

Non-Invertible Symmetries and Scattering Amplitudes

Shota Komatsu



Based on works with Christian Copetti (Oxford) Lucía Córdova (CERN)

2403.04835: “Non-Invertible Symmetries, Anomalies and Scattering Amplitudes”

To appear soon: “S-Matrix Bootstrap and Non-Invertible Symmetries”

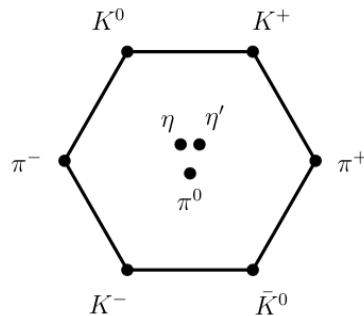
Why are Symmetries Important?

- **Symmetries**: Fundamental concepts in physics.
- Constrain the Renormalization Group flows.

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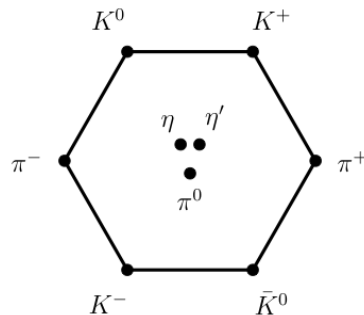
Symmetries: organizing principles for particles and resonances.



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Symmetries: organizing principles for particles and resonances.



- A variety of modern refinements / generalizations.

higher-form symmetries, non-invertible symmetries, 2-groups, ...

[Gaiotto, Kapustin, Seiberg, Willet], [Frolich, Fuchs, Runkel, Schwiebert], [Tachikawa, Bhardwaj], [Chang, Lin, Shao, Wang, Yin], [Benini, Cordova, Hsin], [Cordova, Dumitrescu, Intriligator]..... and many others

Question

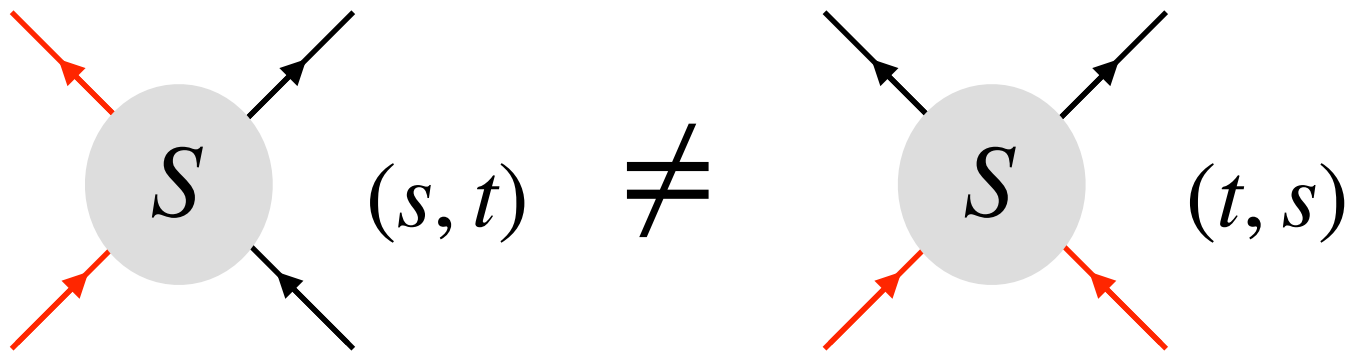
What are implications of modern **symmetries** on
scattering amplitudes?

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Punchlines of the talk

1. In the presence of certain* **non-invertible symmetries** or **anomalies**, **crossing symmetry** of S-matrix is modified.



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What are implications of modern **symmetries** on **scattering amplitudes**?

Punchlines of the talk

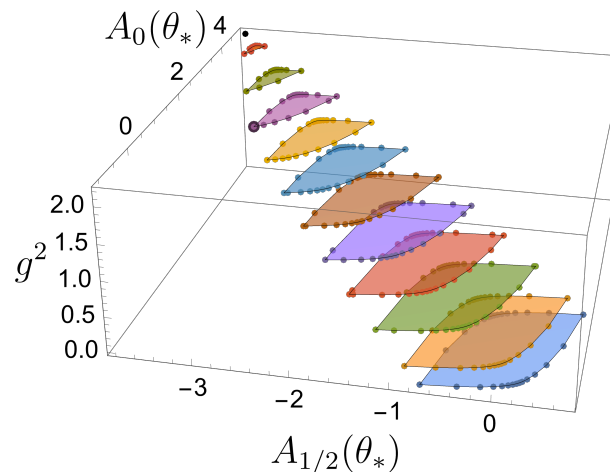
1. In the presence of certain* **non-invertible symmetries** or **anomalies**, **crossing symmetry** of S-matrix is modified.
 - RG flows to gapped phases in 1+1 dimensions. (Comments on higher d at the end)
 - Use integrable flows to check the claim but applies to non-integrable theories as well.

Question

What are implications of modern **symmetries** on
scattering amplitudes?

Punchlines of the talk

2. (Numerical) **S-matrix bootstrap** with modified crossing
constrains the space of theories with **non-invertible symmetries**



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What are implications of modern **symmetries** on **scattering amplitudes**?

Punchlines of the talk

2. (Numerical) **S-matrix bootstrap** with **modified crossing** constrains the space of theories with **non-invertible symmetries**

- Future applications / ongoing works: theories with Haagerup fusion category, monopole scattering, scattering in 3d quantum gravity.... (Comments at the end)

Plan

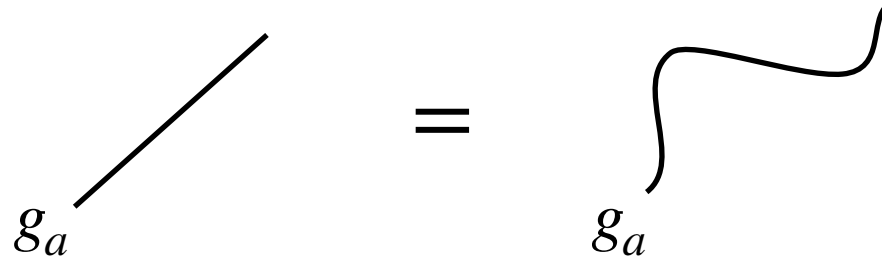
1. Non-invertible symmetries in $1+1$ dim
2. Integrable flow from tricritical Ising and S-matrix
3. Derivation of modified crossing rules
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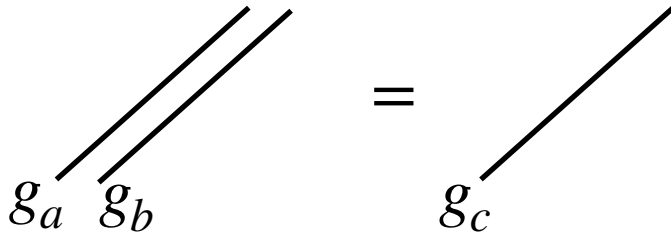
Non-Invertible Symmetries in 1+1 dim

- Symmetries in QFT $\subset$ (d-1)-dim topological operators



A diagram illustrating a symmetry operator g_a . On the left, a straight line segment is labeled g_a at its bottom-left end. This is followed by an equals sign. On the right, a wavy line segment is labeled g_a at its bottom-left end. The wavy line starts with a vertical segment, then curves to the right and back to the left, ending with a vertical segment.

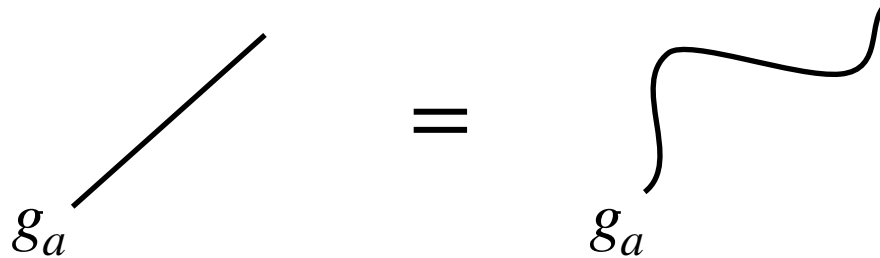
- Usual symmetries (group): $g_a g_b = g_c$ $g_{a,b,c} \in G$



A diagram illustrating the composition of two symmetry operators g_a and g_b . On the left, two parallel line segments are shown, with the left one labeled g_a and the right one labeled g_b at their bottom-left ends. This is followed by an equals sign. On the right, a single line segment is labeled g_c at its bottom-left end.

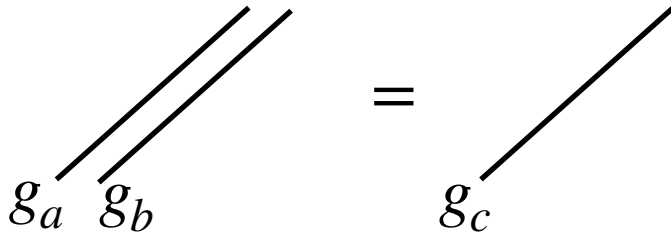
Non-Invertible Symmetries in 1+1 dim

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A diagram illustrating a symmetry operator g_a in 1+1 dimensions. On the left, a straight line segment is labeled g_a at its bottom-left end. This is followed by an equals sign. On the right, a wavy line segment is labeled g_a at its bottom-left end. This represents the fact that the operator is topological and its action is independent of the path it takes.

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A diagram illustrating the composition of two symmetry operators g_a and g_b . On the left, two parallel line segments are shown, with the left one labeled g_a and the right one labeled g_b . This is followed by an equals sign. On the right, a single line segment is labeled g_c . This represents the group multiplication of the symmetries.

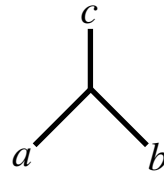
- Non-invertible symmetries (in 1+1 dim):

Fusion category $\mathcal{L}_a \mathcal{L}_b = \sum_{\mathcal{L}_c} N_{ab}^c \mathcal{L}_c$ $(N_{ab}^c \in \mathbb{Z}_{\geq 0})$

Data characterizing fusion categories

$$\mathcal{L}_a \mathcal{L}_b = \sum_{\mathcal{L}_c} N_{ab}^c \mathcal{L}_c$$

- Fusion coefficient N_{ab}^c :



If $N_{ab}^c \neq 0$

- Quantum dimension :

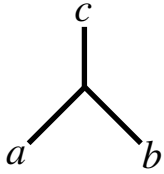
$$\langle \mathcal{L}_a \rangle = \text{loop diagram} = d_a \begin{cases} 1 & \text{group-like} \\ \sqrt{2} & \text{etc. for non-inv.} \end{cases}$$

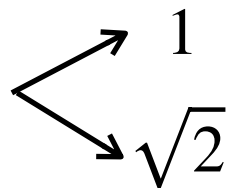
- F -symbol:

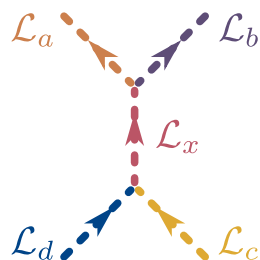
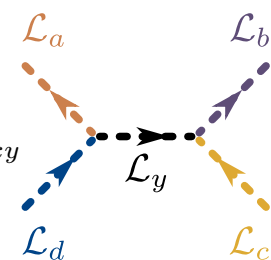
$$= \sum_y F_d^{abc}_{xy}$$

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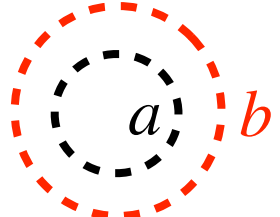
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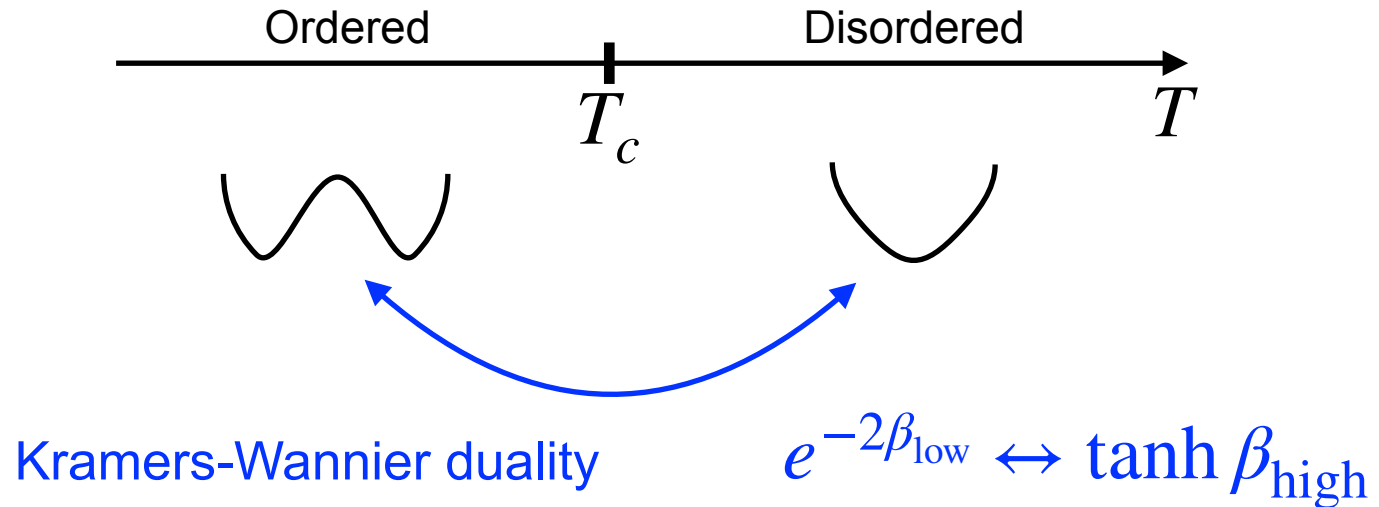
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They satisfy various consistency conditions.

e.g.  $\mapsto d_a d_b = \sum_c N_{ab}^c d_c$

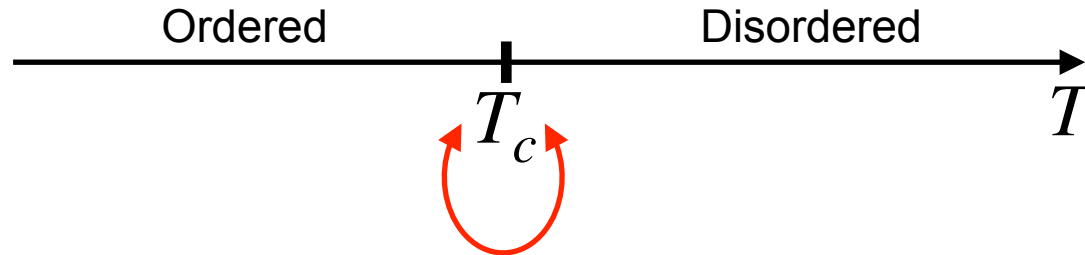
Non-Invertible Kramers-Wannier

- 2d Ising model:
$$Z = \sum_{\{\sigma\}} \exp \left[\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j \right]$$



Non-Invertible Kramers-Wannier

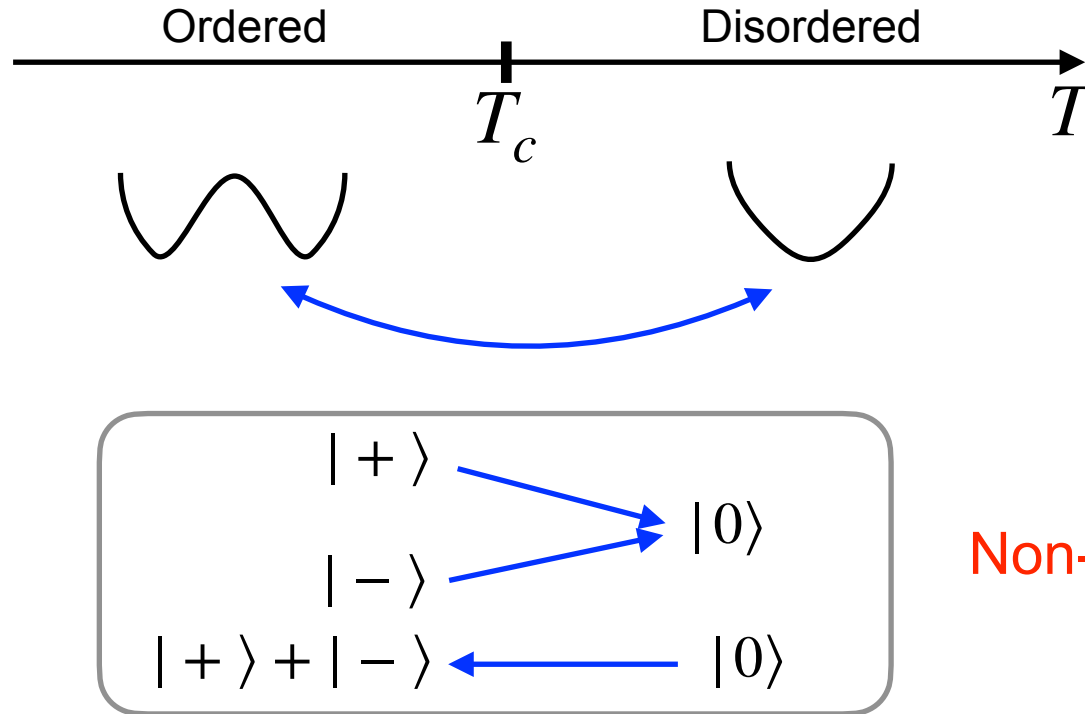
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Kramers-Wannier **symmetry**

Non-Invertible Kramers-Wannier

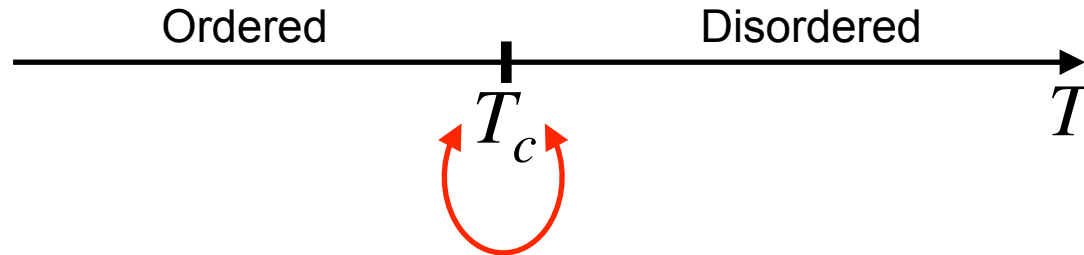
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Non-invertible

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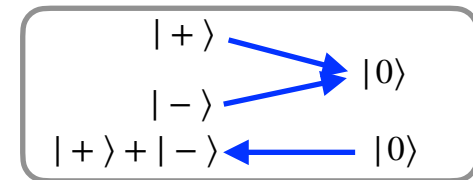


Kramers-Wannier **symmetry** $\mathcal{N}$

- Symmetry of 2d Ising CFT

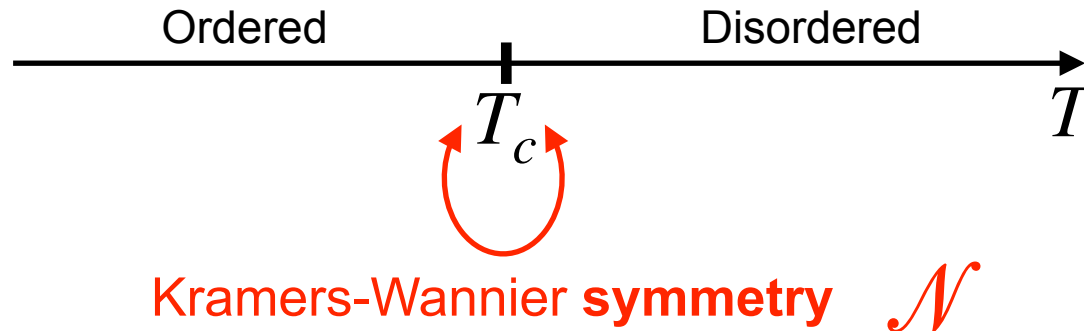
$$\eta^2 = 1, \quad \mathcal{N}^2 = 1 + \eta, \quad \mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N}$$

$$(d_1 = d_\eta = 1, \quad d_{\mathcal{N}} = \sqrt{2})$$



Non-Invertible Kramers-Wannier

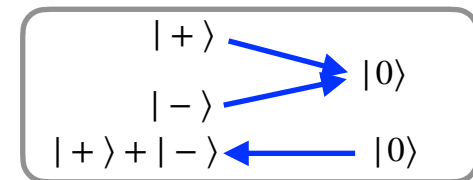
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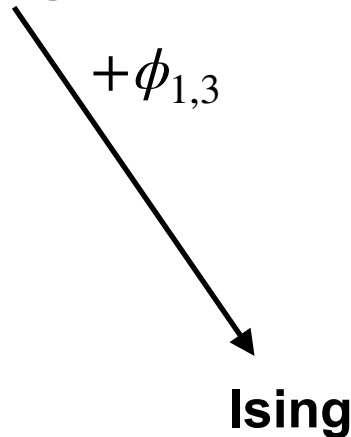


- In Ising CFT, KW symmetry **broken** by relevant perturbations
- RG flow preserving KW symmetry...?

Flow from tricritical Ising

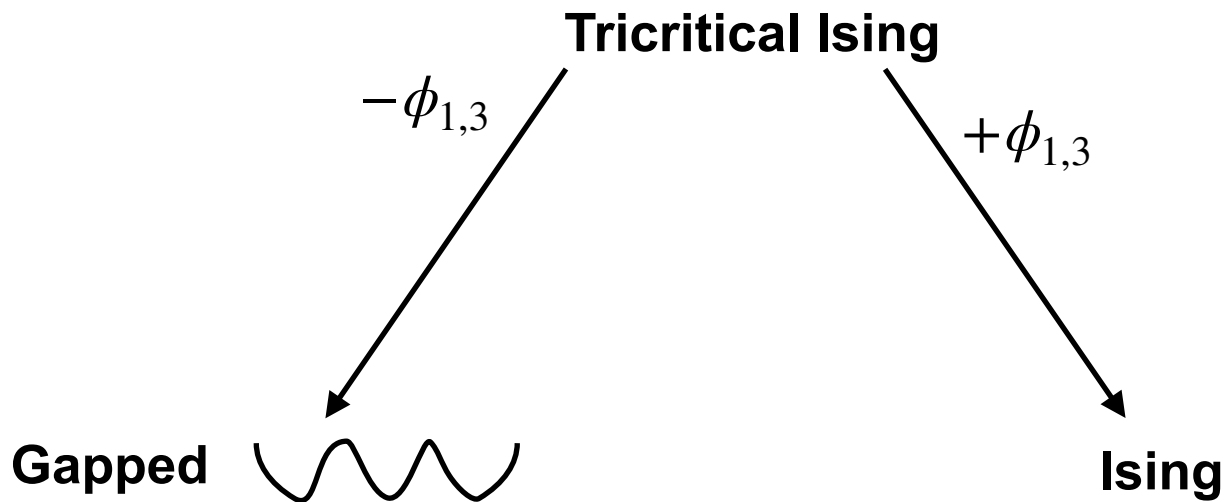
- Ising CFT is the simplest unitary minimal model ($\mathcal{M}_{3,4}$).
- Next simplest is **tricritical Ising** CFT ($\mathcal{M}_{4,5}$).
- Non-invertible symmetric RG flow from tricritical: [Chang, Lin, Shao, Wang, Yin]

Tricritical Ising

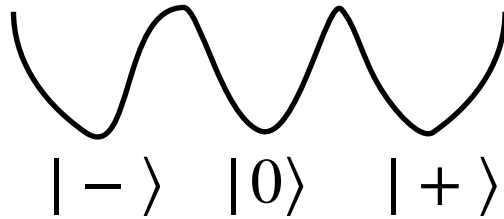


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Action on vacua



- $\mathbb{Z}_2$ -defect exchanges $| + \rangle$ and $| - \rangle$

$$\eta : | + \rangle \leftrightarrow | - \rangle$$

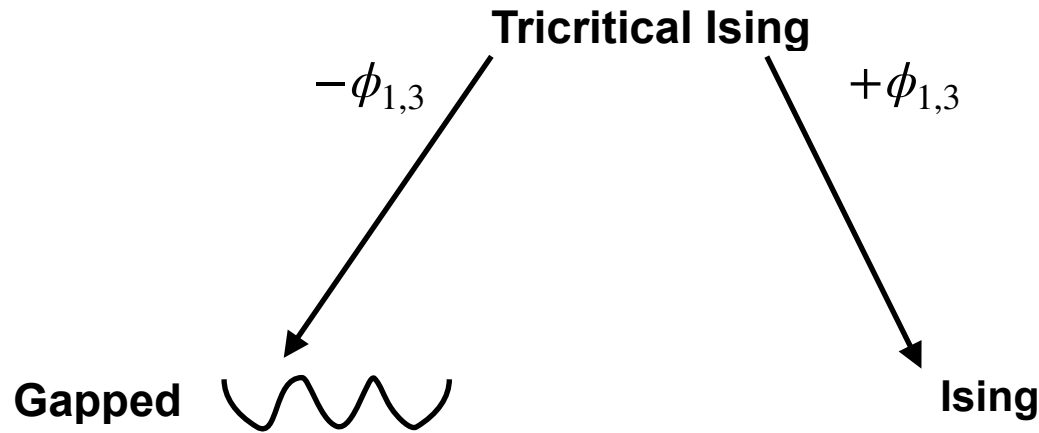
- $\mathcal{N}$ -defect:

$$\mathcal{N} | 0 \rangle = | + \rangle + | - \rangle$$

$$\mathcal{N} | + \rangle = \mathcal{N} | - \rangle = | 0 \rangle$$

- “Superposition” of disordered and ordered vacua in Ising

Remarks



- 3 is the **minimal** number of vacua allowed by non-invertible sym.

$$\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Entries need to be non-negative integers

$$\eta^2 = 1, \quad \mathcal{N}^2 = 1 + \eta,$$

$$\mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N}$$

- Similar patterns for higher minimal models cf. [Tanaka. Nakayama 2407.21353]

$$\mathcal{M}_{n,n+1} - \phi_{1,3} \rightarrow \text{gapped with } (n-1) \text{ vacua}$$

$$\mathcal{M}_{n,n+1} + \phi_{1,3} \rightarrow \mathcal{M}_{n-1,n}$$

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Integrability along the flow

Tricritical Ising $\xrightarrow{-\phi_{1,3}}$ 

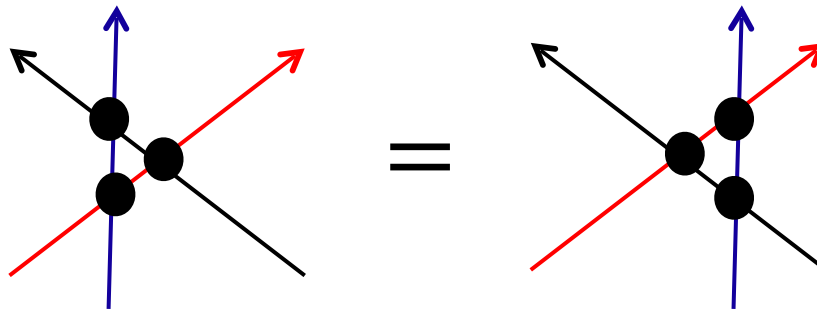
- At UV CFT fixed point, there exist ∞ many higher spin charges.
because of Virasoro
- Perturbation by $\phi_{1,3}$, $\phi_{2,1}$ or $\phi_{1,2}$ preserve higher spin charges.

[Zamolodchikov 1989]

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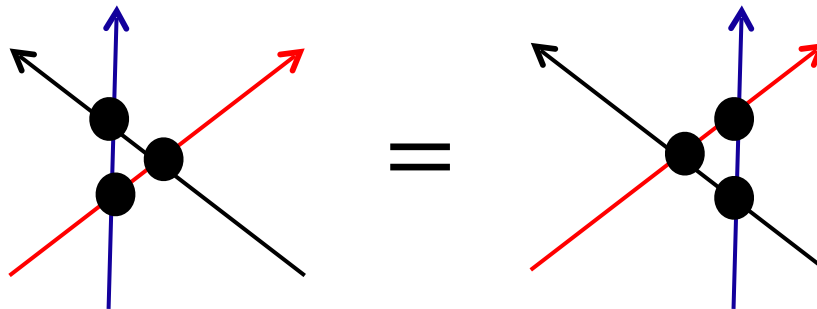
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- Multi-particle S-matrix factorizes to $2 \rightarrow 2$ S-matrices.
[Shankar, Witten] [Parke]
- $2 \rightarrow 2$ S-matrices satisfy the Yang-Baxter equation.



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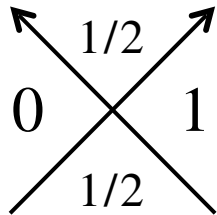
- At UV CFT fixed point, there exist ∞ many **higher spin** charges.
because of Virasoro
- Perturbation by $\phi_{1,3}$, $\phi_{2,1}$ or $\phi_{1,2}$ preserve **higher spin** charges.
[Zamolodchikov 1989]
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[Shankar, Witten] [Parke]
- $2 \rightarrow 2$ S-matrices satisfy the **Yang-Baxter** equation.



- Imposing unitarity, crossing & YB, the S-matrix can be almost uniquely **“bootstrapped”**.
[Zamolodchikov 1991]

S-matrix by Zamolodchikov

- Particles in IR = **kinks** interpolating between adjacent vacua.

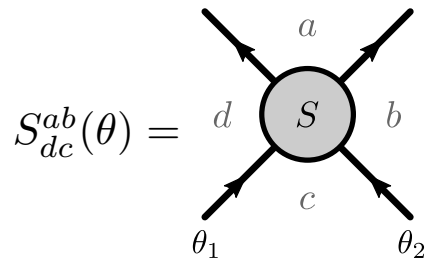


etc .



*Warning: Change of notation!

- It also depends on Mandelstam variable $s = (p_1 + p_2)^2 = 4m^2 \cosh^2(\theta/2)$

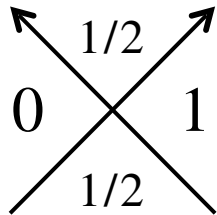


$$\theta = \theta_1 - \theta_2$$

$$s + t = 4m^2 \quad t = -4m^2 \sinh^2(\theta/2)$$

S-matrix by Zamolodchikov

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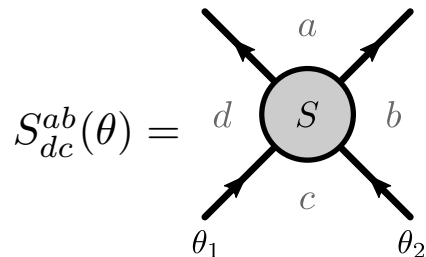


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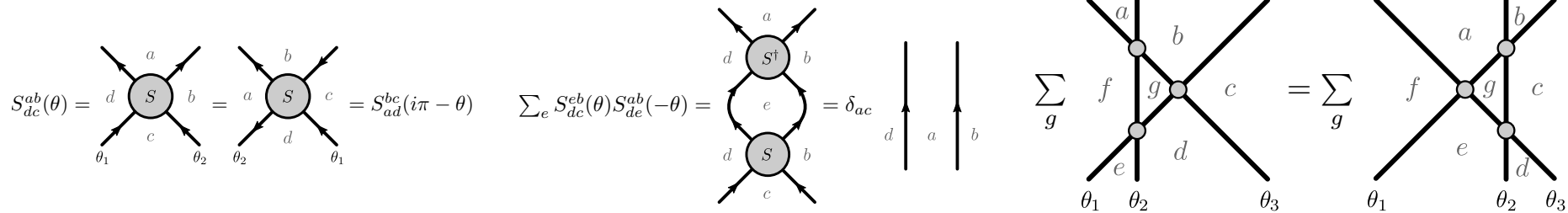
- Crossing symmetry & unitarity

$$S_{dc}^{ab}(\theta) = \text{Diagram 1} = \text{Diagram 2} = S_{ad}^{bc}(i\pi - \theta)$$

$$\sum_e S_{dc}^{eb}(\theta) S_{de}^{ab}(-\theta) = \text{Diagram 3} = \delta_{ac}$$

S-matrix by Zamolodchikov

- Imposing crossing & unitarity & YB,



we can fix S-matrix uniquely (up to overall “CDD” factor) as

$$\hat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d} \right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

$$(n = 3, \quad d_0 = d_1 = 1, \quad d_{1/2} = \sqrt{2})$$

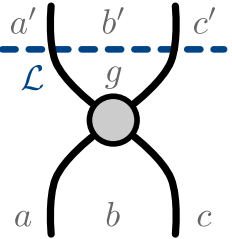
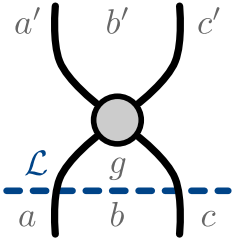
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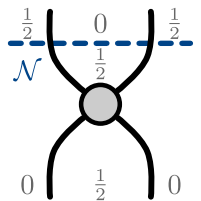
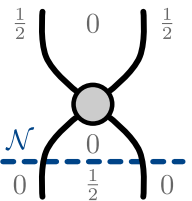
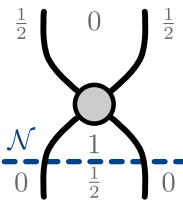
- We expect that it also preserves non-invertible symmetries:



Paradox

$$\sum_g \text{Diagram 1} = \sum_g \text{Diagram 2} \quad ? \quad \hat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d} \right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$



- We found that $\hat{S}$ commutes with $\eta (\mathbb{Z}_2)$, but **not with $\mathcal{N}$** .

$$\text{Diagram 3} \stackrel{?}{=} \text{Diagram 4} + \text{Diagram 5}$$




$$\mathcal{N} : S_{0\frac{1}{2}}^{\frac{1}{2}0}(\theta) \stackrel{?}{=} S_{\frac{1}{2}0}^{0\frac{1}{2}}(\theta) + S_{\frac{1}{2}1}^{0\frac{1}{2}}(\theta)$$

Paradox

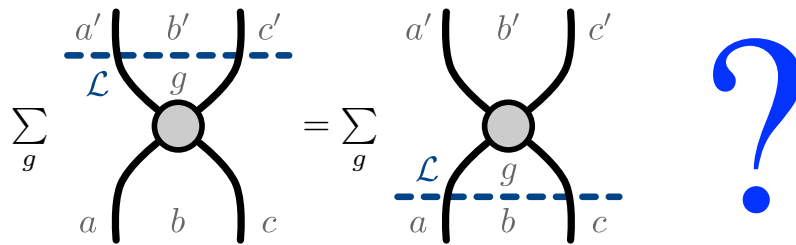
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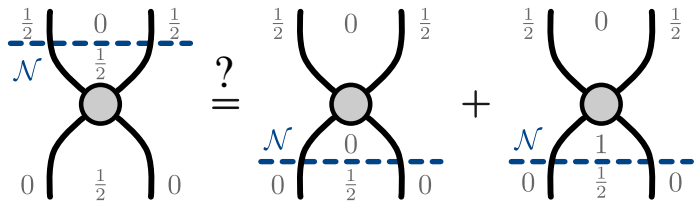
$$\text{Diagram 1} \stackrel{?}{=} \text{Diagram 2} + \text{Diagram 3} \quad \mathcal{N} : \quad \cancel{S_{0\frac{1}{2}}^{\frac{1}{2}0}(\theta)} \stackrel{?}{=} \cancel{S_{\frac{1}{2}0}^{0\frac{1}{2}}(\theta)} + \cancel{S_{\frac{1}{2}1}^{0\frac{1}{2}}(\theta)}$$

- The following 4 properties are mutually incompatible.
 - Unitarity
 - Crossing
 - Integrability (YB)
 - Non-invertible symmetry

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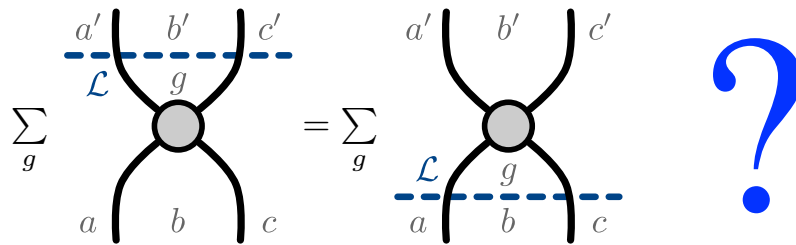
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$$\text{Diagram 1} \stackrel{?}{=} \text{Diagram 2} + \text{Diagram 3} \quad \mathcal{N} : \quad \cancel{S_{0\frac{1}{2}}^{\frac{1}{2}0}(\theta)} \stackrel{?}{=} \cancel{S_{\frac{1}{2}0}^{0\frac{1}{2}}(\theta)} + \cancel{S_{\frac{1}{2}1}^{0\frac{1}{2}}(\theta)}$$


- The following 4 properties are mutually incompatible.

- Unitarity  basic principle
- Crossing
- Integrability (YB)
- Non-invertible symmetry

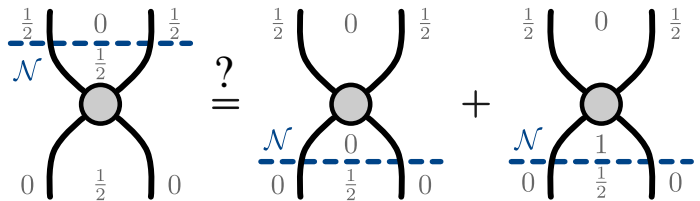
Paradox



$$\sum_g \text{Diagram 1} = \sum_g \text{Diagram 2} \quad ?$$

$$\hat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d} \right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

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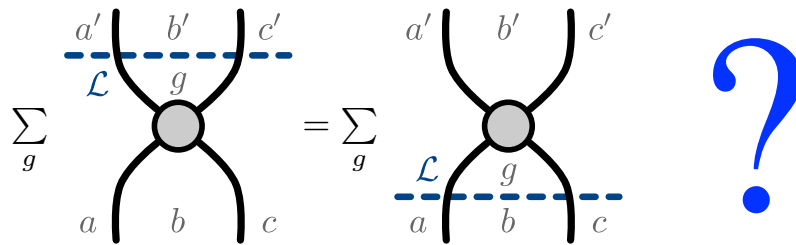
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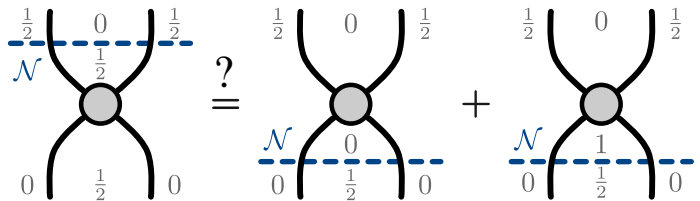
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- The following 4 properties are mutually incompatible.

- Unitarity  basic principle
- Crossing**
- Integrability (YB)  deformation of CFT
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- The only viable option is to give up **crossing**.

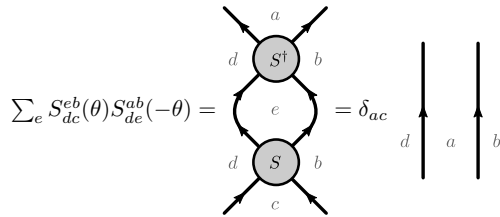
New proposal

$$S_{dc}^{ab}(\theta) = Z(\theta) \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

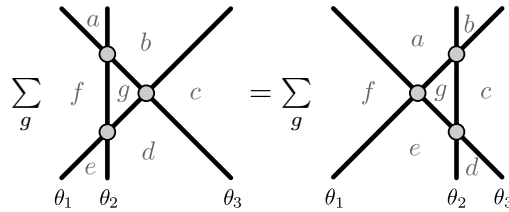
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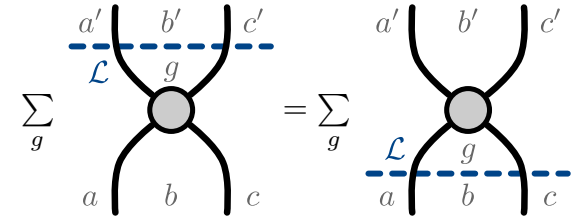
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Unitarity



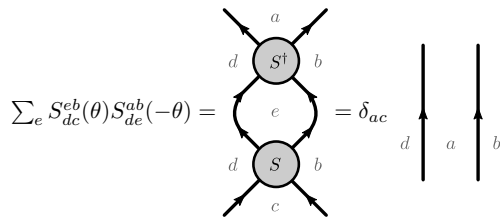
Yang-Baxter



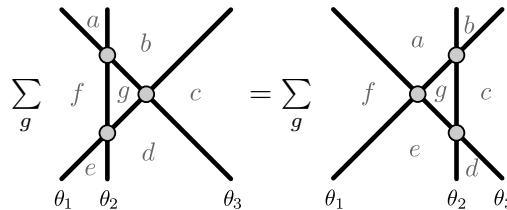
Non-invertible

New proposal

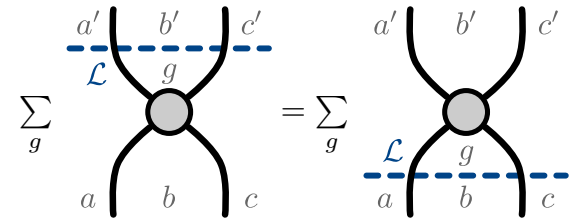
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Unitarity



Yang-Baxter



Non-invertible

- Crossing symmetry is modified:

$$S_{dc}^{ab}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{ad}^{bc}(i\pi - \theta)$$

- Physical origin? (The topic of the rest of the talk)

1. Non-invertible symmetries in 1+1 dim

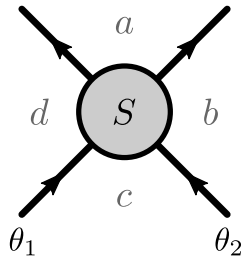
2. Integrable flow from tricritical Ising and S-matrix

3. Derivation of modified crossing rules

4. S-matrix bootstrap

5. Conclusion

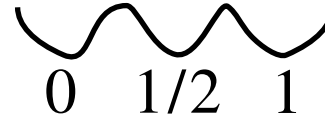
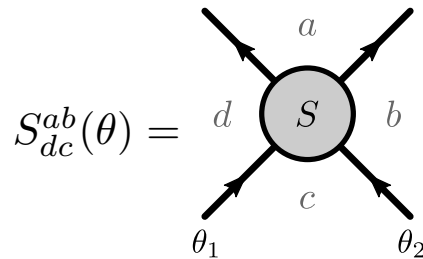
Key physical input

$$S_{dc}^{ab}(\theta) =$$



$$0 \quad 1/2 \quad 1$$

- The IR dynamics is described by a **nontrivial TQFT**.

Key physical input

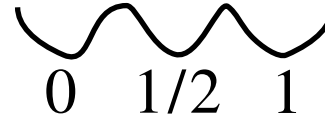
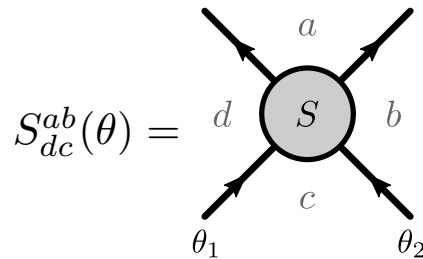


- The IR dynamics is described by a **nontrivial TQFT**.

1. In the IR, the action of a **kink** on vacua = the action of **symmetry line**
 $\mathcal{N} (=: \nu)$

$$\mathcal{N} |0\rangle = |1/2\rangle, \quad \mathcal{N} |1/2\rangle = |0\rangle + |1\rangle, \quad \mathcal{N} |1\rangle = |1/2\rangle.$$

Key physical input



- The IR dynamics is described by a **nontrivial TQFT**.

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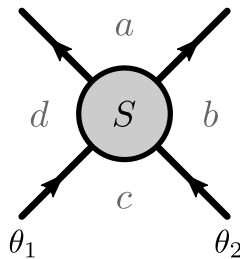
2. The **vacua** are in 1-to-1 correspondence with **symmetry lines**.

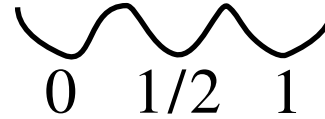
$$|0\rangle \leftrightarrow \mathbf{1} \quad |1/2\rangle \leftrightarrow \mathcal{N} \quad |1\rangle \leftrightarrow \eta$$

("Adjoint" representation of fusion category)

$$\mathcal{N} |1/2\rangle = |0\rangle + |1\rangle \quad \leftrightarrow \quad \mathcal{N}^2 = \mathbf{1} + \eta$$

Key physical input

$$S_{dc}^{ab}(\theta) =$$




- The IR dynamics is described by a **nontrivial TQFT**.

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(“Adjoint” representation of fusion category)

3. All the vacua can be obtained from $|0\rangle$ by acting symmetry lines.

$$\mathcal{N}|0\rangle = |1/2\rangle$$

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Key physical input

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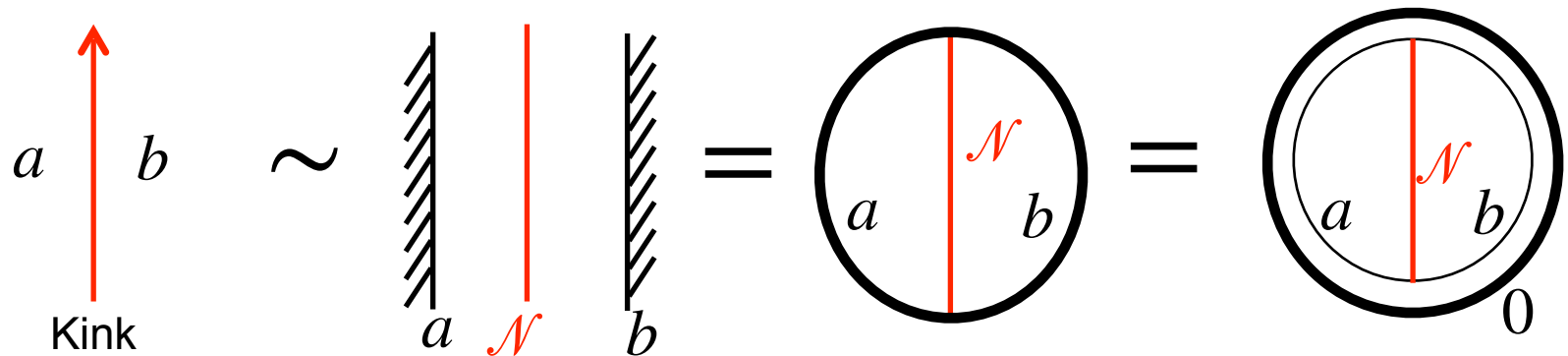
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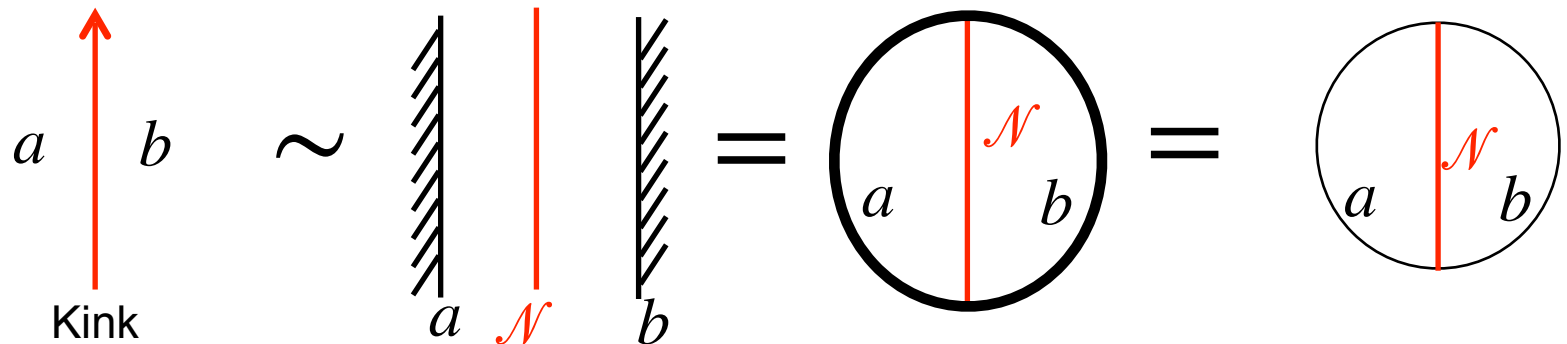
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kink states $\leftrightarrow$ network of lines

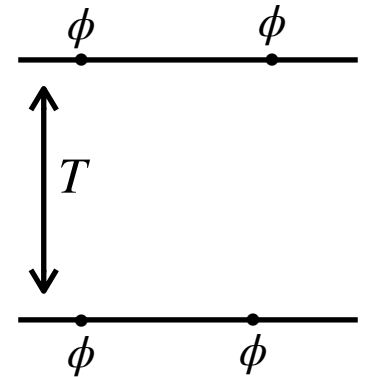
Modified crossing for S-matrix

- Kink creation operators are **non-local** (boundary cond. changing op)

Unclear how to get **S-matrix** from **LSZ** reduction.

- Use alternative (discussed in Itzykson-Zuber Ch.5)

$$S(\{p_i\}) = \lim_{T \rightarrow \infty} \int \prod_j dv_j e^{ip_j x(v_j)} \prod_k (n_k \cdot \overset{\leftrightarrow}{\partial}_{x_k}) G(\{x(v_k)\})$$



- In our case,

$$S_{dc}^{ab}(\theta) \propto \left(\text{disk correlator of bcc ops.} \right) \Big|_{\text{analyt. cont.}}$$

- But we need to take into account normalization.

$$|\text{in}\rangle\rangle = \text{diagram with two vertical dashed lines labeled } v \text{ and } v, \text{ and four colored arcs labeled } d, c, b, a$$

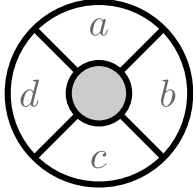
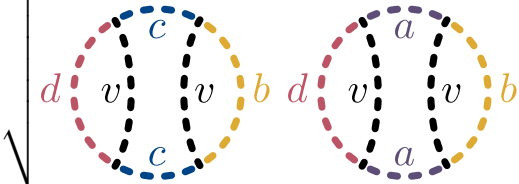
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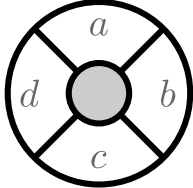
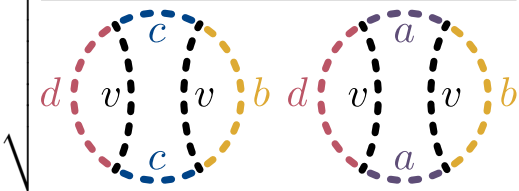
- Thus the correct expression is

$$S_{dc}^{ab}(\theta) = \frac{\text{Diagram 1}}{\sqrt{\text{Diagram 2}}} \Big|_{\text{analyt. cont.}}$$



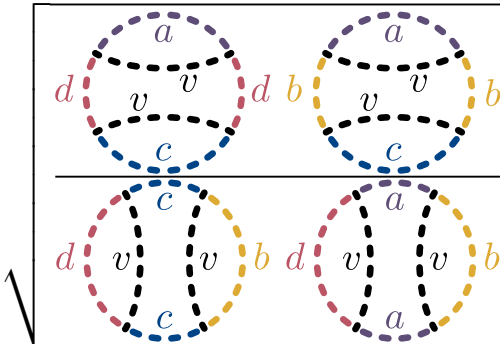
- Numerator : disk correlation function, **crossing symmetric**.
- Denominator : depend on the **channel** we consider.

Modified crossing for S-matrix

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- Numerator : disk correlation function, **crossing symmetric**.
- Denominator : depend on the **channel** we consider.
- Modified crossing:

$$S_{dc}^{ab}(\theta) = \sqrt{\frac{\text{channel dependent diagrams}}{\text{channel dependent diagrams}}} S_{ad}^{bc}(i\pi - \theta)$$


$$S_{dc}^{ab}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{ad}^{bc}(i\pi - \theta)$$

Summary: Physical origin of modified crossing

- The IR dynamics is described by a **nontrivial TQFT**.
- In the IR TQFT, **multi-kink states** correspond to **networks of lines**.
- Networks of lines have nontrivial VEV, which correct **normalizations** of in- and out-states.
- Corrections depend on the **channels** we consider.

1. Non-invertible symmetries in 1+1 dim
2. Integrable flow from tricritical Ising and S-matrix
3. Derivation of modified crossing rules
- 4. S-matrix bootstrap**
5. Conclusion

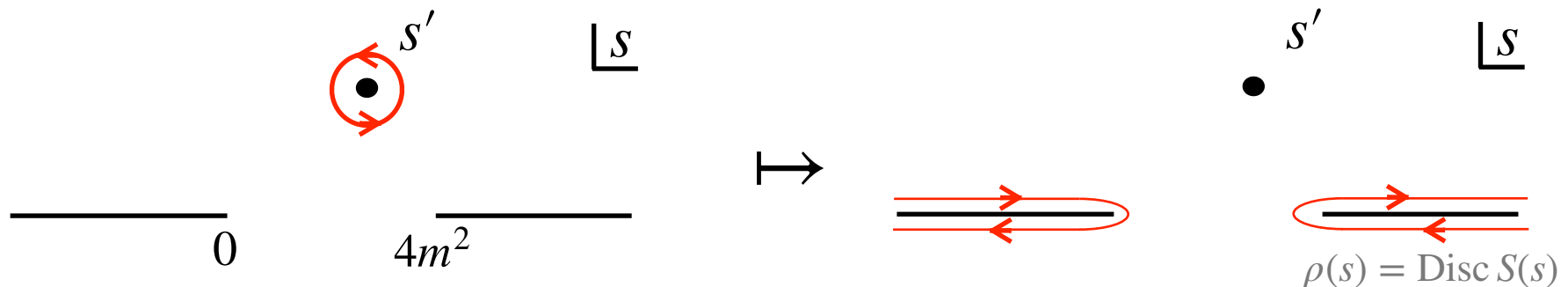
S-matrix bootstrap

Basic idea: Constrain the space of S-matrices from **crossing**, **symmetry** and **unitarity**.

Numerical implementation (“primal” bootstrap):

1. Dispersion relation:

$$S(s) = \frac{1}{2\pi i} \oint ds' \frac{S(s')}{s' - s} = S_\infty + \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \left[\frac{1}{s' - s} + \frac{1}{s' - 4m^2 + s} \right] \rho(s')$$



Trivializes **crossing**. One can also incorporate **symmetry** structures.

$$\underbrace{\frac{1}{s' - s}}_{s\text{-channel}} + \underbrace{\frac{1}{s' - 4m^2 + s}}_{t\text{-channel}} \mapsto \frac{\mathbf{1}_{n \times n}}{s' - s} + \frac{\mathbf{K}_{n \times n}}{s' - 4m^2 + s}$$

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$$\rho(s) = \frac{(s - s_j - 1)a_j + (s_j - s)a_{j-1}}{s_j - s_{j-1}} \quad (s_{j-1} \leq s \leq s_j) \quad \mapsto \quad S[s; a_j]$$

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3. Impose **unitarity** at discretized points.

$$\left| S[s_j; a_j] \right|^2 \leq 1 \quad (\text{at every } s_j)$$

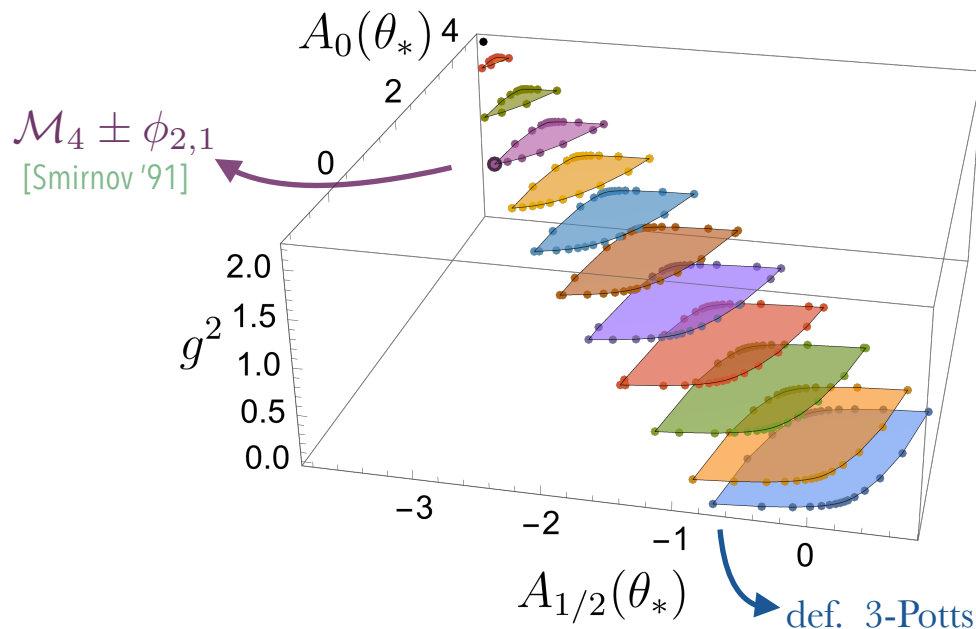
$\mapsto$ Constrain allowed values of a_j , hence $S(s)$.

S-matrix bootstrap with modified crossing

- Straightforward to generalize it to **modified crossing** (change of $\mathbf{K}_{n \times n}$)
- Fibonacci fusion category: $\mathcal{W}^2 = 1 + \mathcal{W}$

- IR: gapped phase with two asymmetric vacua
- Particles: kink, anti-kink, breather, all have the same mass

[Cordova, Garcia-Sepulveda, Holfester '24]



Integrable theories at the cusps of allowed region

1. Non-invertible symmetries in 1+1 dim
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Conclusion

$$S_{dc}^{ab}(\theta) = \sqrt{\text{diagram}} S_{ad}^{bc}(i\pi - \theta)$$

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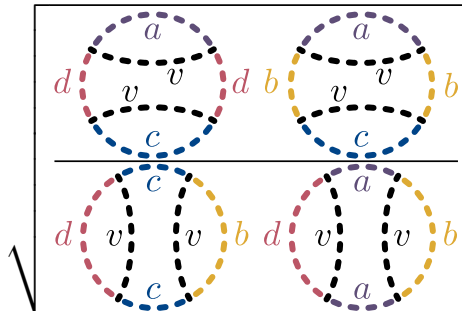
- In RG flows to gapped phase in 1+1 dim, non-invertible symmetries and anomalies lead to **modified crossing symmetry**.
- Physically, it comes from **corrections to norms** of in- and out-states due to TQFT dynamics.
- Similar modified crossing observed in Chern-Simons matter in 2+1 d.

[Mehta, Patel, Prakash, Minwalla, Sharma],...

Lessons from our examples:

1. **Anionic nature of particles is not important.**
2. **TQFT d.o.f can be more hidden.**

Ongoing / Future

$$S_{dc}^{ab}(\theta) = \sqrt{\begin{array}{c} \text{diagram 1} \\ \hline \text{diagram 2} \end{array}} S_{ad}^{bc}(i\pi - \theta)$$


$$S_{dc}^{ab}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{ad}^{bc}(i\pi - \theta)$$

- Integrable bootstrap with Haagerup fusion category.

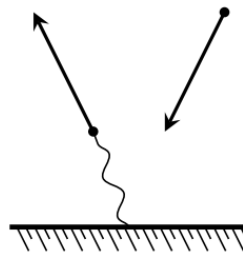
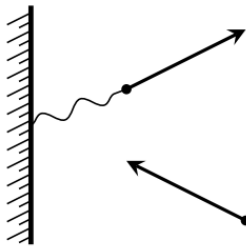
cf. [Huang, Lin, Ohmori, Tachikawa, Tezuka]

[Vanhove, Lootens, Van Damme, Osborne, Verstraete]

- Modified crossing and bootstrap for 3d gravity in flat space

(Relevant TQFT: (flat-space) limit of Virasoro TQFT)

- Modified crossing in monopole scattering?



[Csaki, Hong, Shirman, Telem, Terning, Waterbury]

[van Beest, Boyle Smith, Delmastro, Komargodski, Tong]

[van Beest, Boyle Smith, Delmastro, Mouland, Tong]

- Toy model for IR effects (Faddeev-Kulish) in gravity and gauge theory?

Back ups

Warm up: action of symmetry line

- Consider a path integral on a large disk with BC “0”

$$\langle\langle 0|0\rangle\rangle = \bigcirc_0 = 1 \quad (\text{State in open Hilbert space of TQFT})$$

- Other vacua can be obtained by acting symmetry lines.

$$|a\rangle\rangle = \text{arc with dashed red line and label } a, \quad \langle\langle a|a\rangle\rangle = \bigcirc_a = d_a$$

$$\langle\mathcal{L}_a\rangle = \text{circle with dashed blue line and label } \mathcal{L}_a = d_a \text{ quantum dim}$$

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- Matrix element of symmetry line between normalized states

$$\frac{\langle\langle b|\mathcal{L}_\varphi|a\rangle\rangle}{\sqrt{\langle\langle a|a\rangle\rangle\langle\langle b|b\rangle\rangle}} = \frac{1}{\sqrt{d_a d_b}} \bigcirc_{a,b,\varphi}$$

$$\bigcirc_{a,b,\varphi} = \sqrt{d_a d_b d_\varphi} N_{\varphi a}^b$$

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$$\bigcirc_{a,b,\varphi} = \sqrt{d_a d_b d_\varphi} N_{\varphi a}^b$$

- To correctly realize the symmetry algebra, we need “renormalization”

$$\hat{\mathcal{L}}_\varphi = \frac{1}{\sqrt{d_\varphi}} \text{---}\varphi\text{---}$$

Warm up: action of symmetry line

- Now consider a state with a kink interpolating a and b vacua

$$|a; b\rangle\rangle = \text{Diagram: A semi-circular arc with a vertical dashed line in the center. The left half of the arc is red and labeled 'a', and the right half is yellow and labeled 'b'. The vertical dashed line is labeled 'v'.$$

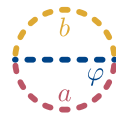
$$\langle\langle a; b | a; b \rangle\rangle = \sqrt{d_a d_b d_v}$$

$$\text{Diagram: A circle with a horizontal dashed line. The top half is yellow and labeled 'b', and the bottom half is red and labeled 'a'. The horizontal dashed line is labeled 'φ'.} = \sqrt{d_a d_b d_φ} N_{φ a}^b$$

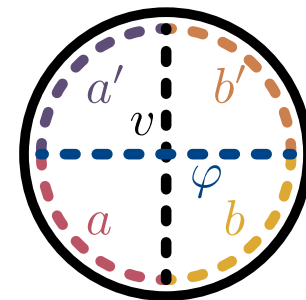
Warm up: action of symmetry line

- Now consider a state with a kink interpolating a and b vacua

$$|a; b\rangle\rangle = \text{Diagram} \quad \langle\langle a; b | a; b \rangle\rangle = \sqrt{d_a d_b d_v}$$


$$\text{Diagram} = \sqrt{d_a d_b d_\varphi} N_{\varphi a}^b$$


- Matrix element between single-kink states:

$$\frac{\langle\langle a'; b' | \hat{\mathcal{L}}_\varphi | a; b \rangle\rangle}{\sqrt{\langle\langle a; b | a; b \rangle\rangle \langle\langle a'; b' | a'; b' \rangle\rangle}} = \left(d_a d_{a'} d_b d_{b'} d_v^2 d_\phi^2 \right)^{-\frac{1}{4}} \text{Diagram}$$


$$= (d_a d_{a'} d_b d_{b'})^{1/4} \begin{bmatrix} \varphi & a' & a \\ v & b & b' \end{bmatrix}$$

Warm up: action of symmetry line

- Now consider a state with a kink interpolating a and b vacua



$$\langle\langle a; b | a; b \rangle\rangle = \sqrt{d_a d_b d_v}$$

A diagram showing a symmetry line action. It features a circle with a thick black boundary. Inside, there are two dashed semi-circles: a red one on the left labeled a and a yellow one on the right labeled b . A horizontal dashed line with a dot in the center, labeled φ , passes through the circle. The equation $\sqrt{d_a d_b d_\varphi} N_{\varphi a}^b$ is written to the right of the diagram.

- Matrix element between single-kink states:

$$\frac{\langle\langle a'; b' | \hat{\mathcal{L}}_\varphi | a; b \rangle\rangle}{\sqrt{\langle\langle a; b | a; b \rangle\rangle \langle\langle a'; b' | a'; b' \rangle\rangle}} = \left(d_a d_{a'} d_b d_{b'} d_v^2 d_\phi^2 \right)^{-\frac{1}{4}}$$

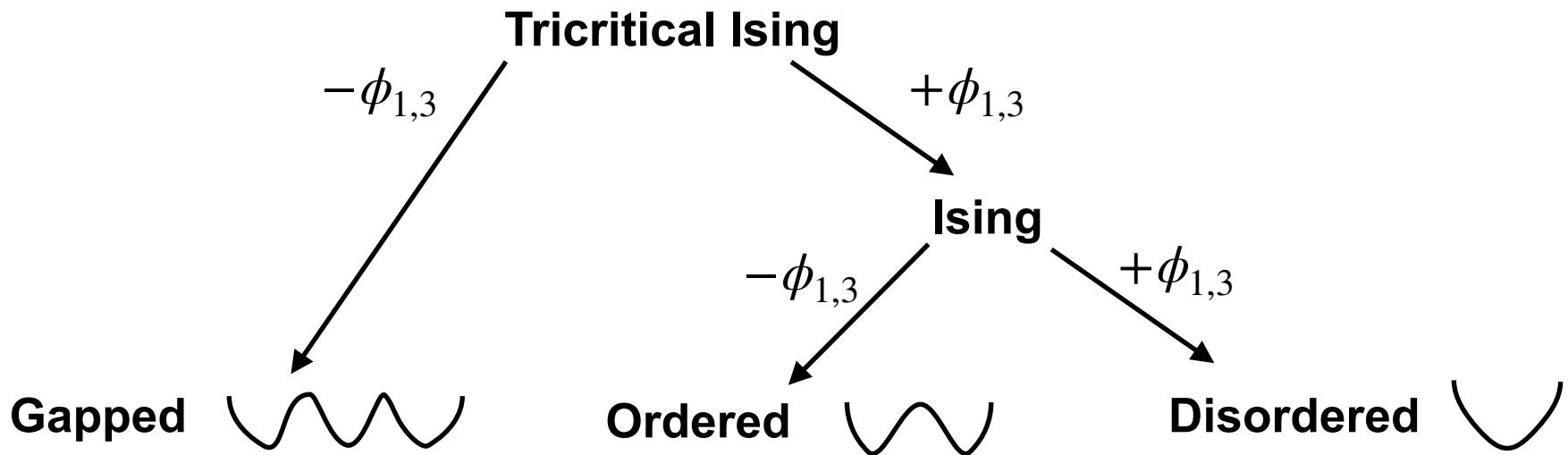
A diagram showing a symmetry line action. It features a circle with a thick black boundary. Inside, there are two dashed semi-circles: a red one on the left labeled a and a yellow one on the right labeled b . A horizontal dashed line with a dot in the center, labeled φ , passes through the circle. A vertical dashed line with a dot at the bottom, labeled v , also passes through the circle. The top of the circle has two dashed semi-circles: a purple one on the left labeled a' and an orange one on the right labeled b' .

$$= (d_a d_{a'} d_b d_{b'})^{1/4} \begin{bmatrix} \varphi & a' & a \\ v & b & b' \end{bmatrix}$$

- Our S-matrix commutes with this action of non-invertible symmetries.

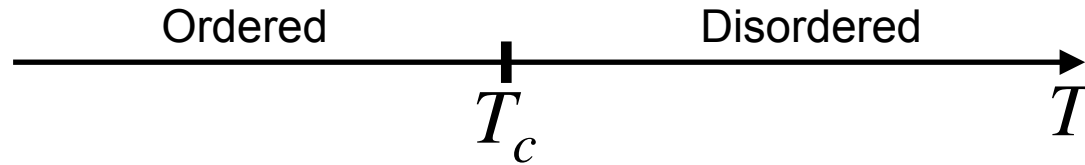
Flow from tricritical Ising

- Ising CFT is the simplest unitary minimal model ($\mathcal{M}_{3,4}$).
- Next simplest is **tricritical Ising** CFT ($\mathcal{M}_{4,5}$).
- RG flow from tricritical:

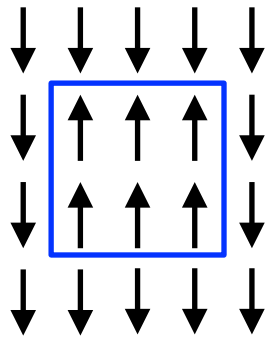


Non-Invertible Kramers-Wannier

- 2d Ising model:
$$Z = \sum_{\{\sigma\}} \exp \left[\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j \right]$$



Low T expansion

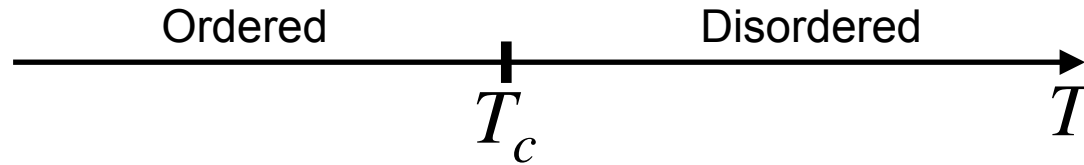


$$Z = \sum_{\text{domain wall}} e^{-2\beta L_{\text{wall}}}$$

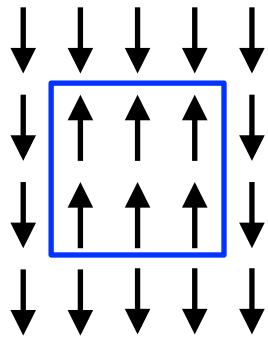


Non-Invertible Kramers-Wannier

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Low T expansion



$$Z = \sum_{\text{domain wall}} e^{-2\beta L_{\text{wall}}}$$

High T expansion

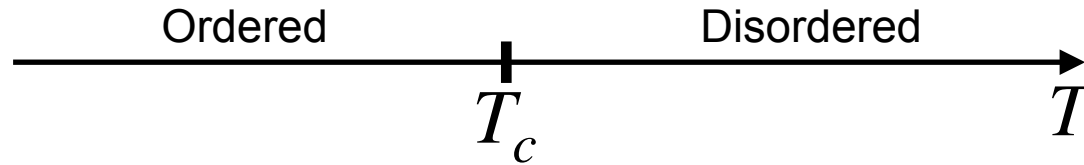
$$Z = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} \left(\cosh \beta + \sigma_i \sigma_j \sinh \beta \right)$$

$$\propto \sum_{\text{loop}} (\tanh \beta)^{L_{\text{loop}}}$$

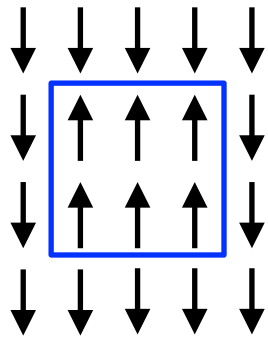
- KW duality:
$$e^{-2\beta_{\text{low}}} \leftrightarrow \tanh \beta_{\text{high}}$$

Non-Invertible Kramers-Wannier

- 2d Ising model:
$$Z = \sum_{\{\sigma\}} \exp \left[\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j \right]$$



Low T expansion



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$$\propto \sum_{\text{loop}} (\tanh \beta)^{L_{\text{loop}}}$$

- KW duality:
$$e^{-2\beta_{\text{low}}} \leftrightarrow \tanh \beta_{\text{high}}$$
- Strictly speaking, there is some **subtlety**....