Spread and Spectral Complexity in Quantum Spin Chains: from Integrability to Chaos

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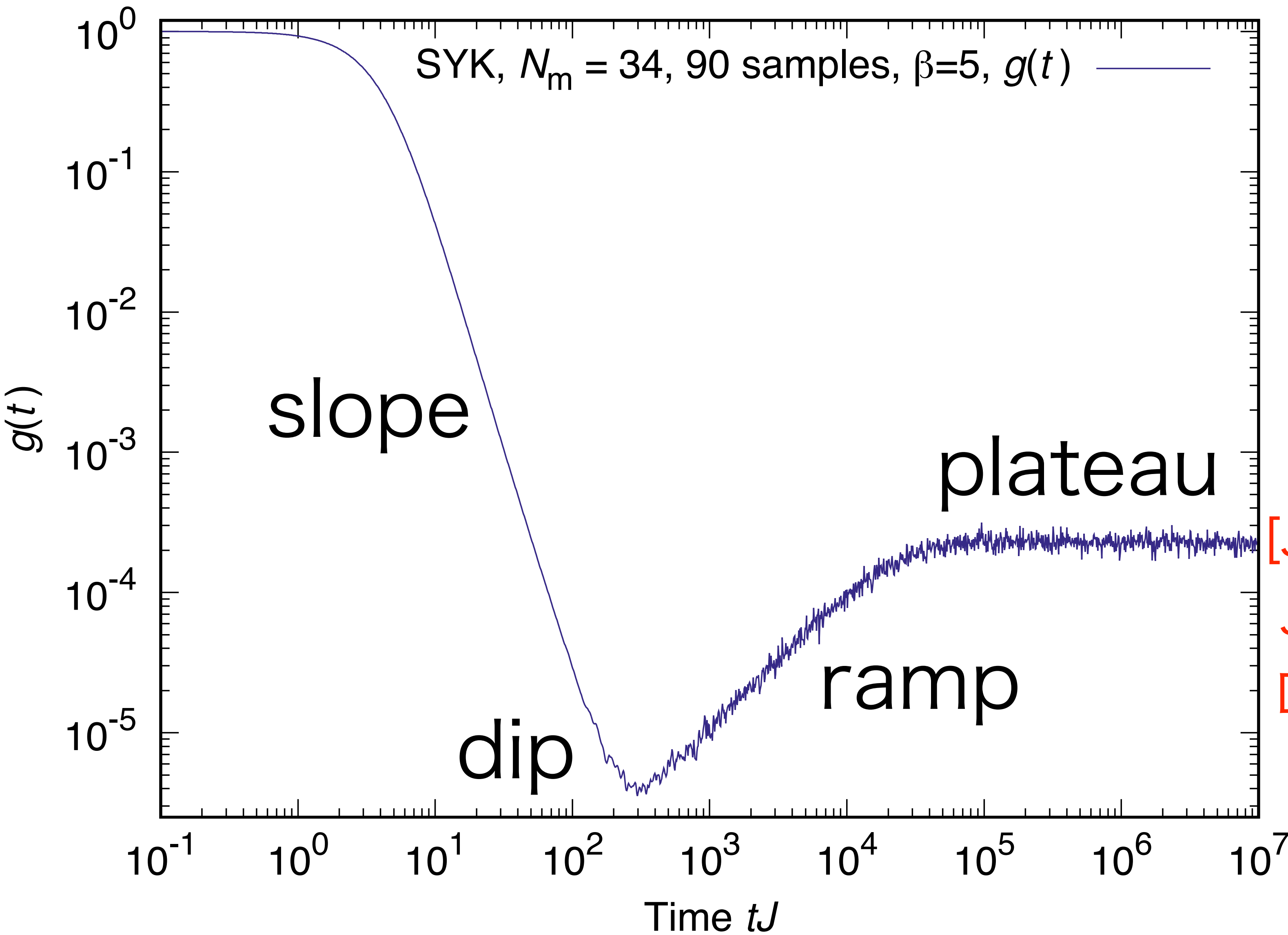
[arXiv:2405.11254]

with Hugo A. Camargo, Kyoung-Bum Huh,
Viktor Jahnke, Hyun-Sik Jeong, Keun-Young Kim

Spectral form factor (SFF) is a measure of quantum chaos.

SFF: $$\frac{|Z(\beta+it)|^2}{|Z(\beta)|^2} = \frac{1}{|Z(\beta)|^2}\sum_{i,j} e^{-\beta(E_i+E_j)} e^{it(E_i-E_j)}$$



Ramp behavior is a sign of chaos.

"Black Holes and Random Matrices"

[J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, M. Tezuka, 2016]

Spread and Spectral Complexity

Spread Complexity: Krylov state complexity for spreading in Krylov subspace

[V. Balasubramanian, P. Caputa, J. M. Magan, Q. Wu, 2022]

$$C(t) := \sum_{n=0} n|\phi_n(t)|^2 \qquad |\Psi(t)\rangle = \sum_{n=0} \phi_n(t)|K_n\rangle$$

Spectral Complexity: Proposed as the dual of wormhole length

[L. V. Iliesiu, M. Mezei, G. Sárosi, 2021]

$$C_S(t) := \frac{1}{d\,Z(2\beta)} \sum_{\substack{n,m \\ E_n \neq E_m}} \left(\frac{\sin\left(t(E_m - E_n)/2\right)}{(E_m - E_n)/2} \right)^2 e^{-\beta(E_m + E_n)}$$

These complexities of TFD state are related to SFF, and the connection with JT gravity has been discussed.

[E. Rabinovici, A. Sánchez-Garrido, R. Shir, J. Sonner, 2023] [L. V. Iliesiu, M. Mezei, G. Sárosi, 2021]

我々の研究動機

Spin chain modelsの
Spread, Spectral Complexity

- 重力双対の有無にかかわらず, SFF, Spread Complexity, Spectral Complexity は定義・計算できる。

- SYKやランダム行列理論のSpread, Spectral Complexity は
  [V. Balasubramanian, P. Caputa, J. M. Magan, Q. Wu, 2022] [J. Erdmenger, S.-K. Jian, Z.-Y. Xian, 2023]
  などで詳しく調べられている。

- 他のspin chain modelsでの
  Spread, Spectral Complexityの振る舞いは？

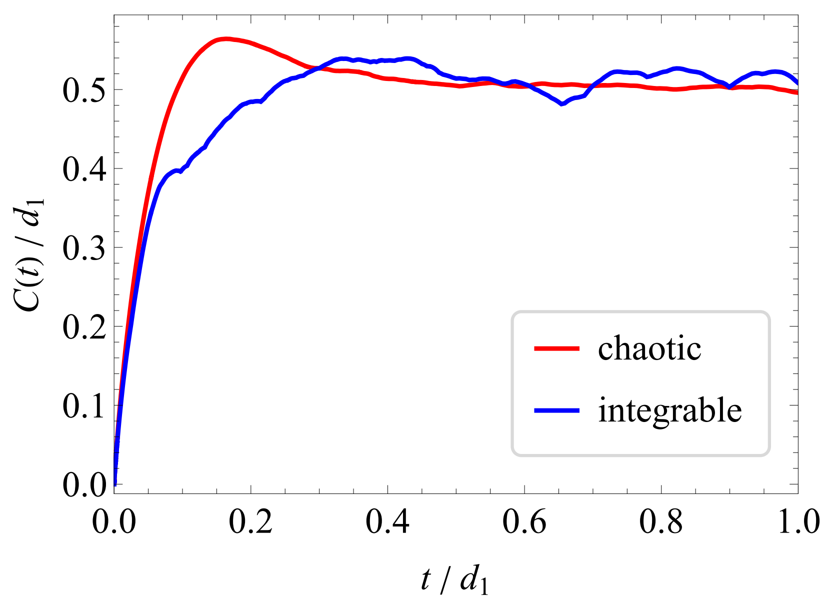
我々の結果

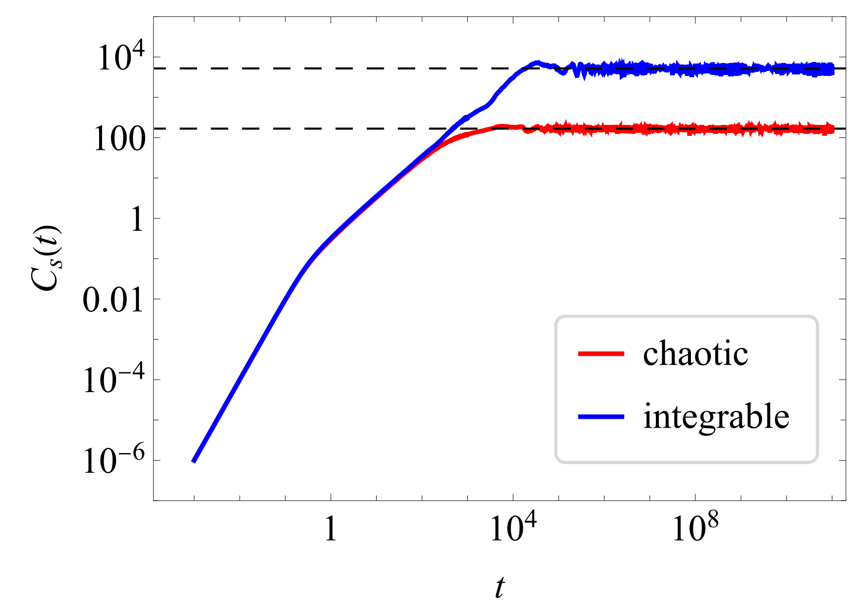
Mixed-field Isingとnext-to-nearest-neighbor (NNN) deformed XXZ を調べた。

Spread Complexity

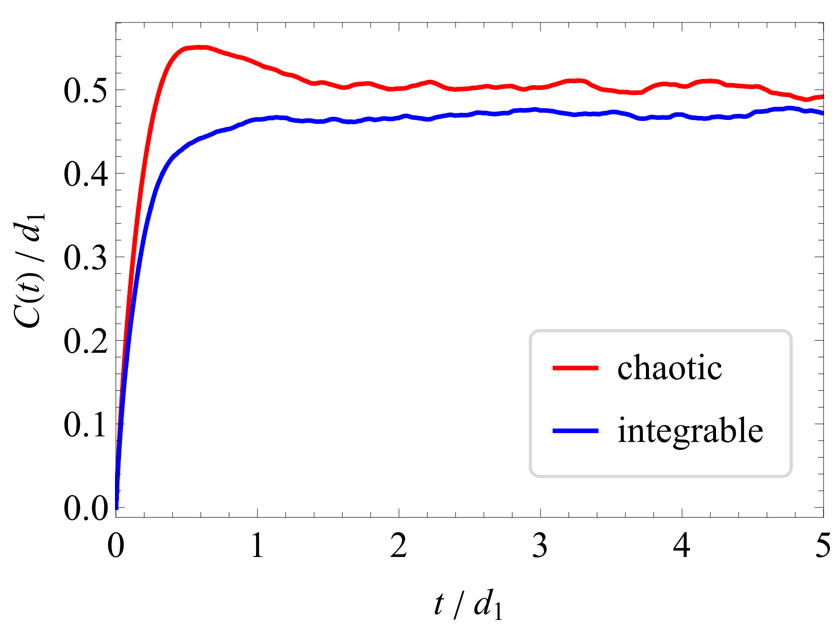
Spectral Complexity

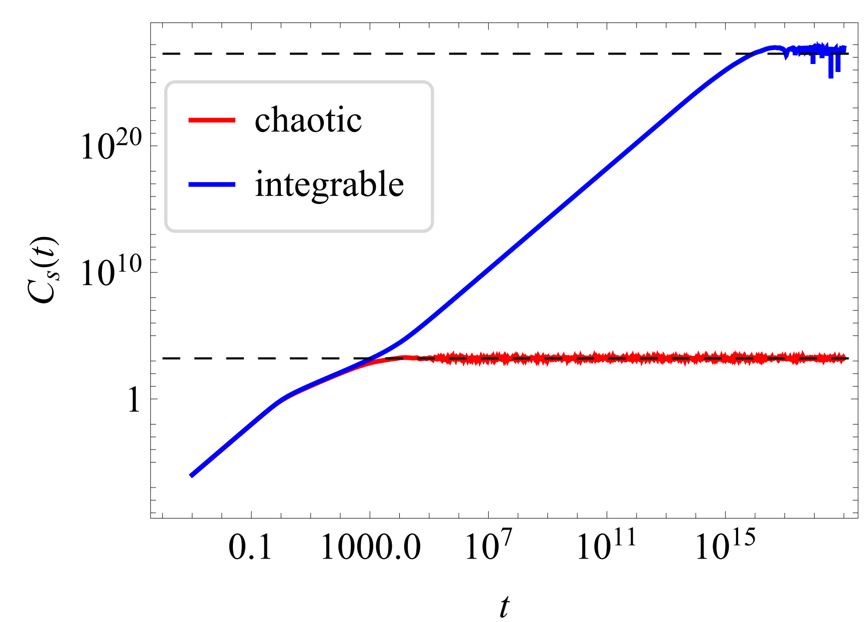
Mixed-field Ising





NNN deformed XXZ





Outline

1. Review of random matrix, Spectral form factor, Spread complexity, Spectral complexity

2. Numerical computations of spin chain models

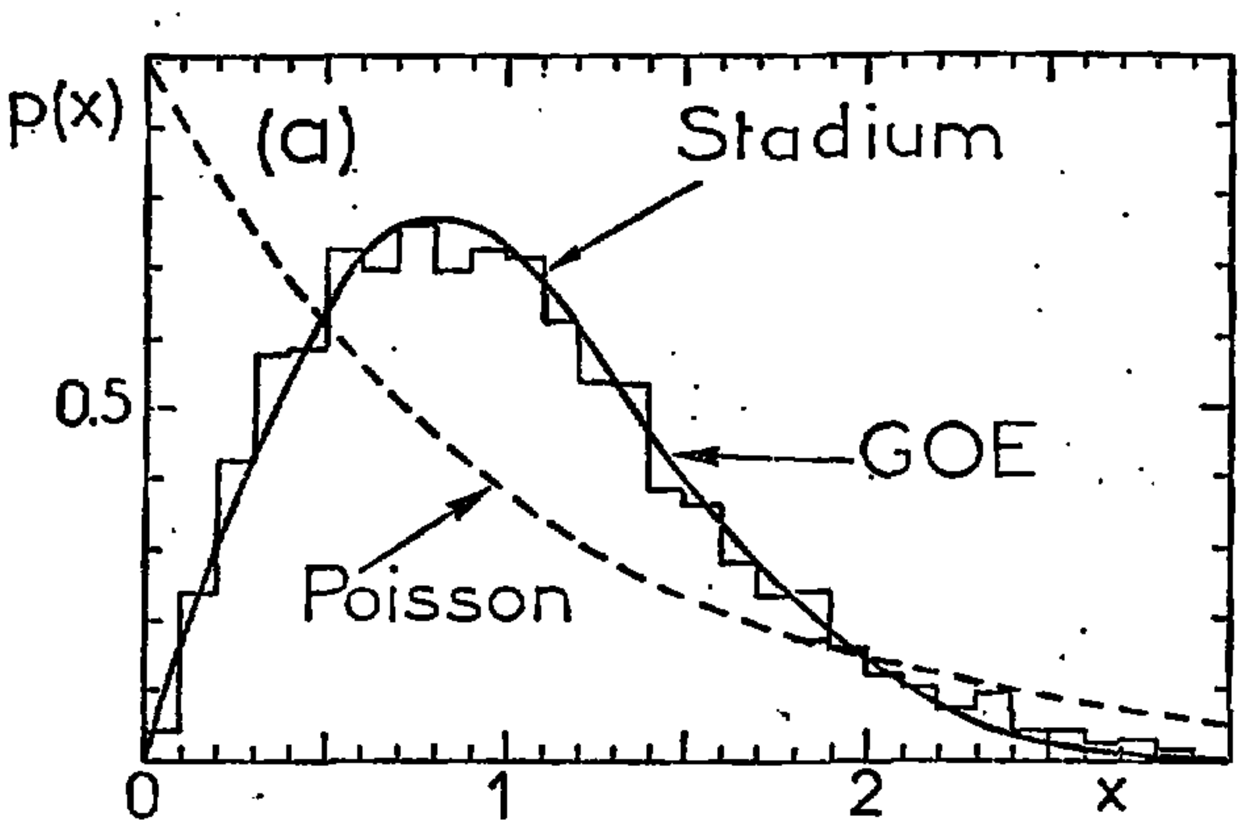
Random matrix and quantum chaos

BGS conjecture [O. Bohigas, M. J. Giannoni, C. Schmit, 1984]

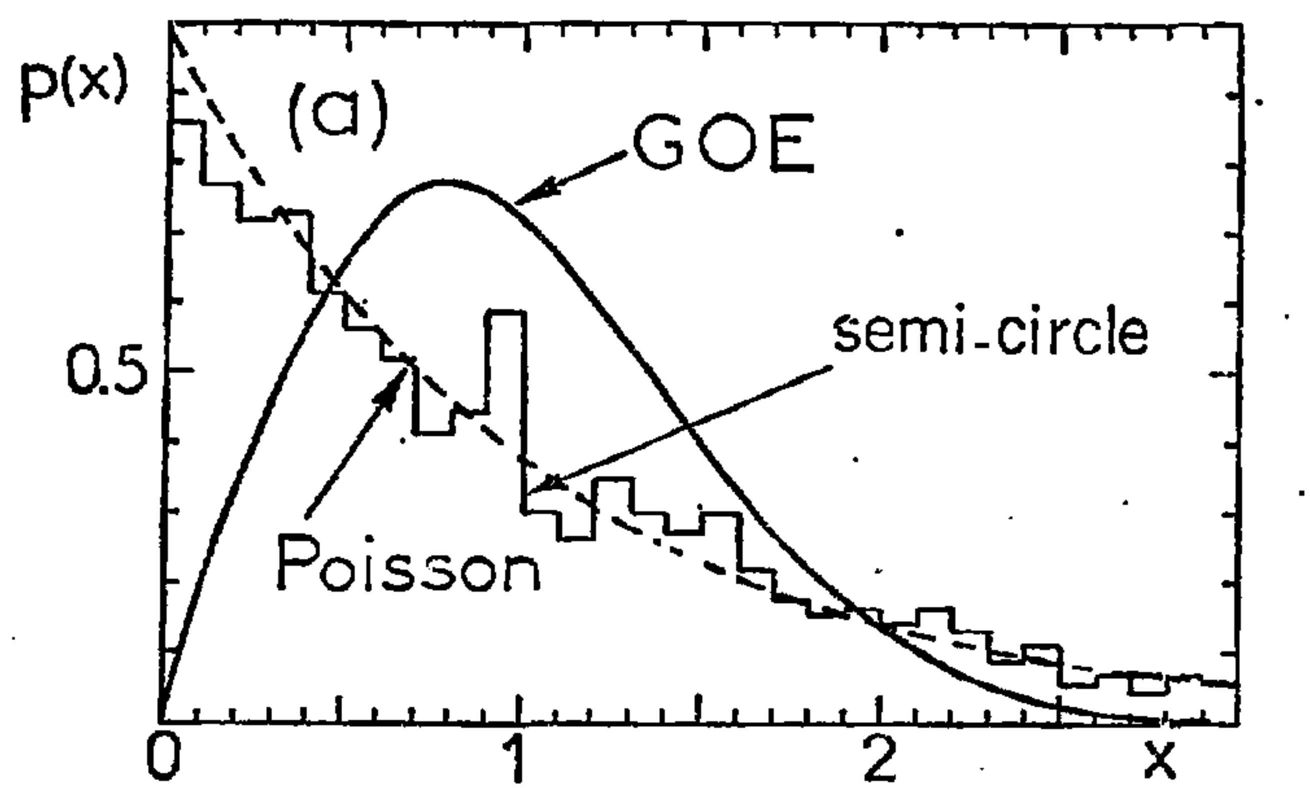
Quantum spectrum of chaotic systems have same properties as eigenvalues of random matrix.

Nearest-neighbor level spacing distribution $P(\Delta E)$

Chaotic



Integrable



Gaussian Unitary Ensemble (GUE)

Matrix element distribution

$$P(H_{ij}) \propto e^{-\frac{1}{2}\mathrm{Tr}H^2}$$

Eigenvalue distribution

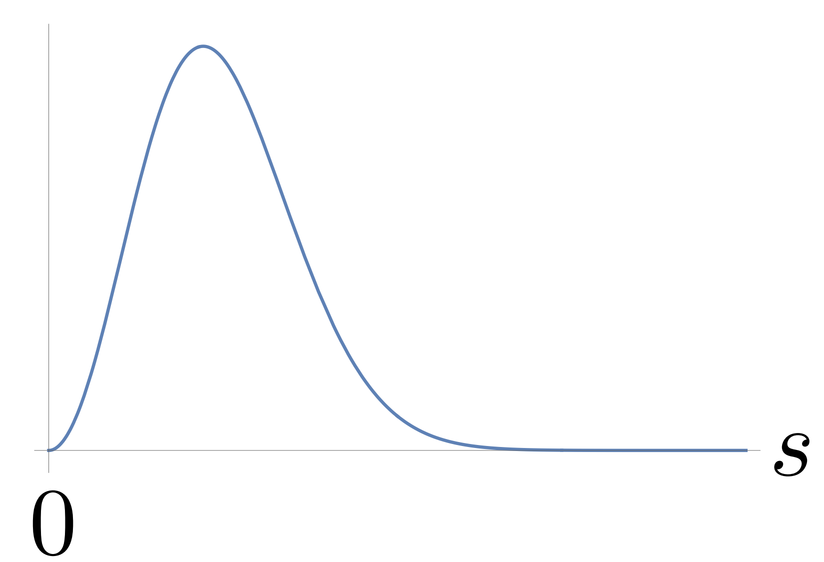
$$P(E_i) \propto e^{-\frac{1}{2}\sum_i E_i^2} \prod_{i<j} |E_i - E_j|^2$$

Correlation

Wigner-Dyson level spacing distribution

$s = \Delta E$

$$P(s) \propto s^2 e^{-\frac{s^2}{4}}$$



SFF with eigenvalue correlation

Two-point function of spectral density (GUE)

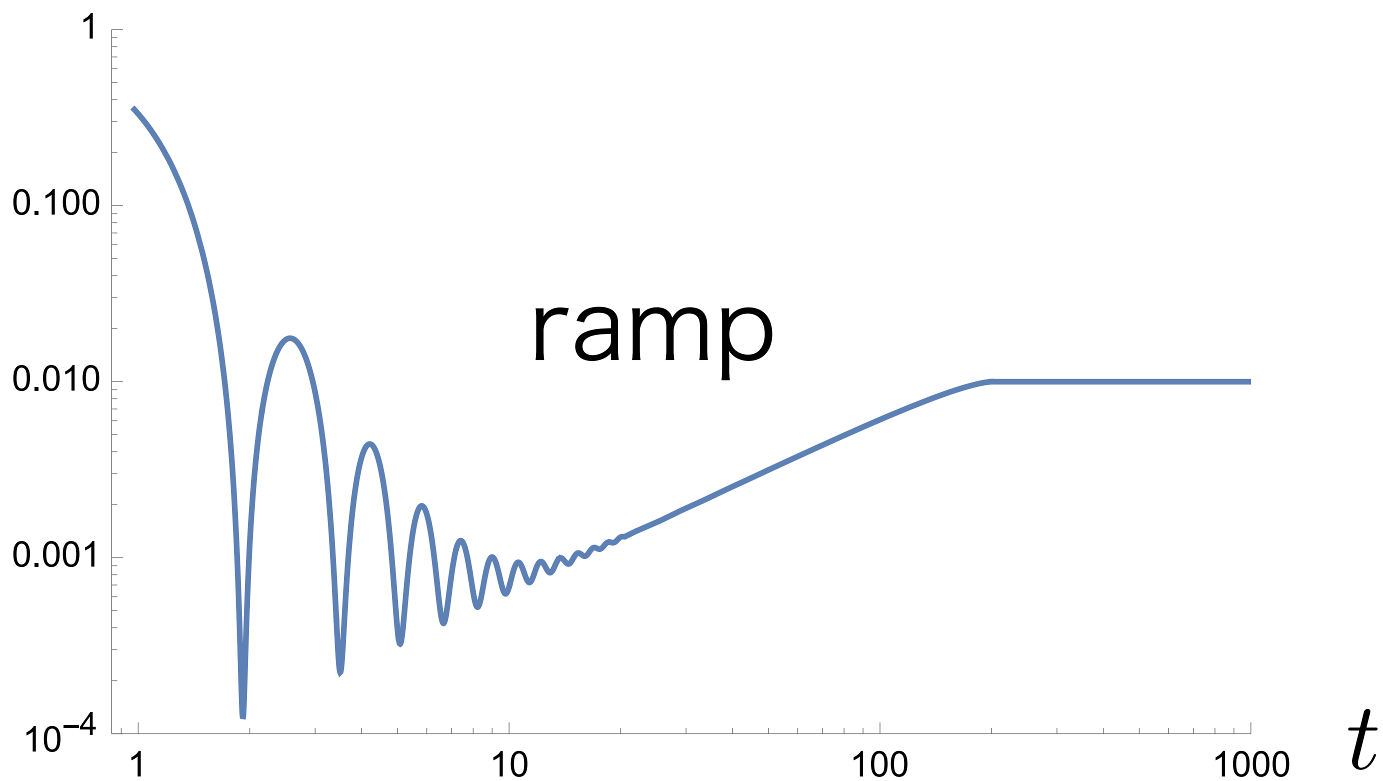
$$\langle \rho(E_1)\rho(E_2)\rangle = \langle \rho\,(E)\rangle \delta(s) + \langle \rho(E_1)\rangle\langle \rho(E_2)\rangle \left[1 - \frac{\sin^2(\pi\langle \rho(E)\rangle s)}{(\pi\langle\rho(E)\rangle s)^2}\right]$$

SFF at infinite temperature $\beta = 0$

$$\frac{\langle |Z(it)|^2\rangle}{d^2} = \mathrm{Re}\left[\frac{J_1(2t)^2}{t^2} + \frac{t}{d^2\pi}\sqrt{1-\frac{t^2}{4d^2}} + \frac{2}{d\pi}\arccos\left(\sqrt{1-\frac{t^2}{4d^2}}\right)\right]$$

SFF with

$d = 100$



SFF without eigenvalue correlation

Two-point function without the sine kernel

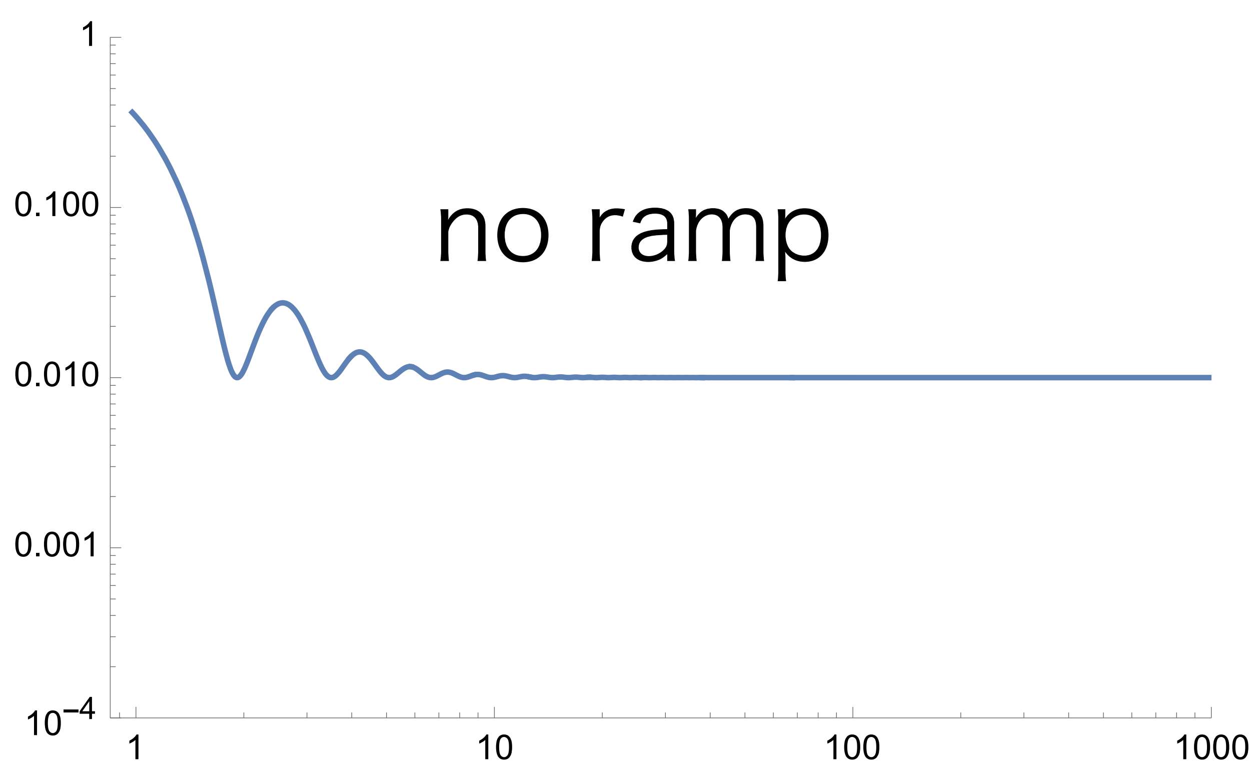
$$\langle \rho(E_1)\rho(E_2)\rangle = \langle \rho\,(E)\rangle\delta(s) + \langle \rho(E_1)\rangle\langle \rho(E_2)\rangle$$

SFF at infinite temperature $\beta = 0$

$$\frac{\langle |Z(it)|^2\rangle}{d^2} = \frac{1}{d} + \frac{|I_1(2it)|^2}{t^2}$$

SFF with

$d = 100$



Spread Complexity

[V. Balasubramanian, P. Caputa, J. M. Magan, Q. Wu, 2022]

Time evolution of an initial state $|\Psi\rangle$

$$|\Psi(t)\rangle = e^{-iHt}|\Psi\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} H^n |\Psi\rangle$$

1. Lanczos algorithm for orthonormalization

$$\{H^n|\Psi\rangle\} \to \{|K_n\rangle\}$$

2. Expand $|\Psi(t)\rangle$ by Krylov basis $|K_n\rangle$

$$|\Psi(t)\rangle = \sum_{n=0} \phi_n(t)|K_n\rangle$$

3. Define Spread Complexity $C(t) := \sum_{n=0} n|\phi_n(t)|^2$

There is an algorithm to compute Spread Complexity of TFD state $|\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_i e^{-\frac{\beta E_i}{2}} |i\rangle \otimes |i\rangle$ from SFF.

Spectral Complexity [L. V. Iliesiu, M. Mezei, G. Sárosi, 2021]

Spectral Complexity was motivated from the wormhole length in JT gravity.

$$C_S(t) := \frac{1}{d\,Z(2\beta)} \sum_{\substack{n,m \\ E_n \neq E_m}} \left(\frac{\sin\left(t(E_m - E_n)/2\right)}{(E_m - E_n)/2} \right)^2 e^{-\beta(E_m + E_n)}$$

Spectral Complexity is directly related to SFF.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} C_S(t) = \frac{2Z(\beta)^2}{d\ Z(2\beta)} \mathrm{SFF}(t) - \frac{2}{d}$$

Long-time average

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, C_S(t) = \frac{2}{d\,Z(2\beta)} \sum_{\substack{n,m \\ E_n \neq E_m}} \frac{e^{-\beta(E_m + E_n)}}{(E_m - E_n)^2}$$

Outline

1. Review of Spectral form factor,
   Spread complexity, Spectral complexity

2. Numerical computations of spin chain models

Spin chain models

NNN deformed XXZ

$$H = H_{XXZ} + H_{NNN}$$

$$H_{XXZ} = \sum_{i=1}^{N-1} J\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right) + J_{zz} S_i^z S_{i+1}^z$$

$$H_{NNN} = \sum_{i=1}^{N-2} J_c S_i^z S_{i+2}^z$$

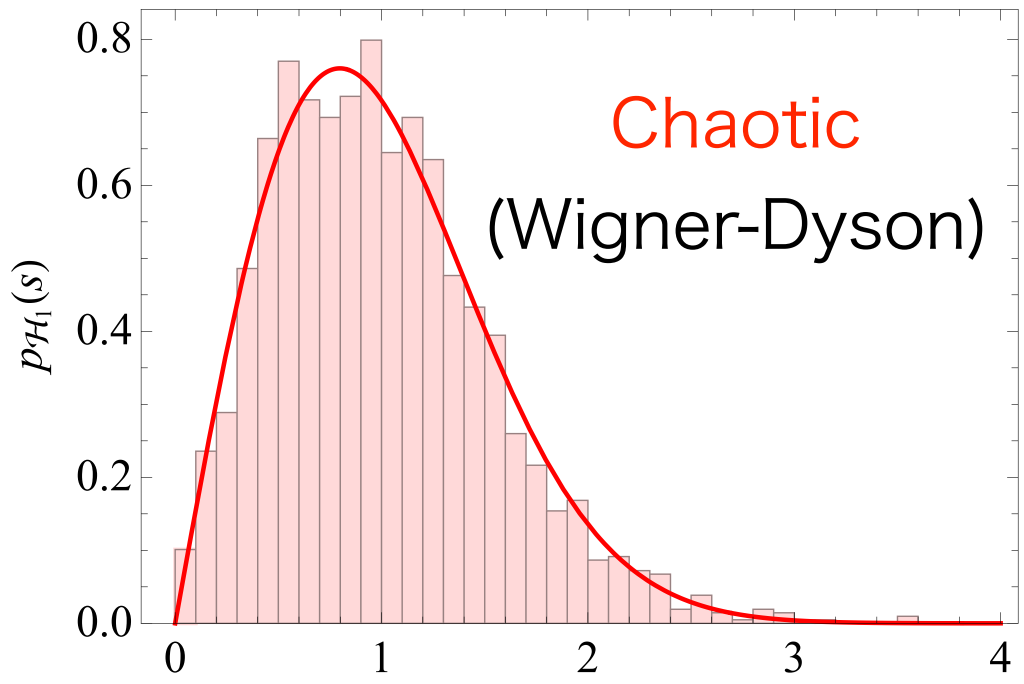
$$(J,\, J_{zz},\, J_c) = \begin{cases} (1,\, 0.5,\, 1)\,, & \text{(Chaotic)} \\ (1,\, 0.5,\, 0)\,, & \text{(Integrable)} \end{cases}$$

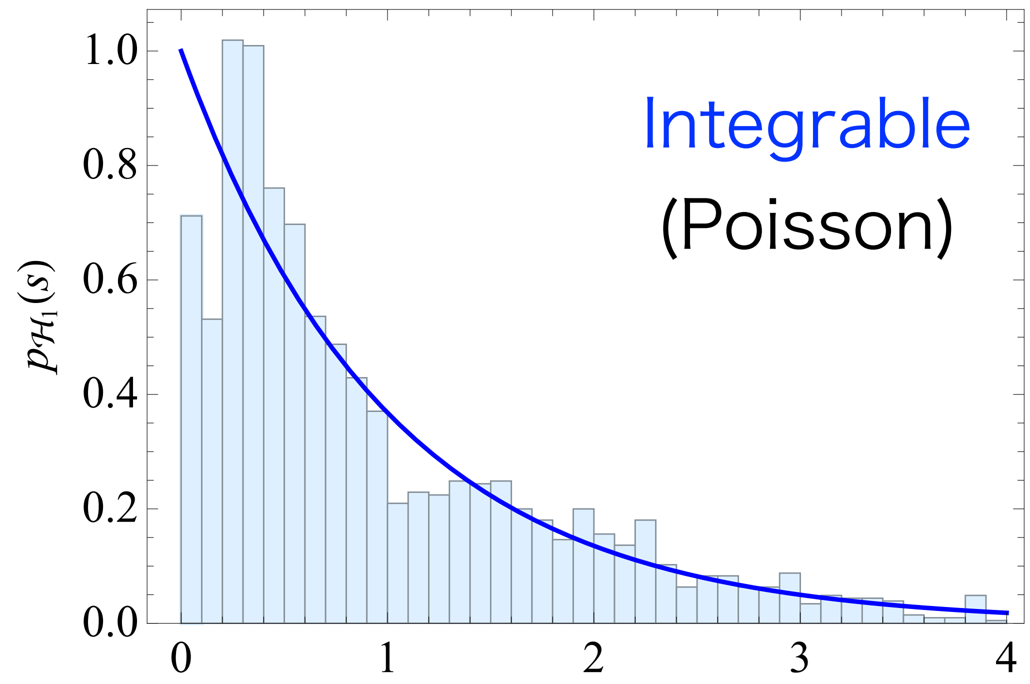
mixed-field Ising

$$H = J \sum_{i=1}^{N-1} S_i^z S_{i+1}^z + \sum_{i=1}^{N} \left(h_x S_i^x + h_z S_i^z\right)$$

$$(J,\, h_x,\, h_z) = \begin{cases} (-1,\, 1.05,\, -0.5)\,, & \text{(Chaotic)} \\ (-1,\, 1,\, -0.001)\,, & \text{(Integrable)} \end{cases}$$

Energy level spacing (mixed-field Ising)





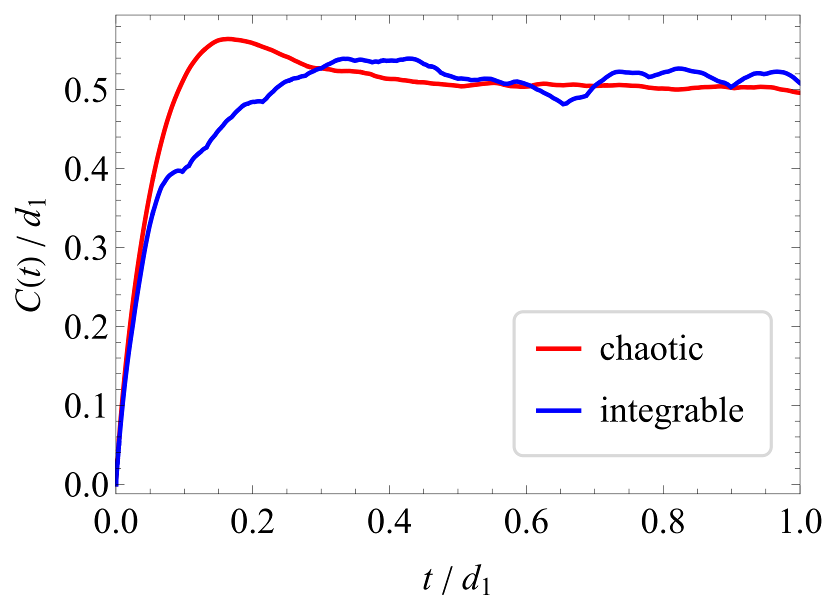
Our numerical results

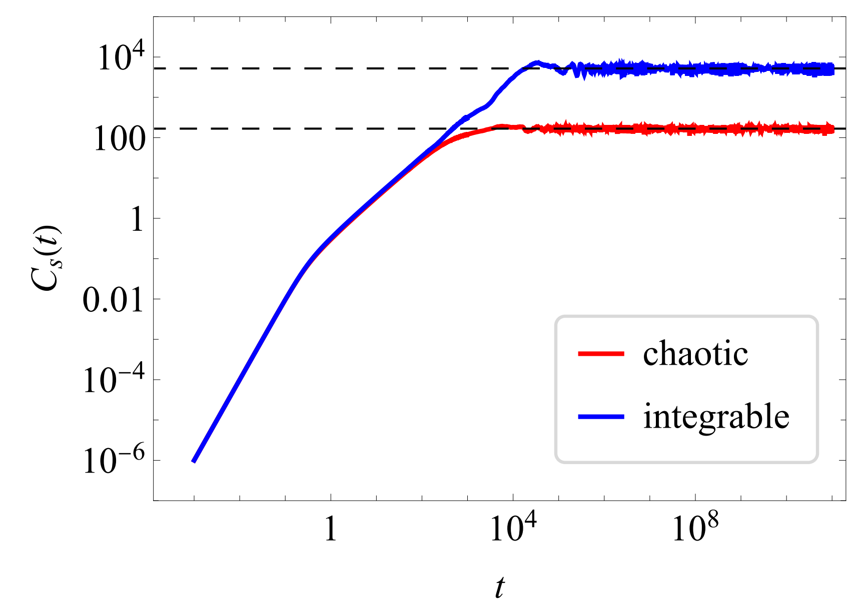
Spread Complexity in chaotic models has a peak.

Spectral Complexity in chaotic models saturates early.

Mixed-field Ising

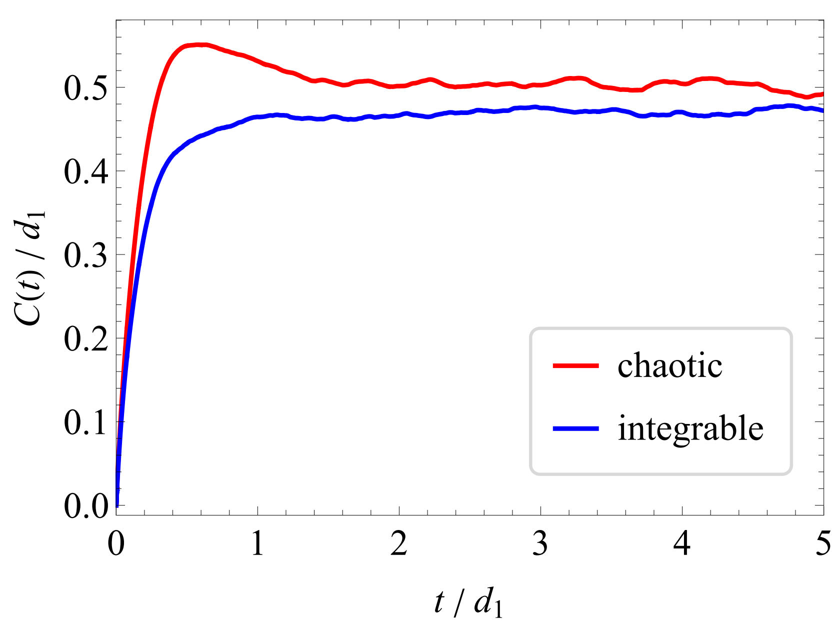
($N = 12, \beta = 0,$ even parity)

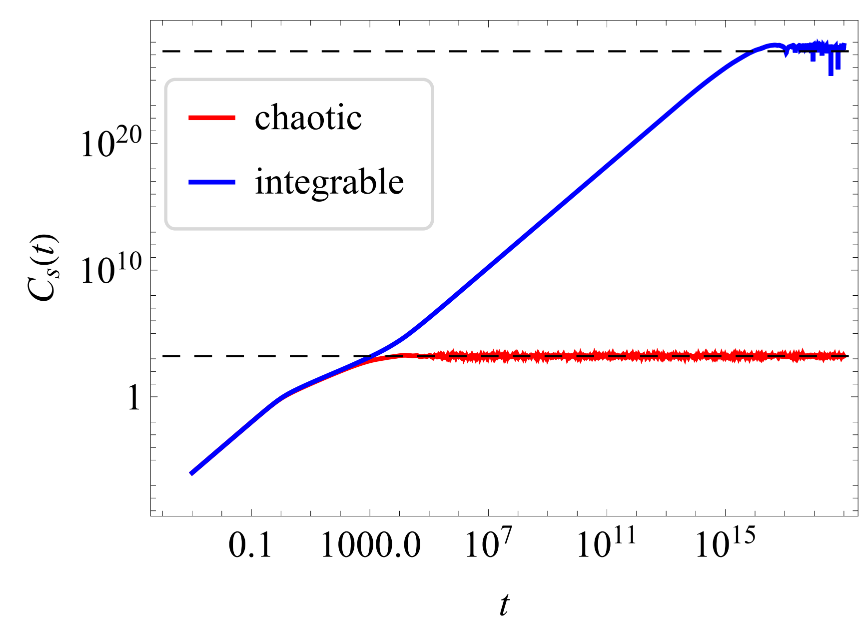




NNN deformed XXZ

($N = 15, \beta = 0,$ even parity, fixed total spin $M_z = -5/2$)





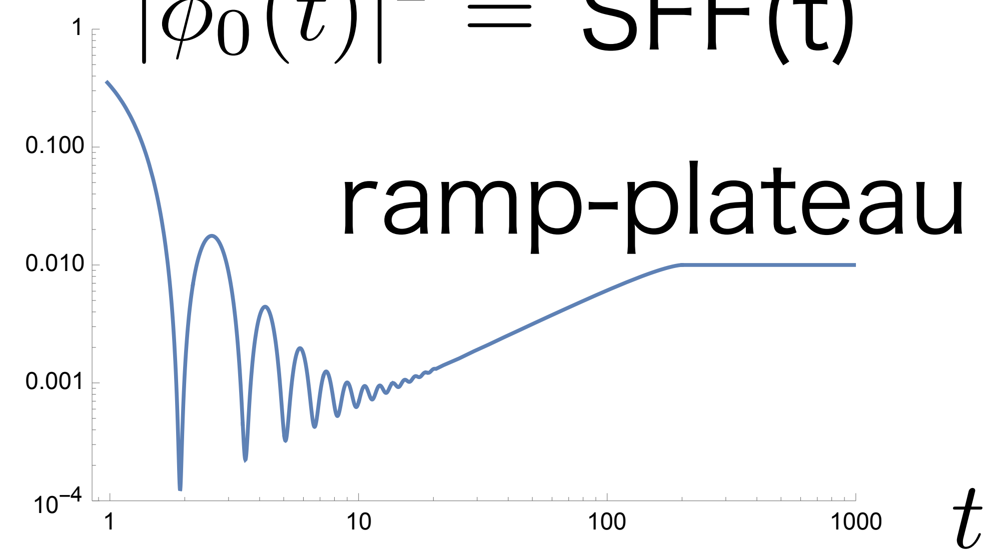
Why Spread Complexity in chaotic models has a peak?

This was observed in the original paper.

[V. Balasubramanian, P. Caputa, J. M. Magan, Q. Wu, 2022]

$$C(t) := \sum_{n=0} n|\phi_n(t)|^2 \quad |\Psi(t)\rangle = \sum_{n=0} \phi_n(t)|K_n\rangle \quad \sum_{n=0} |\phi_n(t)|^2 = 1$$

$|\phi_0(t)|^2 = \text{SFF(t)}$



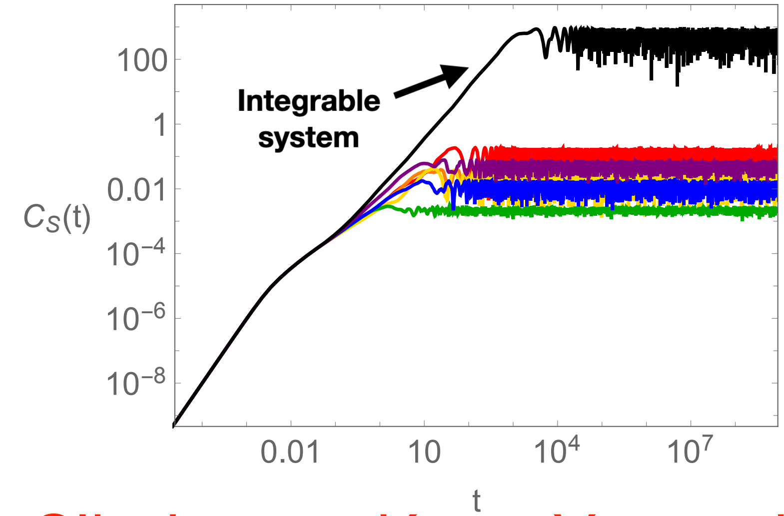
$|\phi_0(t)|^2$ is smaller than the plateau value in the middle time.

A peak does not appear if the ramp is not linear (SYK_2 , our integrable case).

[J. Erdmenger, S.-K. Jian, Z.-Y. Xian, 2023]

Why Spectral Complexity in chaotic models saturates early?

This was observed in our previous paper.

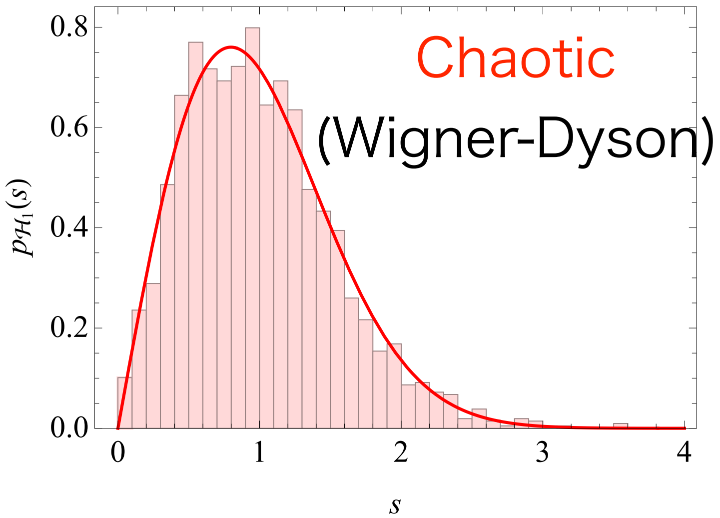


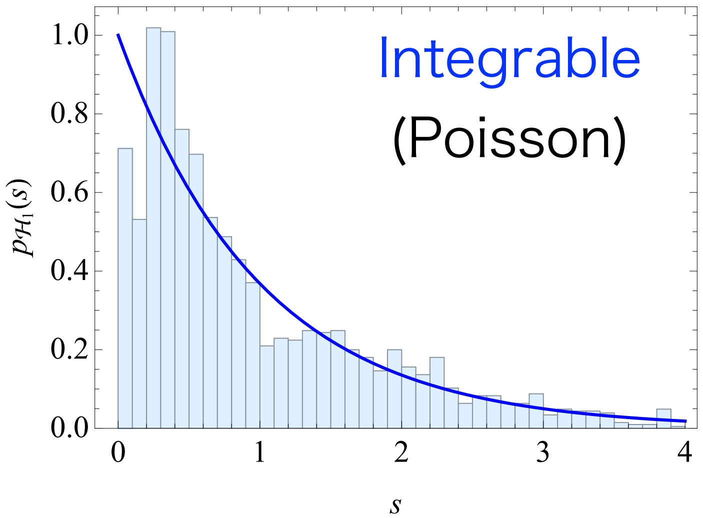
[Hugo A. Camargo, Viktor Jahnke, Hyun-Sik Jeong, Keun-Young Kim, MN, 2023]

Long-time average of Spectral complexity at $\beta = 0$

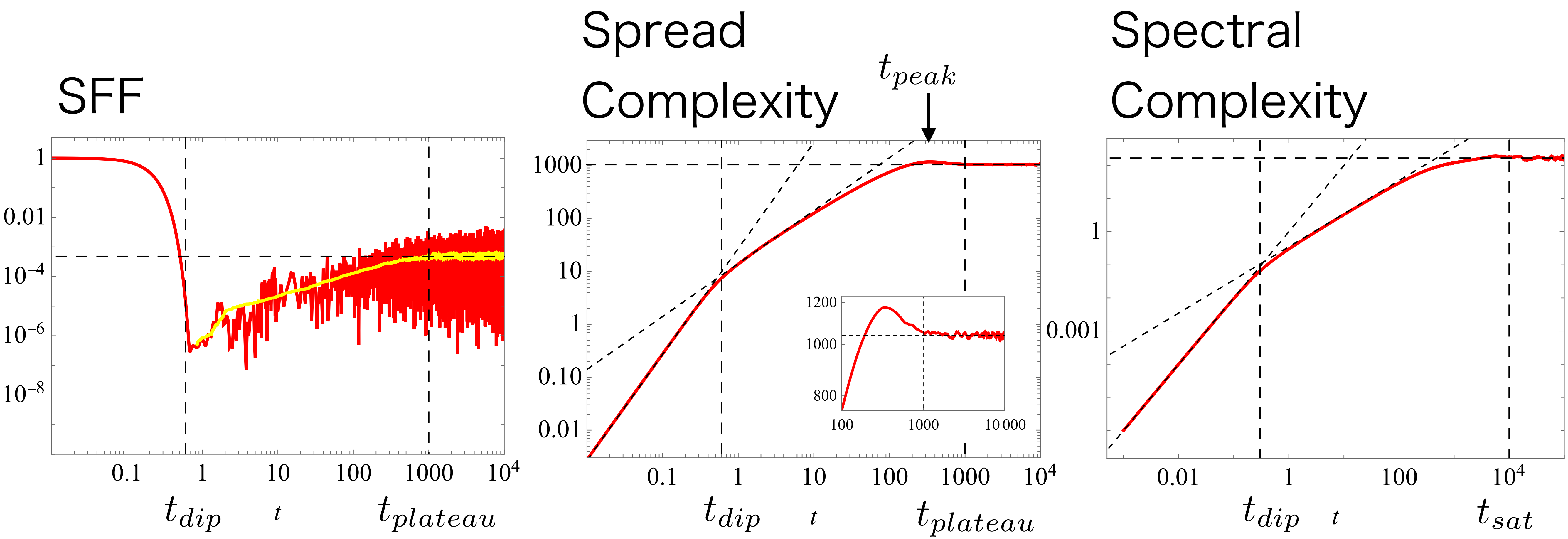
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\, C_S(t) = \frac{2}{d^2}\sum_{E_n\neq E_m}\frac{1}{(E_m-E_n)^2}$$

In chaotic models, ΔE is not small due to eigenvalue correlation.





Timescales in chaotic mixed-field Ising



From $t = t_{dip}$, Spread and Spread Complexities change their behaviors from t^2 to t .

We observe $t_{dip} < t_{peak} \lesssim t_{plateau} < t_{sat}$

$t_{dip} \sim \mathcal{O}(1)$, which is different from RMT case $t_{dip} \sim d^{1/2}$

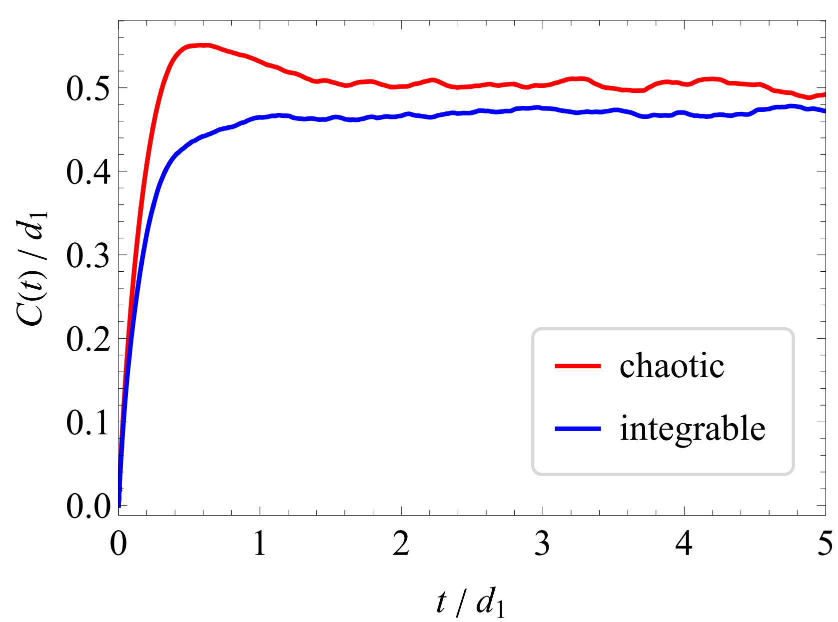
まとめ

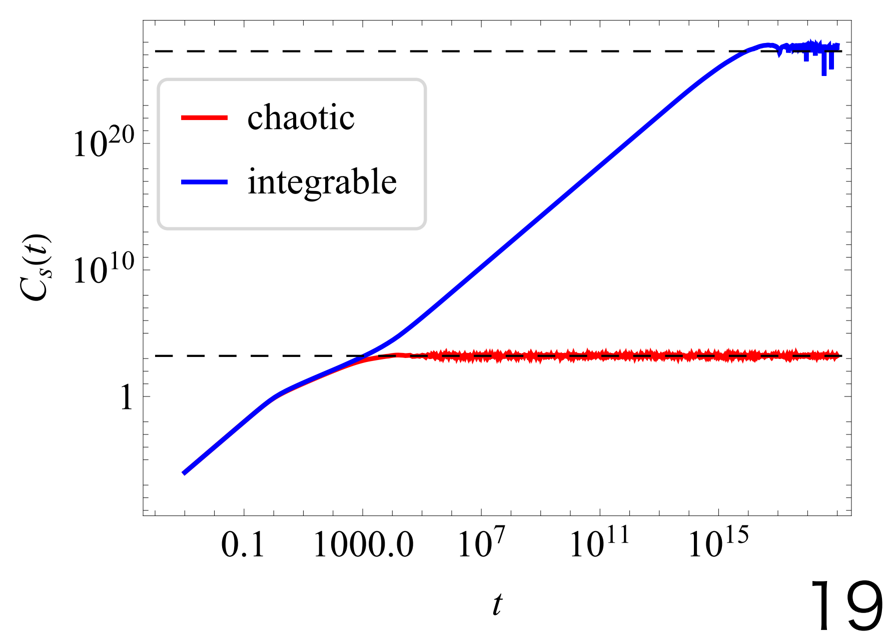
- Mixed-field Isingと NNN deformed XXZ の Spread, Spectral Complexityを調べた。

- Chaoticな場合のSpread Complexityは peakを持つことを確認した。

- Chaoticな場合のSpectral Complexityは 早く飽和することを確認した。

NNN deformed XXZ





今後の課題

- SFFの $t_{dip} \sim \mathcal{O}(1)$ がRMTの $t_{dip} \sim d^{1/2}$ と違う理由は？
  $\rho(E)$ がWigner 半円ではなくGaussianだから？

- 我々が調べたのはTFDのSpread, Spectral Complexity.
  他の状態の Complexity はどうなる？

- RMTではSpread ComplexityとSpectral Complexityが
  early time で一致する。[J. Erdmenger, S.-K. Jian, Z.-Y. Xian, 2023]
  我々のspin modelsでは一致しない。一般には？