Hawking-Page and entanglement phase transition in 2d CFT on curved backgrounds

Akihiro Miyata (KITS)

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Based on the joint work arXiv:2406.06121 (to appear in JHEP)
with Masahiro Nozaki (KITS), Kotaro Tamaoka (Nihon U.) and Masataka Watanabe (Nagoya U.)

- We consider a two-dimensional holographic CFT defined on curved backgrounds, and by varying the CFT background metric we observe that Hawking-Page like phase transition happens.

- We also consider the gravity dual.

Contents of this talk

1. Introduction
    - Hawking-Page transition
    - The inhomogeneous Hamiltonian and the Hamiltonian on a curved space
2. Global and Local properties of the thermal state
3. Gravity dual to the CFT

Summary and Future directions

1. Introduction

Hawking-Page transition

- For simplicity, focus on AdS_3/CFT_2.

- In AdS semi-classical gravity $G_N \ll 1$, dominates saddle geometries are given by either thermal AdS or AdS black holes depending on the temperature $T = \beta^{-1}$ and system size L; there is a phase transition between them, ie., Hawking-Page transition [Hawking-Page '83,…],

Low temperature $L/\beta < 1 \rightarrow$ thermal AdS

High temperature $L/\beta > 1 \rightarrow$ AdS (BTZ) black holes

- In CFT, this phase transition corresponds to a first-order confinement-deconfinement phase transition [Witten '98, Maldacena '98, …].

Deformation of the CFT Hamiltonian

- Start with the deformation of the the system on the spatial circle with circumstance L

$$H_0 = \int_0^L \frac{dx}{2\pi}(T(x) + \overline{T}(x)) = \frac{2\pi}{L}\left[L_0^z + \overline{L}_0^{\bar{z}} - \frac{c}{12}\right] \qquad (z, \bar{z}) = \left(e^{\frac{2\pi(ix+\mathcal{T})}{L}}, e^{\frac{2\pi(-ix+\mathcal{T})}{L}}\right)$$

$$\begin{aligned}\to H_{q-\text{ Möbius}} &= \int_0^L \frac{dx}{2\pi}\left[1 - \tanh 2\theta\left(1 - 2\sin^2\left(\frac{q\pi x}{L}\right)\right)\right](T(x) + \overline{T}(x)) \\ &= \frac{2\pi}{L}\left[L_0^z + \overline{L}_0^z - \frac{\tanh 2\theta}{2}\left(L_q^z + \overline{L}_q^z + L_{-q}^z + \overline{L}_{-q}^z\right) - \frac{c}{12}\right]\end{aligned}$$

q : Positive integer, θ : non-negative real parameter

- The spatial Inhomogeneity is introduced by the enveloping function

$$f(x,\theta) = 1 - \tanh 2\theta\left(1 - 2\sin^2\left(\frac{q\pi x}{L}\right)\right) = 1 - \tanh 2\theta \cdot \cos\left(\frac{2q\pi x}{L}\right)$$

- At $\theta = 0$, the Hamiltonian reduces to the uniform one, and at the large-θ, it becomes the q-deformed Sine-square deformed (q-SSD) Hamiltonian.

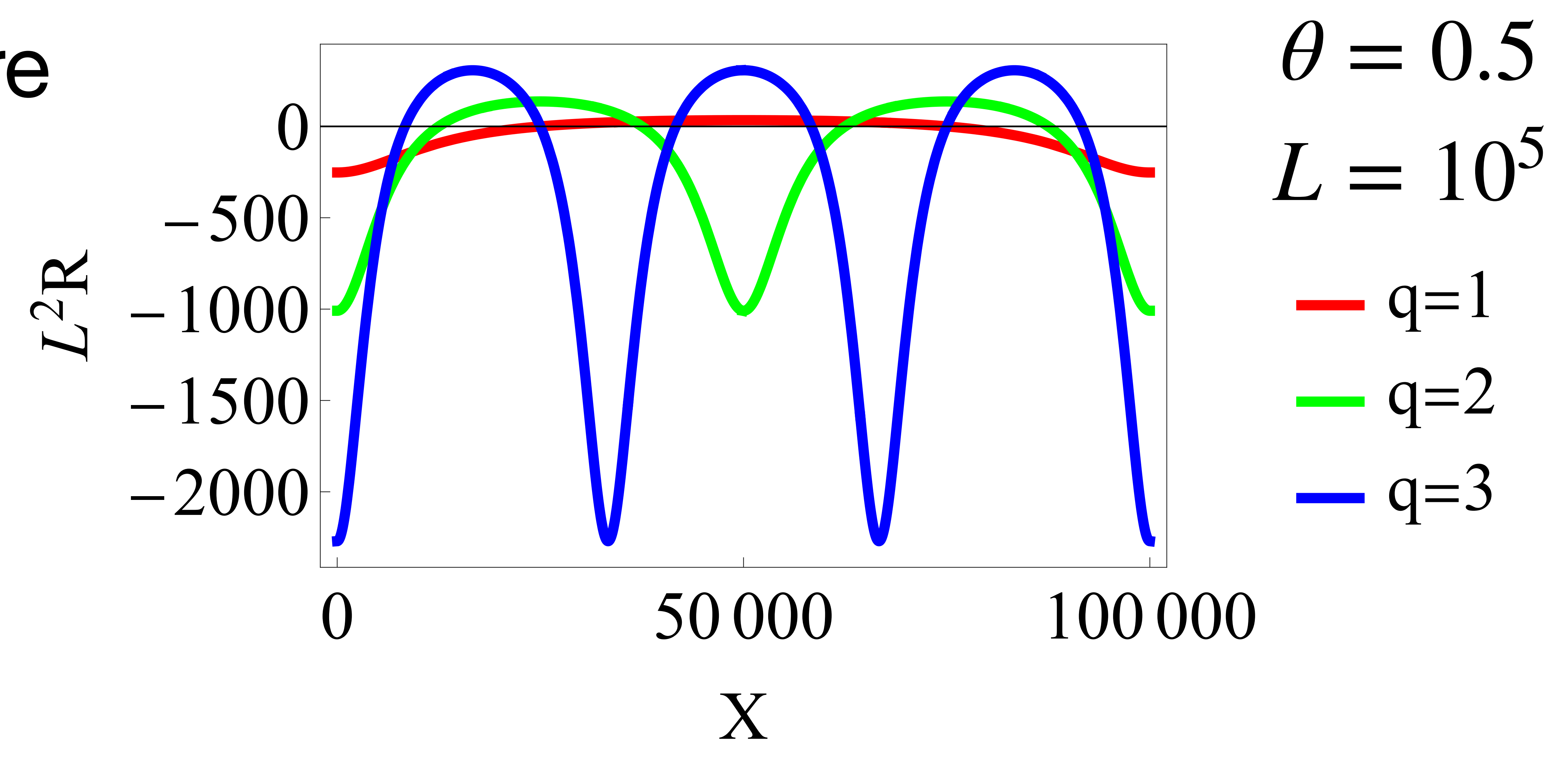
CFT on the curved background

- The inhomogeneous Hamiltonian is equivalent to the uniform Hamiltonian on the curved background with the metric

$$\boxed{\begin{aligned} H_{q-\text{Möbius}} &= \int_0^L \frac{dx}{2\pi} \sqrt{-\det g(x)}(T(x) + \bar{T}(x)), \\ ds^2 &= g_{ab}(x)dx^a dx^b = -f(x,\theta)^2 dt^2 + dx^2 \end{aligned}}$$

- The background has the Ricci curvature

$$R = -\frac{2\partial_x^2 f(x,\theta)}{f(x,\theta)} = \frac{8\pi^2 q^2 \tanh(2\theta)\cos\left(\frac{2\pi q x}{L}\right)}{L^2\left(\tanh(2\theta)\cos\left(\frac{2\pi q x}{L}\right) - 1\right)}.$$



Thermal state for the Hamiltonian

- Consider the thermal state of using the Hamiltonian, not having the form of the uniform one $\rho = e^{-\beta H_{q-\text{ Möbius}}} / \mathbf{tr}\, e^{-\beta H_{q-\text{ Möbius}}}$

- By the conformal maps

$$(z, \bar{z}) = \left(e^{\frac{2\pi(ix+\mathcal{T})}{L}}, e^{\frac{2\pi(-ix+\mathcal{T})}{L}}\right), \chi = \left(\frac{\cosh(\theta)z^q - \sinh(\theta)}{\cosh(\theta) - \sinh(\theta)z^q}\right)^{\frac{1}{q}}, \bar{\chi} = \left(\frac{\cosh(\theta)\bar{z}^q - \sinh(\theta)}{\cosh(\theta) - \sinh(\theta)\bar{z}^q}\right)^{\frac{1}{q}},$$

the Hamiltonian is mapped to that having the standard form like the uniform Hamiltonian,

$$H_{\text{q-Möbius}} = \frac{2\pi}{L_{\text{eff}}}\left[L_0^{\chi} + \bar{L}_0^{\bar{\chi}} - \frac{c}{12}\right] + \frac{2\pi c q^2}{12 L_{\text{eff}}} - \frac{2\pi c q^2}{12 L} \qquad L_{\text{eff}} = L\cosh 2\theta$$

$$\rho = e^{-\beta H_{q-\text{ Möbius}}} / \mathrm{tr}\, e^{-\beta H_{q-\text{ Möbius}}} = \exp\left(\frac{-2\pi\beta\left(L_0^{\chi} + \overline{L_0^{\chi}}\right)}{L_{\text{eff}}}\right) / \mathrm{tr} \exp\left(\frac{-2\pi\beta\left(L_0^{\chi} + \overline{L_0^{\chi}}\right)}{L_{\text{eff}}}\right)$$

- Thus, by using the $(\chi, \bar{\chi})$ coordinates,

the thermal state for the deformed inhomogeneous CFT Hamiltonian

(= CFT Hamiltonian on the curved space)

= the thermal state for the uniform CFT Hamiltonian with replacing $\dfrac{\beta}{L}$ with $\dfrac{\beta}{L\cosh 2\theta}$

- In the coordinates, we can use usual techniques for the standard thermal state

Main result

- We studied thermal and entanglement properties of the inhomogeneous thermal state by changing the parameter θ controlling the CFT background metric in a two-dimensional holographic CFT.
- We found that, by changing θ with fixing $L/\beta < 1$, we observed that Hawking-Page like phase transition happens.
- We also consider the gravity dual for the inhomogeneous thermal state.

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2. Global and Local properties of the thermal state

Thermal entropy

- First, we focus on the thermodynamic property of the thermal state.

- In the $(\chi, \bar{\chi})$ coordinates, the thermal state is given by the usual form with the moduli parameter

$$\boxed{\tau_{\text{Mod.}} = \frac{L_{\text{eff}}}{\beta} = \frac{L\cosh(2\theta)}{\beta}}$$

- From the conventional analysis of the Hawking-Page phase transition, there is a first-order phase transition
  - Thermal AdS_3 $\Longleftrightarrow \tau_{\text{Mod.}} < 1$
  - BTZ black hole $\Longleftrightarrow \tau_{\text{Mod.}} > 1$

$$S_{\text{Thermal}} = \begin{cases} \mathcal{O}(1) & \text{for } \tau_{\text{Mod.}} < 1 \\ \frac{c\pi L_{\text{eff}}}{3\beta} & \text{for } \tau_{\text{Mod.}} > 1 \end{cases}$$

- Starting with $L/\beta < 1$, the system exhibits the phase transition at

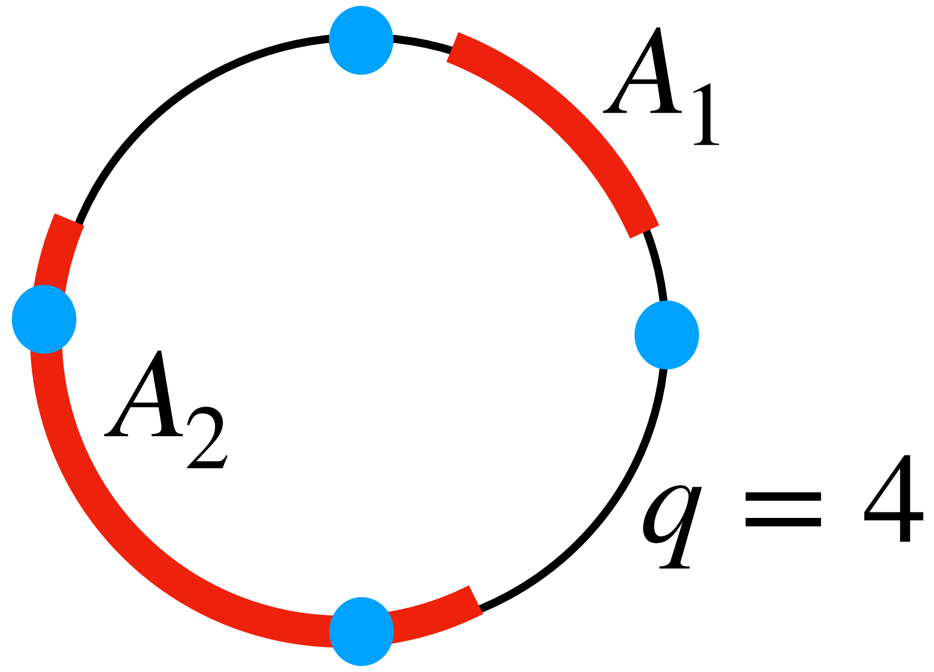
$$\tau_{\text{Mod.}}(\theta_c) = 1 \Longleftrightarrow \theta_c = \frac{1}{2}\cosh^{-1}\left(\frac{\beta}{L}\right) \underset{L/\beta \ll 1}{\approx} \frac{1}{2}\log\left(\frac{2\beta}{L}\right)$$

Entanglement entropy

- Define subsystems A_1, A_2 and compute the entanglement entropies for their subsystems.

$$A_1 = \left\{ x \middle| X_n^f \equiv \frac{n}{q} L < X_2 < x < X_1 < X_{n+1}^f \right\}$$

$$A_2 = \left\{ X_2 < x < X_1 \middle| X_n^f < X_2 < X_{n+1}^f, X_{n+l}^f < X_1 < X_{n+l+1}^f \right\}$$



- To evaluate them, we can use the twist-operator formalism. The Renyi-n entropy is given by

$$S_{A_i}^{(n)} = \frac{1}{1-n} \log \left[\frac{\mathrm{tr}\, \sigma_n(w_1, \bar{w}_1) \bar{\sigma}_n(w_2, \bar{w}_2) e^{-\beta H_{q-\text{ Möbius}}}}{\mathrm{tr}\, e^{-\beta H_{q-\text{ Möbius}}}} \right] \qquad (w, \bar{w}) = (T + ix, T - ix)$$

$$= \frac{1}{1-n} \log \left[\prod_{i=1,2} \left(\frac{d\xi_i}{dw_i} \right)^{h_n} \left(\frac{d\bar{\xi}_i}{d\bar{w}_i} \right)^{\bar{h}_n} \right] + \frac{1}{1-n} \log \left\langle \sigma_n(\xi_1, \bar{\xi}_1) \bar{\sigma}_n(\xi_2, \bar{\xi}_2) \right\rangle_{\text{Torus}}$$

$$\xi = \frac{L_{\text{eff}}}{2\pi} \log[\chi], \bar{\xi} = \frac{L_{\text{eff}}}{2\pi} \log[\bar{\chi}] \qquad \chi = \left(\frac{\cosh(\theta) z^q - \sinh(\theta)}{\cosh(\theta) - \sinh(\theta) z^q} \right)^{\frac{1}{q}}, \bar{\chi} = \left(\frac{\cosh(\theta) \bar{z}^q - \sinh(\theta)}{\cosh(\theta) - \sinh(\theta) \bar{z}^q} \right)^{\frac{1}{q}}$$

Entanglement entropy in low effective temperature

$$\tau_{\mathrm{Mod.}} < 1$$

In the low effective temperature, the entanglement entropy can be evaluated as

$$\begin{aligned} S_{A_{1,2}} &= \frac{c}{6}\log\left[\prod_{i=1,2}\left(1-\tanh 2\theta\cos\left(\frac{2q\pi X_i}{L}\right)\right)\right] + \frac{c}{3}\log\left(\frac{L_{\mathrm{eff}}}{\pi\epsilon}\right) + \frac{c}{6}\log\left|\sinh\left(\frac{\pi(\xi_1-\xi_2)}{L_{\mathrm{eff}}}\right)\right|^2 \\ &= \frac{c}{6}\log\left[\prod_{i=1,2}\left(1-\tanh 2\theta\cos\left(\frac{2q\pi X_i}{L}\right)\right)\right] + \frac{c}{3}\log\left(\frac{L_{\mathrm{eff}}}{\pi\epsilon}\right) + \frac{c}{3}\log\sin\left[\frac{\varphi_1-\varphi_2}{q}\right] \end{aligned},$$

where $\xi = i\dfrac{L_{\mathrm{eff}}}{\pi q}\varphi$

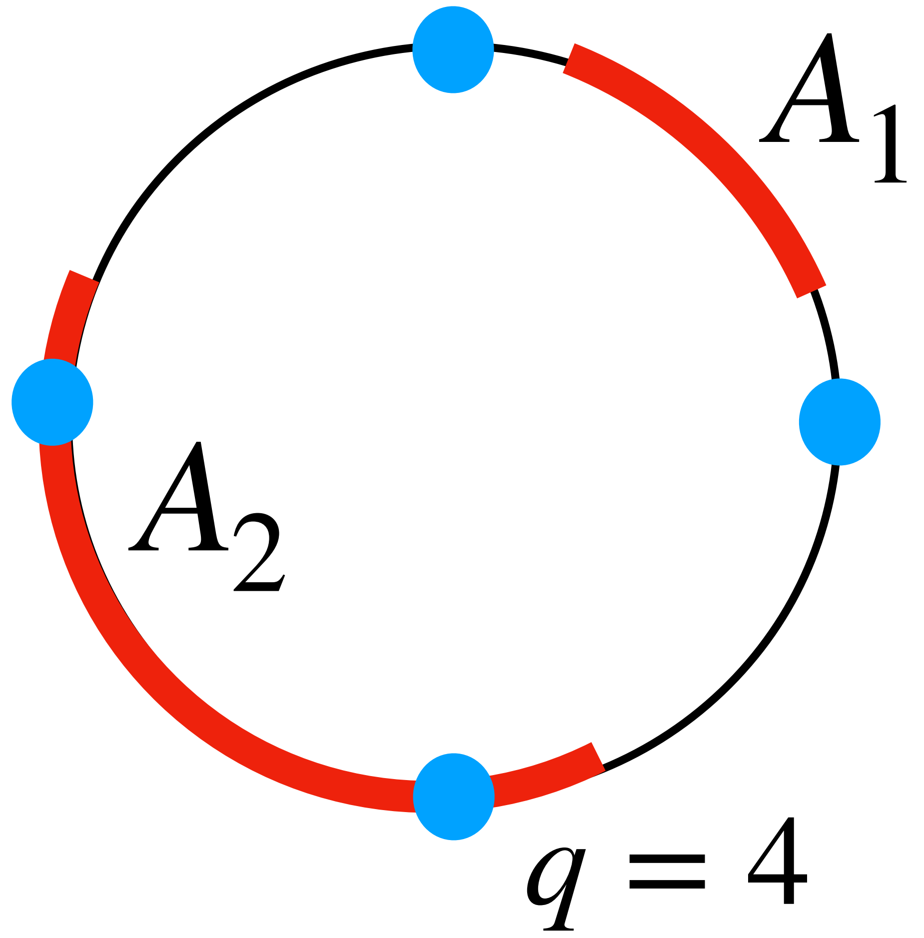
$$e^{i\varphi_i} = \frac{e^{-\theta}\cos\left(\frac{q\pi X_i}{L}\right) + ie^{\theta}\sin\left(\frac{q\pi X_i}{L}\right)}{\sqrt{e^{-2\theta}\cos^2\left(\frac{q\pi X_i}{L}\right) + e^{2\theta}\sin^2\left(\frac{q\pi X_i}{L}\right)}}$$

In the large θ-limit with keeping low effective temperature regime $L \cosh 2\theta/\beta < 1$, the entanglement entropy becomes

$$S_{A_1} \approx \frac{c}{3}\log\left[\frac{L}{\pi q \epsilon}\sin\left[\frac{q\pi\left(\hat{X}_1 - \hat{X}_2\right)}{L}\right]\right]$$

$$X_i = \frac{m_i L}{q} + \hat{X}_i$$

$$S_{A_2} \approx \frac{c}{3}\log\left(\frac{Le^{2\theta}}{2\pi\epsilon}\right) = \frac{2c}{3}\theta + \frac{c}{3}\log\left(\frac{L}{2\pi\epsilon}\right)$$



- If the subsystem does not include points $\sin^2(q\pi x/L) = 0$, the entanglement entropy is given by the vacuum one with system size L/q.

Entanglement entropy in high effective temperature

$$\tau_{\mathrm{Mod.}} > 1$$

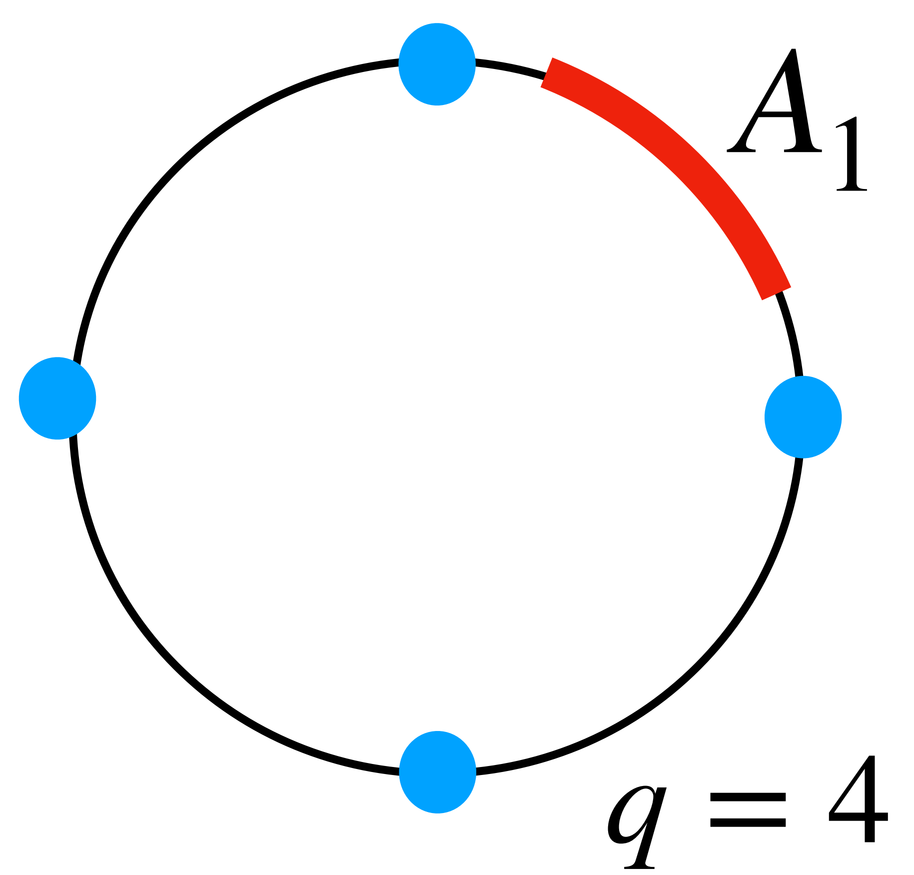
- On the other hand, in the high effective temperature, the entanglement entropy can be evaluated as

$$\begin{aligned}
S_{A_{1,2}} =& \frac{c}{6}\log\left[\prod_{i=1,2}\left(1-\tanh 2\theta\cos\left(\frac{2q\pi X_i}{L}\right)\right)\right]+\frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right) \\
&+\mathrm{Min}\left[\frac{c}{6}\log\left|\sin\left(\frac{\pi(\xi_1-\xi_2)}{\beta}\right)\right|^2, \frac{c}{6}\log\left|\sin\left(\frac{\pi(\pm iL_{\mathrm{eff}}-(\xi_1-\xi_2))}{\beta}\right)\right|^2+\frac{c\pi L_{\mathrm{eff}}}{3\beta}\right] \\
=& \frac{c}{6}\log\left[\prod_{i=1,2}\left(1-\tanh 2\theta\cos\left(\frac{2q\pi X_i}{L}\right)\right)\right]+\frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right) \\
&+\mathrm{Min}\left[\frac{c}{3}\log\left[\sinh\left[\frac{L_{\mathrm{eff}}(\varphi_1-\varphi_2)}{q\beta}\right]\right], \frac{c}{3}\log\left|\sinh\left[\frac{L_{\mathrm{eff}}(\varphi_1-\varphi_2\mp\pi q)}{q\beta}\right]\right|+\frac{c\pi L_{\mathrm{eff}}}{3\beta}\right]
\end{aligned}$$

- In the large-θ limit,

$$S_{A_1} \approx \frac{c}{6}\log\left[4\prod_{i=1,2}\sin^2\left(\frac{q\pi X_i}{L}\right)\right] + \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right) + \frac{c}{3}\log\left[\sinh\left[\frac{L\sin\left[\frac{q\pi(X_1-X_2)}{L}\right]}{2q\beta\sin\left[\frac{q\pi\hat{X}_1}{L}\right]\sin\left[\frac{q\pi\hat{X}_2}{L}\right]}\right]\right]$$

$$\approx \begin{cases} \frac{c}{3}\log\left[\frac{L\sin\left[\frac{q\pi\left(\hat{X}_1-\hat{X}_2\right)}{L}\right]}{q\pi\epsilon}\right] & \text{For } \beta \gg L \\ \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right) + \frac{cL}{6q\beta}\cdot\left[\frac{\sin\left[\frac{q\pi\left(\hat{X}_1-\hat{X}_2\right)}{L}\right]}{\sin\left[\frac{q\pi\hat{X}_1}{L}\right]\sin\left[\frac{q\pi\hat{x}_2}{L}\right]}\right] & \text{For } L \gg \beta \end{cases}$$



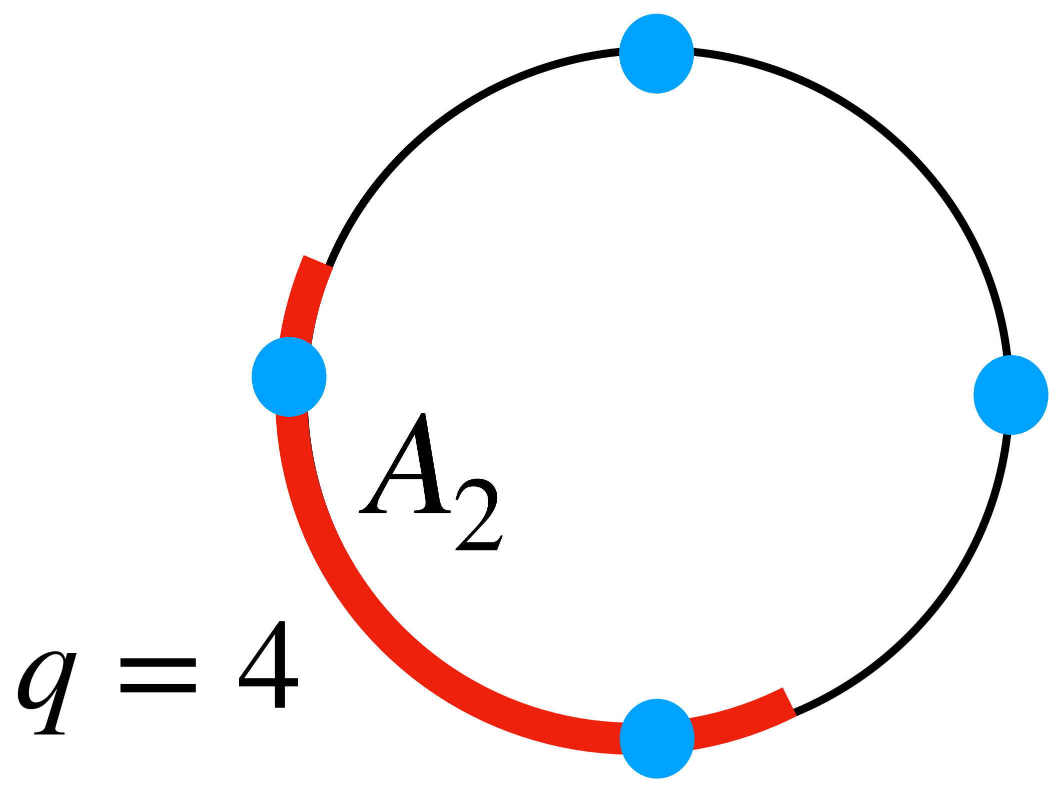
- If $L/\beta \ll 1$, the entanglement entropy reduces to the vacuum one with systems size L/q.
- On the other hand, if $L/\beta \gg 1$, it is proportional to $1/\beta$, but not simply given by thermal entropy.

- In the large-θ limit, for

$$S_{A_2} \approx \frac{c}{6}\log\left[4\prod_{i=1,2}\sin^2\left(\frac{q\pi X_i}{L}\right)\right] + \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right) + \frac{c\pi L e^{2\theta}}{6\beta}\cdot\frac{l}{q} \approx \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\right) + \frac{c\pi L e^{2\theta}}{6\beta}\cdot\frac{l}{q}$$

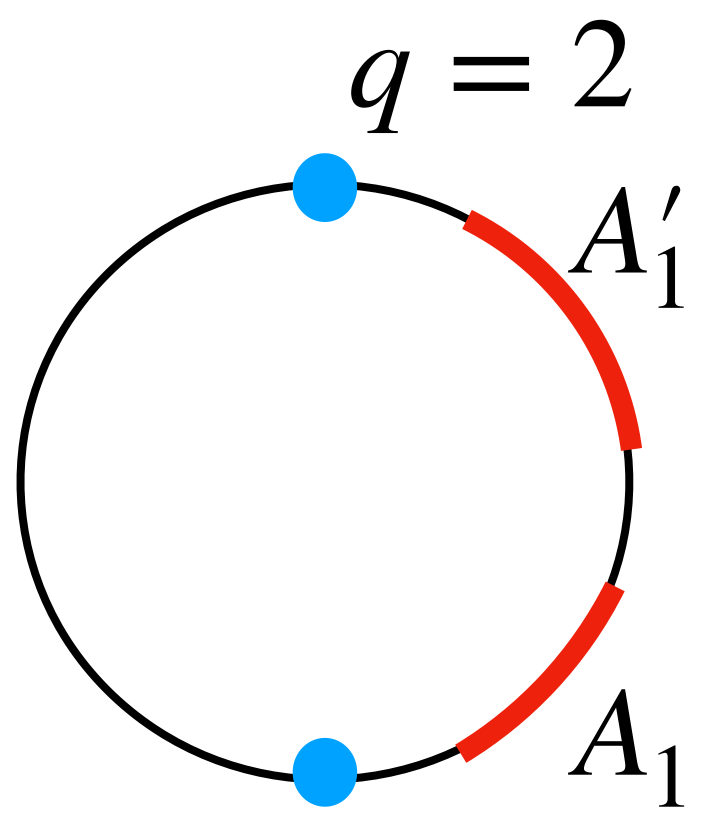
- The entanglement entropy is proportional to number of points $\sin^2(q\pi x/L) = 0$ included in the subsystem A_2, i.e. l , and to the effective system size

$$L_{\mathrm{eff}} \approx L\cdot\frac{e^{2\theta}}{2}$$



Mutual information

- The enveloping function is related to a local temperature.



- Let us see the relation by considering the mutual information between subsystems,

$$I(A_1, A_1') = S_{A_1} + S_{A_1'} - S_{A_1 \cup A_1'}$$

$$A_1 = \left\{ x \middle| X_n^f \equiv \frac{n}{q} L < X_2 < x < X_1 < X_{n+1}^f \right\} \quad A_1' = \left\{ x \middle| X_n^f < X_1 < X_4 < x < X_3 < X_{n+1}^f \right\}$$

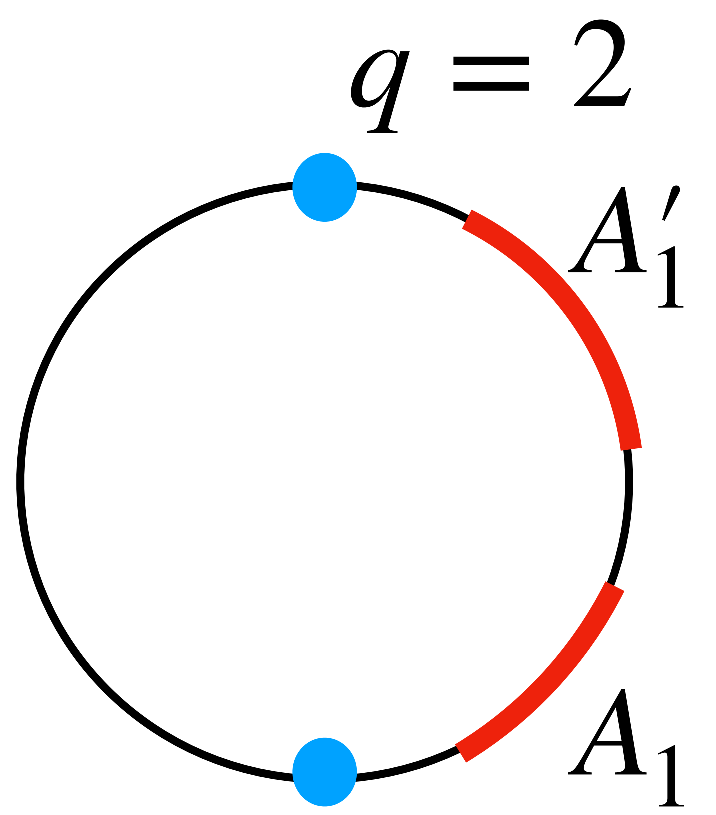
- If we focus on the uniform case with $L/\beta \gg 1, \theta = 0$, the mutual information is just given by

$$I(A_1, A_1')\big|_{L/\beta \gg 1, \mathrm{w}/\theta=0} \approx \max\left\{0, -\frac{c}{3}\log\left[\exp\left[\frac{2\pi}{\beta}(X_4 - X_1)\right] - 1\right]\right\},$$

implying the correlation length (defined by the distance that the mutual information becomes 0) related to the inverse temperature

$$d_{41,\theta=0} = \frac{\beta}{2\pi}\log 2$$

- To find a similar (local) temperature for the inhomogeneous case, we focus on the parameter regions



$$\boxed{\tau_{\text{Mod.}} = L_{\text{eff}}/\beta, L/\beta \gg 1, \theta \gg 1}.$$

- Then, the mutual information can be evaluated as

$$I(A_1, A_1')|_{L/\beta \gg 1 \text{ w/ } \theta \gg 1}$$

$$\approx \max\left\{0, -\frac{c}{3}\log\left[\exp\left[\frac{2L}{q\beta}\cdot\frac{\sin\left(\frac{q\pi}{L}\left(\hat{X}_4 - \hat{X}_1\right)\right)}{\cos\left(\frac{q\pi}{L}\left(\hat{X}_4 - \hat{X}_1\right)\right) - \cos\left(\frac{q\pi}{L}\left(\hat{X}_1 + \hat{X}_4\right)\right)}\right] - 1\right]\right\}.$$

This implies the correlation length related to the "effective local temperature" depending on position

$$\boxed{d_{41,\theta\gg 1} \approx \frac{\beta}{2\pi}\cdot f\left(\frac{\left(\hat{X}_1 + \hat{X}_4\right)}{2}, \theta \to \infty\right)\log 2}$$

$$f\left(\frac{\left(\hat{X}_1 + \hat{X}_4\right)}{2}, \theta \to \infty\right) = 1 - \cos\left(\frac{q\pi}{L}\left(\hat{X}_1 + \hat{X}_4\right)\right)$$

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3. Gravity dual to the CFT

Gravity dual to the deformed CFT

- In the $(\chi, \bar{\chi})$ or $(\xi, \bar{\xi})$ coordinates, the thermal state for the inhomogeneous Hamiltonian is given by the usual form.

- Then, the gravity dual to the thermal state for high effective high temperature $L\cosh(2\theta)/\beta > 1$ is given by the standard BTZ black hole [Banados-Teitelboim-Zanelli '92]

$$\boxed{ds^2_{\text{bulk}} = \frac{dr^2}{r^2 - r_0^2} - r^2\left(\frac{d\xi_\tau - d\bar{\xi}_\tau}{2}\right)^2 + \left(r^2 - r_0^2\right)\left(\frac{d\xi_\tau + d\bar{\xi}_\tau}{2}\right)^2}$$

$$(\xi_\tau, \bar{\xi}_\tau) = (\xi(w) + \tau, \bar{\xi}(\bar{w}) + \tau) \qquad r_0 = \frac{2\pi}{\beta}$$

- In the original coordinates,

$$\boxed{ds^2_{\text{bulk}} = \frac{dr^2}{r^2 - r_0^2} + \left(r^2 - r_0^2\right)d\tau^2 + \frac{r^2}{f(X,\theta)^2}dX^2}$$

$$f(x,\theta) = 1 - \tanh 2\theta\left(1 - 2\sin^2\left(\frac{q\pi x}{L}\right)\right) = 1 - \tanh 2\theta \cdot \cos\left(\frac{2q\pi x}{L}\right)$$

- In the large-r region,

$$ds^2_{\rm bulk}\Big|_{r\gg 1} \approx \frac{dr^2}{r^2} + \frac{r^2}{f(X,\theta)^2}\{f(X,\theta)^2 d\tau^2 + dX^2\}.$$

including the (Euclidean) CFT background metric.

- Or, by introducing the new radial coordinate

$$r'(X,\theta) = \frac{r}{f(X,\theta)}$$

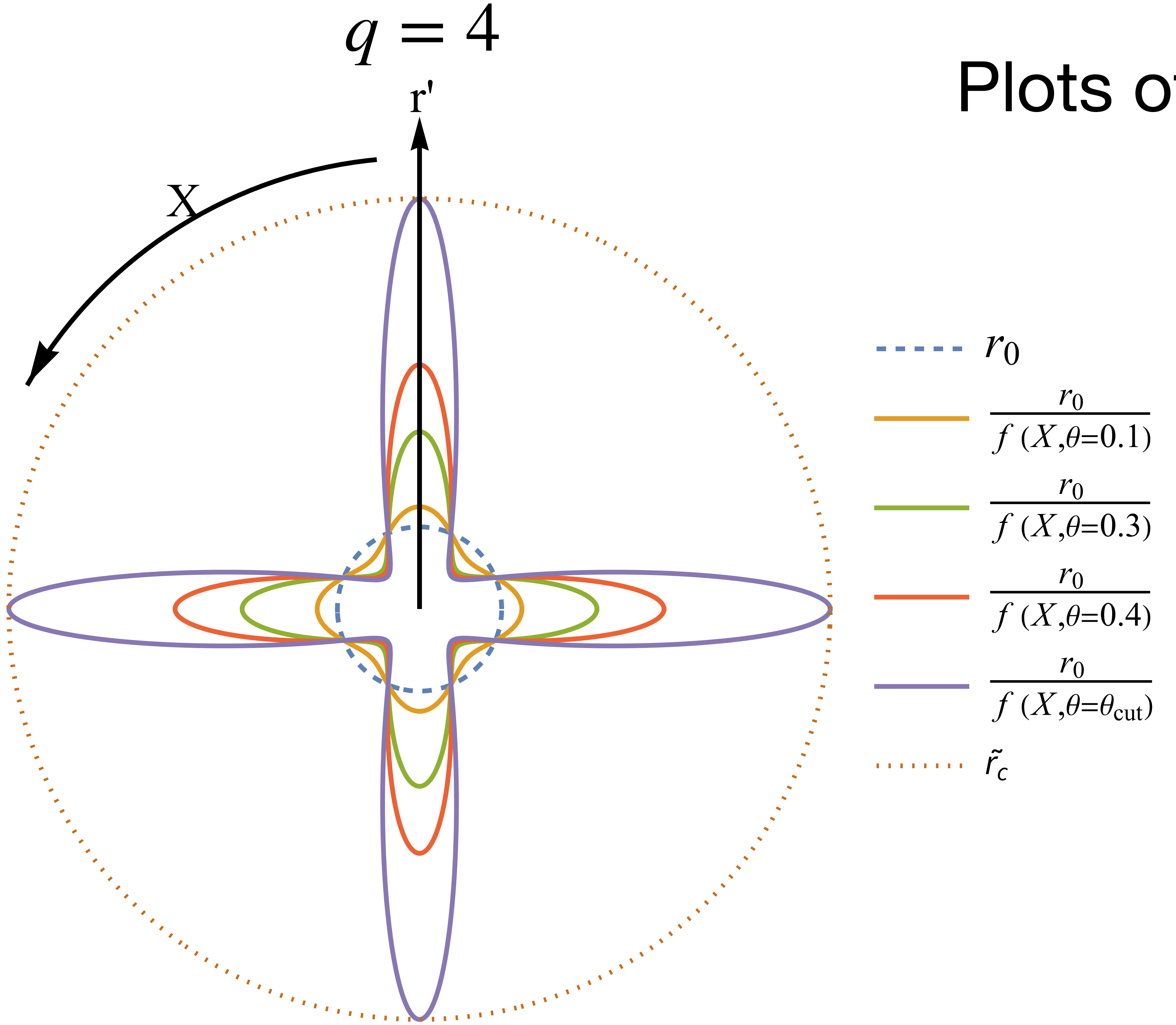
then, we obtain the standard asymptotic metric

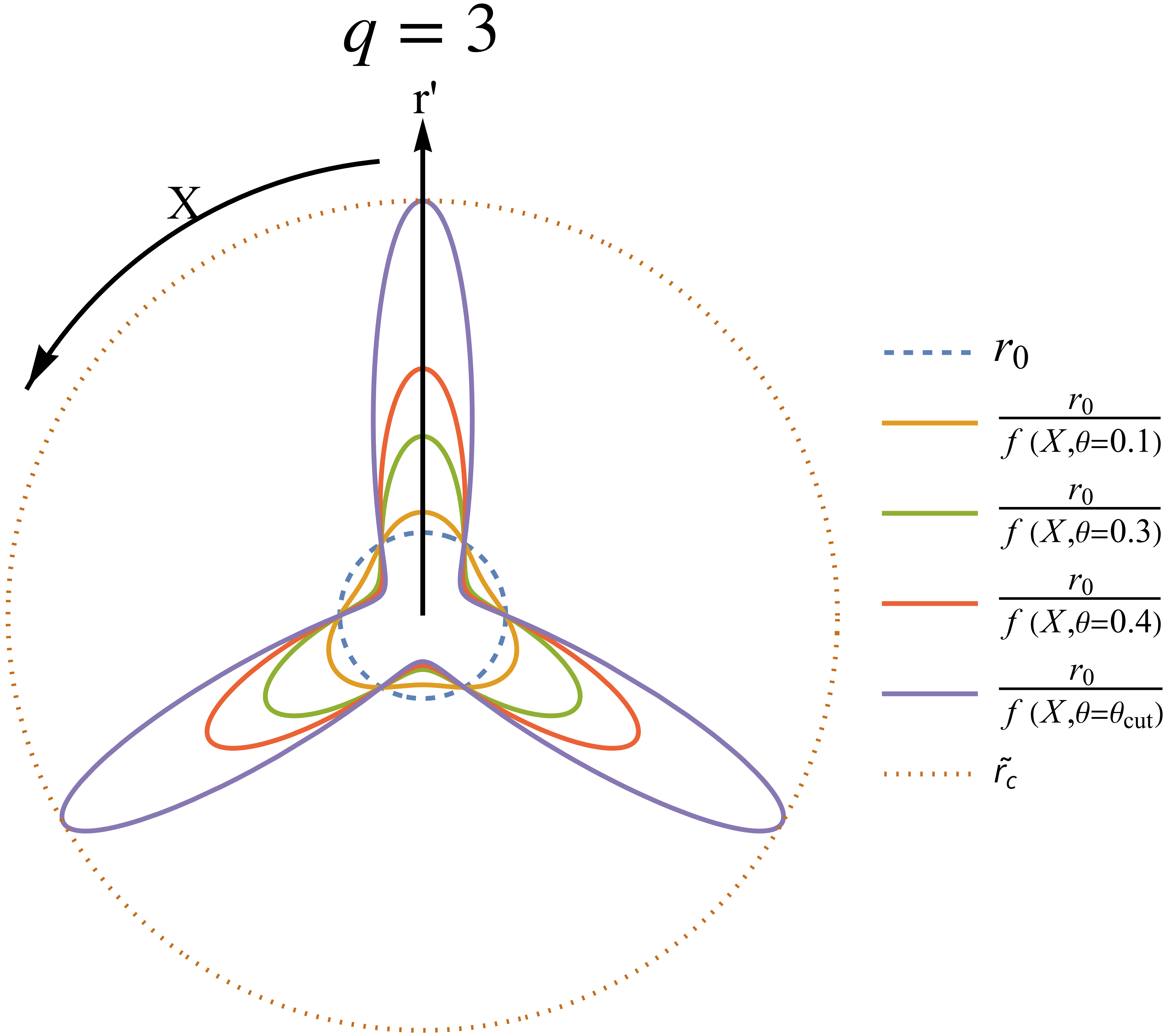
$$ds^2_{\rm bulk}\Big|_{r'\gg 1} \approx \frac{dr'^2}{r'^2} + r'^2\{f(X,\theta)^2 d\tau^2 + dX^2\}$$

- In terms of the new radial coordinates, the horizon radius becomes position-dependent

$$r_0'(X,\theta) = \frac{1}{f(X,\theta)} r_0 \qquad f(x,\theta) = 1 - \tanh 2\theta \cdot \cos\left(\frac{2q\pi x}{L}\right)$$

Plots of r_0'



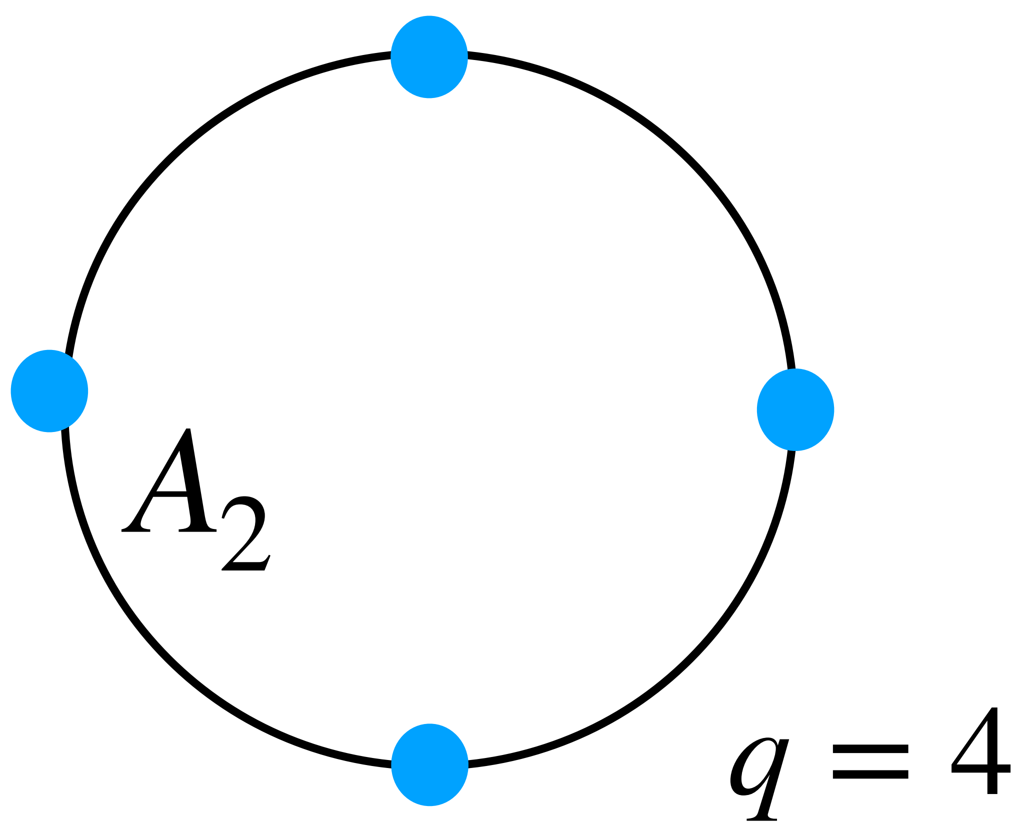


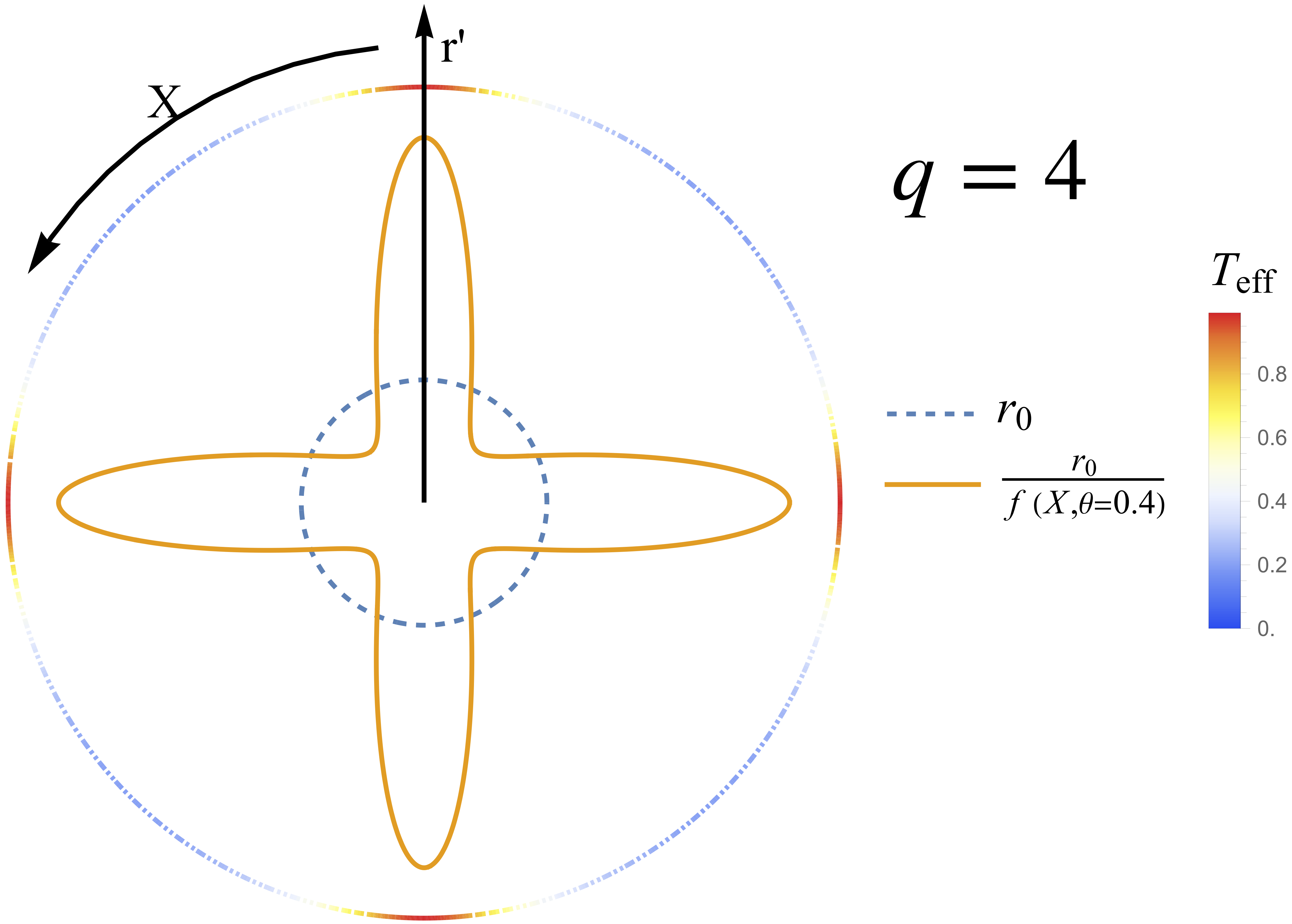
Effective local temperature

- In the new radial coordinates, we can estimate an effective local temperature.

- Let us assume an observer sitting at X; the typical scale of the observer is assumed to be comparable with the re-scaled horizon radius. Such an observer feels the local effective temperature

$$\boxed{\tilde{\beta}_{\mathrm{eff},x=X;\theta} = \frac{2\pi}{r_0'(X,\theta)} = \frac{2\pi}{r_0} f(X,\theta) = \beta f(X,\theta)}$$





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Summary and Future directions

Summary and Future directions

- We studied the thermal state with the CFT Hamiltonian on the curved space
  - The CFT Hamiltonian on the curved space is conformally equivalent to that on the flat space with size $L\cosh(2\theta)$
  - By evaluating the thermal entropy, we found the first-order phase transition by varying the background metric using θ with fixing $L/\beta < 1$.
  - For the entanglement entropy, depending on whether subsystems contain points $\sin^2(q\pi x/L) = 0$ or not, their behavior significantly change.
- Future directions
  - Bulk reconstruction and relative entropy?
  - Entanglement transition induced by the curvature
  - Quantum correlation of 2d Holographic CFTs on the curved spacetime
  - …

Thank you!