem)

♦ **What is the type of von Neumann algebra of the observables of P** if we include **small quadratic gravitational fluctuations** so that the Hilbert space is $\mathcal{H} = \mathcal{H}_{\mathbf{matter}} \otimes \mathcal{H}_{\mathbf{grav}}$?

Answer **Still Type III_1**, just as for an ordinary QFT in a local region $\boldsymbol{P}$.

[42] Borchers, NC 19 (61)787 ; Araki, "A generalization of Borchers theorem", Helv.Phys.Acta.36(63)132.

♦ **What if we treat gravity more intrinsically ?**

Answer Since de Sitter space is **closed** and has compact spatial sections, **the automophisms of dS space have to be treated as gauge constraints.** (Large gauge transformations)

$\Rightarrow$ The correct Hilbert space is $\widehat{\mathcal{H}}$, which is obtained from $\mathcal{H}$ by **imposing de Sitter generators as constraints**.

♦ **How should we actually impose such constraints ?**

The procedure to do so correctly turns out to be a bit subtle.

♦ **A method that works for the whole dS space**:

Introduce a **BRST complex** for the action of $G_{\rm dS}$ on $\mathcal{H}$ and **define $\widehat{\mathcal{H}}$ as the top degree BRST cohomology called "coinvariants"**[43].

(In contrast, the usual $G_{\rm dS}$ invariant subspace is the bottom degree BRST cohomology).

♦ **The method appropriate for the static patch P**:

The global Hilbert space $\widehat{\mathcal{H}}$ constructed by the above method is not accessible to an observer in P.

What is accessible is an **appropriate algebra of observables**,

[43] Higuchi, CQG 8 (1991)1961 & 1983; Marolf-Morrison, "Group averaging for de Sitter free fields", arXiv:0810.5163

which can be obtained by imposing **$G_P(\subset G_{\rm dS})$-constraints on the algebra itself**.

In particular, the most important will be the **time-translation symmetry** and the **observer who can measure time**.

8.6 Choice and characteristic of an observer

- There are several choices, depending on what an observer should be able to measure.

♦ The simplest but **most fundamental choice**:
Observer$=$ A system that can tell time $i.e.\ \mathbb{R}_+$.
Corresponding constraint operator is the Hamiltonia H.

♦ More general choice: Observer that can measure $SO(D-1)$ quantum numbers as well.

8.7 How to include a time-telling observer

General remark:

- As already mentioned, imposing gauge constraints on the states can be done using BRST cohomology method.

However this is rather cumbersome as **it involves states with various ghost numbers**.

- **It is much simpler to focus on the algebra of physical operators**, **which obviously carry no ghost numbers**.

$\Rightarrow$ **Simply restrict to the subalgebra of invariant operators.**

♠ Due to lack of time, we will skip the explanations of all the logic and simply list the results.

List of important results

- The Hilbert space of the observer is

$$\boxed{\begin{aligned} \mathcal{H}_{\mathrm{obs}} &= L^2(\mathbb{R}_+)\,, \quad \mathbb{R}_+ = \text{half line, } q \geq 0 \\ q &= \text{energy of the observer} \end{aligned}}$$

- The algebra of physical observables is the $\widehat{H}$-invariant part

$$\widehat{\mathcal{A}} = \left(\mathcal{A} \otimes \mathcal{B}(L^2(\mathbb{R}_+))\right)^{\widehat{H}}$$
$$\widehat{H} = H + H_{\mathrm{obs}} = H + q$$

$\mathcal{B}(L^2(\mathbb{R}_+))$ = the algebra of **all bounded operators acting on** $L^2(\mathbb{R}_+)$.

- $\widehat{H}\Psi = 0$ is the Wheeler-de Witte equation.

- Properties of the algebra $\widehat{\mathcal{A}}$

Tentatively ignore the constraint $q \geq 0$.

Then the $\widehat{H}$ invariant part

$\mathcal{A}^{\widehat{H}}$ **is generated by** $e^{ipH}\mathbf{a}e^{-ipH}$ **and bounded functions of** q.

- This is nothing but the **"crossed product"**

  $\boxed{\boldsymbol{\mathcal{A}} \rtimes \mathbb{R}}$

  where $\mathbb{R}$ = the 1-parameter family of automorphism group generated by $\boldsymbol{H}$.

It is a "factor" of Type II_∞ and will be denoted by $\boxed{\mathcal{A}_{\mathbf{cr}}}$. ("cr" stands for "crossed".)

- The **constraint $q \geq 0$** can be implemented by imposing the corresponding projection $\mathbf{\Pi}$:

$$\boxed{\widehat{\boldsymbol{\mathcal{A}}} = \mathbf{\Pi}\mathcal{A}_{\mathbf{cr}}\mathbf{\Pi} \qquad (8.8)}$$

- $\widehat{\boldsymbol{\mathcal{A}}}$ acts on the Hilbert space $\mathbf{\Pi}(\boldsymbol{\mathcal{H}} \otimes \boldsymbol{L}^2(\mathbb{R}) = \boldsymbol{\mathcal{H}} \otimes \boldsymbol{L}^2(\mathbb{R}_+)$.

This algebra is of Type II_1.

Thus the constraint $\boldsymbol{q} \geq \mathbf{0}$ is responsible for making

$$\mathbf{II}_{\infty} \Longrightarrow \mathbf{II}_{1}$$

8.8 Bulk formula for the semiclassical entropy

- The analysis is **very similar to the one performed in obtaining QES prescription using the Wall's formula.**

- The **difference** is to include in the relevant formulas for the density matrix and the entropy **the contribution from the DoF of the observer**.

- For the Schwarzschild-de Sitter space in 4-dimensions,
the result reproduces the **Bekenstein formula**
with a shift of $A_{\text{horizon}}/4G_N$ by the contribution from the observer's energy $-\beta E$.
($\beta = 2\pi r_{\text{dS}}$ = inverse de Sitter temperature.)

8.8.1 Comparison with the case of the black hole

<u>**Correspondence**</u>

$\boxed{\text{Observers in the de Sitter}} \Leftrightarrow$ $\boxed{\text{DoF (time translation) at the asymptotic boundaries}}$

- For the black hole, the constraint on $\widehat{h} = h_R - h_L$ removes one of the two factors of $L^2(\mathbb{R}(t))$ (from the left and the right boundaries).

- Thus before imposing the constraints, the two cases are esentially similar.

Crucial difference

- In the case of BH, $\boldsymbol{h_L, h_R}$ are unbounded and need not be positive. $\Longrightarrow$ Type $\mathbf{II_\infty}$.
- In the case of de Sitter, the energies of the observers are bounded and positive. $\Longrightarrow$ Type $\mathbf{II_1}$.

9 What is the essence of "quantum gravity" ?

9.1 Conspicuous characteristics of quantum gravity

[1]

- We usually believe that the gravitational force is a **"gauge" force**, with the gauge group related to **diffeomorphism**.

- However, in contrast to the gauge theories of internal gauge group describing other three forces, **the diffeomorphism group contains the actions which move the Cauchy surface** as well as the ones restricted to that surface.

- The gauge actions **within the Cauchy surface** lead to the constraints of the vanishing of the wave function by the action of generators $\{H, \vec{P}\}$, namely **the Wheeler-de Witt equation and momentum constraints.**
- They however form only a **commuting algebra, not the diffeomorphism algebra**.

Would the **entire non-abelian diffeomorphsim algebra** tell us about unexplored feature of gravity ?

[2]

- A naive but extremely important feature of gravity is that its **sources are only of one sign** and hence, at least classically, gravitational force is always **attractive**.

This means that

— **sources cannot be neutralized within a local region.**

— **gravitational flux cannot terminate in a local region.**

In the framework of von Neumann algebra, this can be expressed as the **grave problems**, namely

- ♦ **lack of the so called "split property"**
- ♦ **lack of the notion of subregions**

Below we will briefly discuss these issues, based on the works of Haag, Swieca, Borchers, Doplicher, Longo, and others.

9.2 Split property, notion of subregions and quantum gravity

We will discuss the following topics[44]

(1) Essence of the **split property** and the notion of **subregions**

(2) **A von Neumann algebraic characterization of the split property**

(3) Notion of **"nuclearity"** which provides **a sufficient condition** for the existence of the split property

[44] For necessary basic knowledge, see Haag's book.

(4) There are theories for which the **split property is likely to be absent**

☐ **Essence of split property and the notion of subregions**:

- The notion of split property is a **strong form of the "locality principle"** and was originally conceived as

the condition for the theory to be interpreted as containing **asymptotic particle states** $\simeq$ the condition to **behave as a "normal" thermodynamic system**, obeying the KMS condition.

One characterization of split property

Let $\mathcal{U}_1, \mathcal{U}_2$ be **spacelike separated regions** and $\mathcal{A}_i(\mathcal{U}_i)$ denote a net of algebras defined on $\mathcal{U}_i$.

Then, the theory is said to possess the **split property** if

- ♦ $[\mathcal{O}_1(\mathcal{U}_1), \mathcal{O}_2(\mathcal{U}_2)] = 0$, for $\mathcal{O}_i \in \mathcal{A}(\mathcal{U}_i)$, **and further**
- ♦ the two algebras $\mathcal{A}_i$ are **"statistically independent"**.

Def. of statistical independence:

Let φ_i $(i = 1, 2)$ be normal states of the algebras $\mathcal{A}_i(\mathcal{U}_i)$. These algebras are said to be statistically independent if

- ♦ there exists a normal state[45] φ extended to $\mathcal{A}_1(\mathcal{U}_1) \vee \mathcal{A}_2(\mathcal{U}_2)$ such that the **measurements by φ of the operators in $\mathcal{A}_i(\mathcal{U}_i)$ do not have correlations**, *i.e.*

$$\varphi(\mathbf{a}_1\mathbf{a}_2) = \varphi(\mathbf{a}_1)\varphi(\mathbf{a}_2)\,, \quad \mathbf{a}_i \in \mathcal{A}_i(\mathcal{U}_i) \tag{9.1}$$

- ♦ and further if φ_i are faithful states, φ should be a faithful state of $\mathcal{A}_1(\mathcal{U}_1) \vee \mathcal{A}_2(\mathcal{U}_2)$.

- When the split property exists, such spacelike separated regions $\mathcal{U}_i$ are independent **subregions** (of the total space).

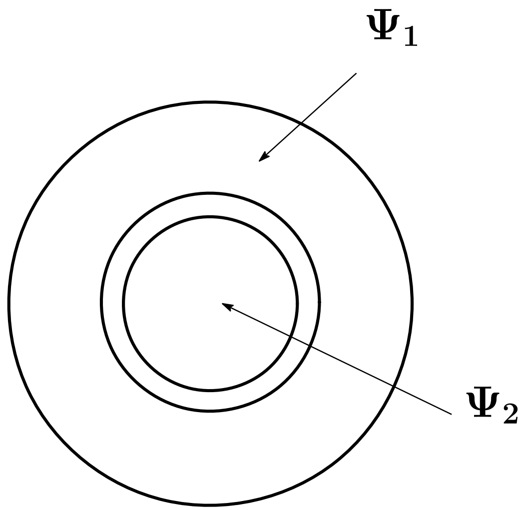
[45] Recall ϕ is normal if $\phi(\mathbf{a}) = \sup \phi(\mathbf{a}_i)$ whenever $\mathbf{a}$ is the strong limit of a monotonically increasing sequence $\{\mathbf{a}_i\}$ in $\mathcal{A}_+$.

Rough comparison of QED and Quantum Gravity

- QED has a split property, essentially due to the microcausality for the charge density operators $[\rho(x), \rho(x')] = 0$, for x and x' on the same spatial slice. This allows the situation

$$\Psi_1 \neq \Psi_2,$$

$$\Psi_i = \text{wave functions}$$



Specifying the integrated charge does not determine the local charge density hence the wave function.

- **For quantum gravity, there are several arguments against the existence of split property.**

  - In contrast to the charge density of QED, there exist the famous Schwinger term (commutator anomaly)[46] and in general the stress tensor components do not commute: [47]

$$[T_{\mu\nu}(x), T_{\rho\sigma}(x')] = \text{Schwinger term} \neq 0 \tag{9.2}$$

  - For gravity, the "energy" is the "charge" and due to the uncertainty principle, a state with a definite energy is delocalized in the temporal direction.

    There is no such property for the QED charge.

---

[46] Deser and Boulware, JMP 8 (67) 1468.

[47] **In lattice regularization, the stress tensors at adjacent space-like points do not commute**.

♦ For gravity, there is only one sign for the “charge”.

Unless there can be localized regions with negative energy somehow, the energy flux must continue to ∞ (or to a singularity ?) and no diffeo-invariant subregions can exist.

□ **von Neumann algebraic characterization of the split property**:

- **There are several equivalent definitions of split property.**

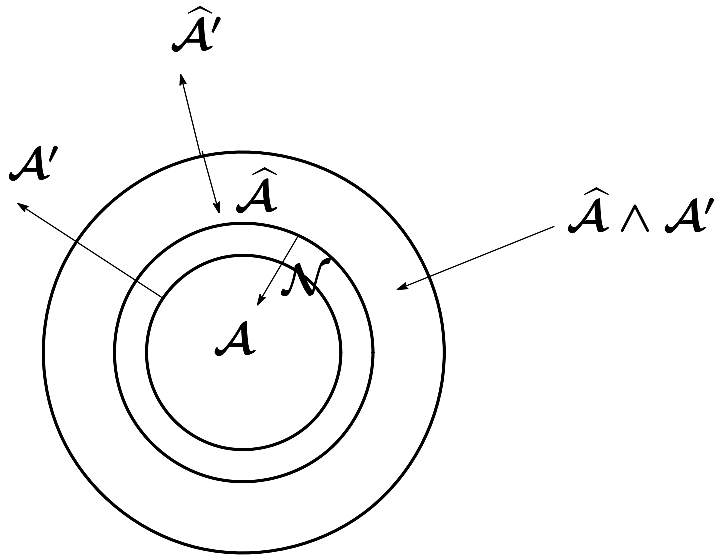
Among them, the following one is clear in the language of von Neumann algebra and stimulating:

Set up

$\mathcal{A}, \widehat{\mathcal{A}}$ = von Neumann factors acting on $\mathcal{H}$.

$\mathcal{A} \subset \widehat{\mathcal{A}}$

Ψ = a cyclic and separating vector for $\mathcal{A}, \widehat{\mathcal{A}}'$ and $\mathcal{A} \vee \widehat{\mathcal{A}}'$



- In this set up, **the theory has a split property** if $\exists$ a **type I factor $\mathcal{N}$** such that

$$\boxed{\mathcal{A} \subset \mathcal{N} \subset \widehat{\mathcal{A}}}$$

This relation is called the **"standard split inclusion"**

□ **Notion of "nuclearity", which provides a sufficient condition for split property**:

- One important sufficient condition for the existence of the split property is so-called the **"nuclearity" condition**[48].

- Roughly, this condition is an **inequality**, bounding from above a certain quantity called **"nuclear index"**, which reflects the **energy density of a localized system at a finite temperature**.

Nuclearity condition

- $\mathcal{L}_r(\subset \mathcal{H})$ = a set of states of the excitation of the vacuum, localized, at a fixed time, **in a ball of radius r**.

[48] The simplest example of this concept was first discussed by Grothendieck (1955)

- Given the Hamiltonian H of the system, one can define the **"nucliear index"** $\mathcal{V}\left(e^{-\beta H}\mathcal{L}_r\right)$, which has the property

$$\mathcal{V}\left(e^{-\beta H}\mathcal{L}_r\right) \leq \mathrm{Tr}\left(e^{-\beta H}\right).$$

- Then, the von Neumann algebra $\mathcal{A}$ relevant to the system is called **"nuclear"**

if ${}^\forall\beta > 0, {}^\forall r > 0$ there exist constants c, n, r_0, β_0 such that

$$\mathcal{V}\left(e^{-\beta H}\mathcal{L}_r\right) \leq e^{cr^3\beta^{-n}}, \quad {}^\forall r \geq r_0,\ 0 < \beta \leq \beta_0$$

- **<u>Theorem:</u>**

Nuclearity $\Rightarrow$ Split property

- The β-dependence of the nuclear index is determined by the **energy density at high energy**.

- Correspondingly, for the proof of the theorem, **what is important is the existence of the upper bound for the level density of the localized states at high energy.**

□ **Examples of theories which are likey to lack split property**:

- Doplicher and Longo[49] gave some examples of theories which are likely to lack the split property because the nuclearity condition is not met.

[49] Invent. math. 73 (1984) 493

♦ **Theory which has ∞ number of particles and the number of species grows very rapidly with the mass.**

$\Rightarrow$ No reasonable notion of temperature at ultra-high energy, $i.e.$ existence of **Hagedorn temperature**.

- This suggests:

string theory may belong to this class.

- **This is in good accord with AdS/CFT** since the dual of CFT is a string theory, not just a theory with massless gravitational field and some matter fields.

But this points to an extremely important question

How is the Planck scale structure related to the existence of massless gravitonational field obeying ordinary diffeomorphism invariance ?

9.3 Important problems for investigating quantum gravity in the framework of von Neumann algebra

- **Is the lack of split property a key feature of QG ?**

  - It may be important to improve or simplify the inequality for the nuclearity condition which leads to split property.

  - **Find a way to see if string theory has the split property or not.**

■ **More generally, how should we study string theory in the framework of von Neumann algebra ?**

Another intriguing and essential question we wish to understand is

■ **What is the difference of the role of entanglement in ordinary QFT in flat space and in QG ?**

♦ **Reeh-Schlieder theorem** says that **the vacuum, the "smoothest state", is maximally entangled at all scales.**

$\Rightarrow$ **A certain type of disentanglement would destroy this smoothness and produce curvature and nontrivial geometry**[50]. Can we characterize such type of **disentanglement** ?

[50]See, for example, M. van Raamsdonk, "Building up spacetime with quantum entanglement", Gen.Rel.Grav. 42 (2010) 2323-2329

Looking through the window of von Neumann algebra , we may find many other intersting and essential questions on Quantum Gravity