Deformation Quantization of $G_{2,4}(\mathbb{C})$ and Related Topics

Taika Okuda (Tokyo University of Science); E-mail: warriors.aboot@gmail.com / 1722702@ed.tus.ac.jp

Joint work with Akifumi Sako (Tokyo University of Science)

1. Deformation Quantization

For a Poisson manifold M, deformation quantization $(C^\infty(M)[[\hbar]], *)$ is defined by a pair of the ring of formal power series $C^\infty(M)[[\hbar]]$ and a star product $*$ denoted by $f * g = \sum_k C_k(f,g)\hbar^k$ satisfying the following conditions:

1. For any $f, g, h \in C^\infty(M)[[\hbar]]$, $f * (g * h) = (f * g) * h$.
2. For any $f \in C^\infty(M)[[\hbar]]$, $f * 1 = 1 * f = f$.
3. For any $f, g \in C^\infty(M)$, $C_k(f,g) = \sum_{I,J} a_{I,J}\partial^I f \partial^J g$, where I, J are multi-indices.
4. $C_0(f,g) = fg, \quad C_1(f,g) - C_1(g,f) = \{f, g\}$.

In particular, for a Kähler manifold, the star product $*$ is separation of variables if $*$ satisfies $a * f = af$ and $f * b = fb$ for an arbitrary (anti-)holomorphic function a (and b).

2. Previous Results for Kähler Manifolds

Deformation quantization (not necessarily separation of variables)

- Moreno ('86)
- Omori-Maeda-Miyazaki-Yoshioka ('98)
- Reshetikhin-Takhtajan ('00)

Deformation quantization with separation of variables

- Karabegov ('96), Gammelgaard ('14)
  $\longrightarrow$ For arbitrary cases.
- Sako-Suzuki-Umetsu ('12), Hara-Sako ('17)
  $\longrightarrow$ For locally symmetric cases (i.e. $\nabla R^{\nabla} = 0$).

3. Construction Method (Hara-Sako ('17))

- For arbitrary N-dimensional locally symmetric Kähler manifold M and $f, g \in C^\infty(M)$, there exists a star product with separation of variables (locally) such that

$$f * g = \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n, \vec{\beta}^*_n} T^n_{\vec{\alpha}_n, \vec{\beta}^*_n} \left(D^{\vec{\alpha}_n} f\right) \left(D^{\vec{\beta}^*_n} g\right), \tag{1}$$

  where $\vec{\alpha}_n, \vec{\beta}_n$ are multi-indices of order n, $D^i := g^{i\bar{j}}\partial_j$, $D^{\bar{i}} := \overline{D^i}$, and $T^n_{\vec{\alpha}_n, \vec{\beta}^*_n}$ is the coefficient of $f * g$ that satisfies some recurrence relations.
- $T^n_{\vec{\alpha}_n, \vec{\beta}^*_n}$ for $n = 0, 1$ are given by $T^0_{\vec{0}, \vec{0}^*} = 1$ and $T^1_{\vec{e}_i, \vec{e}^*_j} = \hbar g_{i\bar{j}}$.
- $T^n_{\vec{\alpha}_n, \vec{\beta}^*_n} := 0$ when $\vec{\alpha}_n$ or $\vec{\beta}^*_n$ has at least one negative component.
- This result states "the existence of star products with separation of variables on arbitrary locally symmetric Kähler manifolds".

4. Recurrence Relations Giving Star Product $f * g$

To give the explicit formulae for $f * g$, we are necessary to determine $T^n_{\vec{\alpha}_n, \vec{\beta}^*_n}$ from the following recurrence relations:

$$\begin{aligned}
&\hbar \sum_{d=1}^{N} g_{\bar{i}d} T^{n-1}_{\vec{\alpha}_n - \vec{e}_d, \vec{\beta}^*_n - \vec{e}^*_i} \\
&= \beta^n_i T^n_{\vec{\alpha}_n, \vec{\beta}^*_n} + \hbar \sum_{k,\rho=1}^{N} \binom{\beta^n_k - \delta_{k\rho} - \delta_{ik} + 2}{2} R_{\bar{\rho}}{}^{\bar{k}\bar{k}}{}_{\bar{i}} T^n_{\vec{\alpha}_n, \vec{\beta}^*_n - \vec{e}^*_\rho + 2\vec{e}^*_k - \vec{e}^*_i} \\
&\quad + \hbar \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{\rho=1}^{N} \left(\beta^n_k - \delta_{k\rho} - \delta_{ik} + 1\right)\left(\beta^n_{k+l} - \delta_{k+l,\rho} - \delta_{i,k+l} + 1\right) \\
&\quad \times R_{\bar{\rho}}{}^{\overline{k+l}\,\bar{k}}{}_{\bar{i}} T^n_{\vec{\alpha}_n, \vec{\beta}^*_n - \vec{e}^*_\rho + \vec{e}^*_k + \vec{e}^*_{k+l} - \vec{e}^*_i}.
\end{aligned} \tag{2}$$

5. The Complex Grassmannians $G_{p,p+q}(\mathbb{C})$

The complex Grassmannian $G_{p,p+q}(\mathbb{C})$ is defined as follows:

$$G_{p,p+q}(\mathbb{C}) := \left\{ V \subset \mathbb{C}^{p+q} \mid V : \text{complex vector subspace } s.t. \dim V = p \right\}.$$

Let U be an open set of $G_{p,p+q}(\mathbb{C})$ such that

$$U := \left\{ Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \in M(p+q, p; \mathbb{C}) \middle| Y_0 \in GL_p(\mathbb{C}),\ Y_1 \in M(q, p; \mathbb{C}) \right\},$$

and $\phi : U \to M(q, p; \mathbb{C})$ be a holomorphic map such that $Y \mapsto \phi(Y) := Y_1 Y_0^{-1}$. Then, we can choose the local coordinates

$$Z := \left(z^I\right) = \left(z^{ii'}\right) = Y_1 Y_0^{-1} (= \phi(Y))$$

by using ϕ, where $I := ii'$ $(i = 1, \cdots, q,\ i' = 1', \cdots, p')$. The Kähler potential Φ of $G_{p,p+q}(\mathbb{C})$ is given by $\Phi = \log\det(\mathrm{Id}_p + Z^\dagger Z)$.

6. The Recurrence Relations for $G_{2,4}(\mathbb{C})$

In this work, we focus on $G_{2,4}(\mathbb{C})$, i.e. $p = q = 2$. For $G_{2,4}(\mathbb{C})$, the recurrence relations (2) are given by

$$\hbar \sum_{D \in \mathcal{I}} g_{\bar{I}D} T^{n-1}_{\vec{\alpha}_n - \vec{e}_D, \vec{\beta}^*_n - \vec{e}^*_I} = \hbar\beta^n_I(\tau_n + \beta^n_{\not{I}}) T^n_{\vec{\alpha}_n, \vec{\beta}^*_n} - \hbar(\beta^n_{i\not{i}'} + 1)(\beta^n_{\not{i}i'} + 1) T^n_{\vec{\alpha}_n, \vec{\beta}^*_n - \vec{e}^*_I + \vec{e}^*_{i\not{i}'} + \vec{e}^*_{\not{i}i'} - \vec{e}^*_{\not{I}}}, \tag{3}$$

where $\tau_n := 1 - n + \frac{1}{\hbar}$, $\vec{\beta}^*_n = (\beta^n_I, \beta^n_{\not{I}}, \beta^n_{i\not{i}'}, \beta^n_{\not{i}i'})$, and $\mathcal{I} := \left\{ I, \not{I}, i\not{i}', \not{i}i' \right\}$. Note that $\not{i}$ (or $\not{i}'$) is the other index which is not i (or not i'). They are uniquely determined when i (or i') is fixed.

To give the star product with separation of variables on $G_{2,4}(\mathbb{C})$, it is necessary to give the explicit form of $T^n_{\vec{\alpha}_n, \vec{\beta}^*_n}$ as the solution of the recurrence relations (3).

7. Main Theorem (O.-Sako ('24))

For $f, g \in C^\infty(G_{2,4}(\mathbb{C}))$, the star product with separation of variables on $G_{2,4}(\mathbb{C})$ is (locally) given by

$$\begin{aligned}
f * g = &\sum_{n=0}^{\infty} \sum_{\substack{J_i \in \{J_i\}_n \\ D_i \in \{D_i\}_n}} \sum_{\substack{k_i = 1 \\ k_i \in \{k_i\}_n}}^{2} \left(\prod_{l=1}^{n} \frac{g_{\overline{k_l j_l'}, D_l} \Upsilon_{l, \{J_i\}_n, \{k_i\}_n}}{\tau_l} \right) \\
&\times \left(\prod_{S \in \mathcal{I}} \prod_{r=1}^{n} \theta\left(\sum_{m=1}^{r-1} d_{S, J_m, k_m} \right) \right) \left(D^{\sum_{m=1}^{n} \overrightarrow{e_{D_m}}} f \right) \left(D^{\sum_{P \in \mathcal{I}} \sum_{m=1}^{n} d_{P, J_m, k_m} \vec{e}^*_P} g \right),
\end{aligned} \tag{4}$$

where $\theta : \mathbb{R} \to \{0, 1\}$ is the step function,

$$d_{j_l \not{j}_l', J_m, k_m} := \delta_{j_l \not{j}_l', J_m} + \delta_{\not{j}_m k_m} \left(\delta_{j_l \not{j}_l', \not{J}_m} - \delta_{j_l \not{j}_l', j_m \not{j}_m'} \right),$$

$$\Upsilon_{l, \{J_i\}_n, \{k_i\}_n} := \frac{\tau_l \delta_{j_l k_l} + 1 + \sum_{m=1}^{l} d_{m, j_l \not{j}_l', J_m, k_m}}{l(\tau_l + 1) + 2\left\{ \sum_{m=1}^{l} \left(\delta_{I J_m} + \delta_{i\not{i}', J_m} \right) \right\} \left\{ \sum_{m=1}^{l} \left(\delta_{\not{I} J_m} + \delta_{\not{i}i', J_m} \right) \right\}},$$

for $l, r = 1, \cdots, n$, $\{J_i\}_n := \{J_1, \cdots, J_n\}$, $\{D_i\}_n := \{D_1, \cdots, D_n\}$, and $\{k_i\}_n := \{k_1, \cdots, k_n\}$.

8. Deriving Main Theorem (1)

We can derive the following recurrence relations (5) from the recurrence relations (3):

$$T^n_{\vec{\alpha}_n, \vec{\beta}^*_n} = \frac{\sum_{J, D \in \mathcal{I}} \sum_{k=1}^{2} \left\{ \left(\tau_n \delta_{jk} + \beta^n_{j\not{j}'} + 1 \right) g_{\overline{kj'}, D} T^{n-1}_{\vec{\alpha}_n - \vec{e}_D, \vec{\beta}^*_n - \vec{e}^*_J - \delta_{\not{j}k}\left(\vec{e}^*_{\not{J}} - \vec{e}^*_{j\not{j}'} \right)} \right\}}{\tau_n \left\{ n(\tau_n + 1) + 2\left(\beta^n_I + \beta^n_{\not{i}i'} \right)\left(\beta^n_{\not{I}} + \beta^n_{i\not{i}'} \right) \right\}}. \tag{5}$$

Furthermore, $T^n_{\vec{\alpha}_n, \vec{\beta}^*_n}$ obtained as the solution of (5) is also the solution of (3). However, it is not easy to solve (5) straightforwardly because the multi-index $\vec{\beta}^*_n - \vec{e}^*_J - \delta_{\not{j}k}\left(\vec{e}^*_{\not{J}} - \vec{e}^*_{j\not{j}'} \right)$ includes subtraction and **addition**.

9. Deriving Main Theorem (2)

We introduce creation and annihilation operators $\mathfrak{a}^\dagger_I$, $\mathfrak{a}_I$, and the linear operator T_n on a Fock space such that $\langle \vec{\alpha}_n | T_n | \vec{\beta}^*_n \rangle = T^n_{\vec{\alpha}_n, \vec{\beta}^*_n}$. By using them, the solution of (5) is explicitly given by

$$\begin{aligned}
T^n_{\vec{\alpha}_n, \vec{\beta}^*_n} = &\sum_{\substack{J_i \in \{J_i\}_n \\ D_i \in \{D_i\}_n}} \sum_{\substack{k_i = 1 \\ k_i \in \{k_i\}_n}}^{2} \delta_{\vec{\alpha}_n, \sum_{m=1}^{n} \overrightarrow{e_{D_m}}} \delta_{\vec{\beta}^*_n, \sum_{P \in \mathcal{I}} \sum_{m=1}^{n} d_{P, J_m, k_m} \vec{e}^*_P} \left(\prod_{S \in \mathcal{I}} \prod_{r=1}^{n} \theta\left(\beta^n_S - \sum_{m=r}^{n} d_{S, J_m, k_m} \right) \right) \\
&\times \left(\prod_{l=1}^{n} \frac{g_{\overline{k_l j_l'}, D_l}}{\tau_l} \right) \left\{ \prod_{l=1}^{n} \frac{\tau_l \delta_{j_l k_l} + \beta^n_{j_l \not{j}_l'} + 1 - \Lambda_{l, j_l \not{j}_l', \{J_i\}_n, \{k_i\}_n}}{l(\tau_l + 1) + 2\left(\beta^n_I + \beta^n_{\not{i}i'} - \Delta_{I, \not{i}i', l, \{J_i\}_n} \right)\left(\beta^n_{\not{I}} + \beta^n_{i\not{i}'} - \Delta_{\not{I}, i\not{i}', l, \{J_i\}_n} \right)} \right\},
\end{aligned} \tag{6}$$

where

$$\Lambda_{r, S, \{J_i\}_n, \{k_i\}_n} := \sum_{m=1}^{n} d_{S, J_m, k_m} - \sum_{m=1}^{r} d_{S, J_m, k_m},$$

$$\Delta_{V, W, l, \{J_i\}_n} := \sum_{m=1}^{n} (\delta_{V J_m} + \delta_{W J_m}) - \sum_{m=1}^{l} (\delta_{V J_m} + \delta_{W J_m}).$$

Substituting (6) into (1), we eventually obtain (4) in our main theorem.

10. Application: (Twisted) Fock Representation of $G_{2,4}(\mathbb{C})$

As an application to physics, (twisted) Fock representation can be given by using a star product with separation of variables on $G_{2,4}(\mathbb{C})$. Let a_I, $\underline{a}_I$ be creation and annihilation operators defined by $a^\dagger_I := z^I$, $\underline{a}_I := \partial_I \Phi / \hbar$. Note that they satisfy the commutation relations $[\underline{a}_I, a^\dagger_J]_* = \delta_{IJ}$, $[a^\dagger_I, a^\dagger_J]_* = [\underline{a}_I, \underline{a}_J]_* = 0$. Then, the (twisted) Fock representation of $G_{2,4}(\mathbb{C})$ is defined by

$$F_{G_{2,4}(\mathbb{C})} := \left\{ \sum_{\vec{M}, \vec{N}} A_{\vec{M}, \vec{N}} |\vec{M}\rangle\underline{\langle\vec{N}|} \middle| A_{\vec{M}, \vec{N}} \in \mathbb{C} \right\}.$$

Here, a vacuum $|\vec{0}\rangle\langle\vec{0}|$ is defined by $|\vec{0}\rangle\langle\vec{0}| := e^{-\Phi/\hbar}$, and

$$|\vec{M}\rangle\underline{\langle\vec{N}|} := \frac{1}{\sqrt{\vec{M}!}} \left(a^\dagger_{11'} \right)^{M_1}_* * \cdots * \left(a^\dagger_{22'} \right)^{M_4}_* * |\vec{0}\rangle\langle\vec{0}| * \left(\underline{a}_{11'} \right)^{N_1}_* * \cdots * \left(\underline{a}_{22'} \right)^{N_4}_* \frac{1}{\sqrt{\vec{N}!}}.$$

Using the (twisted) Fock representation $F_{G_{2,4}(\mathbb{C})}$ of $G_{2,4}(\mathbb{C})$, we can expect to formulate field theories on noncommutative $G_{2,4}(\mathbb{C})$ in the context of deformation quantization. To formulate field theories on noncommutative $G_{2,4}(\mathbb{C})$, it is necessary to give the derivation of the star product and integration formula for $|\vec{M}\rangle\underline{\langle\vec{N}|}$.

11. Outlook: Application for Twistor Theory

As outlooks of the deformation quantization of $G_{2,4}(\mathbb{C})$, we can expect future work related to twistor theory. As an example, we present the noncommutative deformation of twistor correspondence.

$$\begin{array}{ccc}
\boxed{\text{Twistor corresp.}} & & \boxed{\text{N. C. Twistor corresp.}}
\end{array}$$

$$\begin{tikzcd}
 & F_{1,2,4}(\mathbb{C}) \arrow[dl, "\text{fibration}"'] \arrow[dr, "\text{fibration}"] \arrow[rr, dotted, "\text{Noncommutative}", "\text{Deformation}"'] & & & F_{1,2,4}(\mathbb{C})_* \arrow[dl, "*\text{-fibration}"'] \arrow[dr, "*\text{-fibration}"] & \\
G_{2,4}(\mathbb{C}) & & \mathbb{C}P^3 & G_{2,4}(\mathbb{C})_* & & \mathbb{C}P^3{}_*
\end{tikzcd}$$

To attempt the noncommutative deformation of twistor correspondence in twistor theory by using the star product, it is necessary to construct not only the deformation quantization of $G_{2,4}(\mathbb{C})$ (complexified spacetime) and $\mathbb{C}P^3$ (twistor space), but also the complex flag manifold $F_{1,2,4}(\mathbb{C})$.

[1] K. Hara and A. Sako, *"Noncommutative Deformations of Locally Symmetric Kähler manifolds"*, J. Geom. Phys. **114**, (2017), 554–569.

[2] T. Okuda and A. Sako, *"Deformation quantization with separation of variables of $G_{2,4}(\mathbb{C})$"*, arXiv: 2401.00500, (2024), 1–35.

[3] A. Sako and H. Umetsu, *"Twisted Fock Representations of Noncommutative Kähler Manifolds"*, J. Math. Phys. **57(9)**, 093501, (2016), 1–30.