Non-Invertible Symmetries and Scattering Amplitudes

Shota Komatsu

Based on works with Christian Copetti (Oxford) Lucía Córdova (CERN)

2403.04835: "Non-Invertible Symmetries, Anomalies and Scattering Amplitudes"

To appear soon: "S-Matrix Bootstrap and Non-Invertible Symmetries"

Why are Symmetries Important?

- Symmetries: Fundamental concepts in physics.

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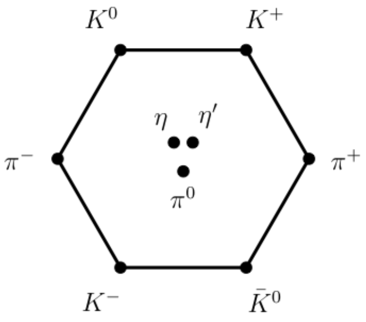

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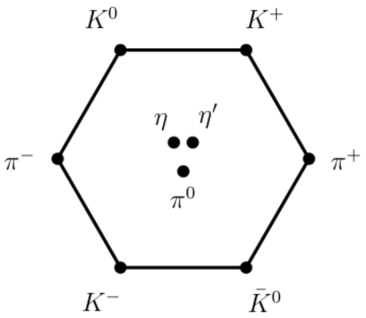

- A variety of modern refinements / generalizations.

  higher-form symmetries, non-invertible symmetries, 2-groups, …

  [Gaiotto, Kapustin, Seiberg, Willet], [Frolich, Fuchs, Runkel, Schwiegert], [Tachikawa, Bhardwaj], [Chang, Lin, Shao, Wang, Yin], [Benini, Cordova, Hsin], [Cordova, Dumitrescu, Intriligator]….. and many others

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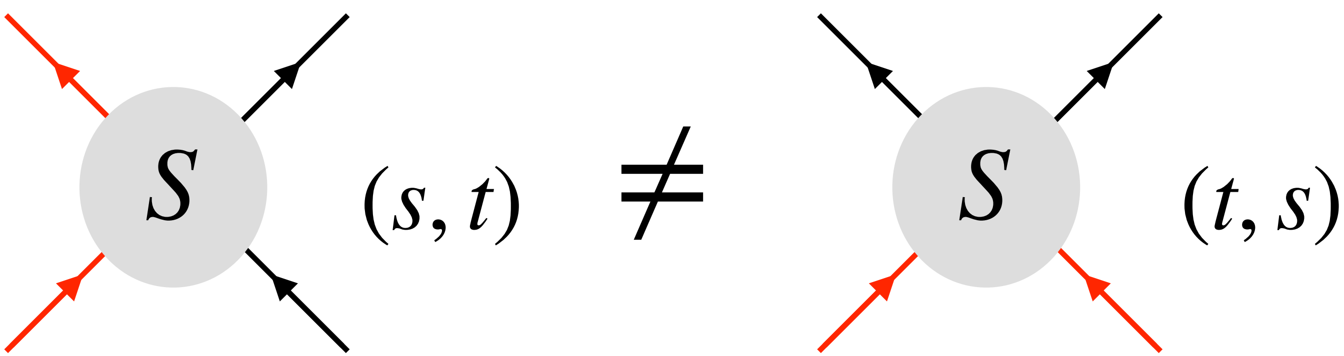
Punchlines of the talk

1. In the presence of certain* non-invertible symmetries or anomalies, crossing symmetry of S-matrix is modified.



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1. In the presence of certain* non-invertible symmetries or anomalies, crossing symmetry of S-matrix is modified.

- RG flows to gapped phases in 1+1 dimensions. (Comments on higher d at the end)
- Use integrable flows to check the claim but applies to non-integrable theories as well.

Question

What are implications of modern symmetries on scattering amplitudes?

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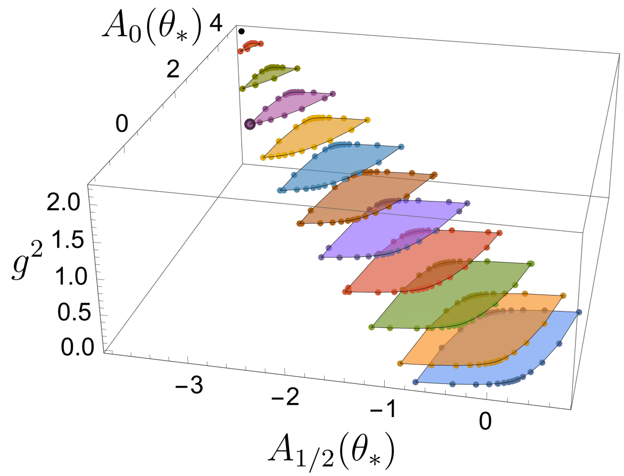
2. (Numerical) S-matrix bootstrap with modified crossing constrains the space of theories with non-invertible symmetries



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- Future applications / ongoing works: theories with Haagerup fusion category, monopole scattering, scattering in 3d quantum gravity…. (Comments at the end)

Plan

1. Non-invertible symmetries in 1+1 dim
2. Integrable flow from tricritical Ising and S-matrix
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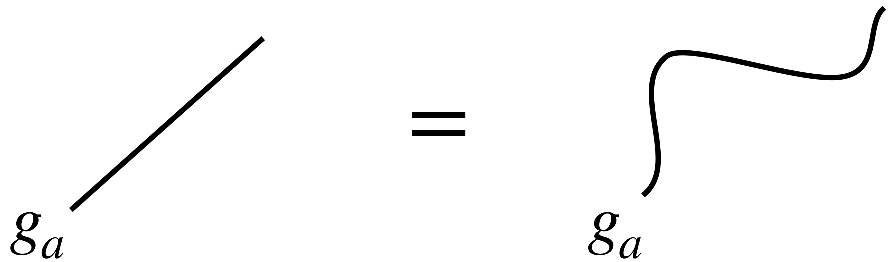
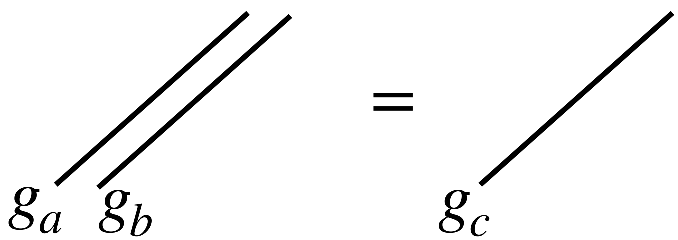
Non-Invertible Symmetries in 1+1 dim

- Symmetries in QFT ⊂ (d-1)-dim topological operators



- Usual symmetries (group): $g_a g_b = g_c$ $\qquad g_{a,b,c} \in G$



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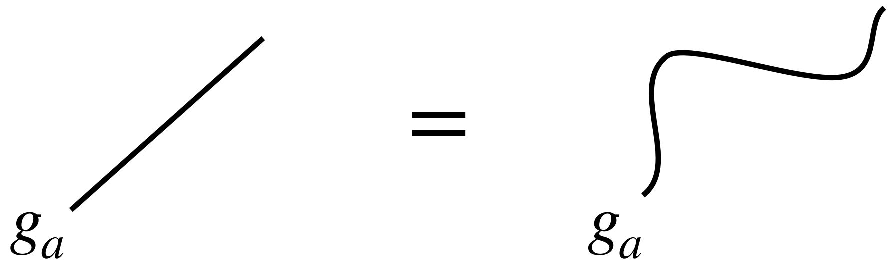

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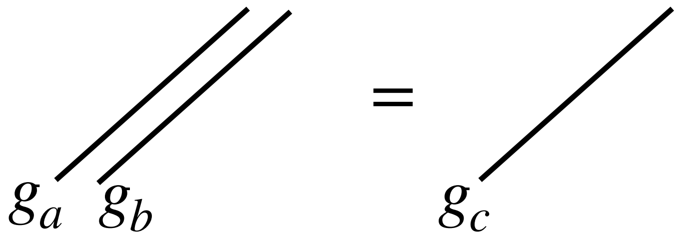


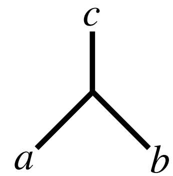
- Non-invertible symmetries (in 1+1 dim):

Fusion category $\quad \mathscr{L}_a \mathscr{L}_b = \sum_{\mathscr{L}_c} N_{ab}{}^c \mathscr{L}_c \qquad (N_{ab}{}^c \in \mathbb{Z}_{\geq 0})$

Data characterizing fusion categories

$$\mathscr{L}_a \mathscr{L}_b = \sum_{\mathscr{L}_c} N_{ab}{}^c \mathscr{L}_c$$

- Fusion coefficient $N_{ab}{}^c$:



If $N_{ab}{}^c \neq 0$

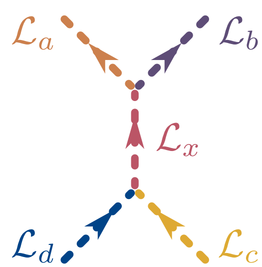
- Quantum dimension : $\langle \mathcal{L}_a \rangle =$

$= \mathrm{d}_a$

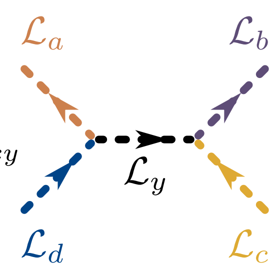
1 group-like

$\sqrt{2}$ etc. for non-inv.

- F-symbol:



$= \sum_y F^{abc}_{d}{}_{xy}$



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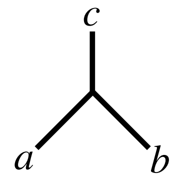

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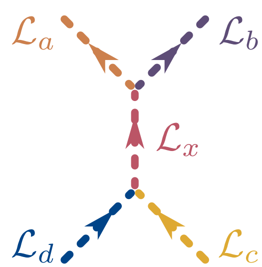

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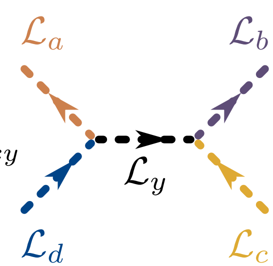
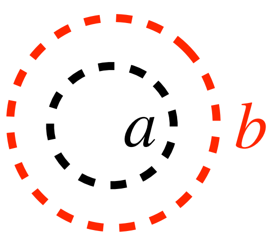


They satisfy various consistency conditions.

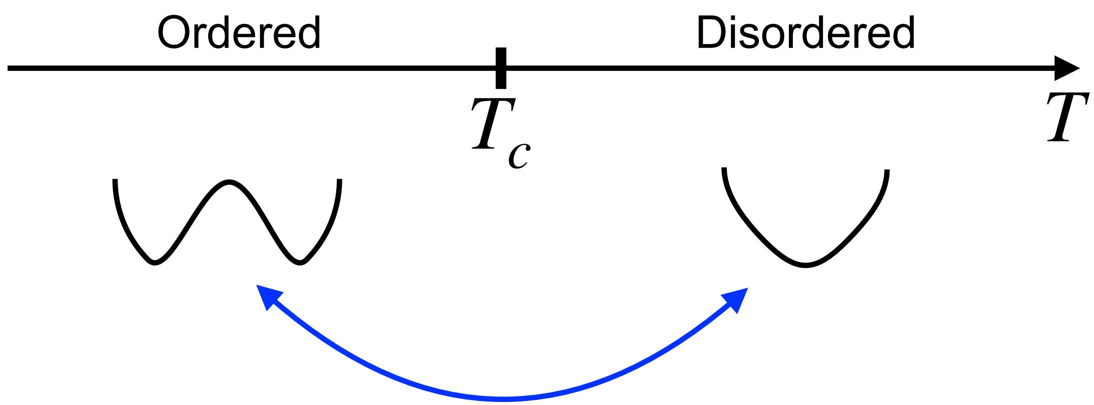
e.g.



$\mapsto \quad \mathrm{d}_a \mathrm{d}_b = \sum_c N_{ab}{}^{c} \mathrm{d}_c$

Non-Invertible Kramers-Wannier

- 2d Ising model: $$Z = \sum_{\{\sigma\}} \exp\left[\beta \sum_{<ij>} \sigma_i \sigma_j\right]$$



Kramers-Wannier duality $\qquad e^{-2\beta_{\text{low}}} \leftrightarrow \tanh \beta_{\text{high}}$

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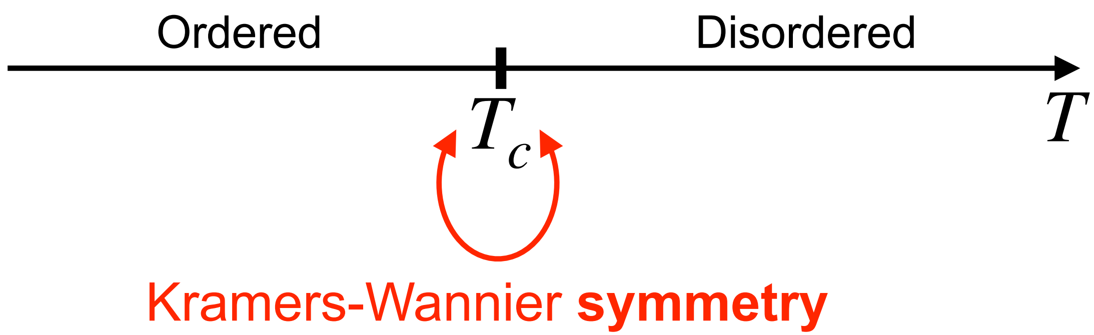

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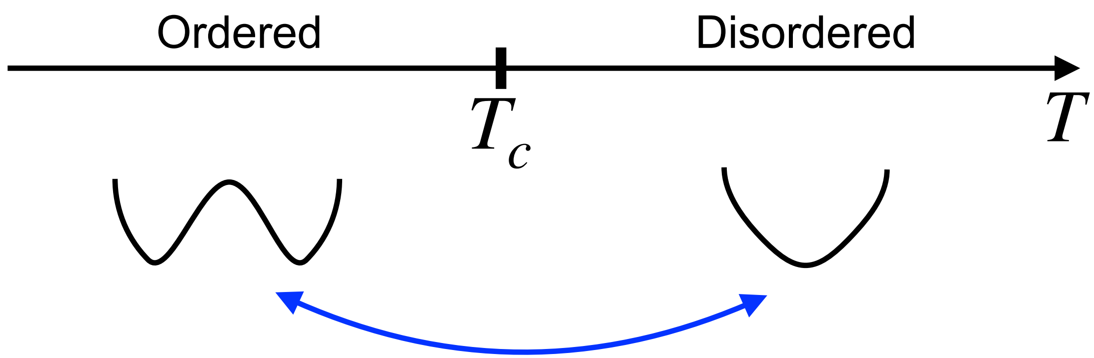

$$|+\rangle \to |0\rangle$$
$$|-\rangle \to |0\rangle$$
$$|+\rangle + |-\rangle \leftarrow |0\rangle$$

Non-invertible

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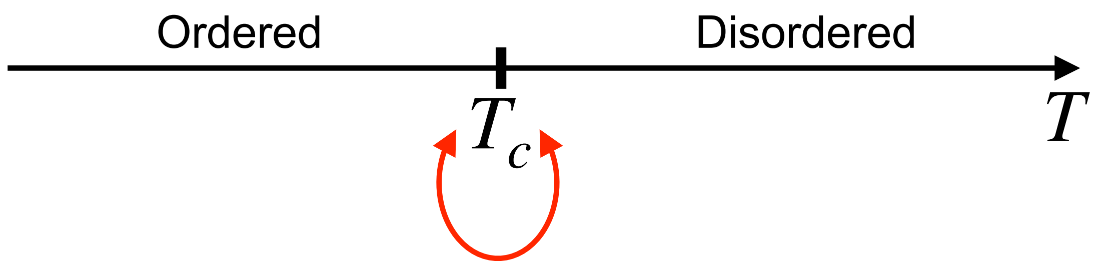
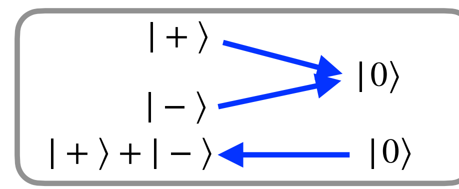


Kramers-Wannier **symmetry** $\mathcal{N}$

- Symmetry of 2d Ising CFT

$$\eta^2 = 1\,, \qquad \mathcal{N}^2 = 1 + \eta\,, \qquad \mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N}$$

$$(\mathrm{d}_1 = \mathrm{d}_\eta = 1, \quad \mathrm{d}_\mathcal{N} = \sqrt{2})$$



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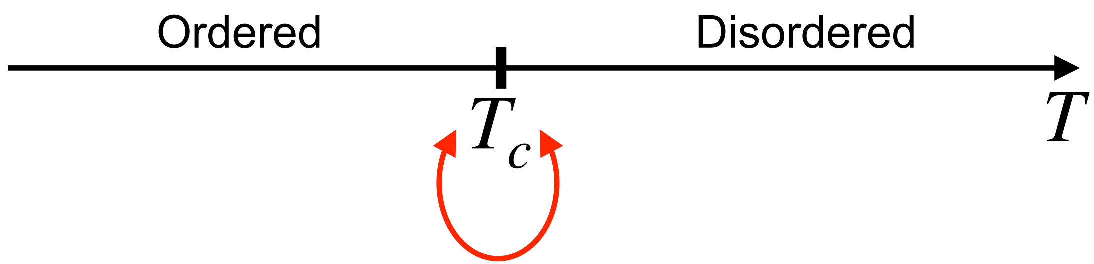

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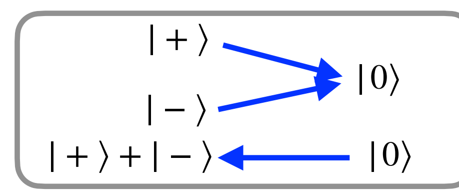

- In Ising CFT, KW symmetry broken by relevant perturbations

- RG flow preserving KW symmetry…?

Flow from tricritical Ising

- Ising CFT is the simplest unitary minimal model ($\mathscr{M}_{3,4}$).

- Next simplest is tricritical Ising CFT ($\mathscr{M}_{4,5}$).

- Non-invertible symmetric RG flow from tricritical: [Chang, Lin, Shao, Wang, Yin]

$$\textbf{Tricritical Ising} \xrightarrow{+\phi_{1,3}} \textbf{Ising}$$

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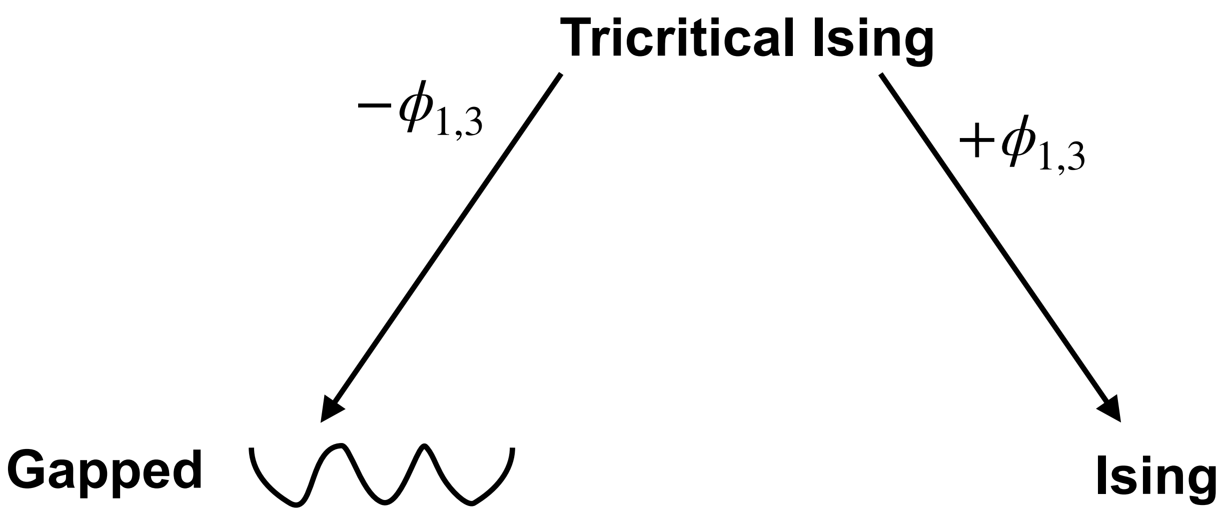
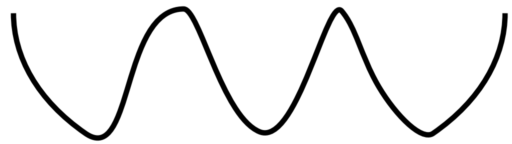


Action on vacua



$|-\rangle \quad |0\rangle \quad |+\rangle$

- $\mathbb{Z}_2$-defect exchanges $|+\rangle$ and $|-\rangle$

$$\eta : |+\rangle \leftrightarrow |-\rangle$$

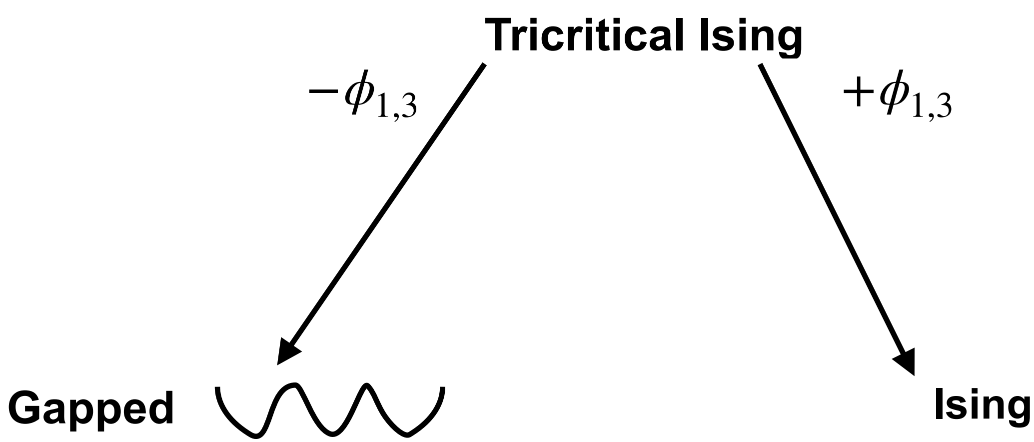
- $\mathcal{N}$-defect:

$$\mathcal{N}|0\rangle = |+\rangle + |-\rangle$$

$$\mathcal{N}|+\rangle = \mathcal{N}|-\rangle = |0\rangle$$

- "Superposition" of disordered and ordered vacua in Ising

Remarks



- 3 is the minimal number of vacua allowed by non-invertible sym.

$$\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{gathered} \eta^2 = 1\,, \qquad \mathcal{N}^2 = 1 + \eta\,, \\ \mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N} \end{gathered}}$$

Entries need to be non-negative integers

- Similar patterns for higher minimal models cf. [Tanaka. Nakayama 2407.21353]

$\mathscr{M}_{n,n+1} - \phi_{1,3} \rightarrow$ gapped with (n-1) vacua

$$\mathscr{M}_{n,n+1} + \phi_{1,3} \rightarrow \mathscr{M}_{n-1,n}$$

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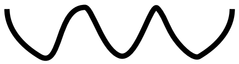
Integrability along the flow

Tricritical Ising $\xrightarrow{-\phi_{1,3}}$ 

- At UV CFT fixed point, there exist ∞ many higher spin charges.

  because of Virasoro

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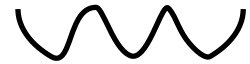
Tricritical Ising $\xrightarrow{-\phi_{1,3}}$ 

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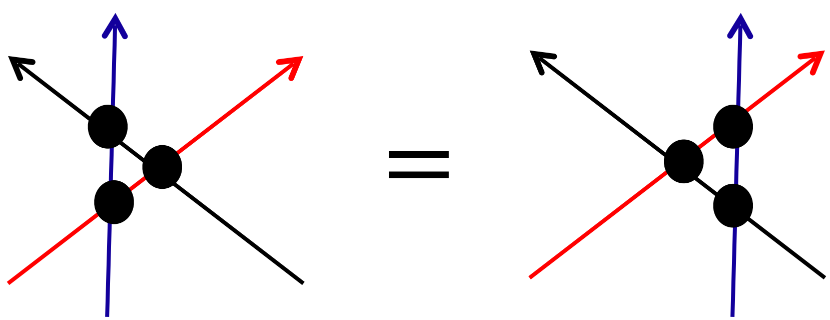
  [Zamolodchikov 1989]

- Multi-particle S-matrix factorizes to 2→2 S-matrices.

  [Shankar, Witten] [Parke]

- 2→2 S-matrices satisfy the Yang-Baxter equation.



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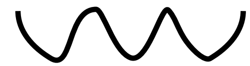
Tricritical Ising $\xrightarrow{-\phi_{1,3}}$ 

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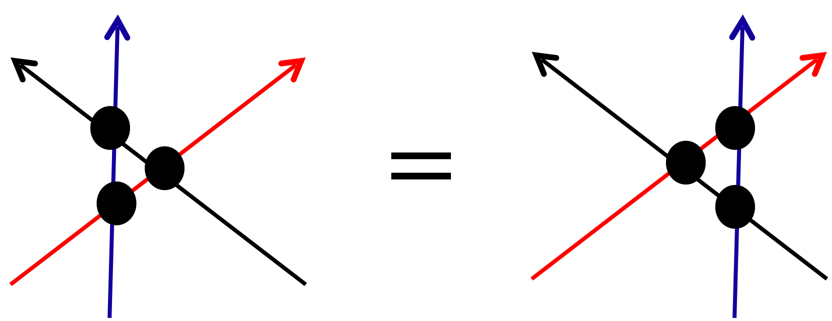
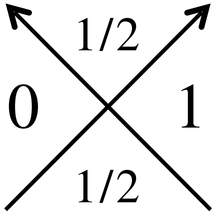


- Imposing unitarity, crossing & YB, the S-matrix can be almost uniquely "**bootstrapped**". [Zamolodchikov 1991]

S-matrix by Zamolodchikov

- Particles in IR = kinks interpolating between adjacent vacua.



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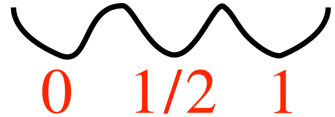
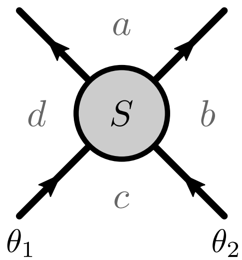


*Warning: Change of notation!

- It also depends on Mandelstam variable $s = (p_1 + p_2)^2 = 4m^2 \cosh^2(\theta/2)$

$S^{ab}_{dc}(\theta) =$



$$\theta = \theta_1 - \theta_2$$

$$s + t = 4m^2 \qquad t = -4m^2 \sinh^2(\theta/2)$$

S-matrix by Zamolodchikov

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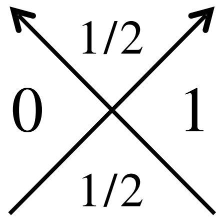

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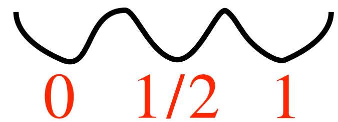

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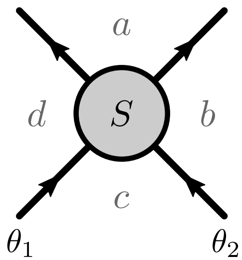
$S^{ab}_{dc}(\theta) =$ 

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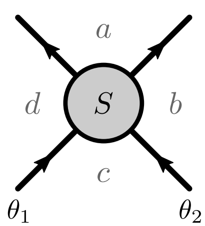
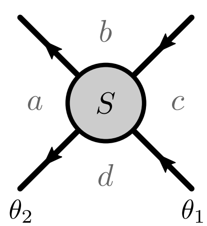
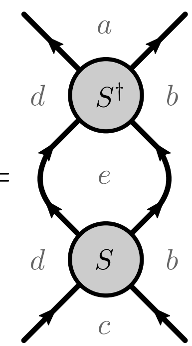
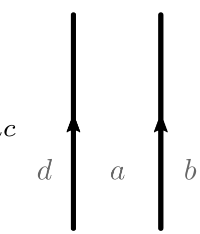
- Crossing symmetry & unitarity

$S^{ab}_{dc}(\theta) =$ 

$=$ 

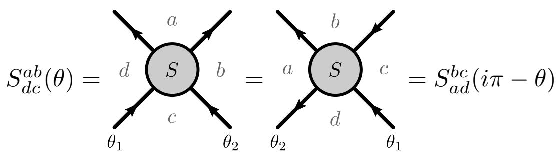
$= S^{bc}_{ad}(i\pi - \theta)$

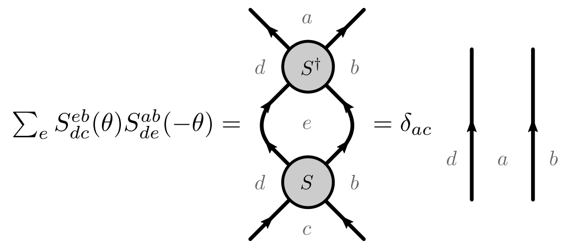
$\sum_e S^{eb}_{dc}(\theta) S^{ab}_{de}(-\theta) =$ 

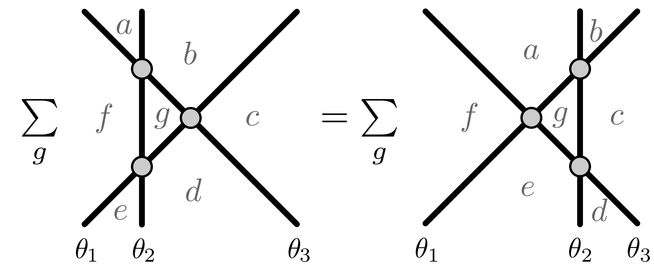
$= \delta_{ac}$ 

S-matrix by Zamolodchikov

- Imposing crossing & unitarity & YB,







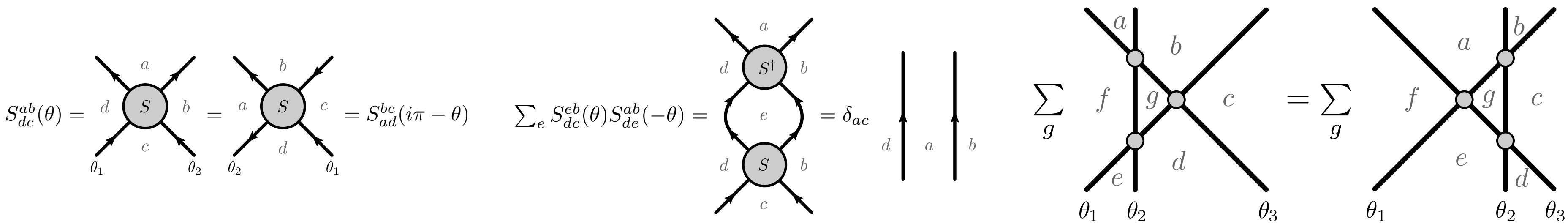
we can fix S-matrix uniquely (up to overall "CDD" factor) as

$$\widehat{S}^{ab}_{dc}(\theta) = Z(\theta) \left(\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}\right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac}\right]$$

$$(n = 3, \quad \mathrm{d}_0 = \mathrm{d}_1 = 1, \quad \mathrm{d}_{1/2} = \sqrt{2})$$

S-matrix by Zamolodchikov

- Imposing crossing & unitarity & YB,

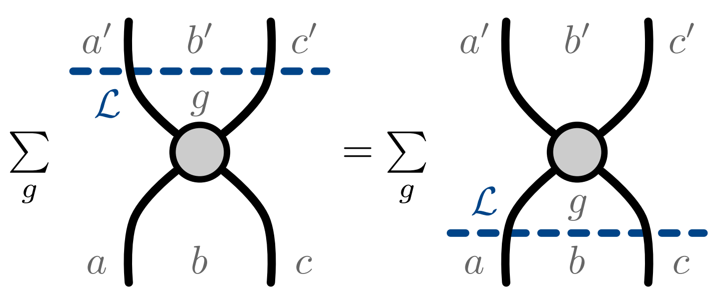


we can fix S-matrix uniquely (up to overall "CDD" factor) as

$$\widehat{S}^{ab}_{dc}(\theta) = Z(\theta)\left(\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}\right)^{\frac{i\theta}{2\pi}}\left[\sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\,\sinh\left(\frac{\theta}{n}\right)\delta_{bd}+\sinh\left(\frac{i\pi-\theta}{n}\right)\delta_{ac}\right]$$

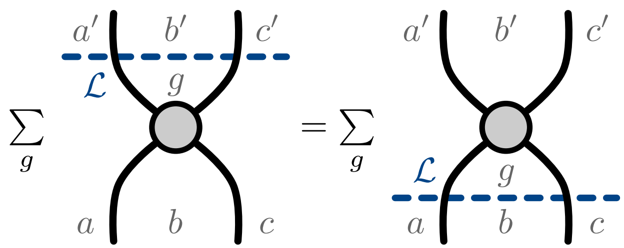
$$(n = 3, \quad \mathrm{d}_0 = \mathrm{d}_1 = 1, \quad \mathrm{d}_{1/2} = \sqrt{2})$$

- We expect that it also preserves non-invertible symmetries:



?

Paradox



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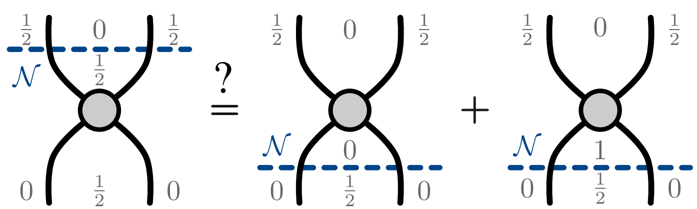
- We found that $\widehat{S}$ commutes with η ($\mathbb{Z}_2$), but not with $\mathcal{N}$.



$$\mathcal{N}: \quad S^{\frac{1}{2}\,0}_{0\,\frac{1}{2}}(\theta) \stackrel{?}{=} S^{0\,\frac{1}{2}}_{\frac{1}{2}\,0}(\theta) + S^{0\,\frac{1}{2}}_{\frac{1}{2}\,1}(\theta)$$

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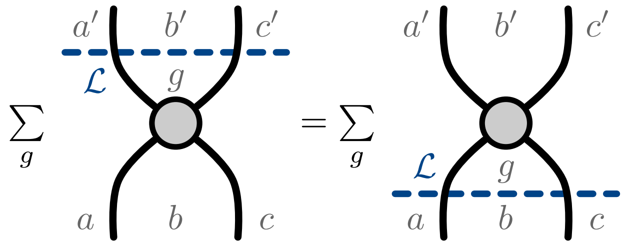

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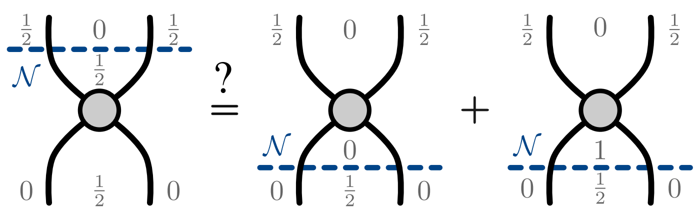

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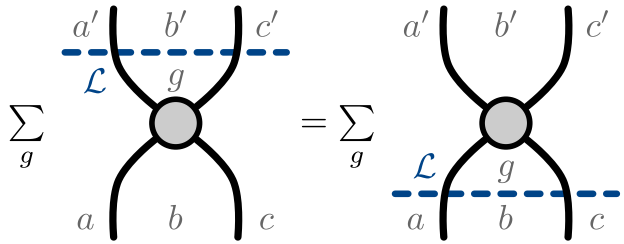
- The following 4 properties are mutually incompatible.
  - Unitarity
  - Crossing
  - Integrability (YB)
  - Non-invertible symmetry

Paradox



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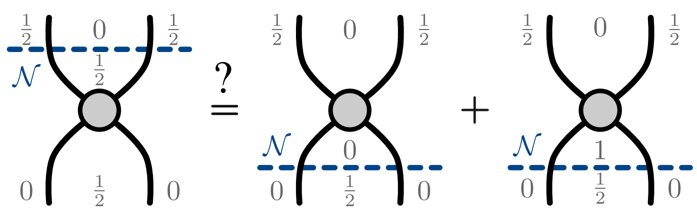

$$\mathcal{N}: \quad \cancel{S^{\frac12 0}_{0\frac12}(\theta) \overset{?}{=} S^{0\frac12}_{\frac12 0}(\theta) + S^{0\frac12}_{\frac12 1}(\theta)}$$

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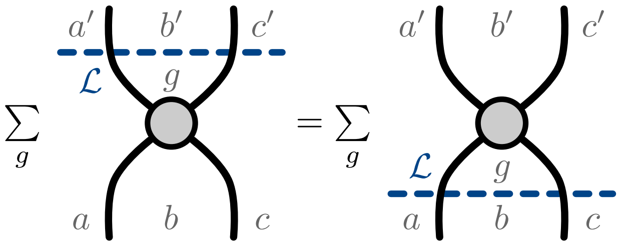
Paradox



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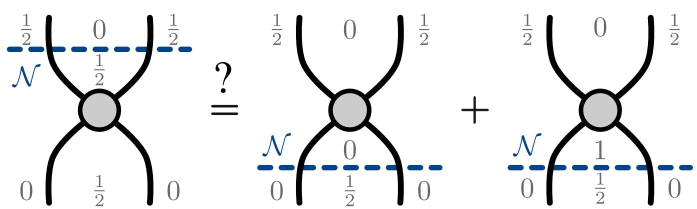

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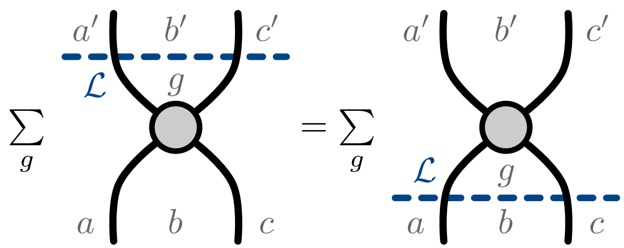
- The following 4 properties are mutually incompatible.
  - Unitarity ✓ basic principle
  - Crossing
  - Integrability (YB) ✓ deformation of CFT
  - Non-invertible symmetry ✓ deformation of CFT

Paradox



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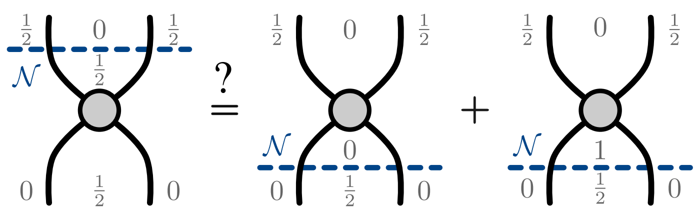

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- The following 4 properties are mutually incompatible.
  - Unitarity ✓ basic principle
  - Crossing
  - Integrability (YB) ✓ deformation of CFT
  - Non-invertible symmetry ✓ deformation of CFT
- The only viable option is to give up crossing.

New proposal

$$S_{dc}^{ab}(\theta) = Z(\theta)\left[\sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\ \sinh\left(\frac{\theta}{n}\right)\ \delta_{bd} + \sinh\left(\frac{i\pi-\theta}{n}\right)\ \delta_{ac}\right]$$

$$\widehat{S}_{dc}^{ab}(\theta) = Z(\theta)\left(\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}\right)^{\frac{i\theta}{2\pi}}\left[\sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\ \sinh\left(\frac{\theta}{n}\right)\ \delta_{bd} + \sinh\left(\frac{i\pi-\theta}{n}\right)\ \delta_{ac}\right]$$

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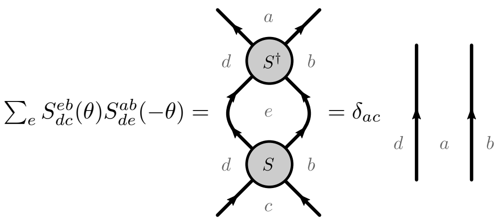
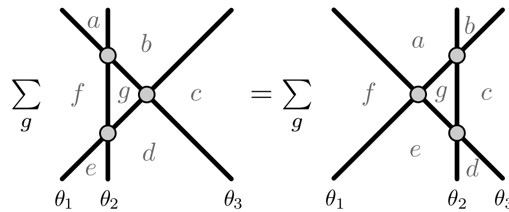
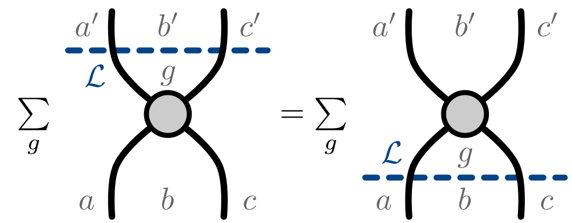


Unitarity



Yang-Baxter



Non-invertible

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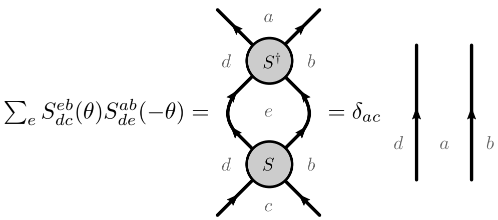
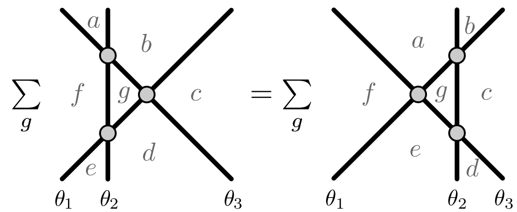


Unitarity



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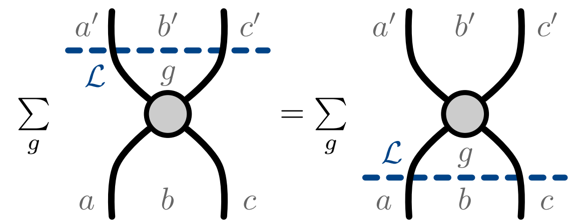


Non-invertible

- Crossing symmetry is modified:

$$S^{ab}_{dc}(\theta) = \sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

- Physical origin? (The topic of the rest of the talk)

1. Non-invertible symmetries in 1+1 dim

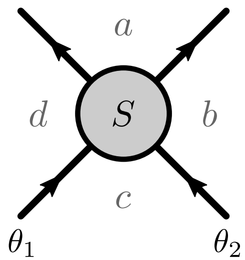
2. Integrable flow from tricritical Ising and S-matrix

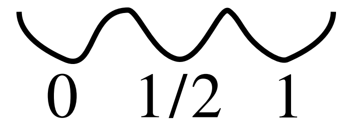
3. Derivation of modified crossing rules

4. S-matrix bootstrap

5. Conclusion

Key physical input

$S^{ab}_{dc}(\theta) =$ 



- The IR dynamics is described by a nontrivial TQFT.

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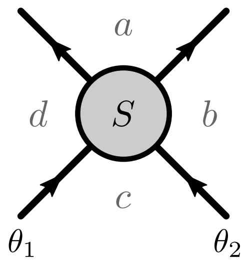
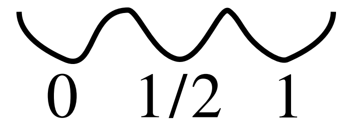
$S_{dc}^{ab}(\theta) =$ 



- The IR dynamics is described by a nontrivial TQFT.

1. In the IR, the action of a kink on vacua = the action of symmetry line $\mathcal{N}\,(=: v)$

$$\mathcal{N}\,|\,0\rangle = |\,1/2\rangle, \qquad \mathcal{N}\,|\,1/2\rangle = |\,0\rangle + |\,1\rangle, \qquad \mathcal{N}\,|\,1\rangle = |\,1/2\rangle\,.$$

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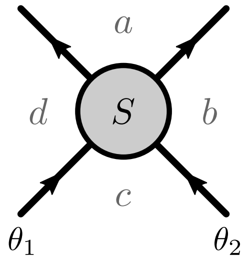
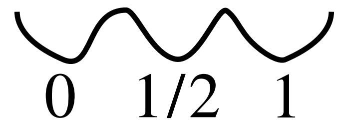
$S^{ab}_{dc}(\theta) =$ 



- The IR dynamics is described by a nontrivial TQFT.

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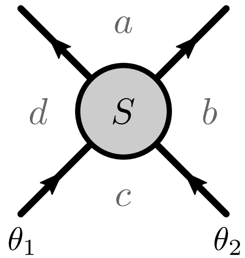
2. The vacua are in 1-to-1 correspondence with symmetry lines.

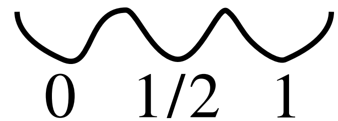
$$|0\rangle \leftrightarrow \mathbf{1} \qquad |1/2\rangle \leftrightarrow \mathcal{N} \qquad |1\rangle \leftrightarrow \eta$$

("Adjoint" representation of fusion category)

$$\mathcal{N}\,|1/2\rangle = |0\rangle + |1\rangle \quad \leftrightarrow \quad \mathcal{N}^2 = \mathbf{1} + \eta$$

Key physical input

$S^{ab}_{dc}(\theta) =$ 



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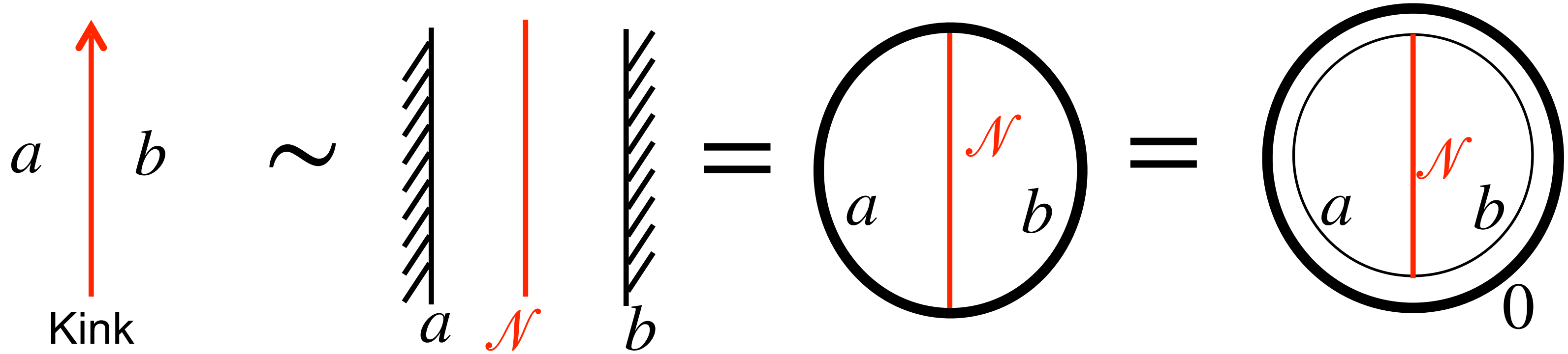

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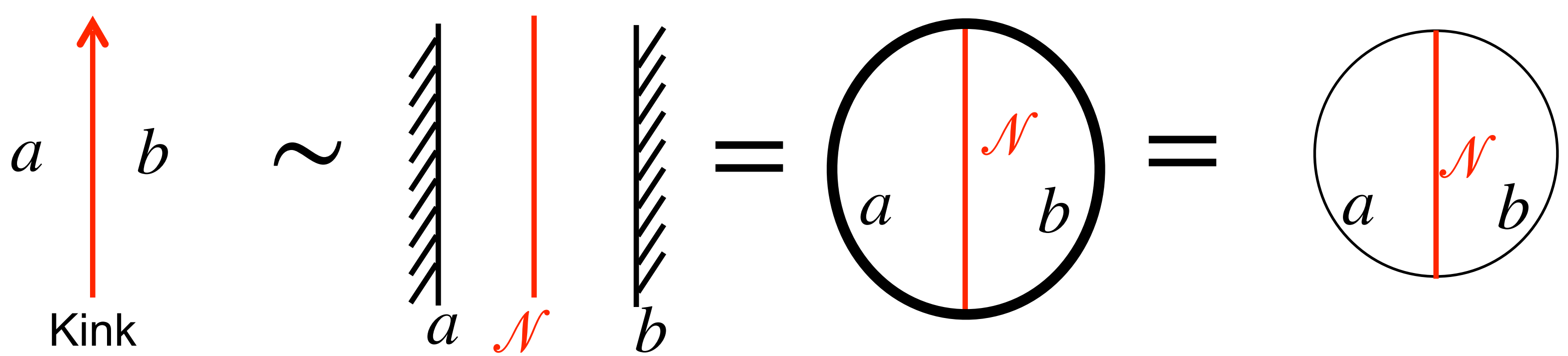


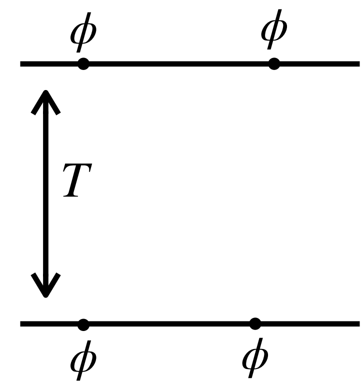
kink states $\leftrightarrow$ network of lines

Modified crossing for S-matrix

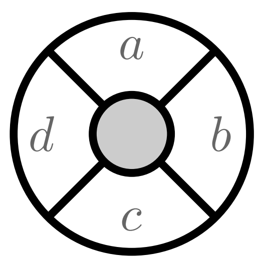
- Kink creation operators are non-local (boundary cond. changing op)

Unclear how to get S-matrix from LSZ reduction.

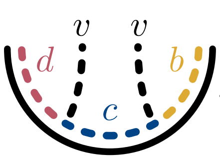
- Use alternative (discussed in Itzykson-Zuber Ch.5)

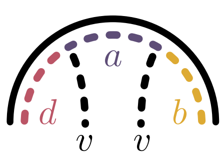


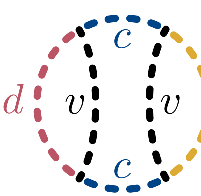
$$S(\{p_i\}) = \lim_{T\to\infty} \int \prod_j dv_j e^{ip_j x(v_j)} \prod_k (n_k \cdot \overleftrightarrow{\partial}_{x_k}) G(\{x(v_k)\})$$

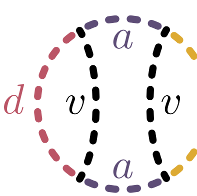
- In our case, $S_{dc}^{ab}(\theta) \propto$  $\Big|_{\text{analyt. cont.}}$ (disk correlator of bcc ops.)

- But we need to take into account normalization.

$|\text{in}\rangle\rangle =$ ,

$|\text{out}\rangle\rangle =$ 

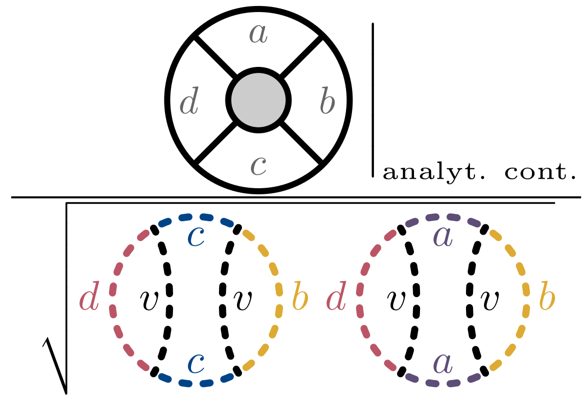
$\langle\langle\text{in}|\text{in}\rangle\rangle = d$  b

$\langle\langle\text{out}|\text{out}\rangle\rangle = d$  b

Modified crossing for S-matrix

- Thus the correct expression is

$$S^{ab}_{dc}(\theta) =$$



- Numerator : disk correlation function, crossing symmetric.
- Denominator : depend on the channel we consider.

Modified crossing for S-matrix

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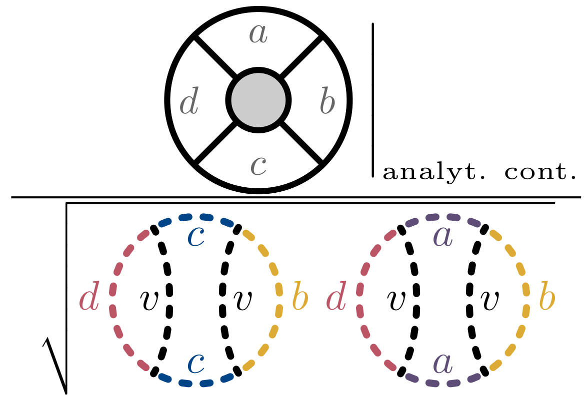

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- Modified crossing:

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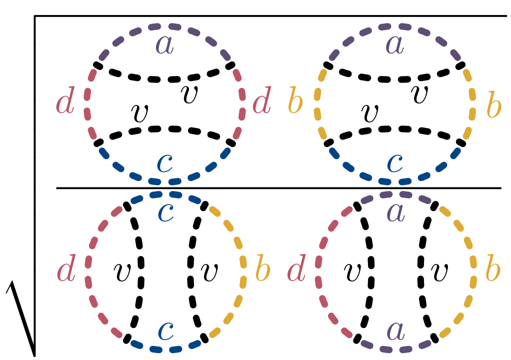


$$S^{bc}_{ad}(i\pi - \theta)$$

$$S^{ab}_{dc}(\theta) = \sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

Summary: Physical origin of modified crossing

- The IR dynamics is described by a nontrivial TQFT.
- In the IR TQFT, multi-kink states correspond to networks of lines.
- Networks of lines have nontrivial VEV, which correct normalizations of in- and out-states.
- Corrections depend on the channels we consider.

1. Non-invertible symmetries in 1+1 dim
2. Integrable flow from tricritical Ising and S-matrix
3. Derivation of modified crossing rules
4. S-matrix bootstrap
5. Conclusion

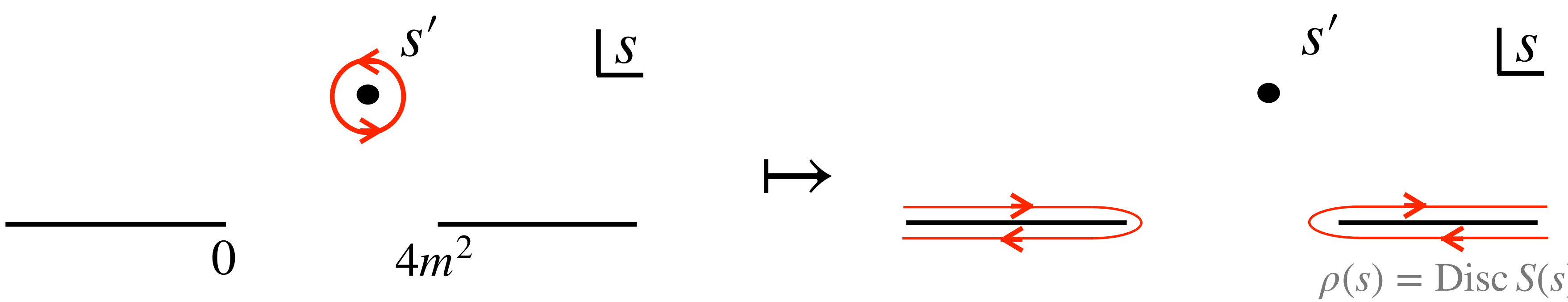
S-matrix bootstrap

Basic idea: Constrain the space of S-matrices from crossing, symmetry and unitarity.

Numerical implementation (“primal” bootstrap):

1. Dispersion relation:

$$S(s) = \frac{1}{2\pi i}\oint ds' \frac{S(s')}{s'-s} = S_\infty + \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \left[\frac{1}{s'-s} + \frac{1}{s'-4m^2+s}\right] \rho(s')$$



Trivializes crossing. One can also incorporate symmetry structures.

$$\underset{s\text{-channel}}{\frac{1}{s'-s}} + \underset{t\text{-channel}}{\frac{1}{s'-4m^2+s}} \quad \mapsto \quad \frac{\mathbf{1}_{n\times n}}{s'-s} + \frac{\mathbf{K}_{n\times n}}{s'-4m^2+s}$$

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2. Approximate ρ by a piece-wise linear function and use dispersion rel.

$$\rho(s) = \frac{(s-s_j-1)a_j + (s_j - s)a_{j-1}}{s_j - s_{j-1}} \qquad (s_{j-1} \le s \le s_j) \qquad \longmapsto \qquad S[s; a_j]$$

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Numerical implementation ("primal" bootstrap):

1. Dispersion relation:

$$S(s) = \frac{1}{2\pi i}\oint ds' \frac{S(s')}{s'-s} = S_\infty + \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \left[\frac{1}{s'-s} + \frac{1}{s'-4m^2+s}\right] \rho(s')$$

2. Approximate ρ by a piece-wise linear function and use dispersion rel.

$$\rho(s) = \frac{(s-s_j-1)a_j + (s_j - s)a_{j-1}}{s_j - s_{j-1}} \qquad (s_{j-1} \le s \le s_j) \qquad \longmapsto \qquad S[s; a_j]$$

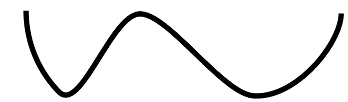
3. Impose unitarity at discretized points.

$$\left| S[s_j; a_j] \right|^2 \le 1 \qquad \text{(at every } s_j\text{)}$$

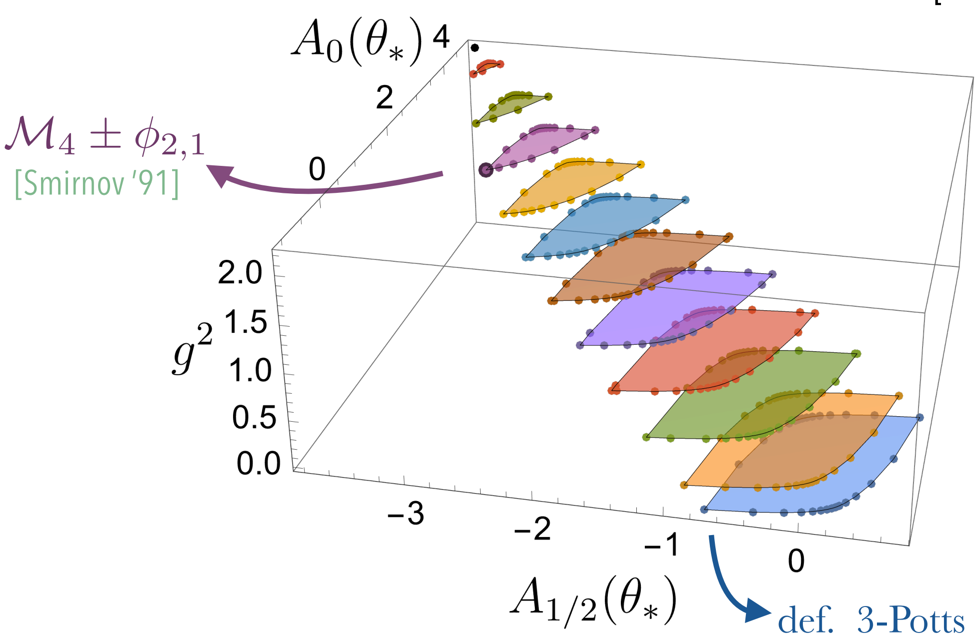
$\longmapsto$ Constrain allowed values of a_j, hence $S(s)$.

S-matrix bootstrap with modified crossing

- Straightforward to generalize it to modified crossing (change of $\mathbf{K}_{n\times n}$)
- Fibonacci fusion category: $\mathscr{W}^2 = 1 + \mathscr{W}$
  - IR: gapped phase with two asymmetric vacua
  - Particles: kink, anti-kink, breather, all have the same mass



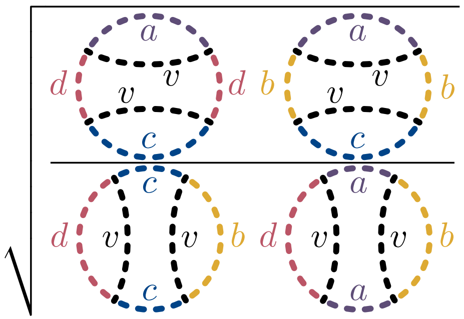
[Cordova, Garcia-Sepulveda, Holfester '24]



Integrable theories at the cusps of allowed region

1. Non-invertible symmetries in 1+1 dim
2. Integrable flow from tricritical Ising and S-matrix
3. Derivation of modified crossing rules
4. S-matrix bootstrap
5. Conclusion

Conclusion

$S^{ab}_{dc}(\theta) =$  $S^{bc}_{ad}(i\pi - \theta)$

$$S^{ab}_{dc}(\theta) = \sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

- In RG flows to gapped phase in 1+1 dim, non-invertible symmetries and anomalies lead to modified crossing symmetry.

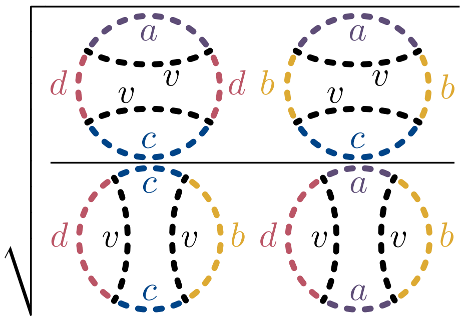
- Physically, it comes from corrections to norms of in- and out-states due to TQFT dynamics.

- Similar modified crossing observed in Chern-Simons matter in 2+1 d.

[Mehta, Patel, Prakash, Minwalla, Sharma],…

Lessons from our examples:
1. Anionic nature of particles is not important.
2. TQFT d.o.f can be more hidden.

Ongoing / Future

$S^{ab}_{dc}(\theta) =$  $S^{bc}_{ad}(i\pi - \theta)$

$$S^{ab}_{dc}(\theta) = \sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

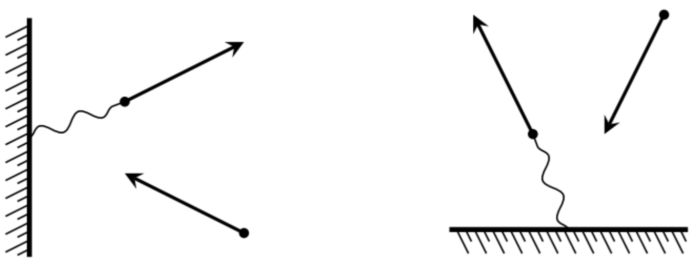
- Integrable bootstrap with Haagerup fusion category.

cf. [Huang, Lin, Ohmori, Tachikawa, Tezuka]
[Vanhove, Lootens, Van Damme, Osborne, Verstraete]

- Modified crossing and bootstrap for 3d gravity in flat space

(Relevant TQFT: (flat-space) limit of Virasoro TQFT)

- Modified crossing in monopole scattering?



[Csaki, Hong, Shirman, Telem, Terning, Waterbury]

[van Beest, Boyle Smith, Delmastro, Komargodski, Tong]

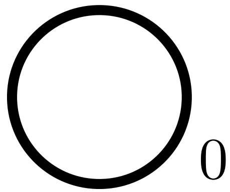
[van Beest, Boyle Smith, Delmastro, Mouland, Tong]

- Toy model for IR effects (Faddeev-Kulish) in gravity and gauge theory?

Back ups

Warm up: action of symmetry line

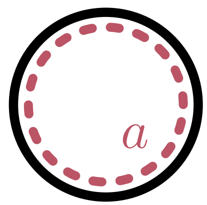
- Consider a path integral on a large disk with BC “ 0 ”

$\langle\!\langle 0|0\rangle\!\rangle =$ 

$= 1$ (State in open Hilbert space of TQFT)

- Other vacua can be obtained by acting symmetry lines.

$|a\rangle\!\rangle =$ 

$,\quad \langle\!\langle a|a\rangle\!\rangle =$ 

$= \mathrm{d}_a$

$\langle \mathcal{L}_a \rangle =$ 

$= \mathrm{d}_a$ quantum dim

Warm up: action of symmetry line

- Consider a path integral on a large disk with BC " 0 "

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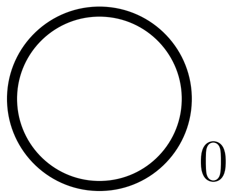

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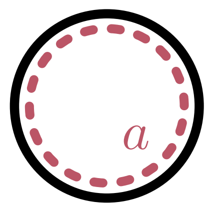

$$= \mathrm{d}_a$$

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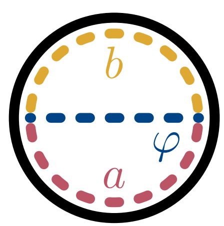
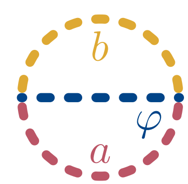
$$= \mathrm{d}_a$$

quantum dim

- Matrix element of symmetry line between normalized states

$$\frac{\langle\!\langle b|\mathcal{L}_\varphi|a\rangle\!\rangle}{\sqrt{\langle\!\langle a|a\rangle\!\rangle\langle\!\langle b|b\rangle\!\rangle}} = \frac{1}{\sqrt{\mathrm{d}_a \mathrm{d}_b}}$$




$$= \sqrt{\mathrm{d}_a \mathrm{d}_b \mathrm{d}_\varphi}\, N_{\varphi\, a}{}^{b}$$

Warm up: action of symmetry line

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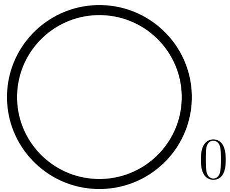
$\langle\langle 0|0\rangle\rangle =$ 

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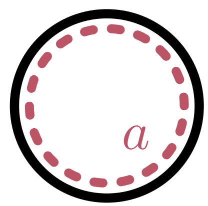
$,\quad \langle\langle a|a\rangle\rangle =$ 

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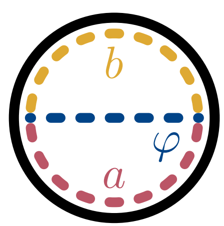
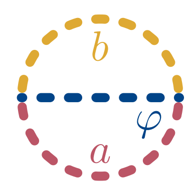
$\frac{\langle\langle b|\mathcal{L}_\varphi|a\rangle\rangle}{\sqrt{\langle\langle a|a\rangle\rangle\langle\langle b|b\rangle\rangle}} = \frac{1}{\sqrt{\mathrm{d}_a\mathrm{d}_b}}$ 



$= \sqrt{\mathrm{d}_a\mathrm{d}_b\mathrm{d}_\varphi}\, N_{\varphi\, a}{}^{b}$

- To correctly realize the symmetry algebra, we need “renormalization”

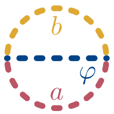
$\hat{\mathcal{L}}_\varphi = \frac{1}{\sqrt{\mathrm{d}_\varphi}}$ 

Warm up: action of symmetry line

- Now consider a state with a kink interpolating a and b vacua

$|a;b\rangle\rangle =$ 

$$\langle\langle a;b\,|\,a;b\rangle\rangle = \sqrt{\mathrm{d}_a \mathrm{d}_b \mathrm{d}_v}$$



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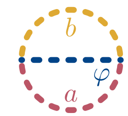
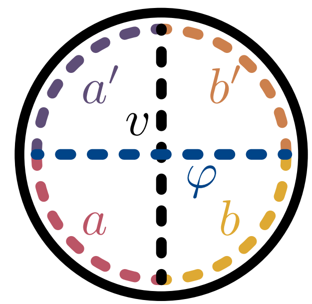


$$= \sqrt{\mathrm{d}_a \mathrm{d}_b \mathrm{d}_\varphi}\, N_{\varphi\, a}{}^{b}$$

- Matrix element between single-kink states:

$$\frac{\langle\langle a';b'\,|\,\hat{\mathcal{L}}_\varphi\,|\,a;b\rangle\rangle}{\sqrt{\langle\langle a;b|a;b\rangle\rangle\,\langle\langle a';b'\,|\,a';b'\rangle\rangle}} = \left(\mathrm{d}_a \mathrm{d}_{a'} \mathrm{d}_b \mathrm{d}_{b'} \mathrm{d}_v^2 \mathrm{d}_\phi^2\right)^{-\frac{1}{4}}$$



$$= \left(\mathrm{d}_a \mathrm{d}_{a'} \mathrm{d}_b \mathrm{d}_{b'}\right)^{1/4} \begin{bmatrix} \varphi & a' & a \\ v & b & b' \end{bmatrix}$$

Warm up: action of symmetry line

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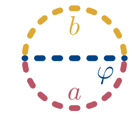

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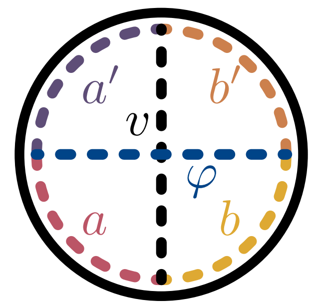
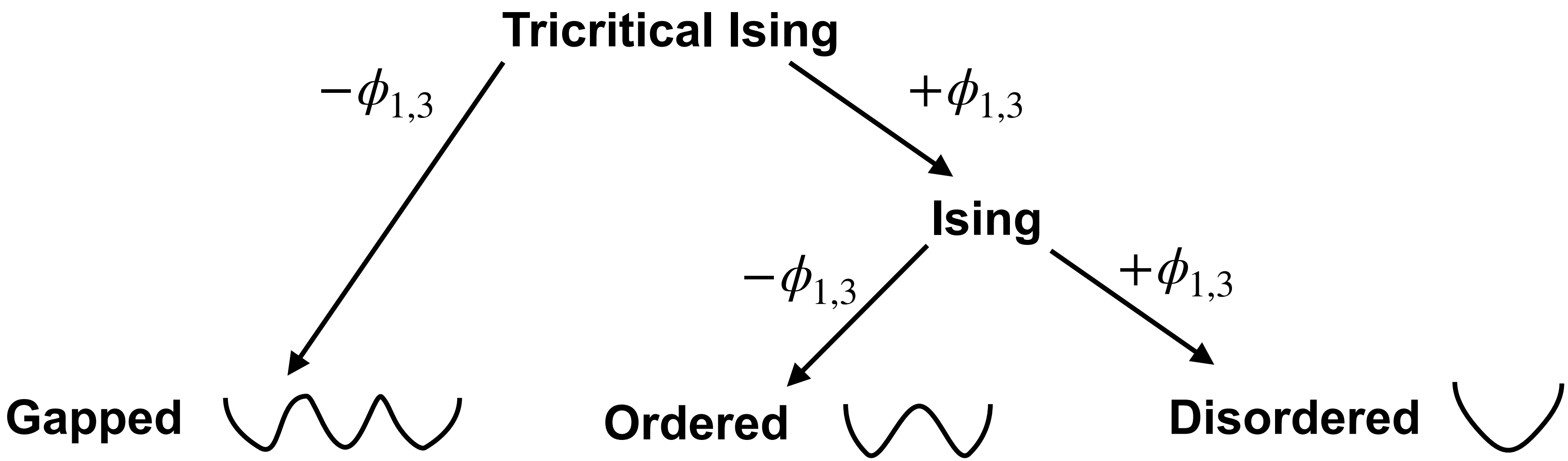


$$= \left(\mathrm{d}_a\mathrm{d}_{a'}\mathrm{d}_b\mathrm{d}_{b'}\right)^{1/4}\begin{bmatrix}\varphi & a' & a\\ v & b & b'\end{bmatrix}$$

- Our S-matrix commutes with this action of non-invertible symmetries.

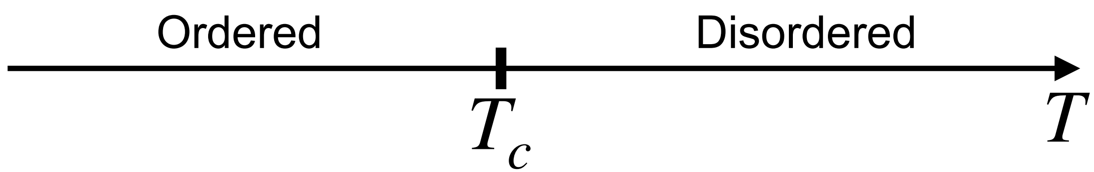
Flow from tricritical Ising

- Ising CFT is the simplest unitary minimal model ($\mathscr{M}_{3,4}$).
- Next simplest is tricritical Ising CFT ($\mathscr{M}_{4,5}$).
- RG flow from tricritical:

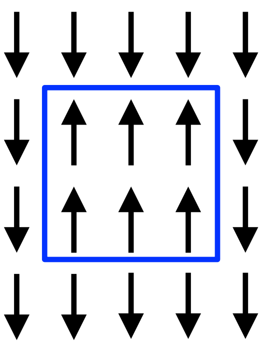


Non-Invertible Kramers-Wannier

- 2d Ising model: $$Z=\sum_{\{\sigma\}}\exp\left[\beta\sum_{<ij>}\sigma_i\sigma_j\right]$$



Low T expansion



$$Z=\sum_{\text{domain wall}} e^{-2\beta L_{\text{wall}}}$$

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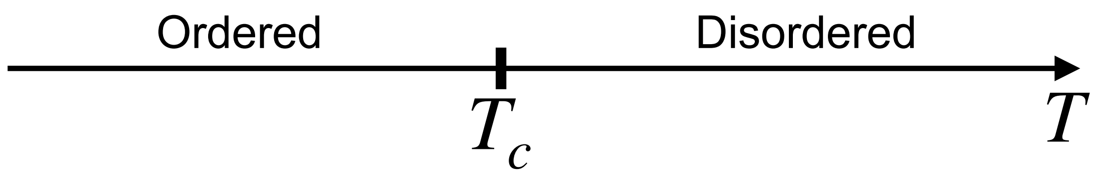

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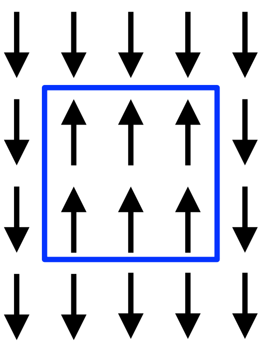

$$Z = \sum_{\text{domain wall}} e^{-2\beta L_{\text{wall}}}$$

High T expansion

$$Z = \sum_{\{\sigma\}} \prod_{<ij>} \left(\cosh\beta + \boxed{\sigma_i \sigma_j \sinh\beta}\right)$$

$$\propto \sum_{\text{loop}} (\tanh\beta)^{L_{\text{loop}}}$$

- KW duality: $$e^{-2\beta_{\text{low}}} \leftrightarrow \tanh\beta_{\text{high}}$$

Non-Invertible Kramers-Wannier

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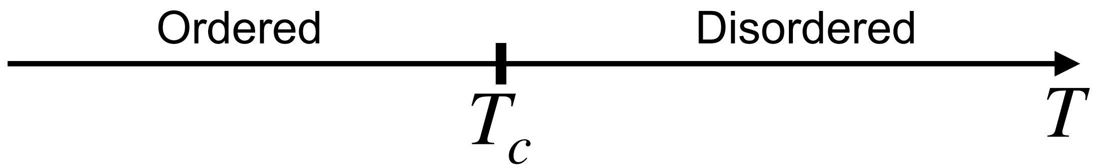

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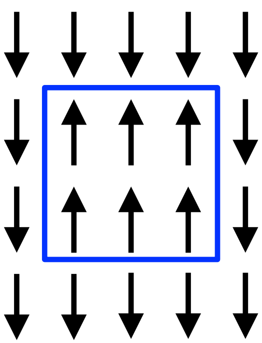

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- Strictly speaking, there is some subtlety….