Nonperturbative correlation functions from homotopy algebras

弦理論と場の理論 2024

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based on arXiv 2405.10935 **K.K.**, Y. Okawa

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Motivations

Analyzing string field theory by the algebraic approach.

String field theory : String version of quantum field theory
fundamental degrees of freedom: string field Ψ

Difficulty in SFT

- String field contains infinitely many spacetime fields.
- Action can have infinitely many interaction terms.

For the systematic construction and treatment,
algebraic approach may help.

Homotopy algebra and string field theory

- A_∞ algebra : open SFT
- L_∞ algebra : closed SFT
- open-closed homotopy algebra (OCHA) : open-closed SFT

etc...

- Relating covariant and light-cone string field theories

  [Erler and Matsunaga, arXiv:2012.09521]
- Calculating scattering amplitudes

  [Kajiura, math/0306332] etc.
- Integrating out fields

  [Sen, arXiv:1609.00459],

  [Erbin, Maccaferri, Schnabl and Vošmera, arXiv:2006.16270],

  [Koyama, Okawa and Suzuki, arXiv:2006.16710] etc.

etc.

Homotopy algebras and quantum field theory

We can also use homotopy algebras to describe ordinary QFT.

The description in terms of homotopy algebra is universal.

Well-known formula

$$\pi_1 \, \mathbf{P} \, \boldsymbol{Q} \, \mathbf{P} + \pi_1 \, \mathbf{P} \; \boldsymbol{m} \, \frac{1}{\mathbf{I} + \boldsymbol{h} \, \boldsymbol{m} + i\hbar \, \boldsymbol{h} \, \mathbf{U}} \, \mathbf{P} \, .$$

We do not need path integrals,
so there is no problem in path integral measure!

Our Strategy

- Understand relations between simple QFT and homotopy algebras.
- The description of homotopy algebra is universal.
- Generalize the relation to the complex QFT or SFT.
- Analyze SFT effectively or find hidden physics we have missed in QFT.

Today's talk

Perturbative correlation functions (for general gauge-fixed theory)

[Okawa (2022)],[Konosu, Okawa (2023)],[Konosu (2023)],[Konosu, Yoshinaka (2024)]

$$\langle\, \Phi^{\otimes n} \,\rangle = \pi_n \, \boldsymbol{f} \, \mathbf{1} \,,$$

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\boldsymbol{m} + i\hbar \, \boldsymbol{h} \, \boldsymbol{U}} = \mathbf{I} + \sum_{n=1}^{\infty} (-1)^n \, (\, \boldsymbol{h} \, \boldsymbol{m} + i \, \hbar \, \boldsymbol{h} \, \mathbf{U} \,)^n$$

Then, I will discuss the extention to the nonperturbative descriptions in 0-dimensional scalar QFT.

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Recipe for A_∞ algebra

- Prepare the action S (BV master action or gauge-fixed action). (For simplicity, we choose to use gauge-fixed action.)
- Consider vector space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$
  $\mathcal{H}_1$ is the space of field and $\mathcal{H}_2$ is the space of antifield.
- Introduce formal basis vectors $e^a(x)$ for $\mathcal{H}_1$ and $\widetilde{e}^a(x)$ for $\mathcal{H}_2$.
  "inner product" ω

$$\omega(e^a(x), \widetilde{e}^b(y)) = \delta_{ab}\, \delta^d(x-y)$$

- Define operators to satisfy $(M_n = Q, m_n)$

$$S = \underbrace{\frac{1}{2}\,\omega(\,\Phi, Q\,\Phi\,)}_{\text{kinetic term}} + \sum_n \frac{1}{n+1}\, \underbrace{\omega\Big(\,\Phi, m_n(\overbrace{\,\Phi\, \otimes \ldots \otimes\, \Phi\,}^{n}\Big))}_{(n+1)-\text{point interaction}},$$

$$\Phi \in \mathcal{H}_1$$

$+ A_\infty$ relations $+$ cyclicity

Homotopy algebras

the 0-dimensional quartic Euclidean theory ($\lambda > 0$)

$$S = \frac{1}{2}\, m^2\, \varphi^2 + \frac{1}{4}\, \lambda\, \varphi^4$$

$$= \underbrace{\frac{1}{2}\, \omega(\, \Phi, Q\, \Phi\,)}_{\text{kinetic term}} + \sum_n \frac{1}{n+1}\, \underbrace{\omega\Big(\, \Phi, m_n(\overbrace{\,\Phi\, \otimes \ldots \otimes\, \Phi\,}^{n}\Big))}_{(n+1)-\text{point interaction}}\,,$$

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\,,$$

c : basis vector of $\mathcal{H}_1$ d : basis vector of $\mathcal{H}_2$

$$\Phi = \varphi\, c\,,$$

$$\begin{pmatrix} \omega\,(\,c\,,c\,) & \omega\,(\,c\,,d\,) \\ \omega\,(\,d\,,c\,) & \omega\,(\,d\,,d\,) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\,1 & 0 \end{pmatrix},$$

$$Q\, c = m^2\, d\,, \quad Q\, d = 0\,,$$

$$m_3(c \otimes c \otimes c) = \lambda\, d\,.$$

Homotopy algebras

We also define additional operators to calculate physical quantity.

P : projection onto the subspace of $\mathcal{H}$
To calculate correlation functions, we define $P = 0$.

h : contracting homotopy (propagator)

$$Q\,h + h\,Q = \mathbb{I} - P\,, \quad h\,P = 0\,, \quad P\,h = 0\,, \quad h^2 = 0\,.$$

$\mathbb{I}$: identity operator on $\mathcal{H}$
In this case,

$$h\,d = \frac{1}{m^2}\,c\,, \quad h\,c = 0\,.$$

U: quantum correction

$$(\,\omega \otimes \mathbb{I}\,)\,(\,\mathbb{I} \otimes U\,) = \mathbb{I}$$

Homotopy algebra and Correlation Function

We then extend the operator on $\mathcal{H}$ to

$$T\mathcal{H} = \mathcal{H}^{\otimes 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \mathcal{H}^{\otimes 3} \oplus \ldots .$$

We denote the operator on $T\mathcal{H}$ by the bold face. For example,

$$\begin{aligned} m = \sum_n m_n \; &\to \; \boldsymbol{m} \\ h \; &\to \; \boldsymbol{h} \\ U \; &\to \; \mathbf{U} \end{aligned}$$

We also define the projection $\pi_n : T\mathcal{H} \to \mathcal{H}^{\otimes n}$.
e.g.)

$$\pi_2 \left(\mathbf{1} + \Phi + \Phi \otimes \Phi + \Phi \otimes \Phi \otimes \Phi \right) = \Phi \otimes \Phi \, .$$

Homotopy algebra and Correlation Function

Then, perturbative correlation functions are given by

$$\langle\, \Phi^{\otimes n} \,\rangle = \pi_n\, \boldsymbol{f}\, \mathbf{1}\,,$$

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \hbar\,\boldsymbol{h}\,\mathbf{U}} = \mathbf{I} + \sum_{n=1}^{\infty} (-1)^n \,(\, \boldsymbol{h}\,\boldsymbol{m} - \hbar\,\boldsymbol{h}\,\mathbf{U}\,)^n\,,$$

where $\mathbf{1}$ is the basis vector of $\mathcal{H}^{\otimes 0}$.

In 0-dimensional case,

$$\langle\, \Phi^{\otimes n} \,\rangle = \langle\, \varphi^n \,\rangle\, c^{\otimes n}\,.$$

Homotopy algebra and Correlation Function

Then, perturbative correlation functions are given by

$$\langle \Phi^{\otimes n} \rangle = \pi_n \, \boldsymbol{f} \, \mathbf{1} \,,$$

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \hbar \boldsymbol{h}\,\mathbf{U}} = \mathbf{I} + \sum_{n=1}^{\infty} (-1)^n \, (\, \boldsymbol{h}\,\boldsymbol{m} - \hbar\,\boldsymbol{h}\,\mathbf{U}\,)^n$$

Does the exact inverse of $\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \hbar\,\boldsymbol{h}\,\mathbf{U}$ exist?

$\rightarrow$ We will examine this fact in 0-dimensional scalar field theory.

Nonperturbative extension

In zero-dimensional case, we can represent the operators of homotopy algebras

$$T\mathcal{H}_1 \to T\mathcal{H}_1$$

by infinite dimensional matrices.
e.g.) $\boldsymbol{h}\,\mathbf{U}$, $\boldsymbol{h}\,\boldsymbol{m}$

We identify the operators with matrices by

$$\mathbf{A}_{ij}\,c^{\otimes i} = \pi_i\,\mathbf{A}\,\pi_j$$

Truncating the size of matrices by N, we can carry out numerical calculations.

Nonperturbative extension

In the quartic Euclidean theory,

$$S = \frac{1}{2}\,\varphi^2 + \frac{1}{4}\,\lambda\,\varphi^4\,,$$

$$\boldsymbol{h}\,\mathbf{U} = \frac{1}{m^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\boldsymbol{h}\,\boldsymbol{m}_3 = \frac{\lambda}{m^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Results in φ^4 theory

In the matrix representation, correlation functions are calculated by

$$\langle\, \varphi^n \,\rangle = \boldsymbol{f}_{n0}\,,$$

where $\boldsymbol{f}$ is the inverse matrix of $\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \hbar\,\boldsymbol{h}\,\mathbf{U}$.

For $N = 25$, $\boldsymbol{f}_{20}$ and $\boldsymbol{f}_{40}$ as functions of λ are given by

$$\boldsymbol{f}_{20} = \frac{1 + 140\lambda + 6660\lambda^2 + 129360\lambda^3 + 957075\lambda^4 + 1853460\lambda^5}{1 + 143\lambda + 7065\lambda^2 + 147420\lambda^3 + 1267350\lambda^4 + 3615885\lambda^5 + 1514205\lambda^6}$$

$$\boldsymbol{f}_{40} = \frac{3 + 405\lambda + 18060\lambda^2 + 310275\lambda^3 + 1762425\lambda^4 + 1514205\lambda^5}{1 + 143\lambda + 7065\lambda^2 + 147420\lambda^3 + 1267350\lambda^4 + 3615885\lambda^5 + 1514205\lambda^6}$$

Nonperturbative extension

For $\lambda = 0.04$ and $N = 100$,

$$\langle \varphi^2 \rangle = 0.906536724355833769\ldots,$$
$$\boldsymbol{f}_{20} = 0.906536724355833771\ldots.$$

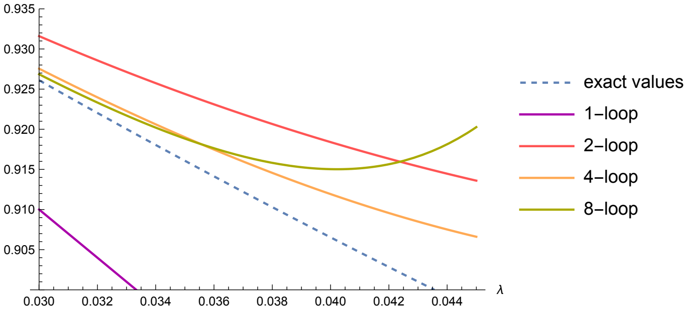

Figure: From arXiv 2405.10935 K. Konosu, Y. Okawa

Nonperturbative extension

Even when $\lambda = 0.2$, the nonperturbative effects dominate.

$$\langle \varphi^2 \rangle = 0.724059025\ldots,$$

and the estimated value from our formula for $N = 100$

$$\boldsymbol{f}_{20} = 0.724059020\ldots.$$

$\langle \varphi^2 \rangle$	$\lambda = 0.04$	$\lambda = 0.2$	$\lambda = 1.5$	$\lambda = 3$
exact	0.9065367244	0.7240590202	0.4066915207	0.3130156270

N	$\lambda = 0.04$	$\lambda = 0.2$	$\lambda = 1.5$	$\lambda = 3$
10	0.9059745348	0.7024793388	0.2685512367	0.1574468085
25	0.9065366639	0.7237546945	0.3751623774	0.2543859219
50	0.9065367244	0.7240552164	0.4002397294	0.2932452874
100	0.9065367244	0.7240590258	0.4072861268	0.3167705780

Table: From arXiv 2405.10935 K. Konosu, Y. Okawa

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Deformation of the formula

In the current formalism, we need to split the action into free part and interaction part.
Is our formula "*background independent*"?
i.e. the results coincide when we choose different free part?

We can deform our formula and obtain

$$\langle\, \Phi^{\otimes n} \,\rangle = \pi_n \, \frac{1}{\mathbf{I} + i\hbar\, \mathbf{H}\, \mathbf{U}} \, \mathbf{P}\, \mathbf{1} \,,$$

$$\mathbf{P} = e^{\boldsymbol{\Phi}_*} \, \pi_0 \,,$$

$$\mathbf{M}\, \mathbf{H} + \mathbf{H}\, \mathbf{M} = \mathbf{I} - \mathbf{P} \,, \qquad \mathbf{H}\, \mathbf{P} = 0 \,, \qquad \mathbf{P}\, \mathbf{H} = 0 \,, \qquad \mathbf{H}^2 = 0 \,.$$

$\boldsymbol{\Phi}_*$: operator associated with classical solution Φ_*
$\mathbf{H}$: constructed from $\boldsymbol{h}$, $\boldsymbol{m}$ and $\boldsymbol{\Phi}_*$

Our conjecture

In our analysis, the direct application to the theory which has multiple saddle points fails.

e.g.)
Airy function

$$S = \, - a\, \varphi - \frac{1}{3}\, \varphi^3$$

with $a < 0$.

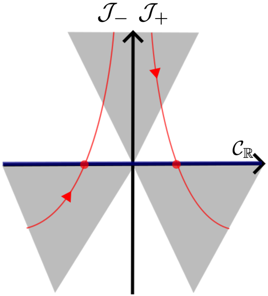
Double-well potential

$$S = \frac{1}{2}\, \varphi^2 - \frac{1}{4} \lambda\, \varphi^4\,.$$

Our conjecture

the Lefschetz thimble [Witten, arXiv:1001.2933]
(Airy function $a < 0$)

$$Z = \int_{\mathcal{C}_{\mathbb{R}}} d\varphi \, e^{\frac{i}{\hbar} S}$$
$$= \int_{\mathcal{J}_-} d\varphi \, e^{\frac{i}{\hbar} S} + \int_{\mathcal{J}_+} d\varphi \, e^{\frac{i}{\hbar} S}$$



Our conjecture

Associated with the deformed formula, we arrive at the conjecture:
The formula describes correlation functions on the *Lefschetz thimble* [Witten, arXiv:1001.2933] associated with the solution.

e.g.) Airy function

$$\langle\, \varphi^n \,\rangle = \frac{Z_-}{Z_- + Z_+}\,\langle\, \varphi^n \,\rangle_- + \frac{Z_+}{Z_- + Z_+}\,\langle\, \varphi^n \,\rangle_+$$

$Z_\pm$: partition function associated with thimble $\pm\sqrt{-a}$
$\langle\,\cdot\,\rangle_\pm$: expectation value associated with thimble $\pm\sqrt{-a}$

$$\langle\, \Phi^{\otimes n} \,\rangle = \pi_n \,\frac{1}{\mathbf{I} + i\hbar\,\mathbf{H}\,\mathbf{U}}\,\mathbf{P}\,\mathbf{1}\,,$$

$$\mathbf{P} = e^{\mathbf{\Phi}_*}\,\pi_0\,,$$

Airy function

$a = -1$ with $N = 50$

$$\langle\,\varphi\,\rangle_- \simeq -1.06944263 + 0.18869689\,i\,,$$
$$\langle\,\varphi\,\rangle_-^{\rm hom} \simeq -1.06944243 + 0.18869651\,i$$

$$\langle\,\varphi^3\,\rangle_- \simeq -1.06944263 - 0.81130311\,i\,,$$
$$\langle\,\varphi^3\,\rangle_-^{\rm hom} \simeq -1.06944243 - 0.81130349\,i\,.$$

$$\langle\,\varphi\,\rangle_+ \simeq 1.06944263 + 0.18869689\,i\,,$$
$$\langle\,\varphi\,\rangle_+^{\rm hom} \simeq 1.06944243 + 0.18869651\,i$$

$$\langle\,\varphi^3\,\rangle_+ \simeq 1.06944263 - 0.81130311\,i\,,$$
$$\langle\,\varphi^3\,\rangle_+^{\rm hom} \simeq 1.06944243 - 0.81130349\,i\,.$$

Similar analysis works well to the double-well potential.

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Summary

I explained the formula for correlation functions

$$\langle\, \Phi^{\otimes n} \,\rangle = \pi_n \, \boldsymbol{f} \, \mathbf{1} \,,$$

$$\begin{aligned} \boldsymbol{f} &= \frac{1}{\mathbf{I} + \boldsymbol{h}\boldsymbol{m} + i\hbar\boldsymbol{h}\mathbf{U}} \\ &= \mathbf{I} + \sum_n (-1)^n (\boldsymbol{h}\boldsymbol{m} + i\hbar\boldsymbol{h}\mathbf{U})^n \,. \end{aligned}$$

In zero-dimensional QFT, we can numerically calculate the "exact" inverse of $\mathbf{I} + \boldsymbol{h}\boldsymbol{m} + i\hbar\boldsymbol{h}\mathbf{U}$.

We test several examples in scalar field theory and confirm that our formula contains the information on nonperturbative effect and have relations to the Lefshetz thimbles.

Future works

- The principle to determine the ratio of the correlation functions with multiple thimbles without path integral?
- How to interpret Lefshetz thimbles in terms of homotopy algebra?
- Extension to non-zero dimensional theory