

弦の散乱の力オスに向けて - 弦の半古典軌道 -

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2024年8月9日@基礎物理学研究所

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T. Yoda, Complex saddles of multi-loop diagrams with string,

公開後、S.Mizera氏らの研究との重複が判明
どうにか出版できないか模索中。

本発表も重複する点があるが、

出来る限り私の動機に基付いた新しい議論を紹介する。



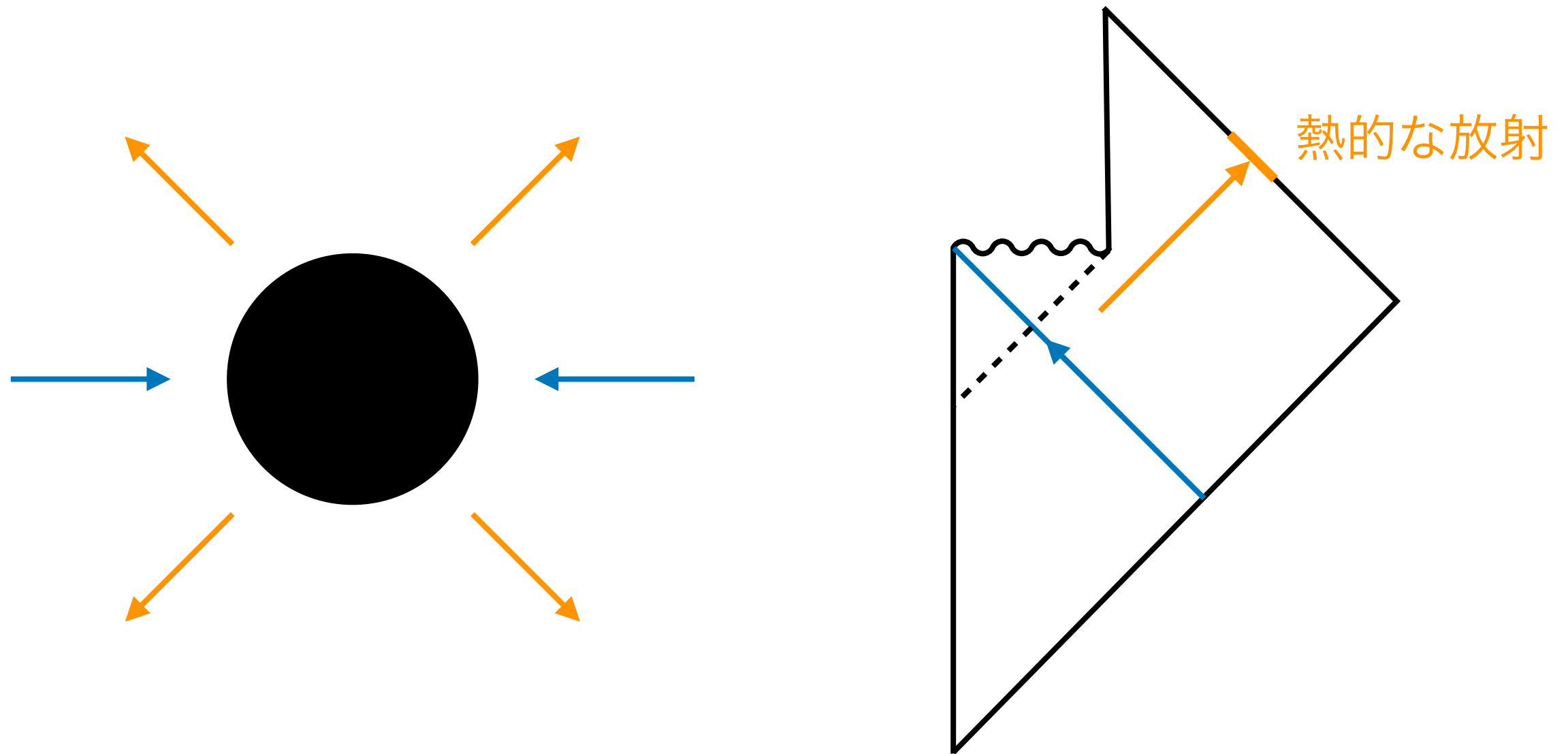
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導入 1/3

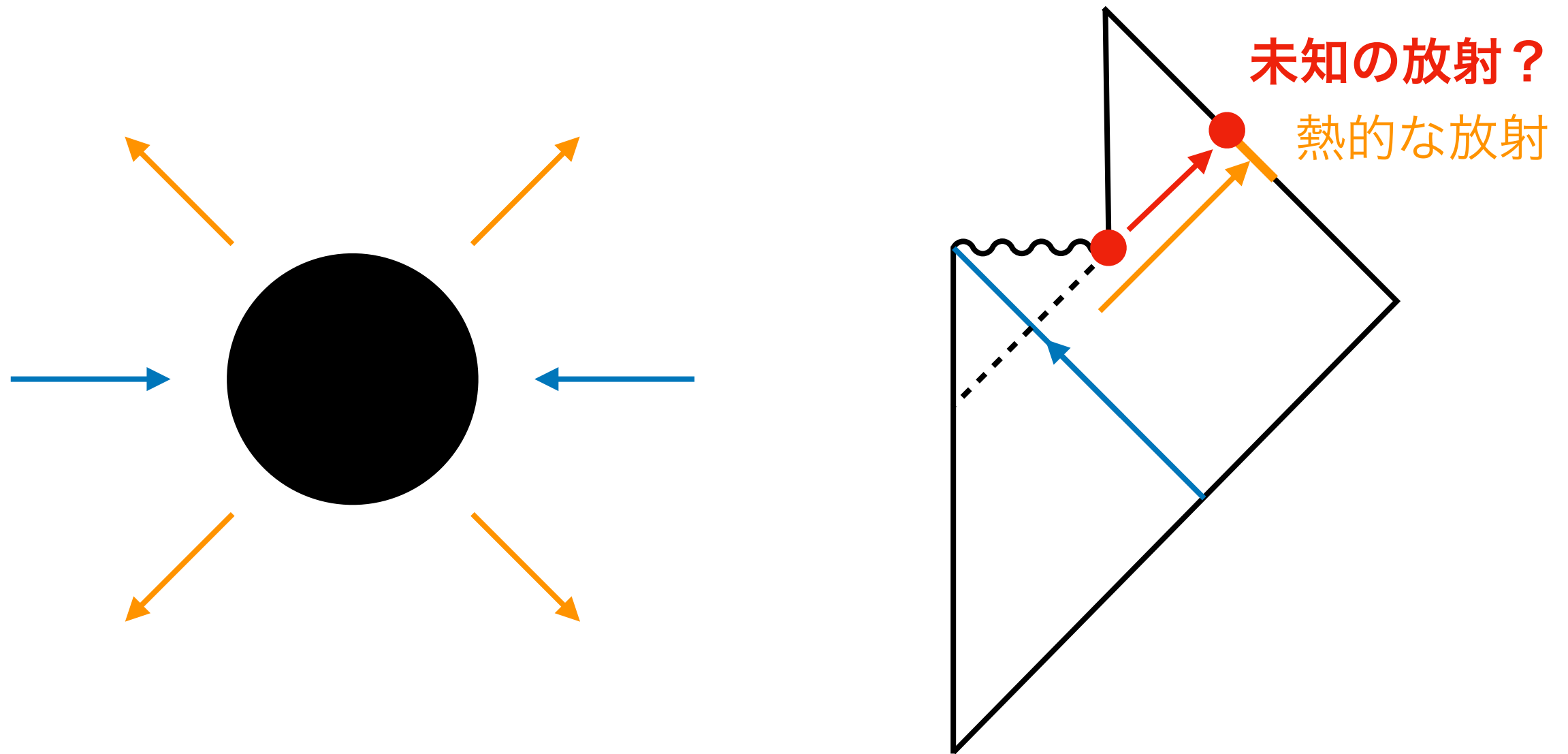
思考実験として、十分高エネルギーな粒子の衝突を考える
量子論・古典重力理論からの予想は、ブラックホールの生成/蒸発/消滅



特に、**蒸発→消滅の一点**を**時間的に分解**したい
このブラックホールを伴う散乱過程を、弦理論で考えられないか？

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思考実験として、十分高エネルギーな粒子の衝突を考える
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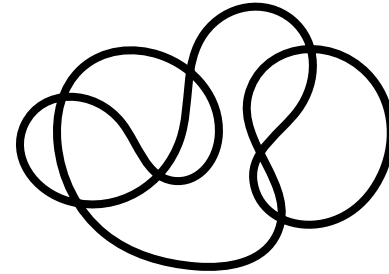
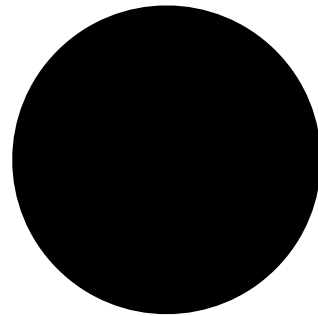
このブラックホールを伴う散乱過程を、弦理論で考えられないか？

導入 2/3

ブラックホールは、弦の状態のアンサンブルとみなせる

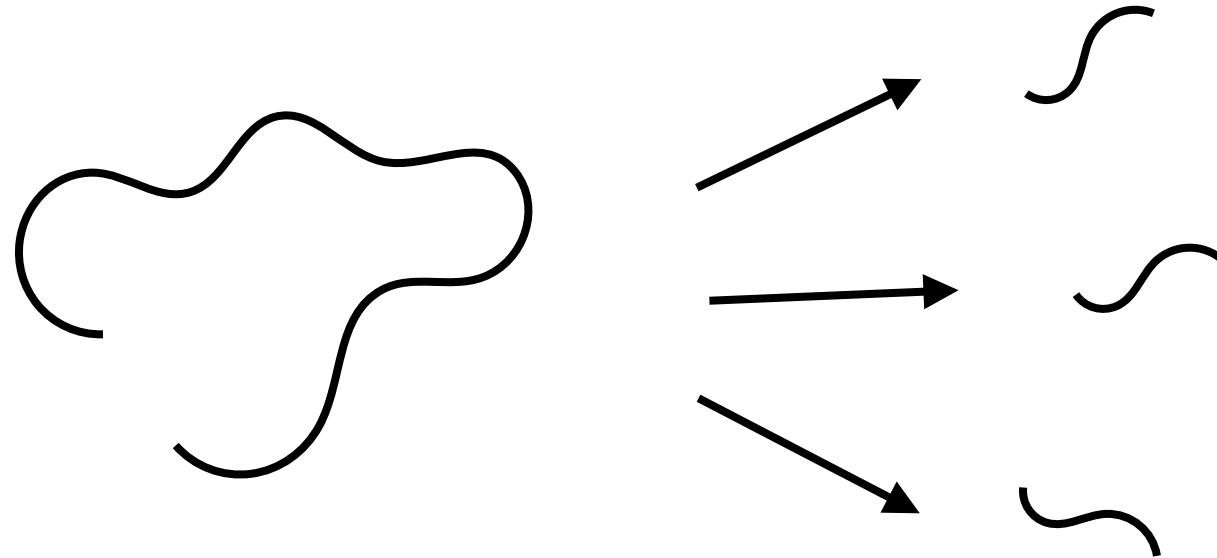
Horowitz-Polchinski対応：ブラックホールエントロピー = 弦のエントロピー

[Horowitz, Polchinski, 97, 98; Chen, Maldacena, Witten, 23]



Amati-Russo：十分重い弦の状態のアンサンブル = 黒体(Hagedorn温度)

[Amati, Russo, 99]



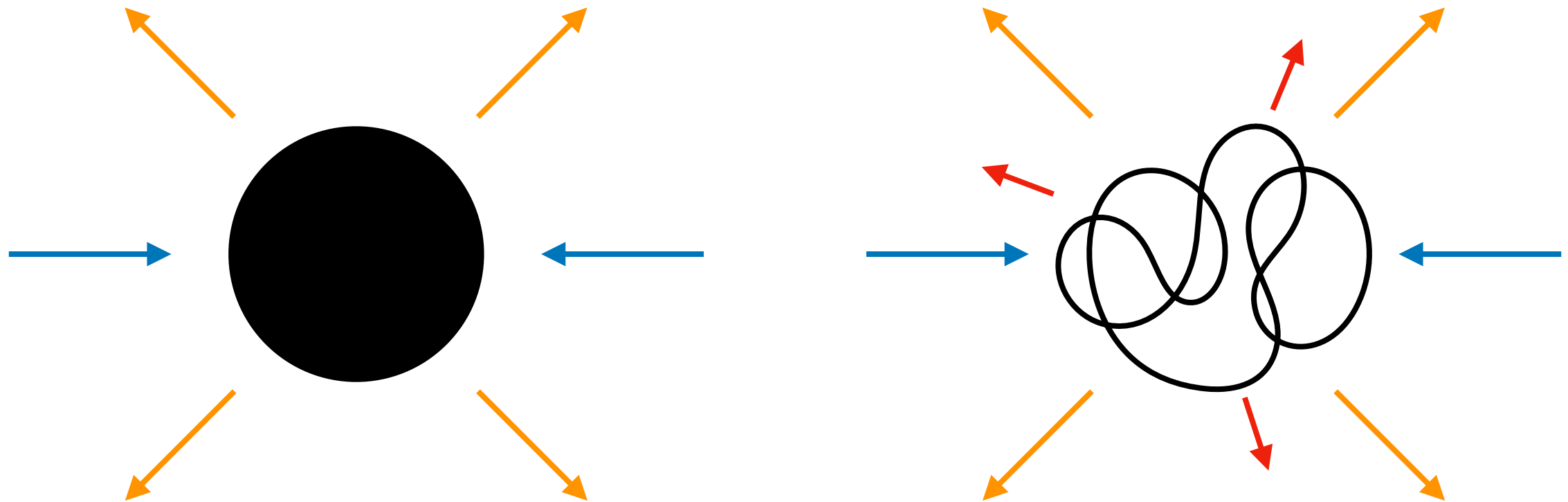
いずれも**弦の状態のアンサンブル**

蒸発→消滅を**時間的に分解するには不向き**

導入 3/3

新しいアプローチ：

弦の散乱振幅から**波束の放出時刻**を計算、**分布の時間発展**を解析



最終目標：

簡単な初期条件から**熱的な分布が生じる機構**

熱的な放射が起きた後、**既存の予想を超えた放射現象**

弦の散乱の力オスに向けて

- 弦の半古典軌道 -

今回

新しいアプローチの詳細、Veneziano振幅を具体例に

- | | |
|----------------|-----|
| 1. 弦の散乱振幅と積分路 | (5) |
| 2. 時間の遅れと半古典軌道 | (7) |
| 3. 議論及び、まとめと展望 | (3) |

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1. 弦の散乱振幅と積分路

1.1. 形式的な積分表示

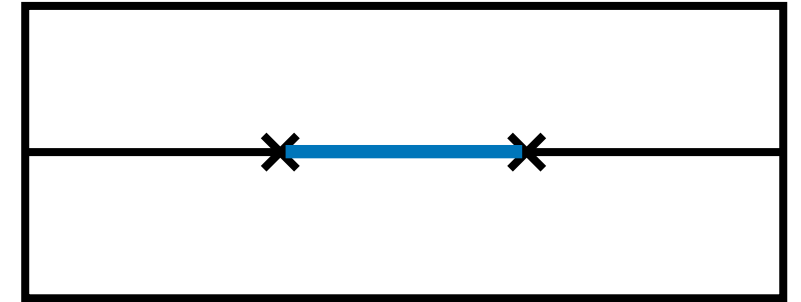
1.2. 積分路の再定義

1.3. シンブル解析

1.1 形式的な積分表示

Veneziano振幅の形式的な積分表示

$$\mathcal{A}_4 = \int_0^1 dy y^{+p_1 p_2} (1 - y)^{+p_1 p_3}$$



典型的な実の運動量で発散

通常は虚の運動量領域で積分、解析的表示を得た後に解析接続

より一般の多点振幅では、

超幾何関数を用いた***単純な解析的表示は知られていない**

発散するため直接的な**数値計算が不可能**

*より一般化された超幾何関数に関する議論は有り

1.2 積分路の再定義

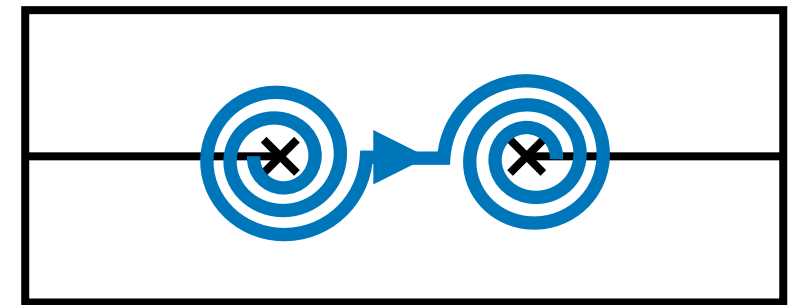
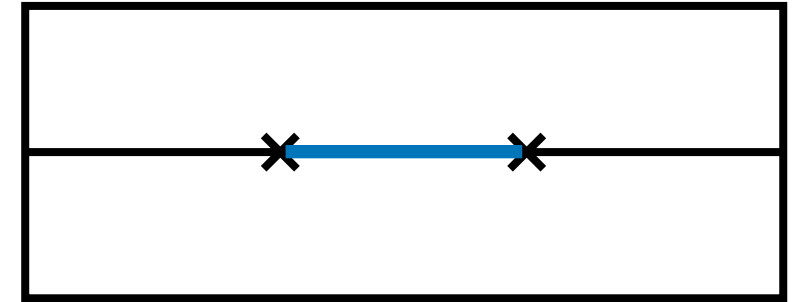
渦巻き状の積分路に取り替える

[Witten, 13]

$$\mathcal{A}_4 = \int_0^1 dy y^{+p_1 p_2} (1 - y)^{+p_1 p_3}$$

↓

$$\tilde{\mathcal{A}} = \int_{C_\epsilon} dy y^{+p_1 p_2} (1 - y)^{+p_1 p_3}$$



積分は収束、良く定義されている
正しい解析接続を与える

1.3 シンブル解析

より一般の多点振幅で積分を評価することを見据え、
敢えて鞍点近似(高エネルギーで良い近似)を行ってみる

良く定義された積分表示：

[Mizera, Pokraka, 20]

[TY, 24]

$$\mathcal{A}_{st}^\epsilon = \int_{C(\epsilon)} dz \, z^{-\alpha(s)-1-i\epsilon} (1-z)^{-\alpha(t)-1-i\epsilon}$$

鞍点：

$$z_{n,m} = \frac{\alpha(s) + 1 + i\epsilon}{\alpha(s) + \alpha(t) + 2 + 2i\epsilon} \quad \text{on the } (n, m)\text{-th Riemann sheet}$$

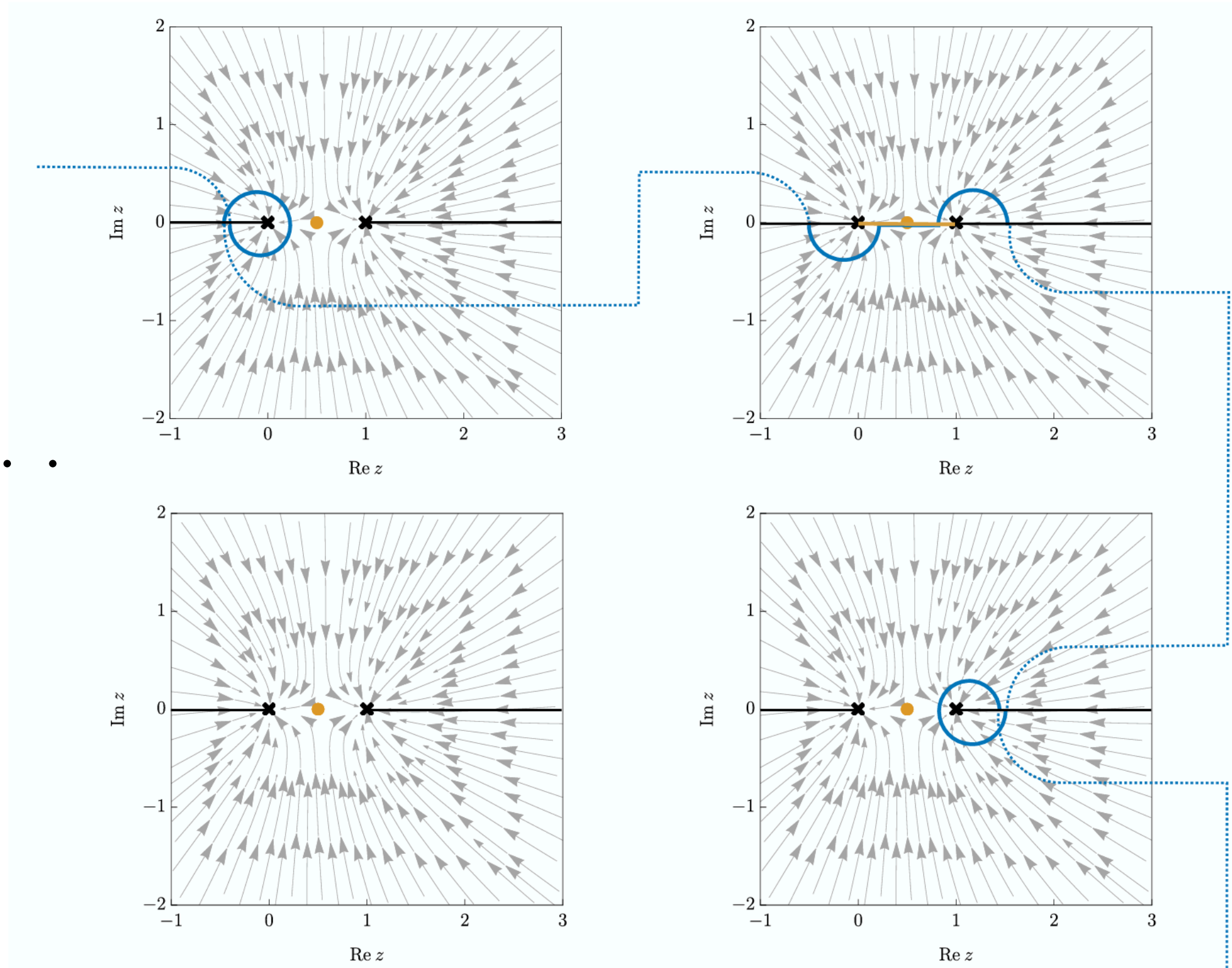
$$\ln z_{n,m} = \ln z_{0,0} + 2\pi i n, \quad \ln(1 - z_{n,m}) = \ln(1 - z_{0,0}) + 2\pi i m.$$

無限枚のリーマン面上に鞍点が見つかる

実/虚の運動量に応じて寄与する鞍点に変化 → **極・零**を生じる

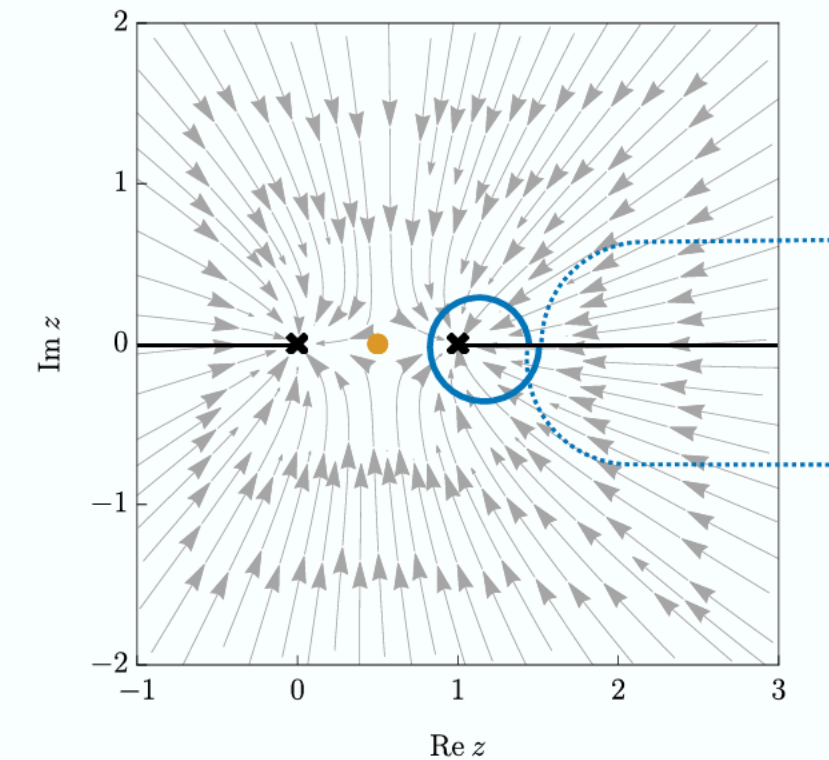
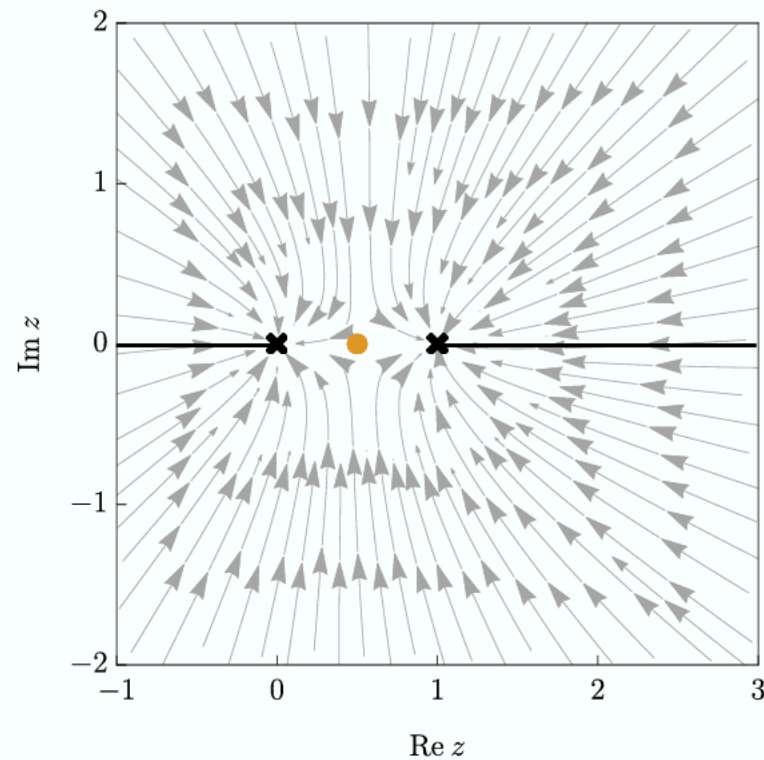
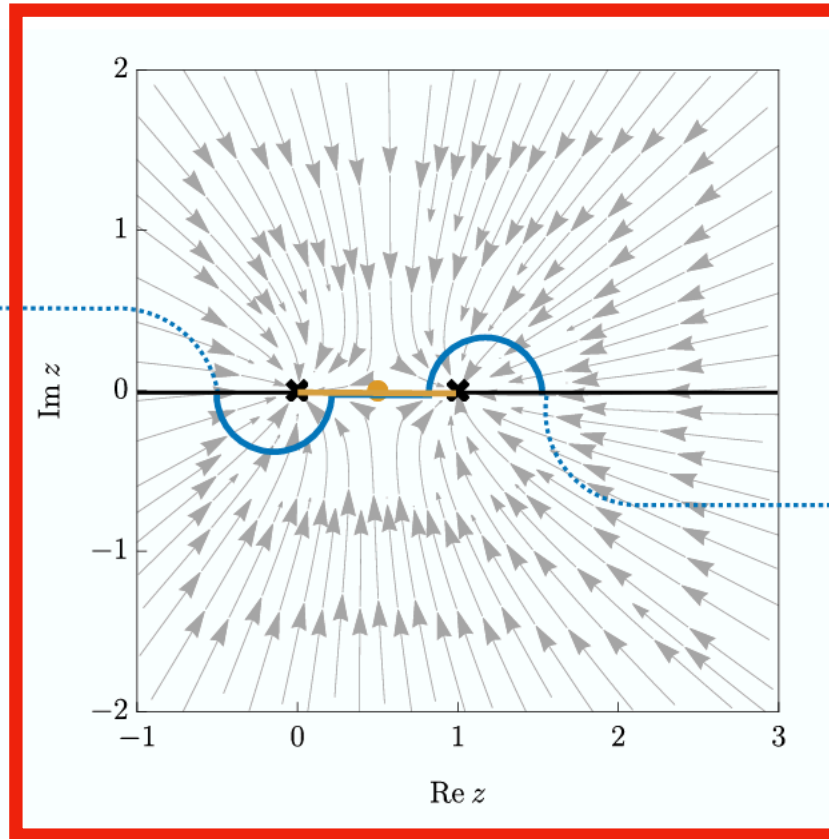
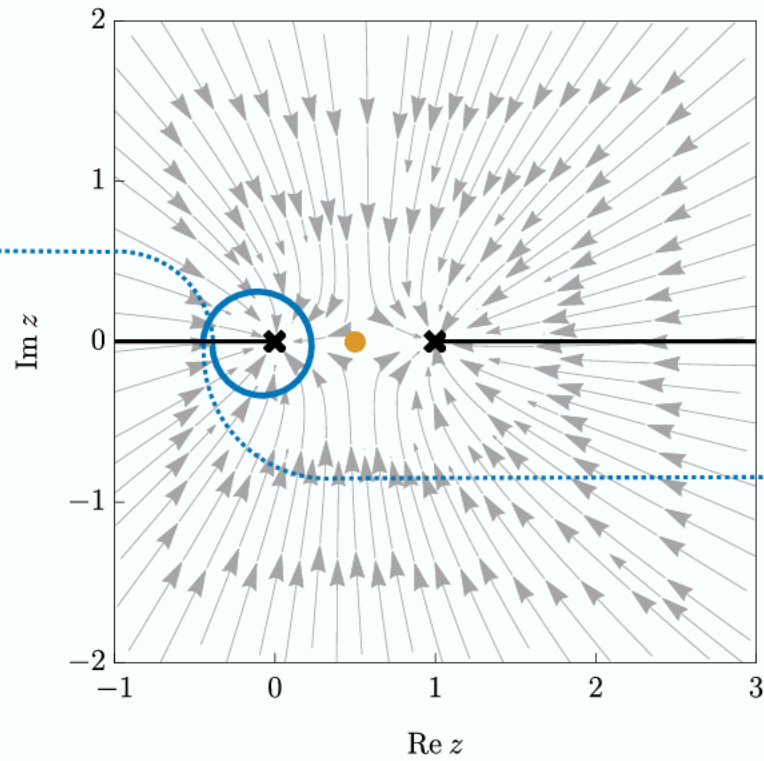
1.3 シンブル解析

非物理的な領域 $\alpha(s), \alpha(t), \alpha(s) + \alpha(t) \ll -1$



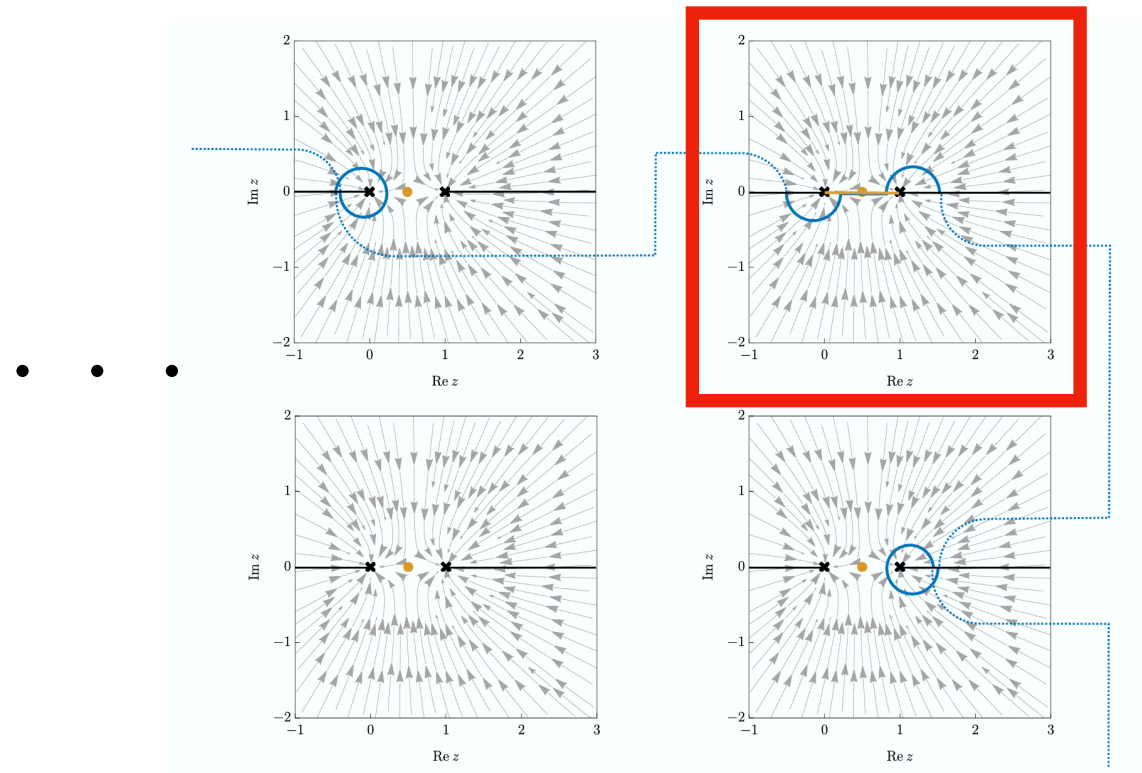
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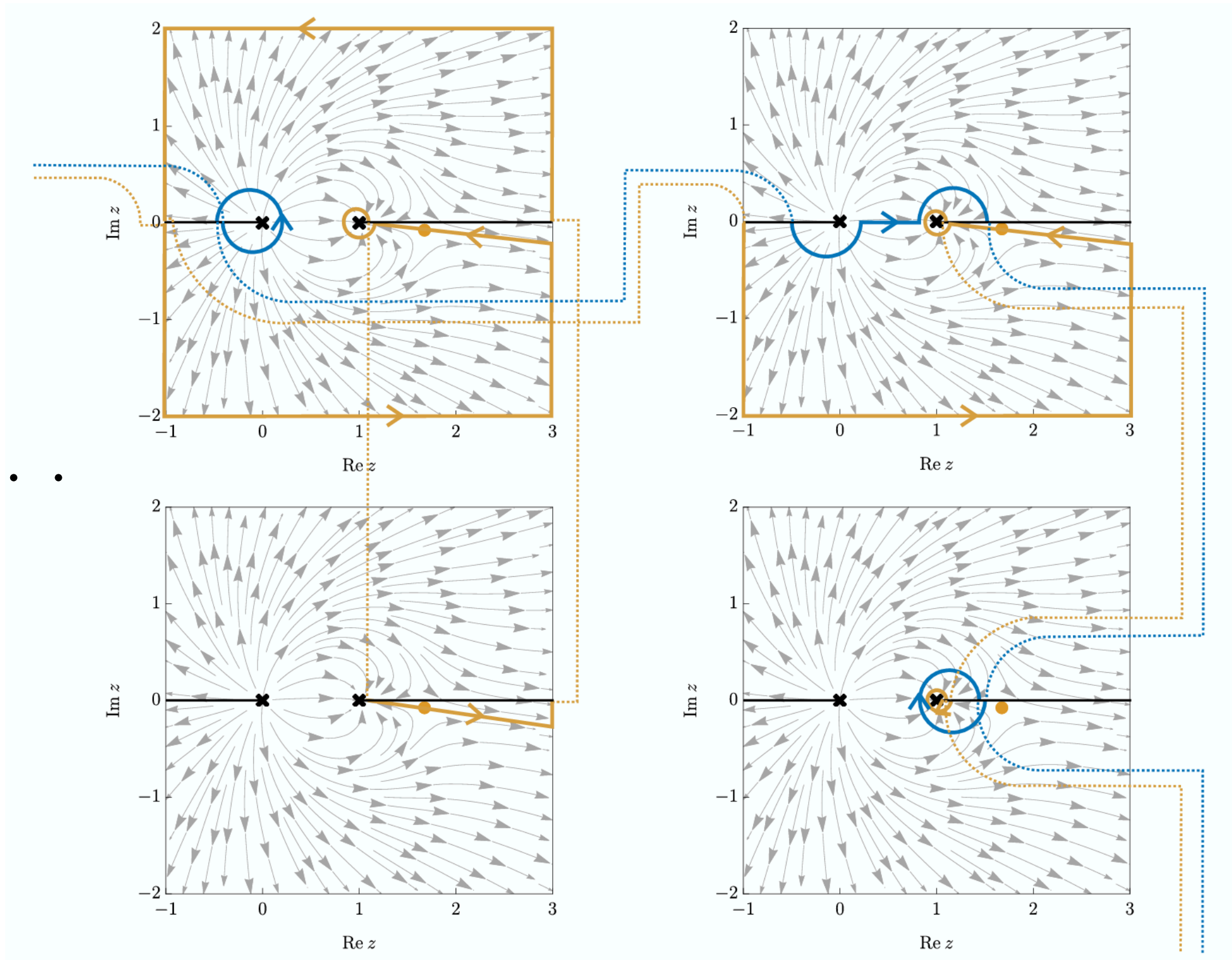


$$\begin{aligned}
 \mathcal{A}_{st}^\epsilon &= \int_{C(\epsilon)} dz \, e^{S(z)} \\
 &\sim \sqrt{\frac{2\pi}{-S''(z_{0,0})}} e^{S(z_{0,0})} \\
 &\sim (stu)^{-3/2} \exp \left[-\frac{1}{2} (s \ln s + t \ln t + u \ln u) \right]
 \end{aligned}$$

極・零なし

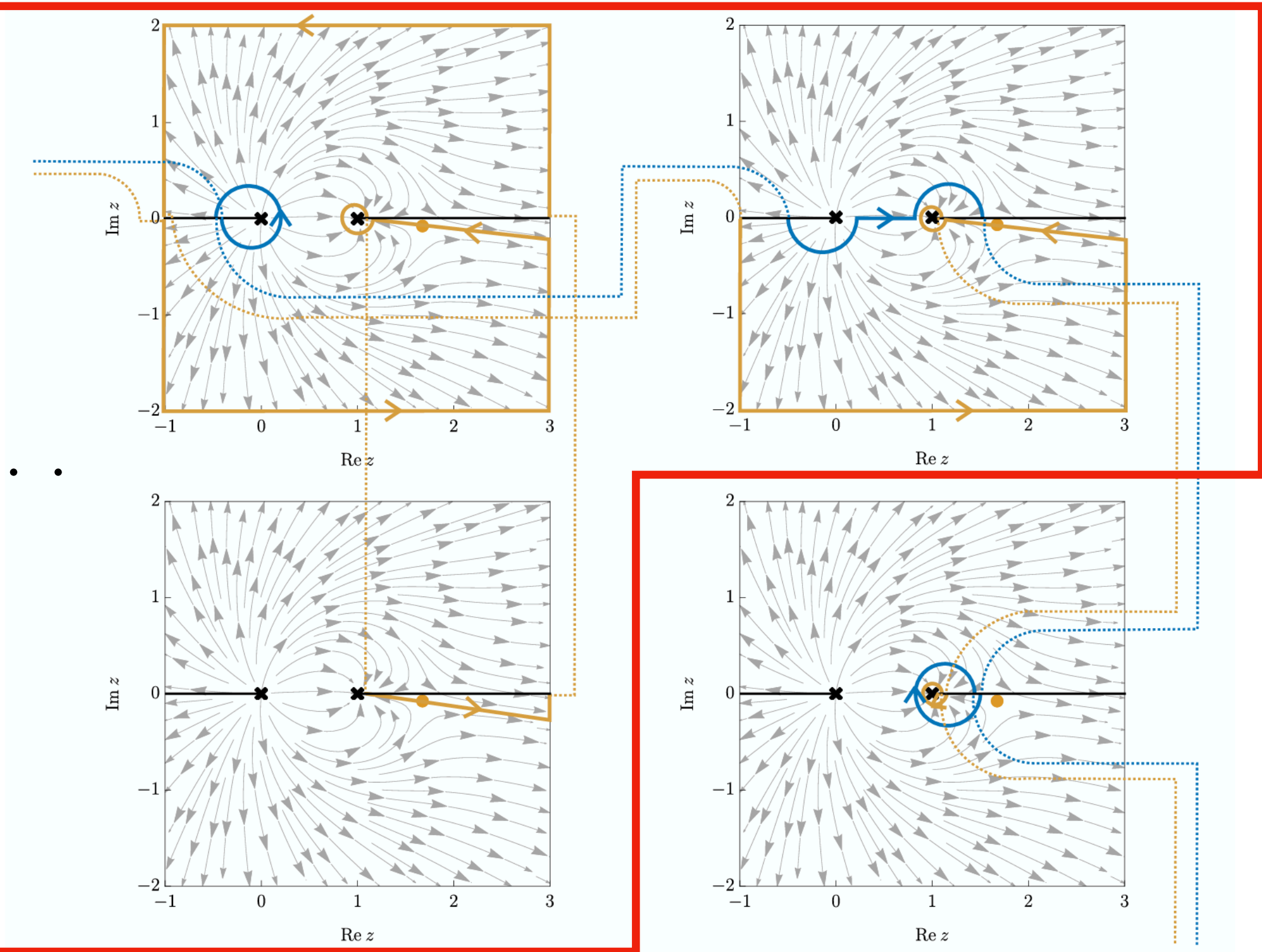
1.3 シンブル解析

物理的な領域 $\alpha(s), \alpha(s) + \alpha(t) \gg 1, \alpha(t) \ll -1$



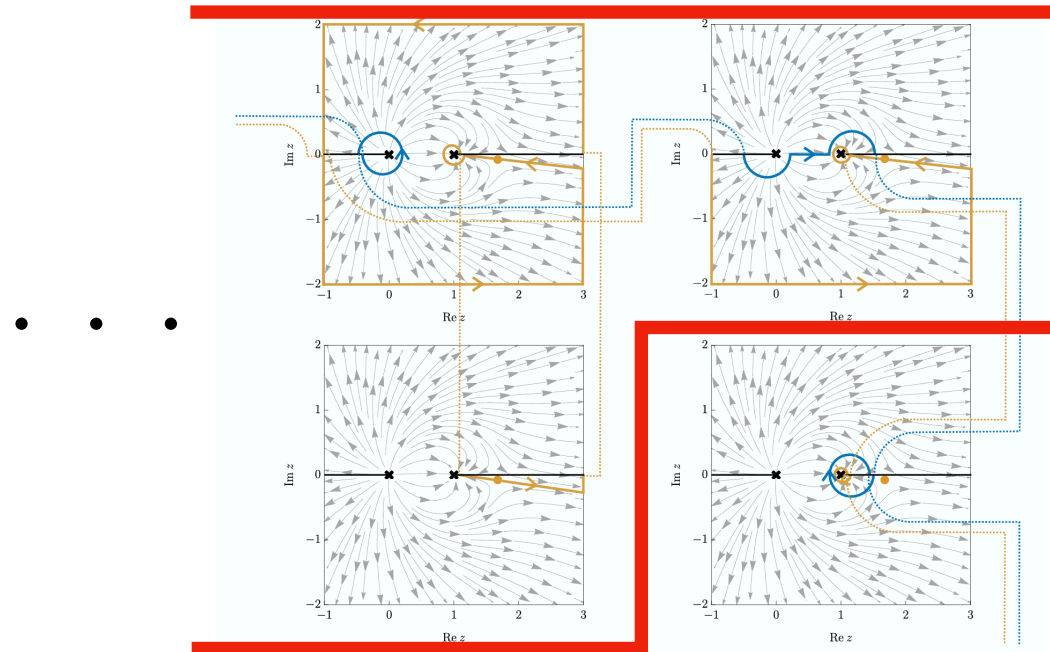
1.3 シンブル解析

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1.3 シンブル解析

物理的な領域 $\alpha(s), \alpha(s) + \alpha(t) \gg 1, \alpha(t) \ll -1$



$$\begin{aligned}
 \mathcal{A}_{st}^\epsilon &= \int_{C(\epsilon)} dz e^{S(z)} \\
 &\sim \sqrt{\frac{2\pi}{-S''(z_{0,0})}} e^{S(z_{0,0})} + \sum_{n \leq -1} \sqrt{\frac{2\pi}{-S''(z_{n,0})}} e^{S(z_{n,0})} - \sum_{n \leq -1} \sqrt{\frac{2\pi}{-S''(z_{n,-1})}} e^{S(z_{n,-1})} \\
 &= \left[1 + \sum_{n \leq -1} e^{-2\pi i n (\alpha(s) + 1 + i\epsilon)} - \sum_{n \leq -1} e^{-2\pi i n (\alpha(s) + 1 + i\epsilon) - 2\pi i (-1) (\alpha(t) + 1 + i\epsilon)} \right] \sqrt{\frac{2\pi}{-S''(z_{0,0})}} e^{S(z_{0,0})} \\
 &\sim \frac{\sin \pi (\alpha(s) + \alpha(t))}{\sin \pi \alpha(s)} (stu)^{-3/2} \exp \left[-\frac{1}{2} (s \ln s + t \ln |t| + u \ln |u|) \right]
 \end{aligned}$$

極・零あり！

1.弦の散乱振幅と積分路

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1.2.積分路の再定義

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2.時間の遅れと半古典軌道

2.1.鞍点の物理的意味 -不可解な点-

2.2.点粒子の振幅 -アナロジーとして-

2.3.弦の振幅

2.4.時間の遅れ

2.5.半古典軌道

*[TY, 24]に基付くが、以下の論文と多くの点で重複する

[Mizera, Pokraka, 20; Eberhardt, Mizera, 23, 24; Caron-Huot, Giroux, Hannesdottir, Mizera, 23]

2.1 鞍点の物理的意味 -不可解な点-

Veneziano振幅の元を辿ると、

$$\begin{aligned}\mathcal{A} &= \int \frac{\prod_{j=1}^4 dz_j}{\text{Vol.}} \int \mathcal{D}X \prod_{j=1}^4 e^{ip_j X(z_j)} e^{-\int_M d^2z (\partial X)^2} \\ &= \int \frac{\prod_{j=1}^4 dz_j}{\text{Vol.}} e^{+ip_i p_j \ln |z_i - z_j|}\end{aligned}$$

経路積分の鞍点：

$$X^\mu(z) = i \sum_{j=1}^4 p_j^\mu \ln |z - z_j|$$

Moduli積分の鞍点：

$$z_{n,m} = \frac{\alpha(s) + 1 + i\epsilon}{\alpha(s) + \alpha(t) + 2 + 2i\epsilon} \quad \text{on the } (n, m)\text{-th Riemann sheet}$$

不可解な点

弦の位置が純虚
無限個の鞍点

素朴にWick回転しても積分が収束し零/極が消滅
これらの意味は？

2.2 点粒子の振幅 -アナロジーとして-

伝播関数： $\longrightarrow$

$$\langle p | \frac{1}{p^2 + m^2} | p' \rangle = \int_0^\infty dT \int \mathcal{D}x \, e^{-ipx(T)} e^{+ip'x(0)} \exp \left[+i \int_0^T dt (\dot{x}^2 - m^2) \right]$$

“Moduli積分” “頂点演算子” Lorentzian作用

実際、逆に辿れる

$$(\text{rhs}) = \int_0^\infty dT \int dx \, dx' \, e^{-ipx} e^{+ip'x'} \int_{x'}^x \mathcal{D}x \, \exp \left[+i \int_0^T dt (\dot{x}^2 - m^2) \right]$$

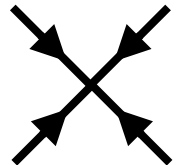
古典経路： $x_{\text{cl}}(t) = x' + (x - x') \frac{t}{T}$ $\longleftarrow$ 終端点での粒子の固有時

$$= \int_0^\infty dT \int dx \, dx' \, e^{-ipx} e^{+ip'x'} \exp \left[+i \left(\frac{x - x'}{T} \right)^2 T - im^2 T \right]$$

$$= \int_0^\infty dT \, e^{-i(p^2 + m^2)T} \delta(p - p')$$

$$= \frac{1}{p^2 + m^2} \delta(p - p')$$

2.2 点粒子の振幅 -アナロジーとして-

4点散乱振幅：

$$g \cdot \prod_{j=1}^4 \frac{1}{p_j^2 + m^2} \cdot \delta\left(\sum_j p_j\right) = g \int_0^\infty \prod_{j=1}^4 dT_j \int \mathcal{D}x \prod_{j=1}^4 e^{+ip_j x_j} \exp \left[+i \int_G dt (\dot{x}^2 - m^2) \right]$$

“Moduli積分” “頂点演算子” Lorentzian作用

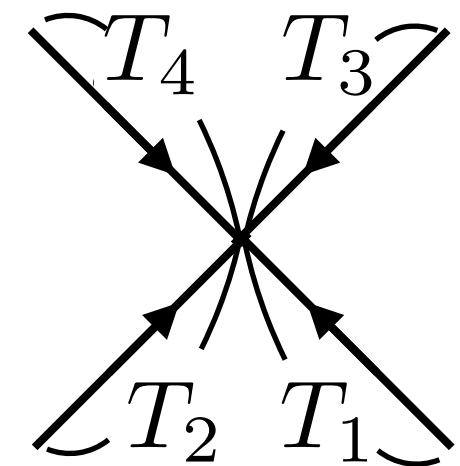
同様、逆に辿れて、

$$\begin{aligned} &= g \int_0^\infty \prod_{j=1}^4 dT_j \int \prod_{j=1}^4 dx_j e^{+ip_j x_j} \cdot \exp \left[+i \sum_{j=1}^4 \left(\left(\frac{x_j}{T_j} \right)^2 - m^2 \right) T_j \right] \cdot \delta\left(\sum_{j=1}^4 p_j\right) \\ &= g \int_0^\infty \prod_{j=1}^4 dT_j \exp \left[-i \sum_{j=1}^4 (p_j^2 + m^2) T_j \right] \delta\left(\sum_{j=1}^4 p_j\right) \end{aligned}$$

運動量保存($p_1^2 = p_1(-p_2 - p_3 - p_4)$ など)を使うと、

$$= g \int_0^\infty \prod_{j=1}^4 dT_j \exp \left[+i \left(\sum_{j < k} p_j p_k (T_j + T_k) - m^2 \sum_{j=1}^4 T_j \right) \right]$$

↑
Veneziano振幅と類似



2. 時間の遅れと半古典軌道

2.3 弦の振幅

再定義された積分路を用いると、

$$\mathcal{A}_4 = \int_{C_\epsilon} \frac{\prod_{j=1}^4 dz_j}{\text{Vol.}} \exp \left[+ \sum_{j < k} p_j p_k \ln(z_j - z_k) \right]$$

類似として点粒子

$$g \int_0^\infty \prod_{j=1}^4 dT_j \exp \left[+i \left(\sum_{j < k} p_j p_k (T_j + T_k) - m^2 \sum_{j=1}^4 T_j \right) \right]$$



弦の固有時刻

$$T_1(z_1) = -i \ln \frac{\prod_{1 < k} (z_1 - z_k)^{1/2}}{\prod_{j < k} (z_j - z_k)^{1/6}}$$

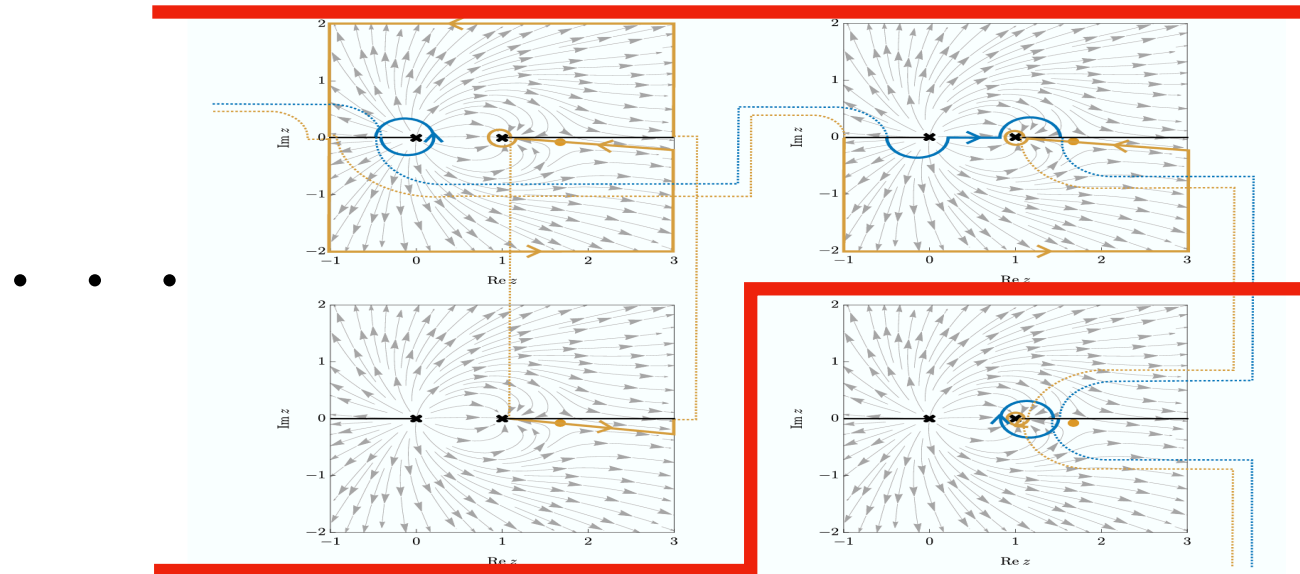
別のリーマン面に移ると、余分に実時間が進む/遅れる

なお、この時間(の平均)は波束の時間とも一致する

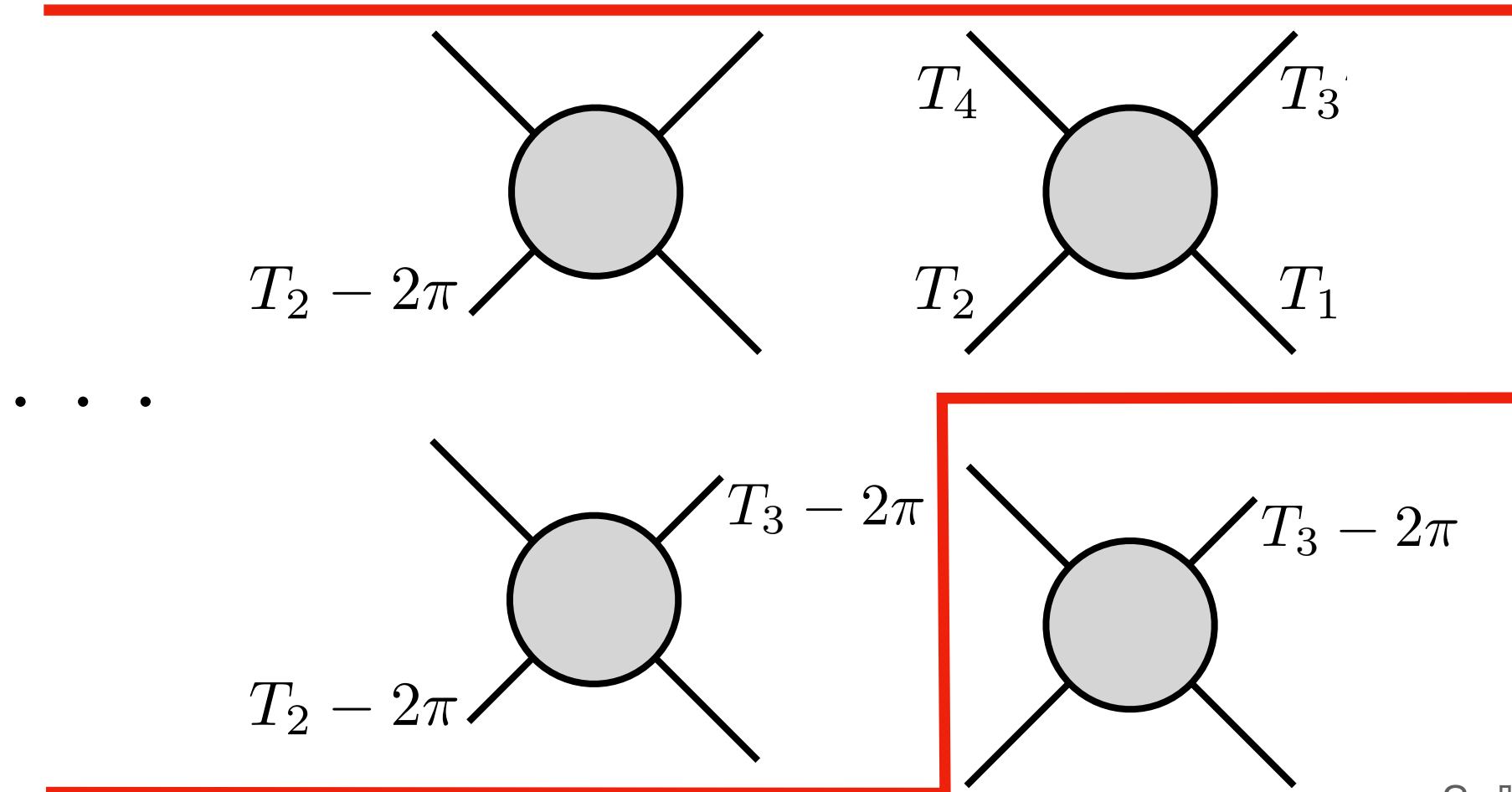
$$\langle \Delta T \rangle := \frac{1}{2\sqrt{s}} \frac{1}{i} \frac{\partial}{\partial E} \ln \mathcal{A}_4$$

2.4 時間の遅れ

寄与していた鞍点を思い出すと、



各鞍点での値を代入すると、



2.4 時間の遅れ

鞍点近似によって得られる時間の遅れの例：

$$\langle \Delta T \rangle \sim i \left(\ln \frac{p_1 p_2}{p_1 p_2 + p_1 p_3} + \frac{p_1 p_3}{p_1 p_2} \ln \frac{-p_1 p_3}{p_1 p_2 + p_1 p_3} \right) - \frac{\pi/2}{(\sin \pi p_1 p_2)^2} + \left(1 + \frac{p_1 p_3}{p_1 p_2} \right) \frac{\pi/2}{(\sin \pi (p_1 p_2 + p_1 p_3))^2}$$

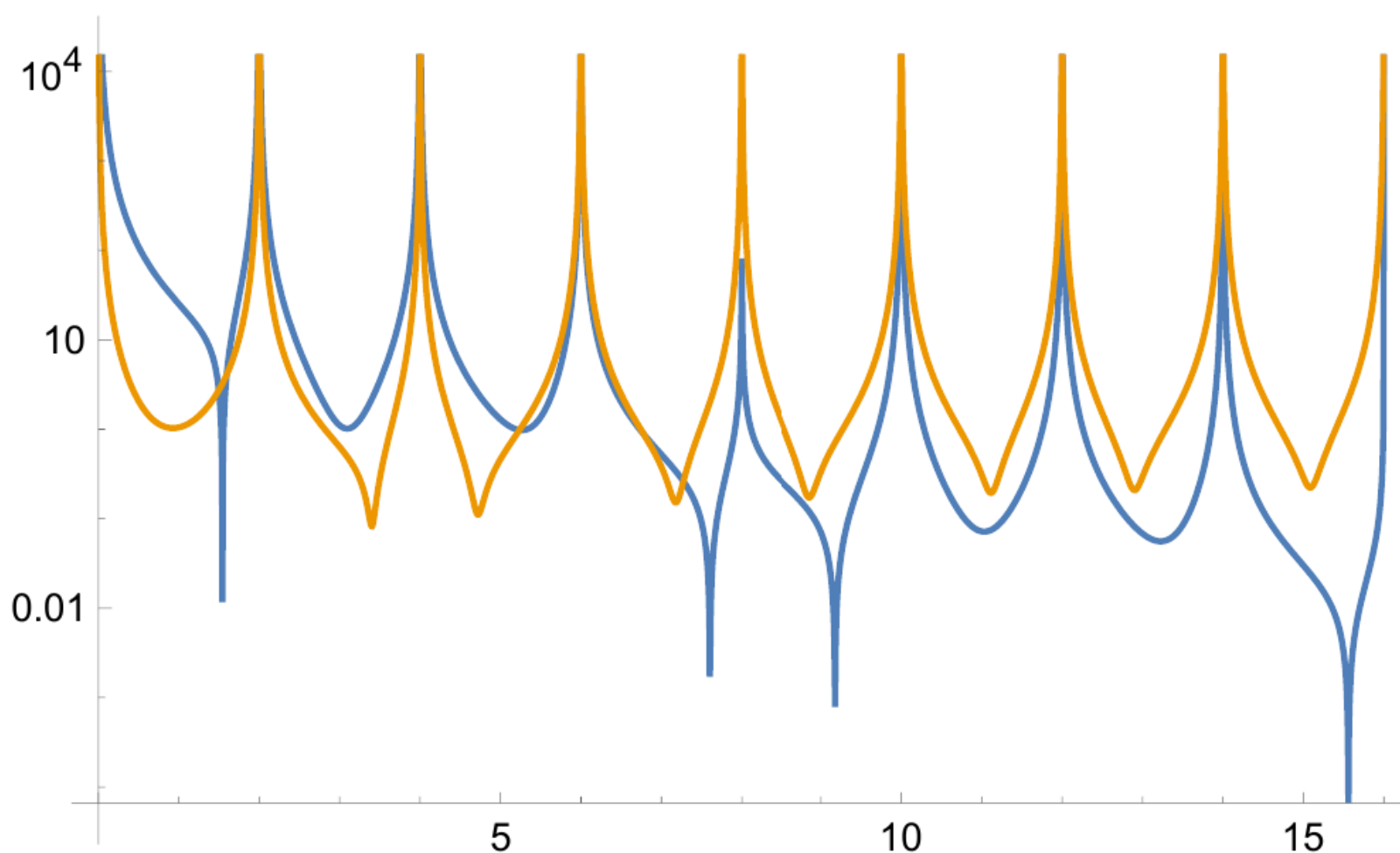


Figure 1: $|\langle \Delta T \rangle|$ with respect to s when $\theta = \pi/2$. Blue: Exact. Orange: Saddle approx.

2.3 半古典軌道

再定義された積分路を用いると、

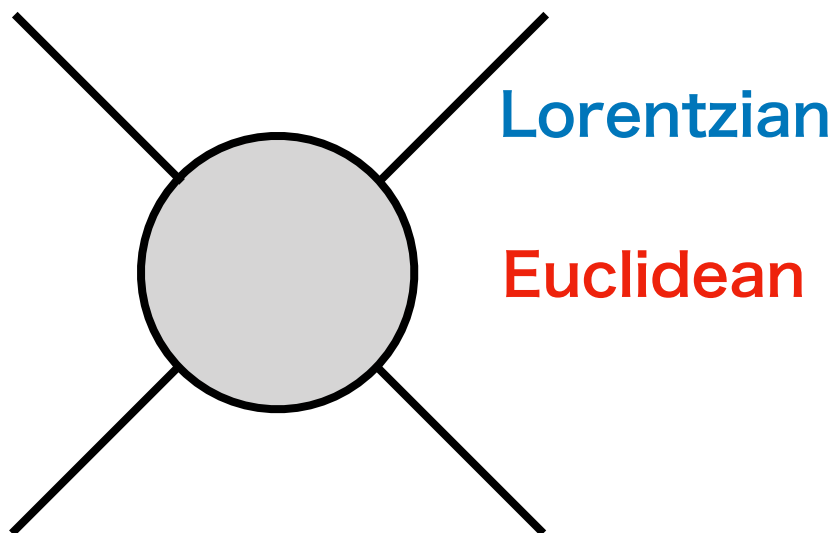
$$\mathcal{A}_4 = \int_{C_\epsilon} \frac{\prod_{j=1}^4 dz_j}{\text{Vol.}} \exp \left[+ \sum_{j < k} p_j p_k \ln(z_j - z_k) \right]$$

半古典軌道(の予想)

$$X^\mu(z) = i \sum_{j=1}^4 p_j^\mu \ln(z - z_j)$$

↑
絶対値でない！

標的空間での弦の位置は、漸近的には**実数**だが、散乱領域では**虚数**



$$\sim \frac{\sin \pi(\alpha(s) + \alpha(t))}{\sin \pi \alpha(s)} (stu)^{-3/2} \exp \left[-\frac{1}{2} (s \ln s + t \ln |t| + u \ln |u|) \right]$$

2.時間の遅れと半古典軌道

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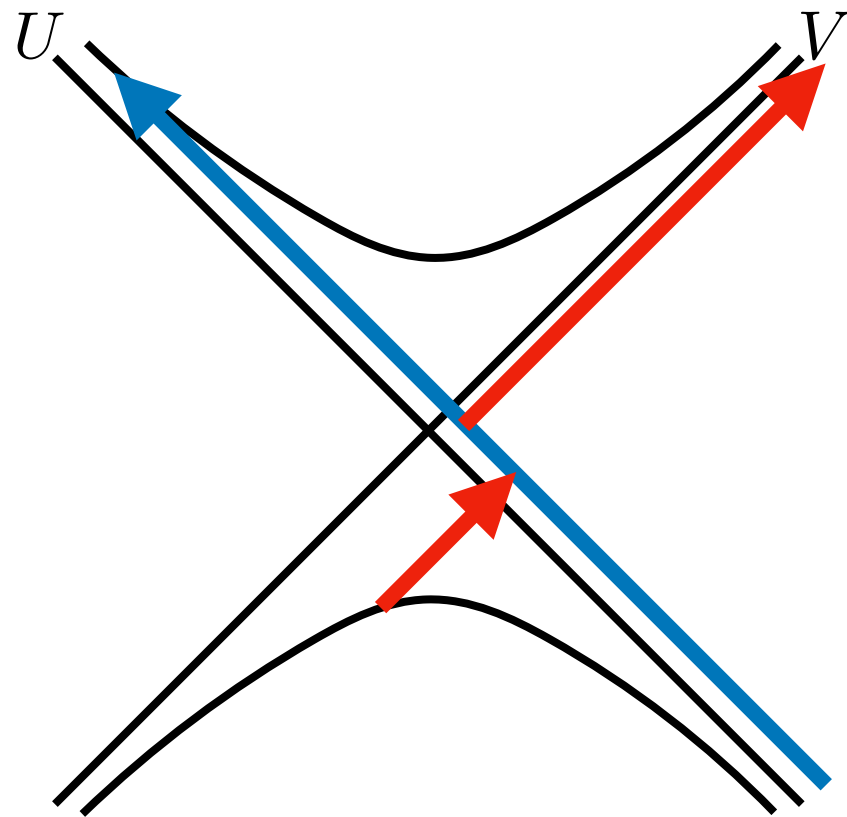
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3.1 カオスとの関係

量子カオス



[Polchinski 15; See also Polchinski 16; Dray, 't Hooft 84; 't Hooft 96]
[Shenker, Stanford; 14, 15. Kitaev (KITP seminar); 14, 15]

OTOCの指数増大を以って量子カオス
増大率がLyapunov指数に相当

Lyapunov指数の重力双対 = 表面重力

$$\Delta t = \kappa^{-1} e^{\kappa(t-r^*)} \Delta U$$

今回の時間の遅れは、表面重力による時間の遅れと比較できる量

3.1 カオスとの関係

古典散乱のカオス

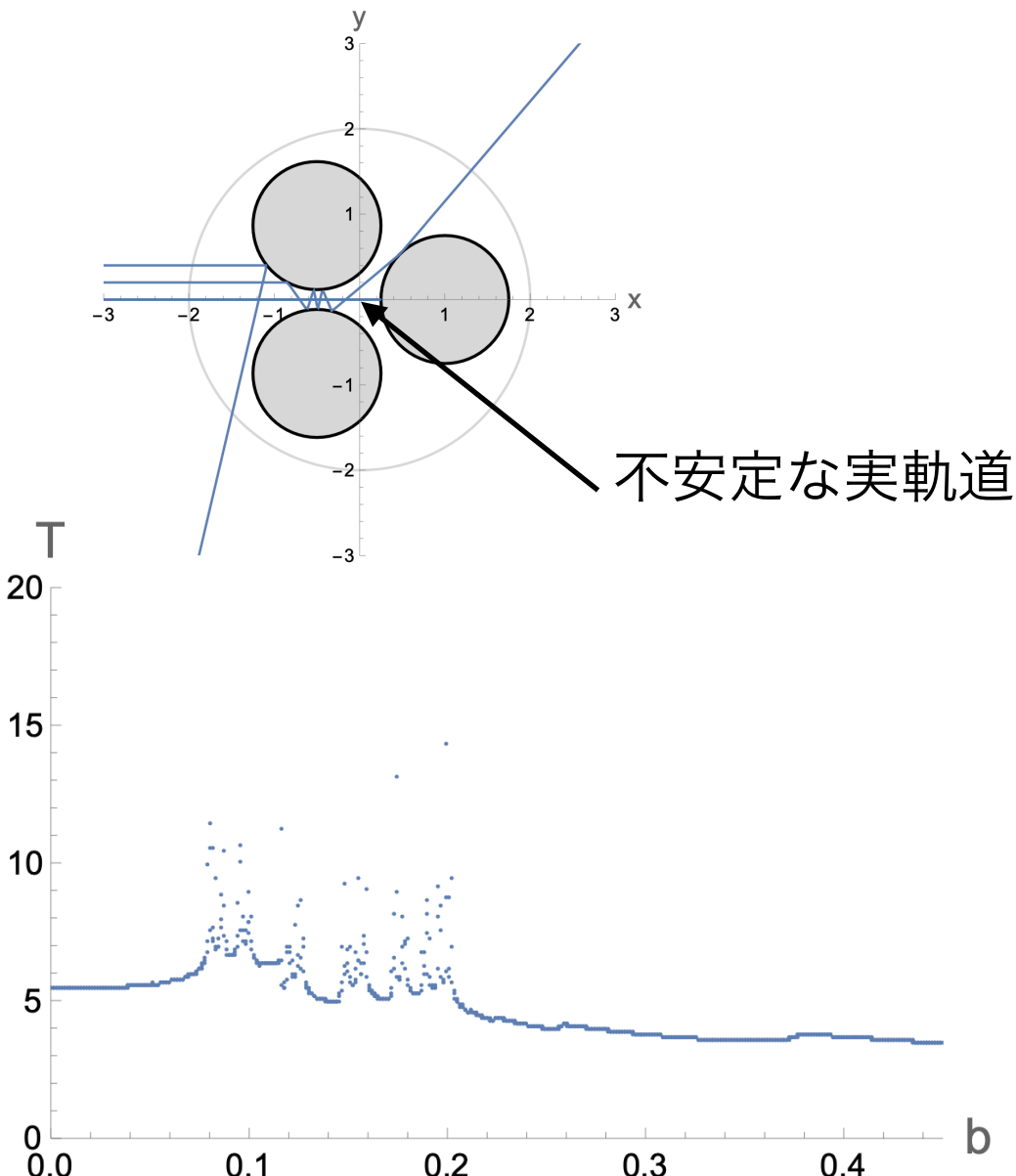
初期条件が忘れられる～熱化の意味では類似するが、直接的な対応は成立せず

古典散乱のカオスの特徴

軌道のズレの引き伸ばし

軌道の折りたたみ

時間の遅れにフラクタル



弦の散乱

複素の半古典軌道

時間の遅れが周期的(ツリーレベル)

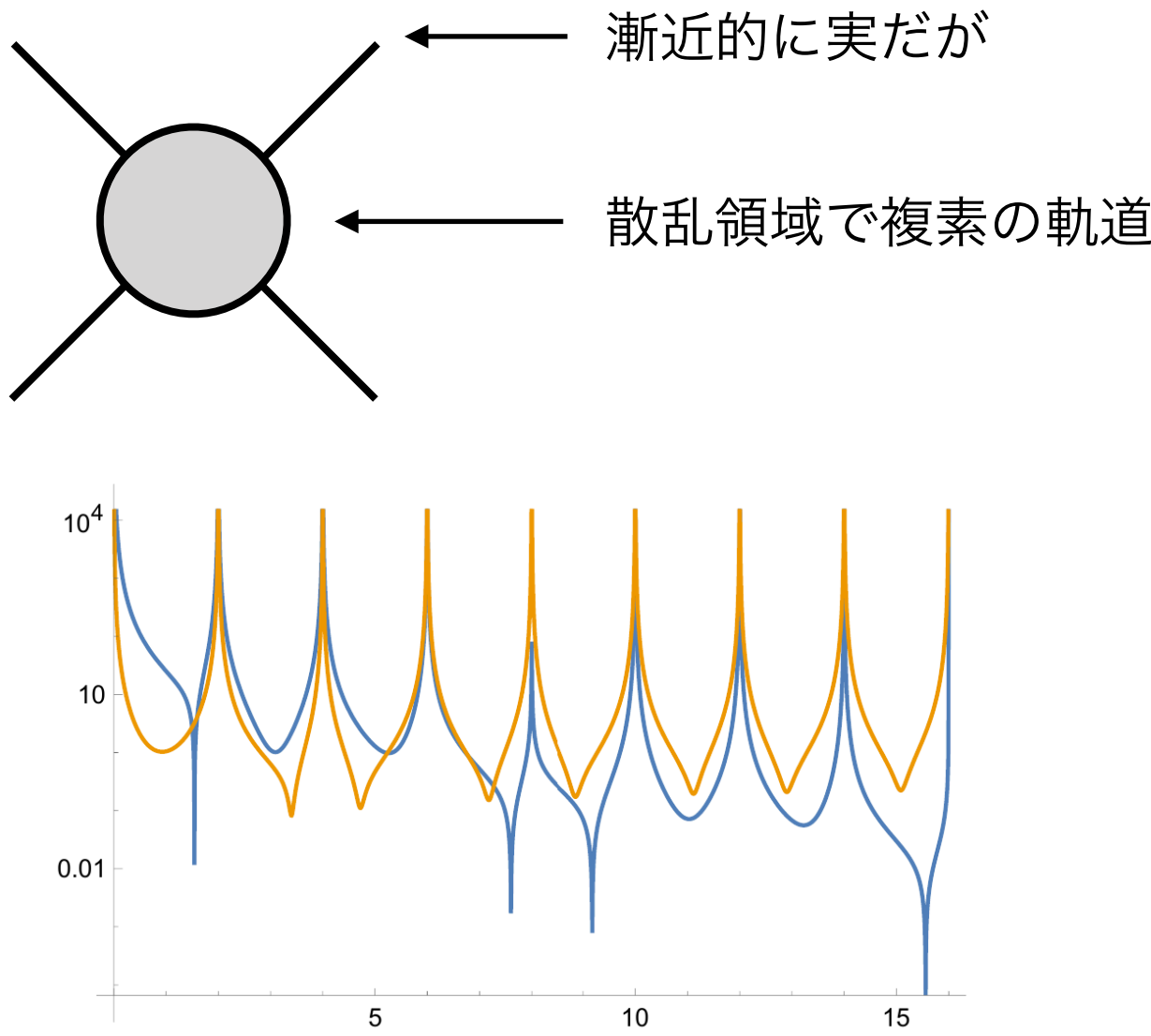


Figure 1: $|\langle \Delta T \rangle|$ with respect to s when $\theta = \pi/2$. Blue: Exact. Orange: Saddle approx.

3.2 まとめと展望

まとめ

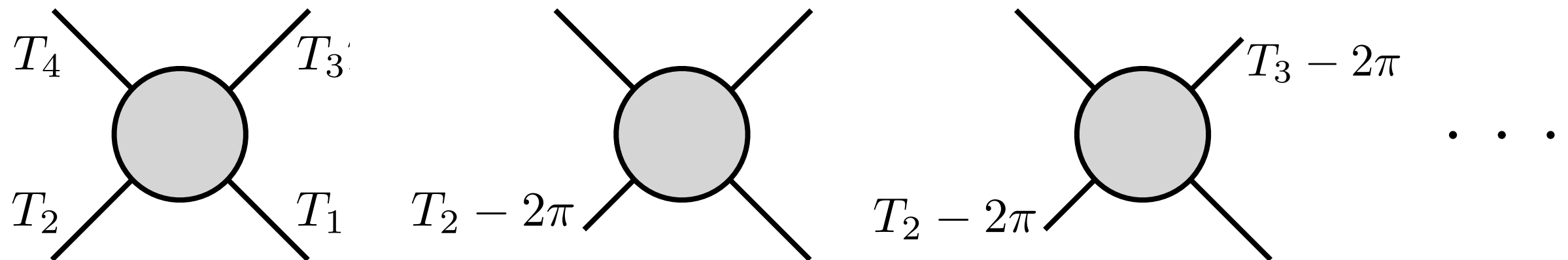
散乱振幅→**波束の放出時刻/分布の時間発展**で、蒸発過程の詳細に迫りたい

簡単な初期条件から熱的な分布が生じる機構？

熱的な放射が起きた後、既存の予想を超えた放射現象？

振幅の再定義・シンブル解析が有用

特にVeneziano振幅の理解：**虚の半古典軌道、異なる実時間の弦の干渉**



展望

より一般の場合に、分布の時間変化をあらわに得たい

シンブル解析で、主要な寄与の物理的意味を明確にしたい

Mende-Ooguriに習って、高次補正のBorel総和を得たい

補助スライド

Veneziano amplitude & asymptotic expansion

Analytic expression:

$$\mathcal{A}_{st} = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}, \quad \alpha(x) = x/2 + 1$$

Unphysical region $\alpha(s), \alpha(t), \alpha(s) + \alpha(t) \ll -1$

$$\mathcal{A}_{st} \sim (stu)^{-3/2} \exp \left[-\frac{1}{2} (s \ln s + t \ln t + u \ln u) \right]$$

Physical region $\alpha(s), \alpha(s) + \alpha(t) \gg 1, \alpha(t) \ll -1$

$$\mathcal{A}_{st} = \frac{\sin \pi(\alpha(s) + \alpha(t))}{\sin \pi\alpha(s)} \frac{\Gamma(\alpha(s) + \alpha(t) + 1)\Gamma(-\alpha(t))}{\Gamma(\alpha(s) + 1)}$$
$$\sim \frac{\sin \pi(\alpha(s) + \alpha(t))}{\sin \pi\alpha(s)} (stu)^{-3/2} \exp \left[-\frac{1}{2} (s \ln s + t \ln |t| + u \ln |u|) \right]$$

reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \neq 0, \pm 1, \dots$$

poles & zeros

Saddle point approx

Suppose that we did not know the analytic expression

Then saddle point approx is a nice method to study asymptotics

Formal integral expression:

[Gross, Mende, 88; Gross Manes, 89; Mende Ooguri, 90]

$$\mathcal{A}_{st} \stackrel{\text{formal}}{=} \int_0^1 dz \, e^{-(\alpha(s)+1) \ln |z| - (\alpha(t)+1) \ln |1-z|}$$

Saddle

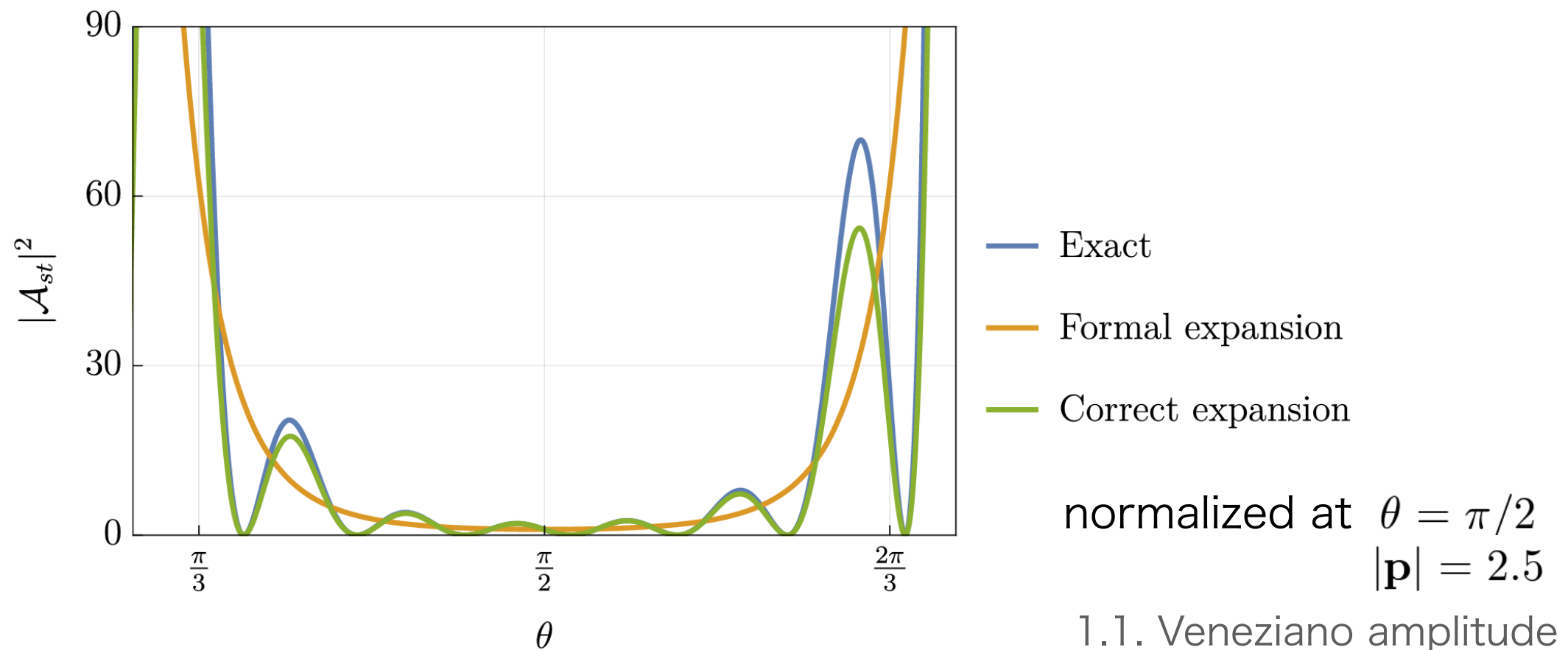
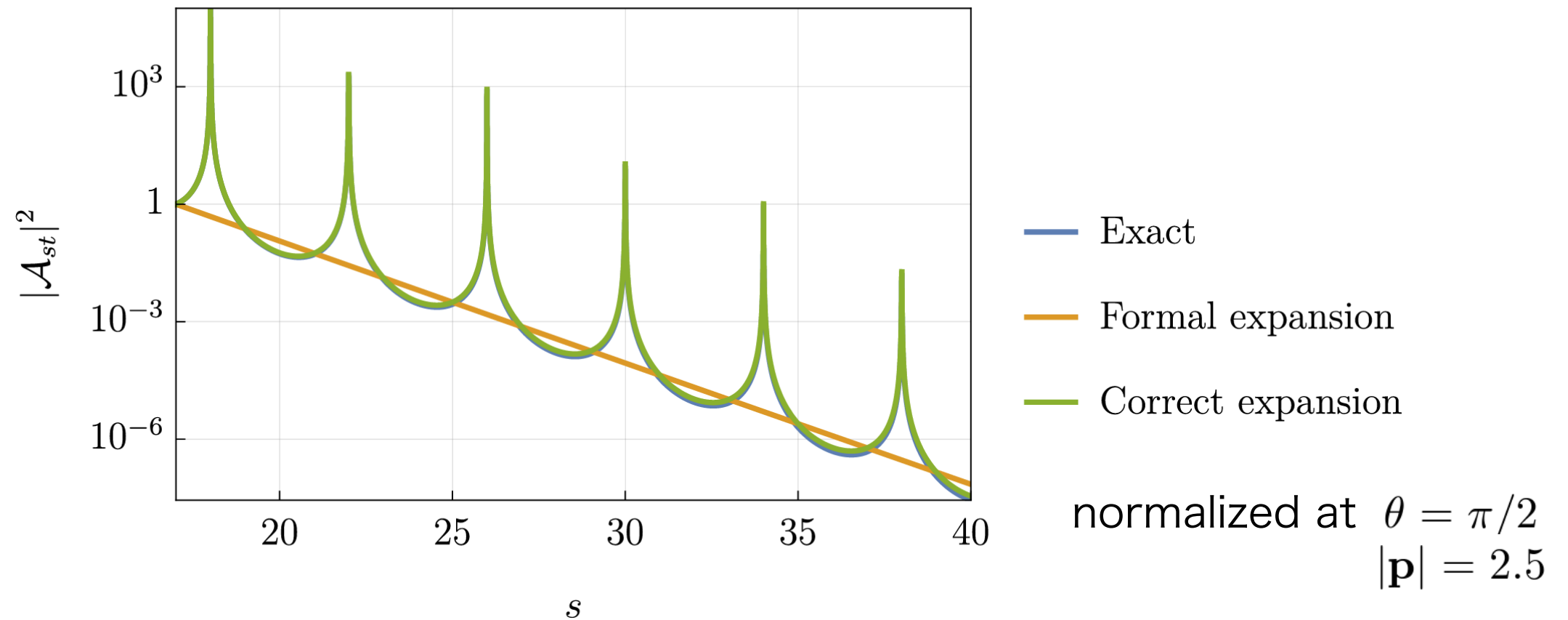
$$z_0 = \frac{\alpha(s) + 1}{\alpha(s) + \alpha(t) + 2}$$

Saddle approx

$$\begin{aligned} \mathcal{A}_{st} &\stackrel{\text{formal}}{\sim} \left(\frac{\alpha(u)^3}{\alpha(s)\alpha(t)} \right)^{-1/2} e^{-(\alpha(s)+1) \ln \alpha(s) - (\alpha(t)+1) \ln \alpha(t) - (\alpha(u)-1) \ln \alpha(u)} \\ &\sim (stu)^{-3/2} \exp \left[-\frac{1}{2} (s \ln s + t \ln t + u \ln u) \right]. \end{aligned}$$

correct fall-off, but missing poles & zeros

Missing poles & zeros

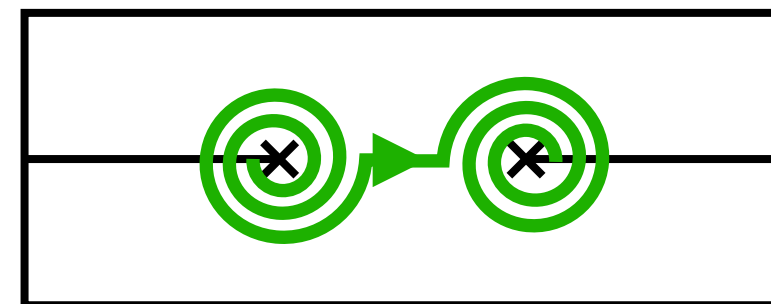


Regularization of the integral

Let us cure the divergences of the formal integral
Feynman- $i\epsilon$:

[Witten, 13]

$$\mathcal{A}_{st}^\epsilon = \int_{C(\epsilon)} dz \, z^{-\alpha(s)-1-i\epsilon} (1-z)^{-\alpha(t)-1-i\epsilon}$$



Now we find infinitely many complex saddles

[Mizera, Pokraka, 20]

[TY, 24]

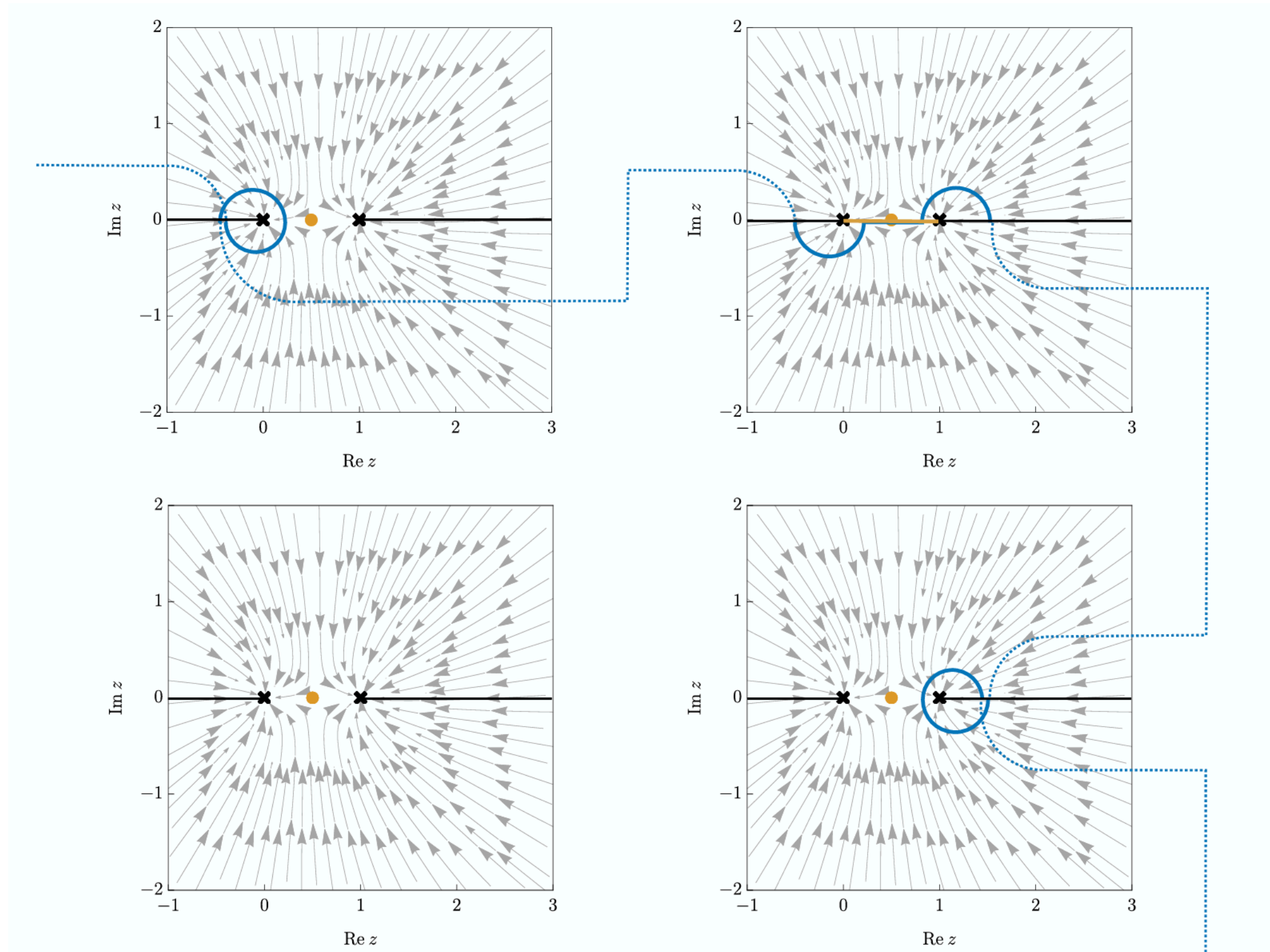
$$z_{n,m} = \frac{\alpha(s) + 1 + i\epsilon}{\alpha(s) + \alpha(t) + 2 + 2i\epsilon} \quad \text{on the } (n, m)\text{-th Riemann sheet}$$

$$\ln z_{n,m} = \ln z_{0,0} + 2\pi i n, \quad \ln(1 - z_{n,m}) = \ln(1 - z_{0,0}) + 2\pi i m.$$

Next we identify which saddles contribute to the integral

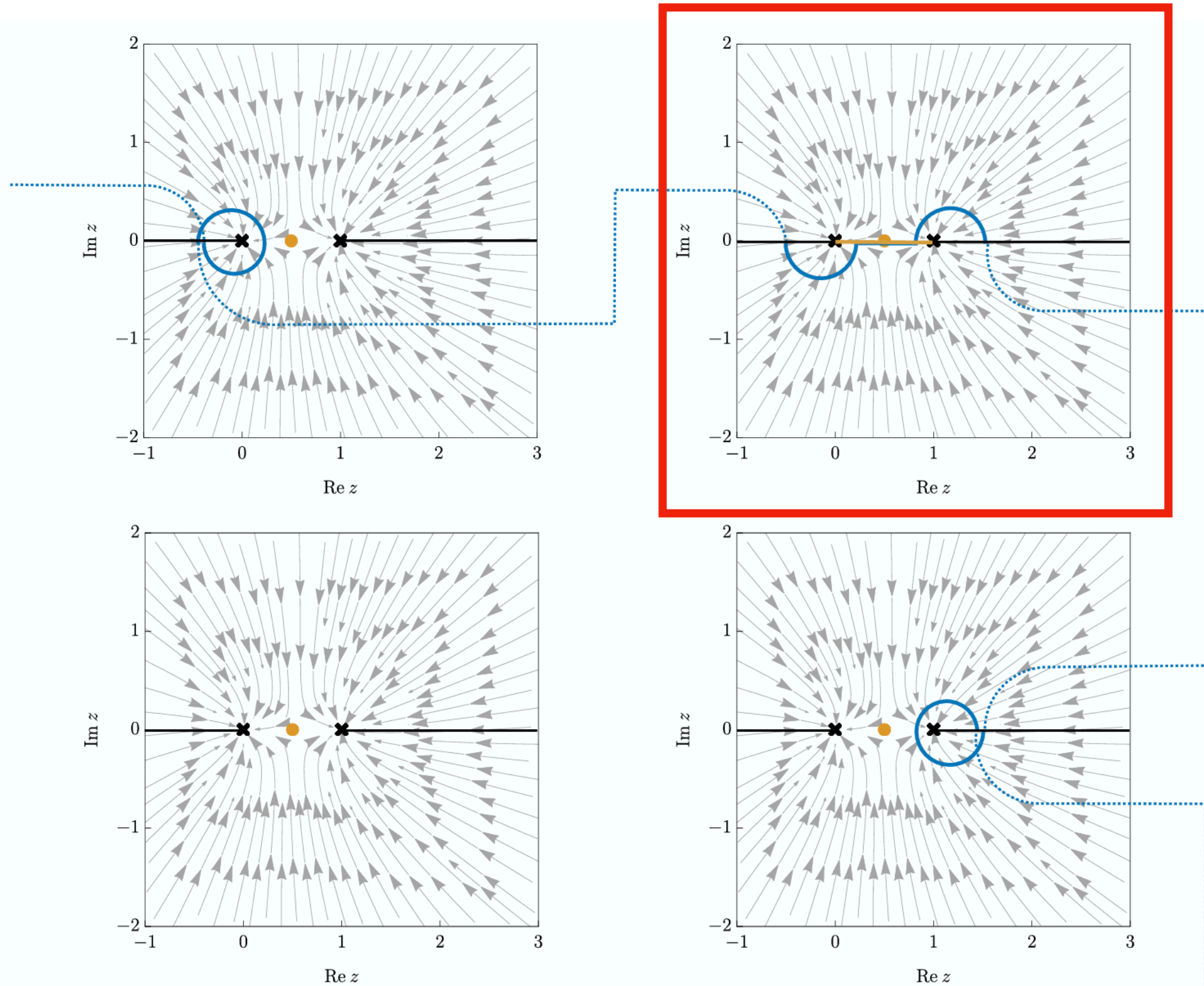
Thimble (steepest-descent) structure

Unphysical region $\alpha(s), \alpha(t), \alpha(s) + \alpha(t) \ll -1$



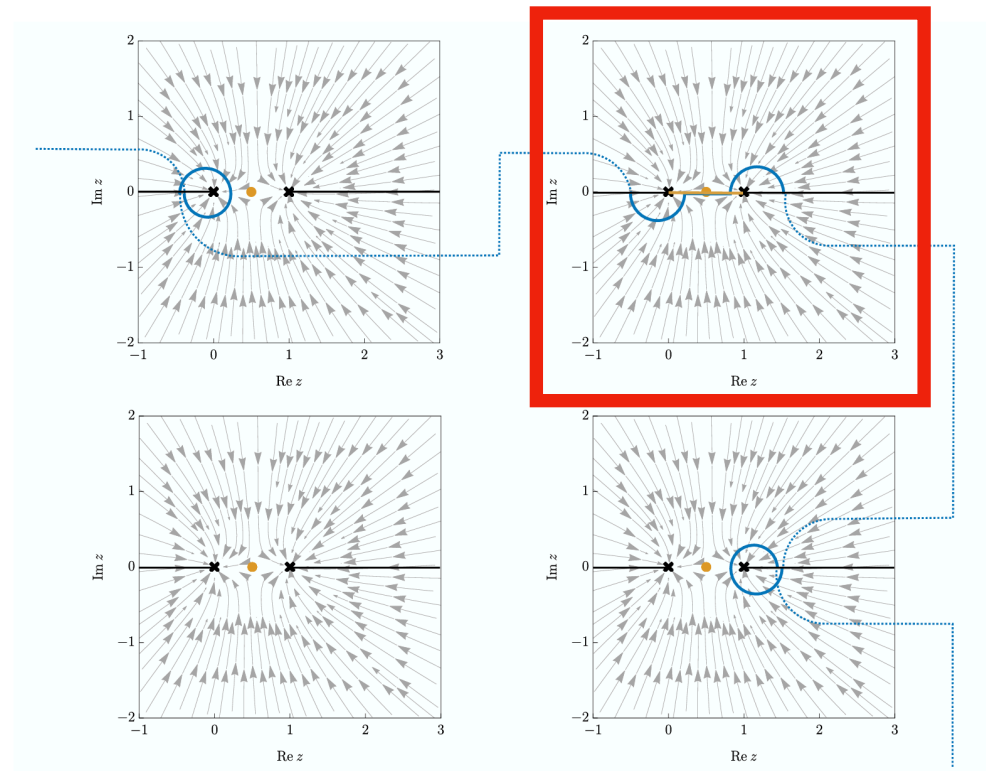
Thimble (steepest-descent) structure

Unphysical region $\alpha(s), \alpha(t), \alpha(s) + \alpha(t) \ll -1$



Thimble (steepest-descent) structure

Unphysical region $\alpha(s), \alpha(t), \alpha(s) + \alpha(t) \ll -1$



$$\mathcal{A}_{st}^\epsilon = \int_{C(\epsilon)} dz e^{S(z)}$$

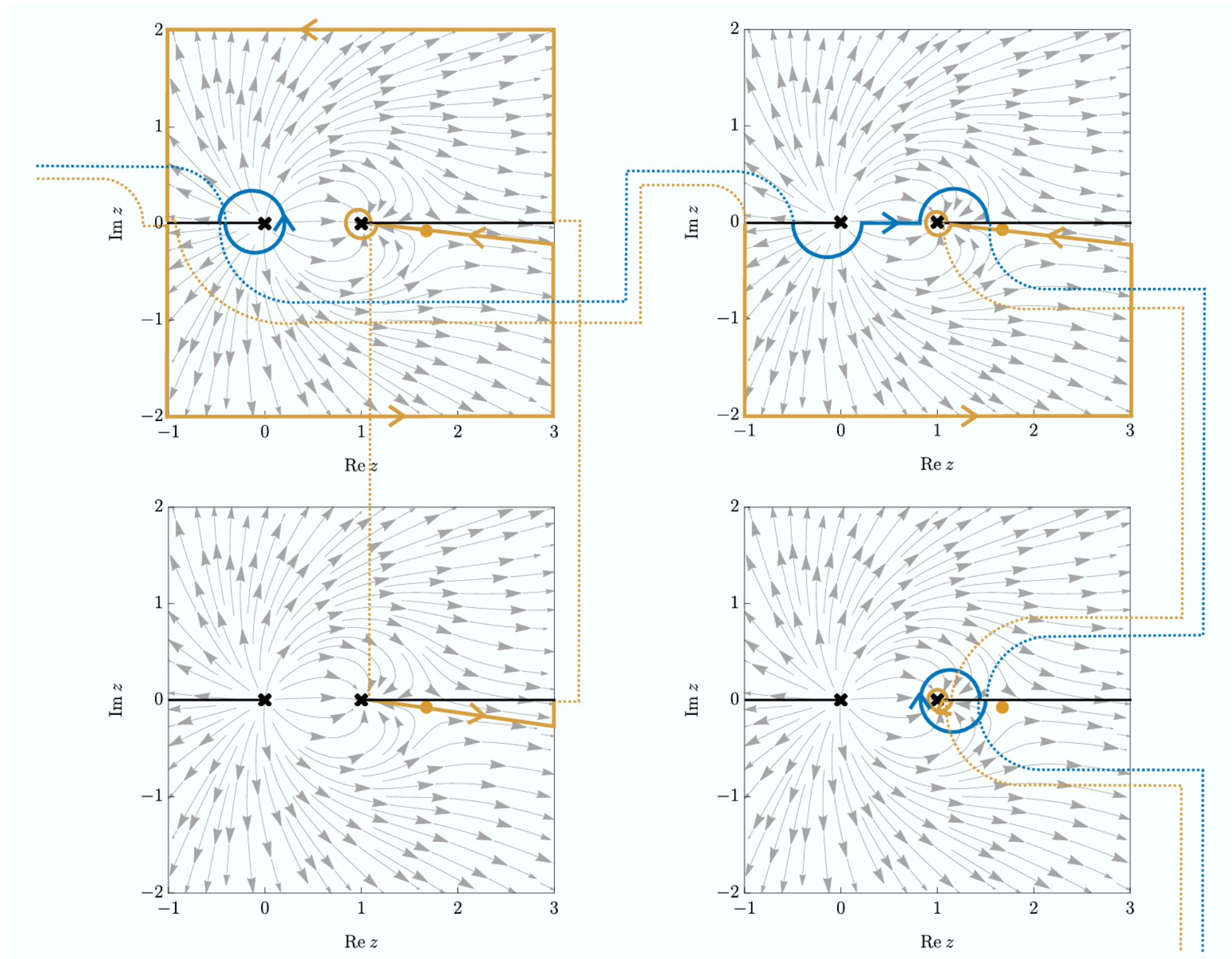
$$\sim \sqrt{\frac{2\pi}{-S''(z_{0,0})}} e^{S(z_{0,0})}$$

$$\sim (stu)^{-3/2} \exp \left[-\frac{1}{2} (s \ln s + t \ln t + u \ln u) \right]$$

correct result

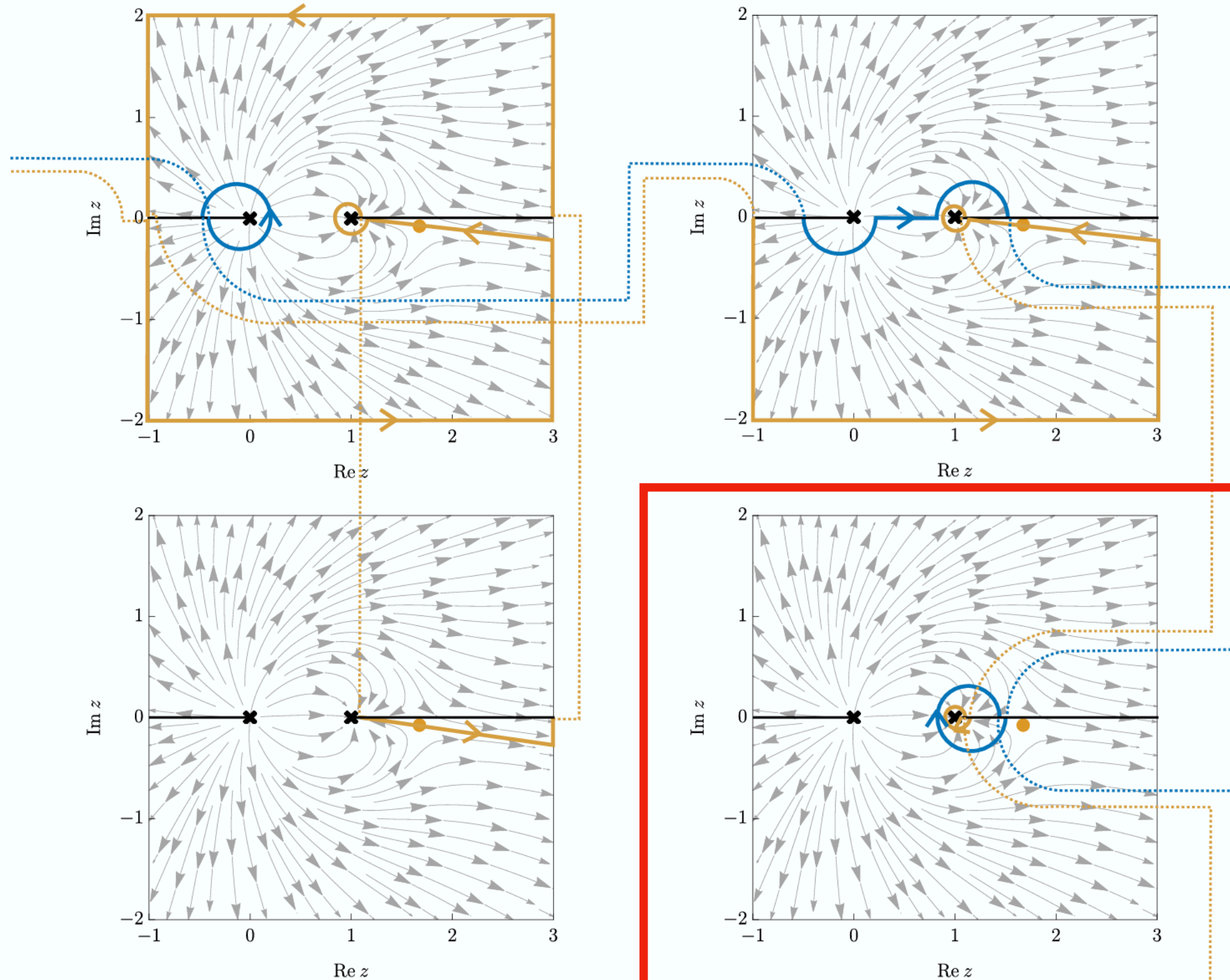
Thimble (steepest-descent) structure

Physical region $\alpha(s), \alpha(s) + \alpha(t) \gg 1, \alpha(t) \ll -1$



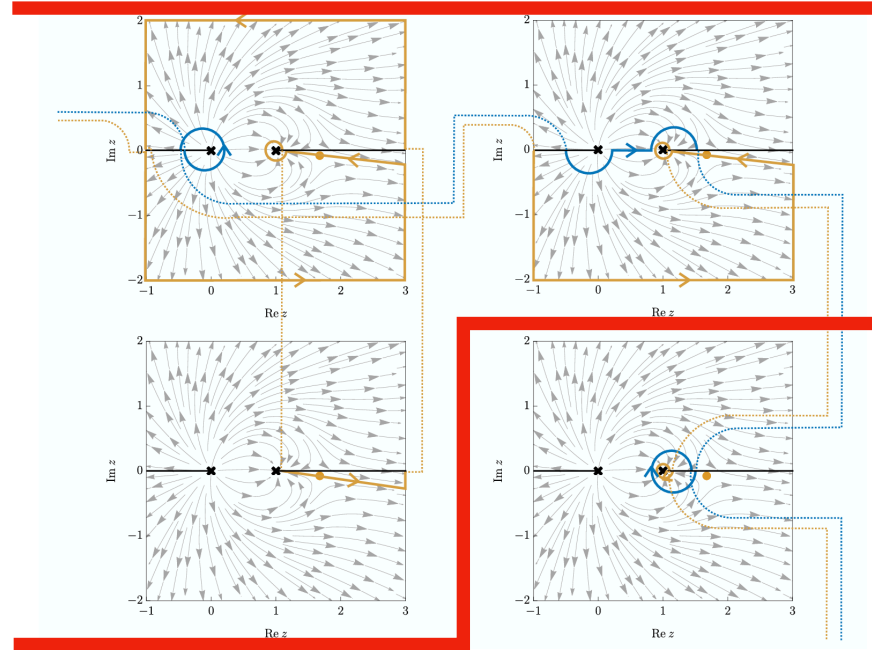
Thimble (steepest-descent) structure

Physical region $\alpha(s), \alpha(s) + \alpha(t) \gg 1, \alpha(t) \ll -1$



Thimble (steepest-descent) structure

Physical region $\alpha(s), \alpha(s) + \alpha(t) \gg 1, \alpha(t) \ll -1$



$$\begin{aligned}
 \mathcal{A}_{st}^\epsilon &= \int_{C(\epsilon)} dz e^{S(z)} \\
 &\sim \sqrt{\frac{2\pi}{-S''(z_{0,0})}} e^{S(z_{0,0})} + \sum_{n \leq -1} \sqrt{\frac{2\pi}{-S''(z_{n,0})}} e^{S(z_{n,0})} - \sum_{n \leq -1} \sqrt{\frac{2\pi}{-S''(z_{n,-1})}} e^{S(z_{n,-1})} \\
 &= \left[1 + \sum_{n \leq -1} e^{-2\pi i n (\alpha(s) + 1 + i\epsilon)} - \sum_{n \leq -1} e^{-2\pi i n (\alpha(s) + 1 + i\epsilon) - 2\pi i (-1) (\alpha(t) + 1 + i\epsilon)} \right] \sqrt{\frac{2\pi}{-S''(z_{0,0})}} e^{S(z_{0,0})} \\
 &\sim \frac{\sin \pi (\alpha(s) + \alpha(t))}{\sin \pi \alpha(s)} (stu)^{-3/2} \exp \left[-\frac{1}{2} (s \ln s + t \ln |t| + u \ln |u|) \right] \quad \text{correct poles \& zeros}
 \end{aligned}$$

Physical interpretation - Semi-classical string path

Semi-classical path of string

[TY, 24]

$$X(z) = \sum_j \underbrace{(-ip_j) \ln |z - z_j|}_{\text{Euclidean time}}$$

Each complex saddle satisfies

$$X(z_{n,m}) - X(z_{0,0}) = \underbrace{p_2(2\pi n) + p_3(2\pi m)}_{\text{Lorentzian time}}$$

and adds an extra phase factor to the amplitude,
reproducing poles & zeros

$$\mathcal{A}_{st}^\epsilon \sim \left[1 + \sum_{n \leq -1} e^{-2\pi i n(\alpha(s)+1+i\epsilon)} - \sum_{n \leq -1} e^{-2\pi i n(\alpha(s)+1+i\epsilon) - 2\pi i(-1)(\alpha(t)+1+i\epsilon)} \right] \sqrt{\frac{2\pi}{-S''(z_{0,0})}} e^{S(z_{0,0})}$$

Thus **complex saddles are associated with optical paths**,
whose interference reproduces poles & zeros of the amplitude

Physical interpretation - Stringy mass tower

Poles are associated with stringy mass tower

[TY, 24]

$$\mathcal{A}_{st} = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}, \quad \alpha(x) = x/2 + 1$$

$\alpha(s) \in \mathbb{Z}_{\geq}$

Such poles were reproduced by infinitely many complex saddles

$$\begin{aligned} \mathcal{A}_{st}^\epsilon &= \int_{C(\epsilon)} dz e^{S(z)} \\ &\sim \sqrt{\frac{2\pi}{-S''(z_{0,0})}} e^{S(z_{0,0})} + \sum_{n \leq -1} \sqrt{\frac{2\pi}{-S''(z_{n,0})}} e^{S(z_{n,0})} - \sum_{n \leq -1} \sqrt{\frac{2\pi}{-S''(z_{n,-1})}} e^{S(z_{n,-1})} \\ &= \left[1 + \sum_{n \leq -1} e^{-2\pi i n(\alpha(s)+1+i\epsilon)} - \sum_{n \leq -1} e^{-2\pi i n(\alpha(s)+1+i\epsilon) - 2\pi i (-1)(\alpha(t)+1+i\epsilon)} \right] \sqrt{\frac{2\pi}{-S''(z_{0,0})}} e^{S(z_{0,0})} \\ &\sim \frac{\sin \pi(\alpha(s) + \alpha(t))}{\sin \pi \alpha(s)} (stu)^{-3/2} \exp \left[-\frac{1}{2} (s \ln s + t \ln |t| + u \ln |u|) \right] \end{aligned}$$

Thus complex saddles capture stringy mass tower

Physical interpretation - Partial wave unitarity

Residue is expanded in the Gegenbauer polynomials

[Arkani-Hamed, Eberhardt, Huang, Mizera, 22]

[Maity, 22]

$$-\text{Res}_n = \sum_{j=0}^n a_{n,j} C_j^\alpha(x), \quad \alpha = \frac{D-3}{2}$$

Partial wave unitarity demands that the coeffs are all positive

$$\begin{aligned} D=26 \quad -\text{Res}_0 &= 2 &= 2C_0^\alpha(x) \\ -\text{Res}_1 &= 4x &= \frac{4}{23}C_1^\alpha(x) \\ -\text{Res}_2 &= -\frac{1}{4} + \frac{25}{4}x^2 &= \frac{1}{46}C_2^\alpha(x) \\ -\text{Res}_3 &= -x + 9x^3 &= \frac{2}{575}C_3^\alpha(x) \\ -\text{Res}_4 &= \frac{3}{64} - \frac{245}{96}x^2 + \frac{2401}{192}x^4 &= \frac{2401}{3601800}C_4^\alpha(x) + \frac{49}{400200}C_2^\alpha(x) + \frac{1}{2700}C_0^\alpha(x) \end{aligned}$$

Complex saddles reproduce appropriate poles, providing approx value of the coeffs

[TY, 24]

$$\begin{aligned} -\text{Res}_2 &\sim -0.250 + 6.71x^2 + \mathcal{O}(x^4) \\ -\text{Res}_3 &\sim -0.900x + 9.27x^3 + \mathcal{O}(x^6) \\ -\text{Res}_4 &\sim +0.0428 - 2.37x^2 + 13.7x^4 + \mathcal{O}(x^8) \end{aligned}$$

Thus **complex saddles are necessary for partial wave unitarity**

Complex saddles & possible chaos in string scatterings

1. Complex saddles in string(y) amplitudes

1.1. Veneziano amplitude (tree) (9)

1.2. Stringy toy model (1-loop, multi-loop) (6)

1.3. Higher-genus corrections (2)

2. Application: finding chaos in string scatterings

2.1. BH chaos & BH-string correspondence (3)

2.2. Chaos in classical scatterings (2)

2.3. Possible chaos in string scatterings (2)

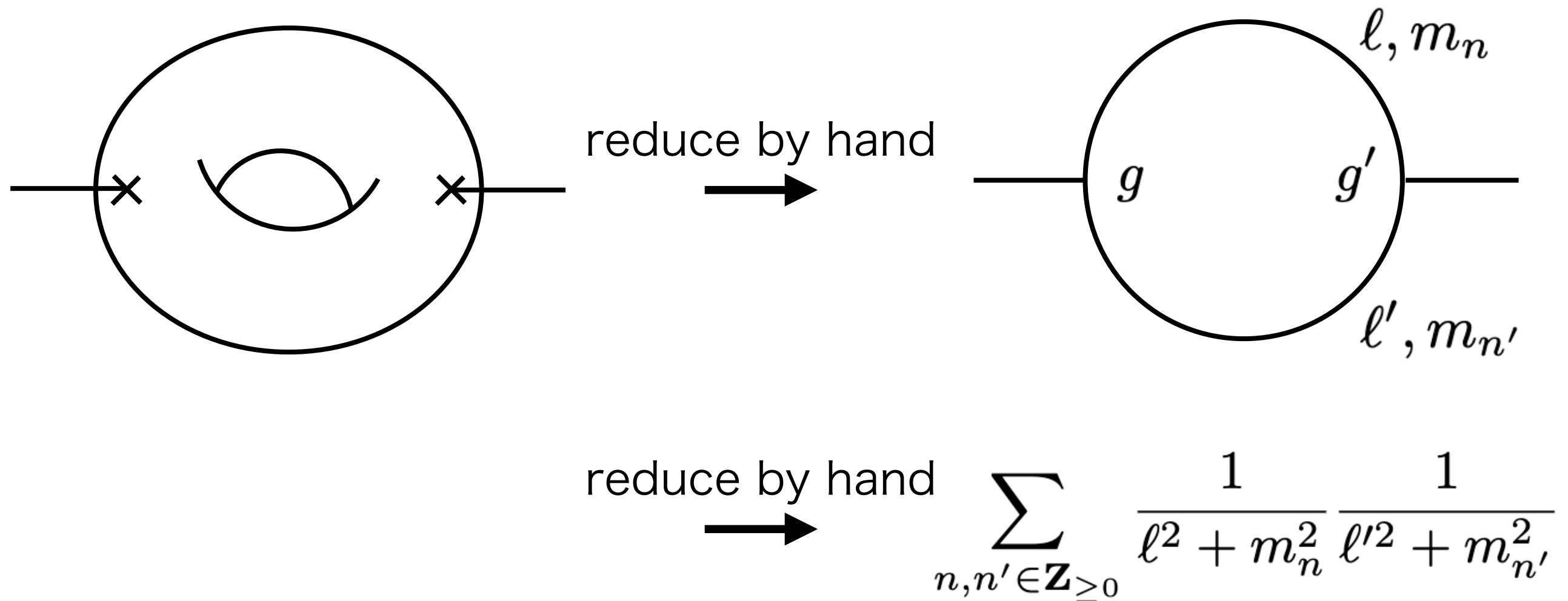
A toy model to mimic stringy mass tower

We want to apply cpx saddle analysis to higher genus corrections

As the first step, let us test how cpx saddles detect characteristic properties of string amplitude

Stringy toy model:

[TY, in preparation]



A toy model to mimic stringy mass tower

Particularly when $m_n = m_{n'}$

It is an integral of the Hurwitz zeta function

$$\begin{aligned}\mathcal{A}(p^2) &= \sum_{m^2 \in \mathbf{Z}_{\geq 0}} \text{---} \bigcirc \text{---} \\ &= \sum_{m^2 \in \mathbf{Z}_{\geq 0}} \int_0^1 dx_0 dx_1 \delta(x_0 + x_1 - 1) \frac{1}{[x_0 x_1 p^2 + m^2]^{-(d/2-2)}} \\ &= \int_0^1 dx_0 dx_1 \delta(x_0 + x_1 - 1) \zeta_{\text{H}}(-(d/2-2), x_0 x_1 p^2)\end{aligned}$$

Oscillatory behavior & stringy mass tower

It is known that the Hurwitz zeta is decomposed into an oscillatory part and a polynomial part

$$\zeta(-k, x) = k! \sum_{n \neq 0} \left(\frac{1}{2\pi i n} \right)^{k+1} e^{2\pi i n x} - \sum_{n=0}^{m-1} \frac{1}{(n+x-m)^{-k}} \quad x-1 \leq m < x.$$

The oscillatory part stems from stringy mass tower m_n

$$\mathcal{A}(p^2) = \sum_{m^2 \in \mathbf{Z}_{\geq 0}} \int_0^1 dx_0 dx_1 \delta(x_0 + x_1 - 1) \frac{1}{[x_0 x_1 p^2 + m^2]^{-(d/2-2)}}$$

└──┘
momentum shift can be absorbed into the sum

Next we show the oscillatory part is captured by complex saddles

Regularization of an integral representation

Schwinger parameter representation

$$\zeta(-k, x) = \frac{1}{\Gamma(-k - i\delta)} \int_{C_\delta} dt \, t^{-k-i\delta-1} \frac{e^{-xt}}{1 - e^{-t}}$$

The divergence at the origin is cured by the Feynman prescription

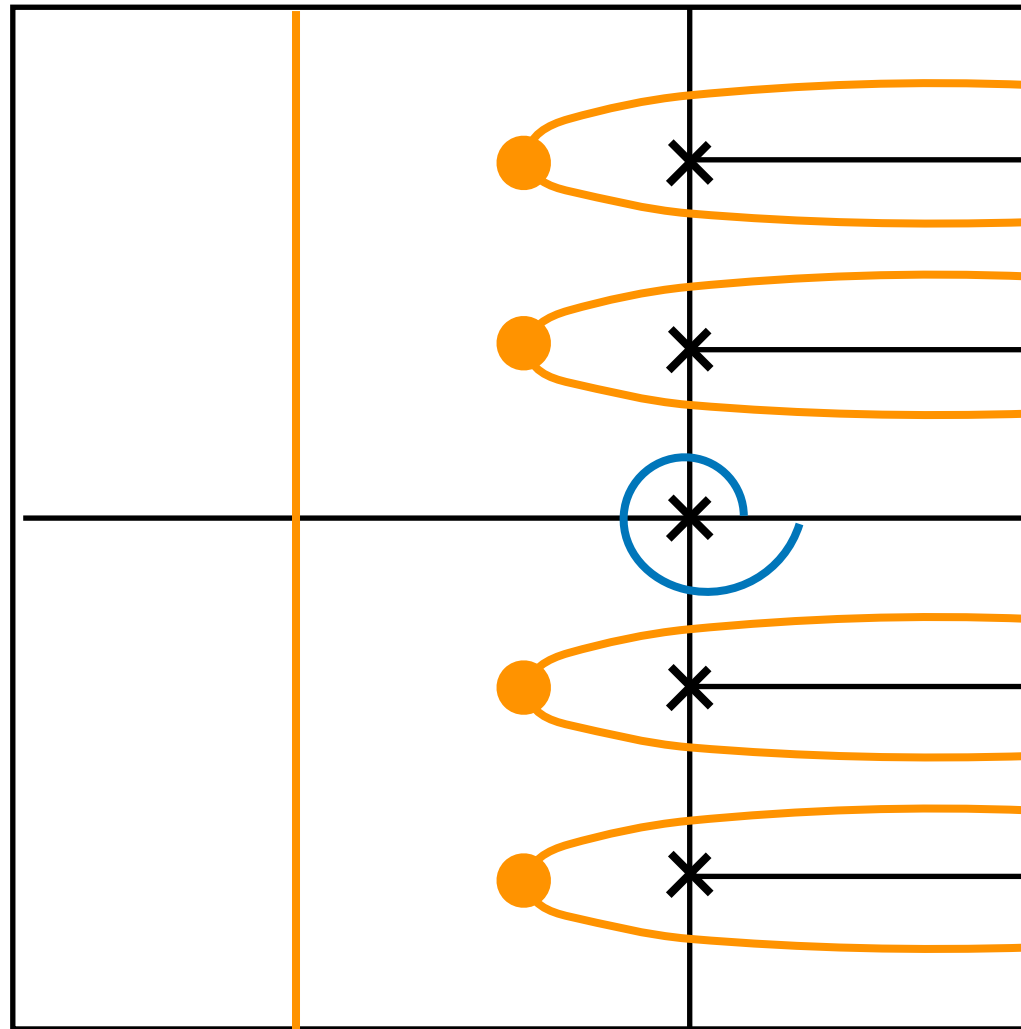
Complex saddles

$$t_n \simeq 2\pi i n - \frac{1}{x + \frac{k}{2\pi i n}}$$

Next we identify which saddles contribute to the integral

Thimble (steepest-descent) structure

When $0 < x < 1$



× infinitely many sheets

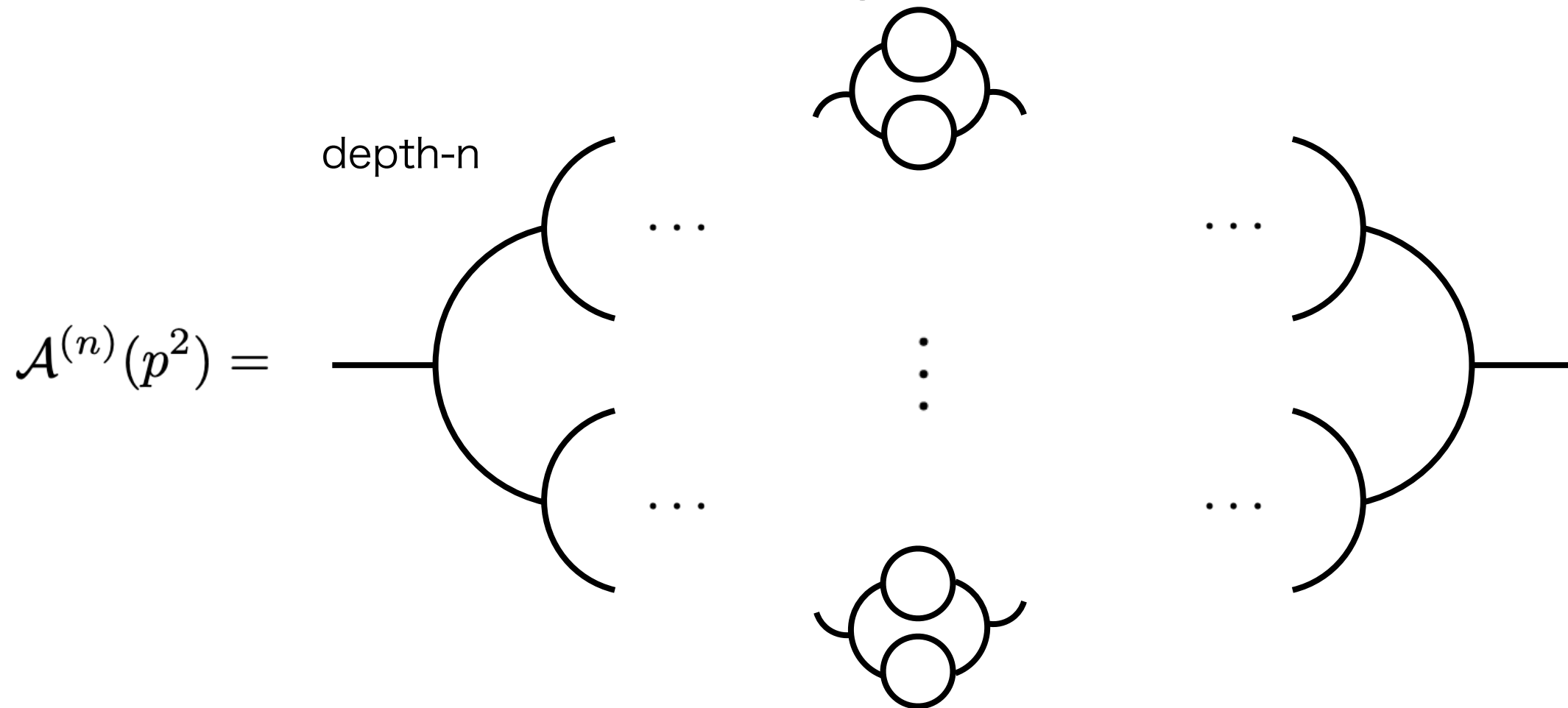
$$\zeta(-k, x) \sim \frac{1}{\Gamma(-k)} \sum_{n \neq 0} \sum_{m=-\infty}^0 (-1)^{\sigma_n} e^{-2\pi i(k+1)m} \sqrt{\frac{2\pi}{-S''(t_n)}} e^{S(t_n)}$$

$$\sim k! \sum_{n \neq 0} \left(\frac{1}{2\pi i n} \right)^{k+1} e^{2\pi i n x}$$

Complex saddles capture oscillatory behaviors associated with stringy mass tower

Multi-loop cases

Consider a nested multi-loop diagram



Symmetry of the diagram allows us to find complex saddles

$$\sim \Gamma(-k) k! \sum_{m \notin 4\mathbb{Z}} \sum_{i=1}^{n+1} c_{4^{n-i+1}m}(\xi, k) \left(\frac{1/4^{n-i+1}}{2\pi i m} \right)^{k+1} e^{-2\pi i m \cdot 4^{n-i+1} \xi} + \dots$$

$$\xi = \frac{3 - 2^{-n+1}}{2^{n+1}} \alpha' p^2 \quad k = (2^{n+1} - 1) \left(\frac{d}{2} - \frac{3 \cdot 2^{n+1} - 4}{2^{n+1} - 1} \right)$$

Complex saddles capture oscillatory behaviors

Complex saddles & possible chaos in string scatterings

1. Complex saddles in string(y) amplitudes

1.1. Veneziano amplitude (tree) (9)

1.2. Stringy toy model (1-loop, multi-loop) (6)

1.3. Higher-genus corrections (2)

2. Application: finding chaos in string scatterings

2.1. BH chaos & BH-string correspondence (3)

2.2. Chaos in classical scatterings (2)

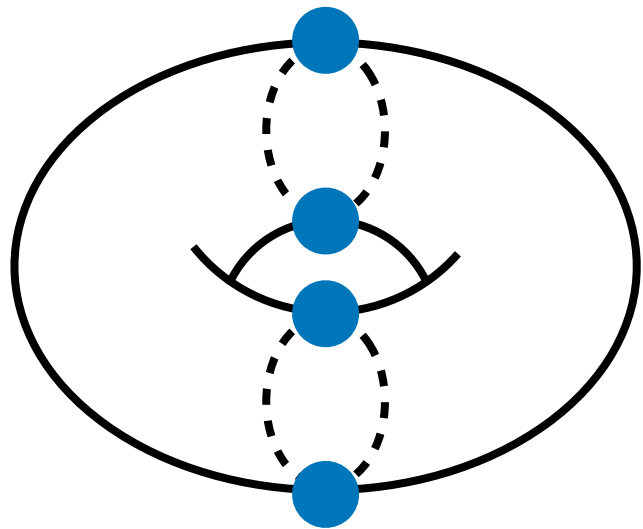
2.3. Possible chaos in string scatterings (2)

Extend complex saddle analysis to higher genus corrections

Original saddle point approx used an analogy to electromagnetic equilibrium

For example, in genus-one case

[Gross, Mende, 88]



$$= g^4 \int d\tau \prod_{i=1}^{4-1} d^2 \nu_i \Omega(\tau) \exp \left[\frac{1}{2} \sum_{i < j} p_i \cdot p_j \ln \chi_{ij} \right]$$

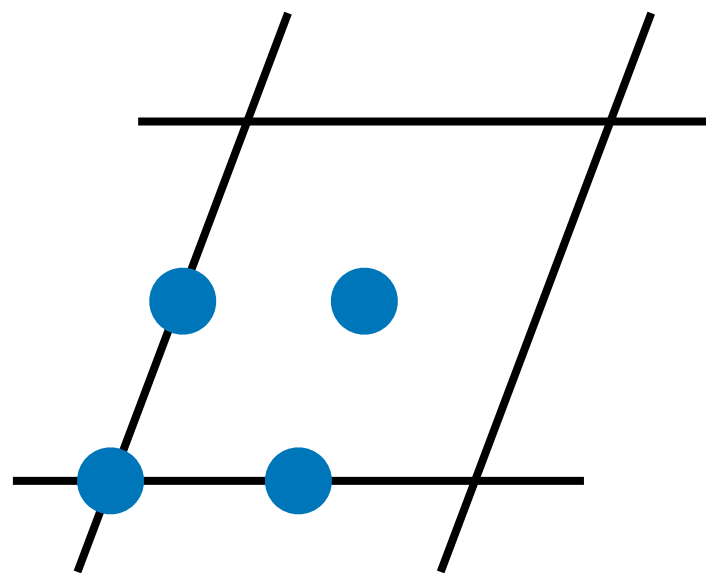
$$\ln \chi_{ij} = \frac{-\pi \operatorname{Im}^2(\nu_i - \nu_j)}{\operatorname{Im} \tau} + \ln \left| \frac{\theta_1(\nu_i - \nu_j | \tau)}{\theta_1'(0 | \tau)} \right|$$

green function on torus

Saddle = EM equilibrium

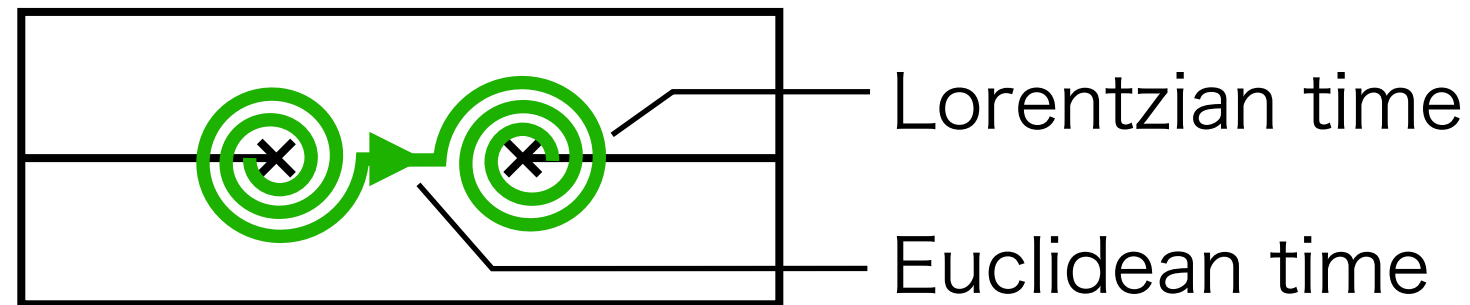
$$0 = \frac{d}{d\tau} \ln \chi_{ij}$$

$$\nu_1 = 1/2, \quad \nu_2 = \tau/2, \quad \nu_3 = (1 + \tau)/2, \quad \nu_4 = 0,$$

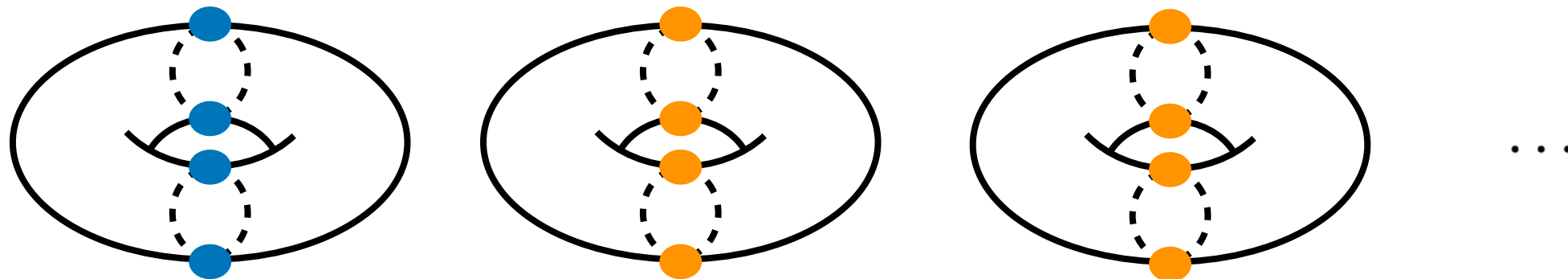


Extend complex saddle analysis to higher genus corrections

Using the Feynman- $i\epsilon$ prescription,
string lives on infinitely many copies of the worldsheet



Then we may find infinitely many cpx saddles on different sheets



Thimble (steepest-descent) structure depends on their topology
Thus we may arrive at more general formula for general #genus

$$\mathcal{A}_g(s, t, u) \sim \left(\sum_n e^{i\theta_n(s, t, u)} \right) \mathcal{A}_{\text{Gross-Mende}}(s, t, u)$$

contributing complex saddles may
be determined only by its topology

Complex saddles & possible chaos in string scatterings

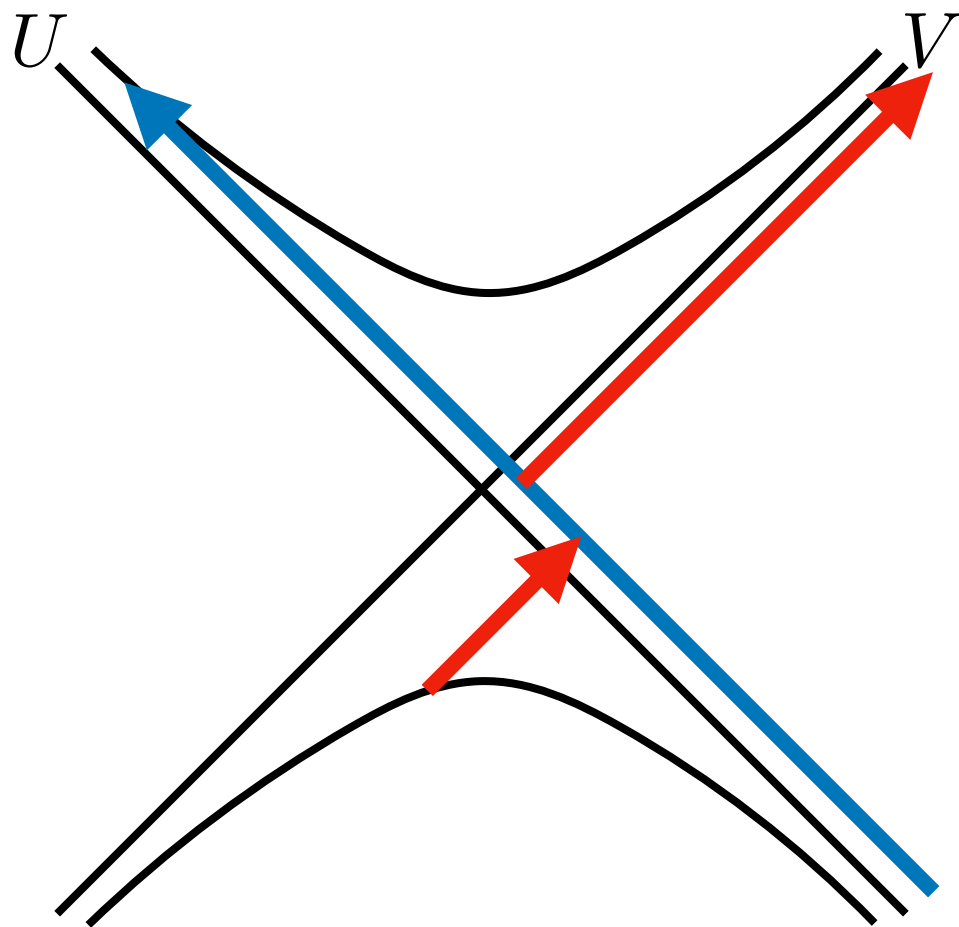
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- 2. Application: finding chaos in string scatterings**
 - 2.1. BH chaos & BH-string correspondence** (3)
 - 2.2. Chaos in classical scatterings (2)
 - 2.3. Possible chaos in string scatterings (2)

Black hole chaos

“Scatterings” by a black hole are sensitive to initial conditions due to the redshift near horizon

[Polchinski 15; See also Polchinski 16; Dray, 't Hooft 84; 't Hooft 96]

[Shenker, Stanford; 14, 15. Kitaev (KITP seminar); 14, 15]



$$\Delta t = \kappa^{-1} e^{\kappa(t-r^*)} \Delta U$$

exp growth

variation of initial condition

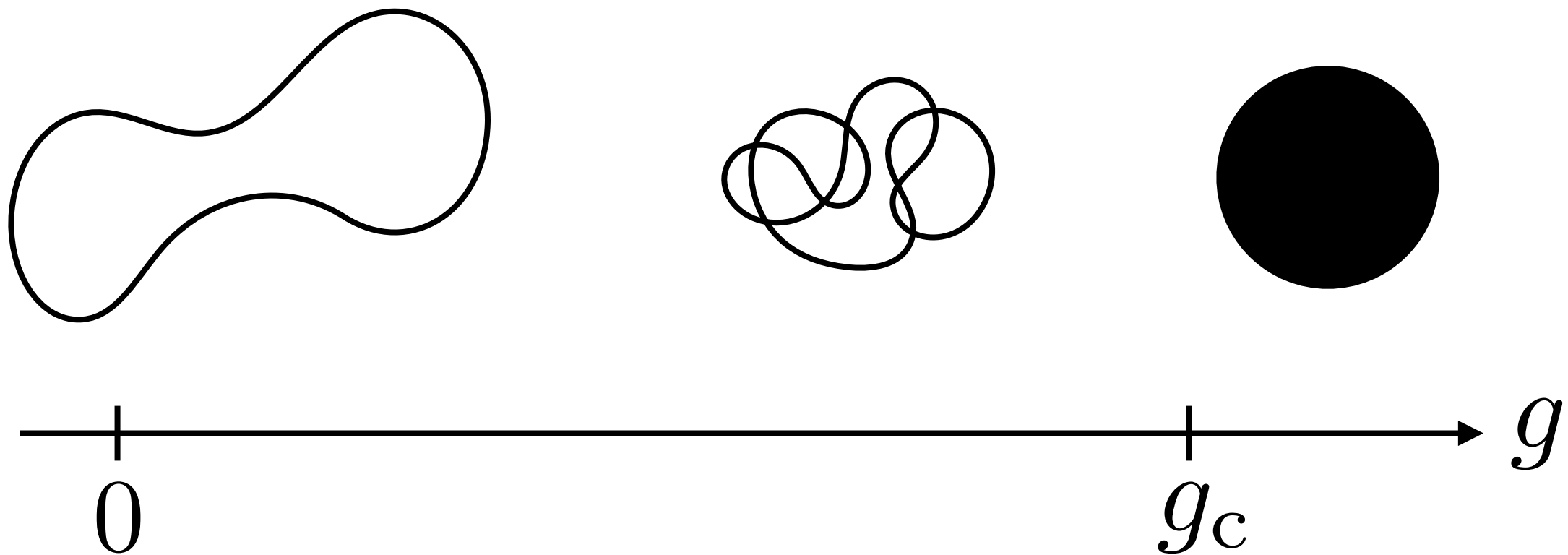
Such sensitivity is mapped to quantum chaos on holographic side
Surface gravity κ is interpreted as the Lyapunov exponent

BH-string correspondence

Self-gravitating string turns into a black hole

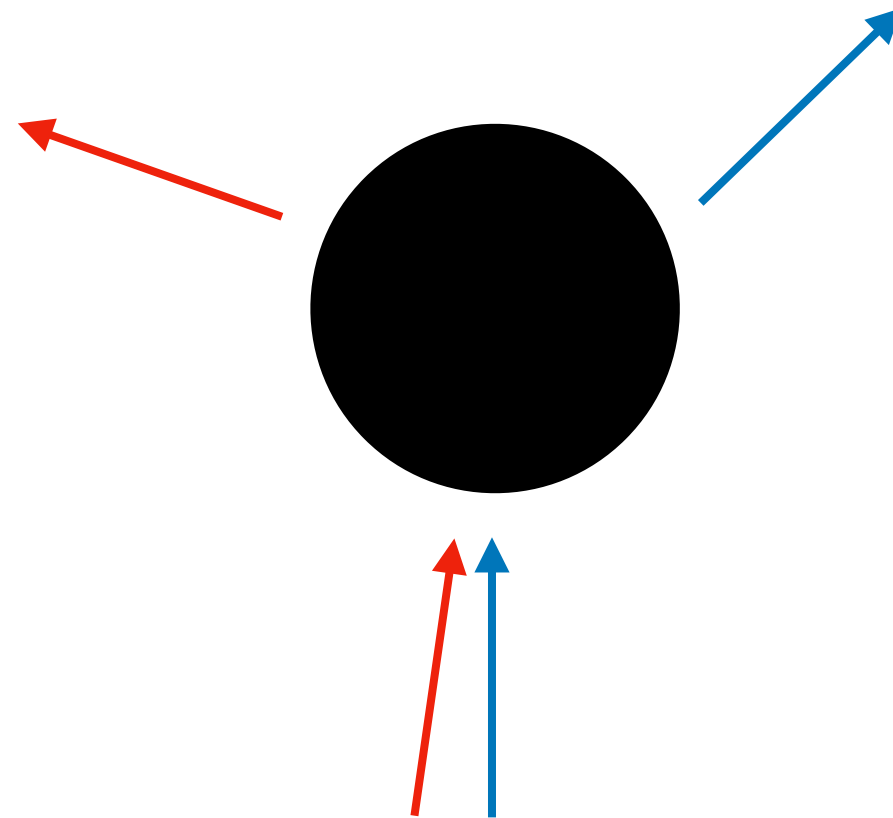
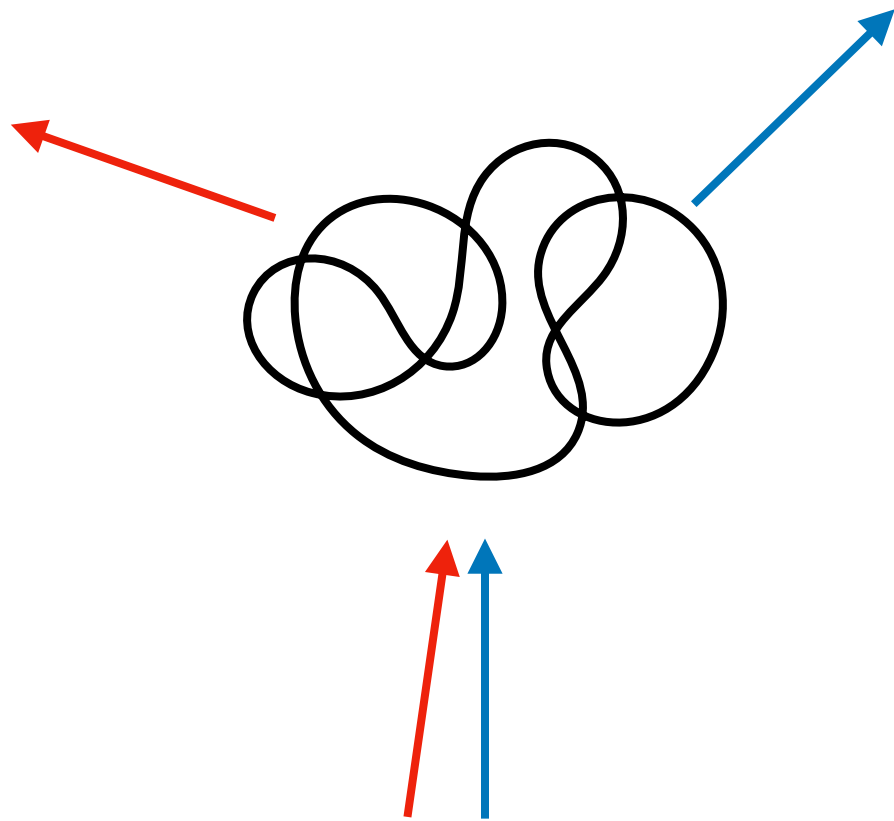
At the critical point, called the corresponding point, the entropy of a string is equal to the entropy of a black hole

[Horowitz, Polchinski, 97, 98; Amati, Russo, 99; Chen, Maldacena, Witten, 21]



Comparison of BH & string as scatterers

Both of black hole and string look like chaotic scatterers



- Initial conditions are forgotten due to complicated scattering processes
- It takes long time for a particle to escape from scattering region

- Initial conditions are forgotten due to Hawking radiation
- It takes long time for a particle to escape due to redshift

These stories motivate us to find chaos in string scatterings and to measure their Lyapunov exponents

Complex saddles & possible chaos in string scatterings

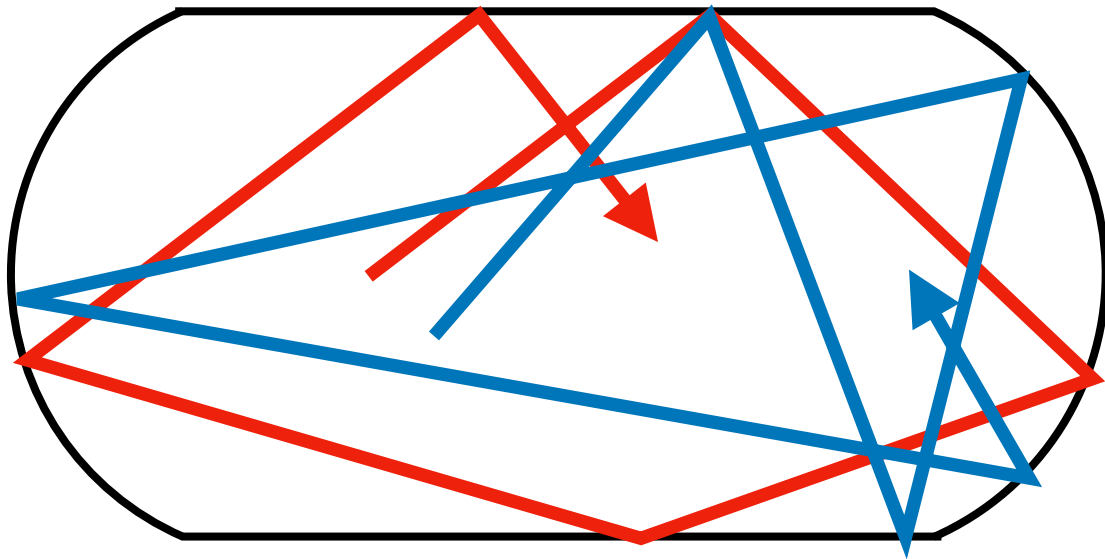
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Chaos in general

Let us make the concept of chaos more solid

[Textbooks e.g. Lai, Tél, 11; Seoane, Sanjuán, 12]

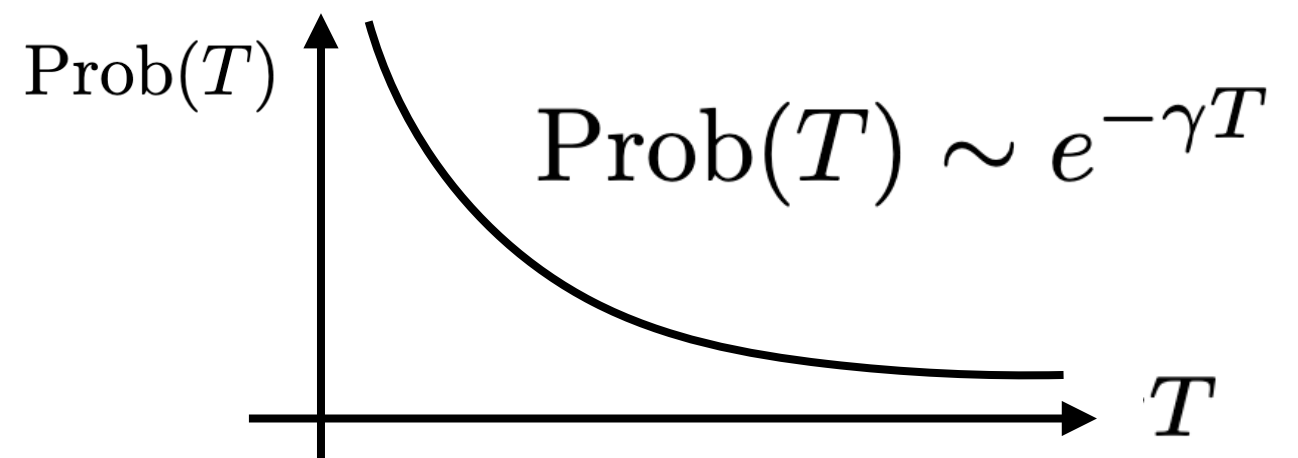
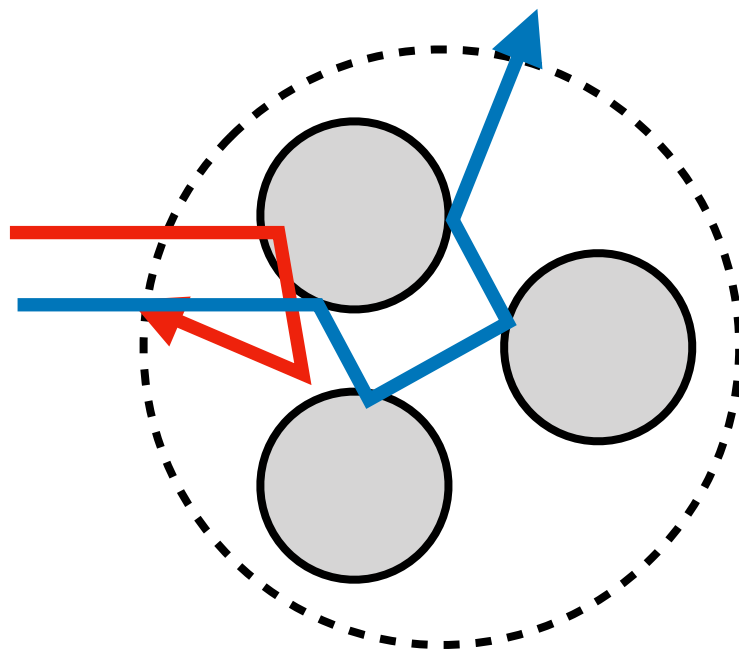
- Sensitive to initial condition



A diagram illustrating the exponential divergence of trajectories. A red arrow labeled $\delta q(0)$ and a blue arrow labeled $\delta q(t)$ originate from the same point. The blue arrow is longer and points in a different direction, representing the growth of the separation over time. The equation $\delta q(t) \sim e^{\lambda_L t} \delta q(0)$ is shown next to the arrows.

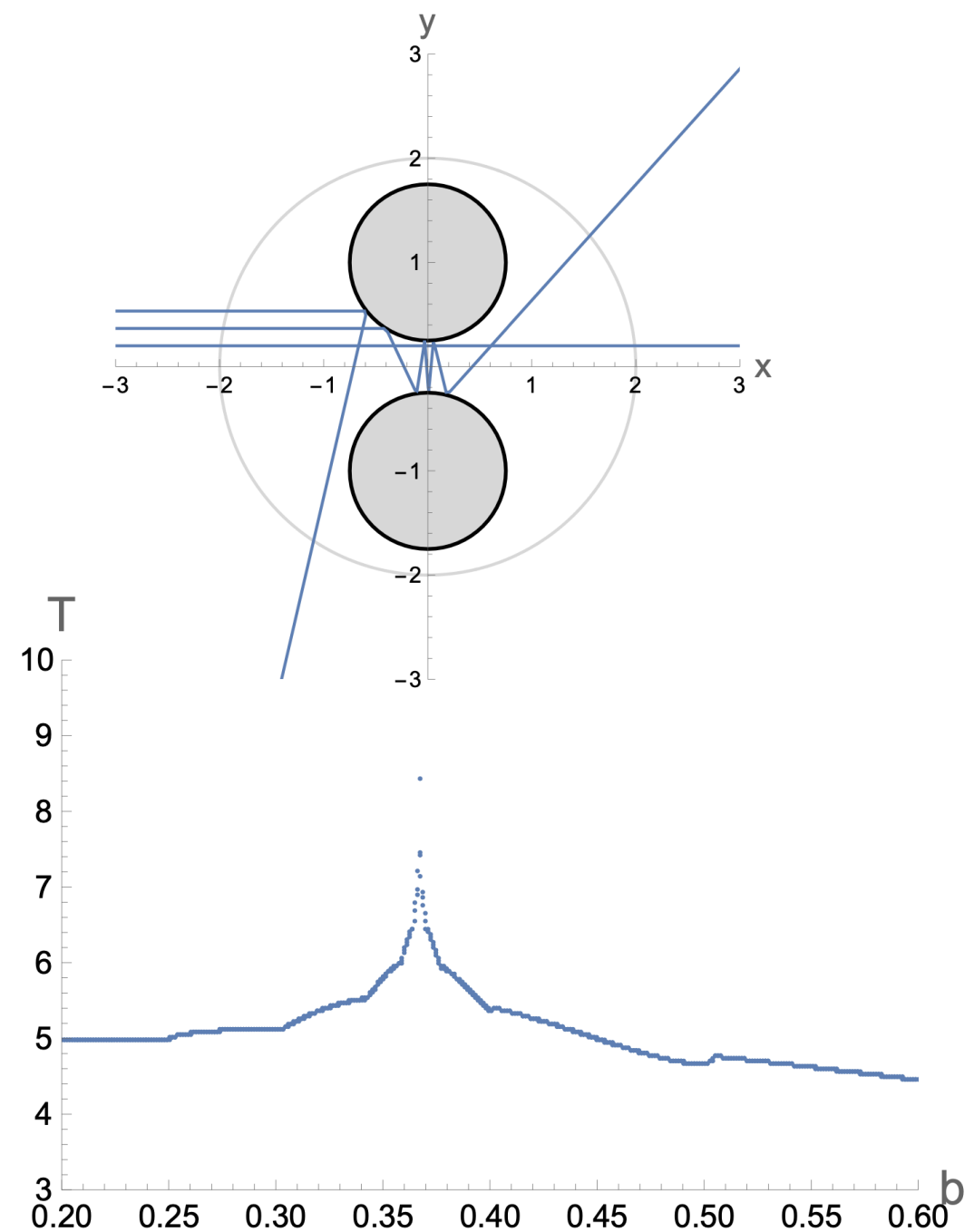
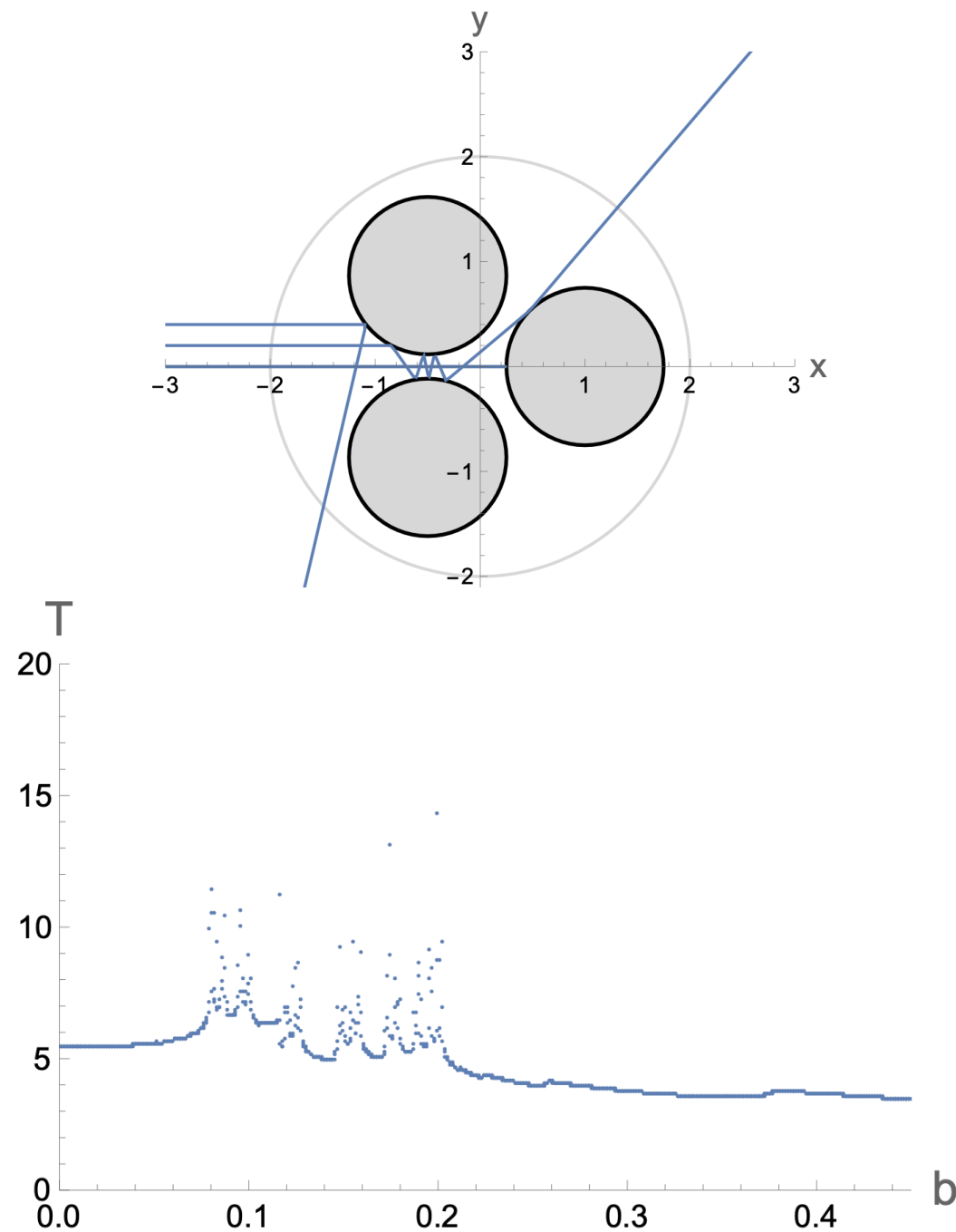
$$\delta q(t) \sim e^{\lambda_L t} \delta q(0)$$

- Long time-delay



Chaos in classical scatterings

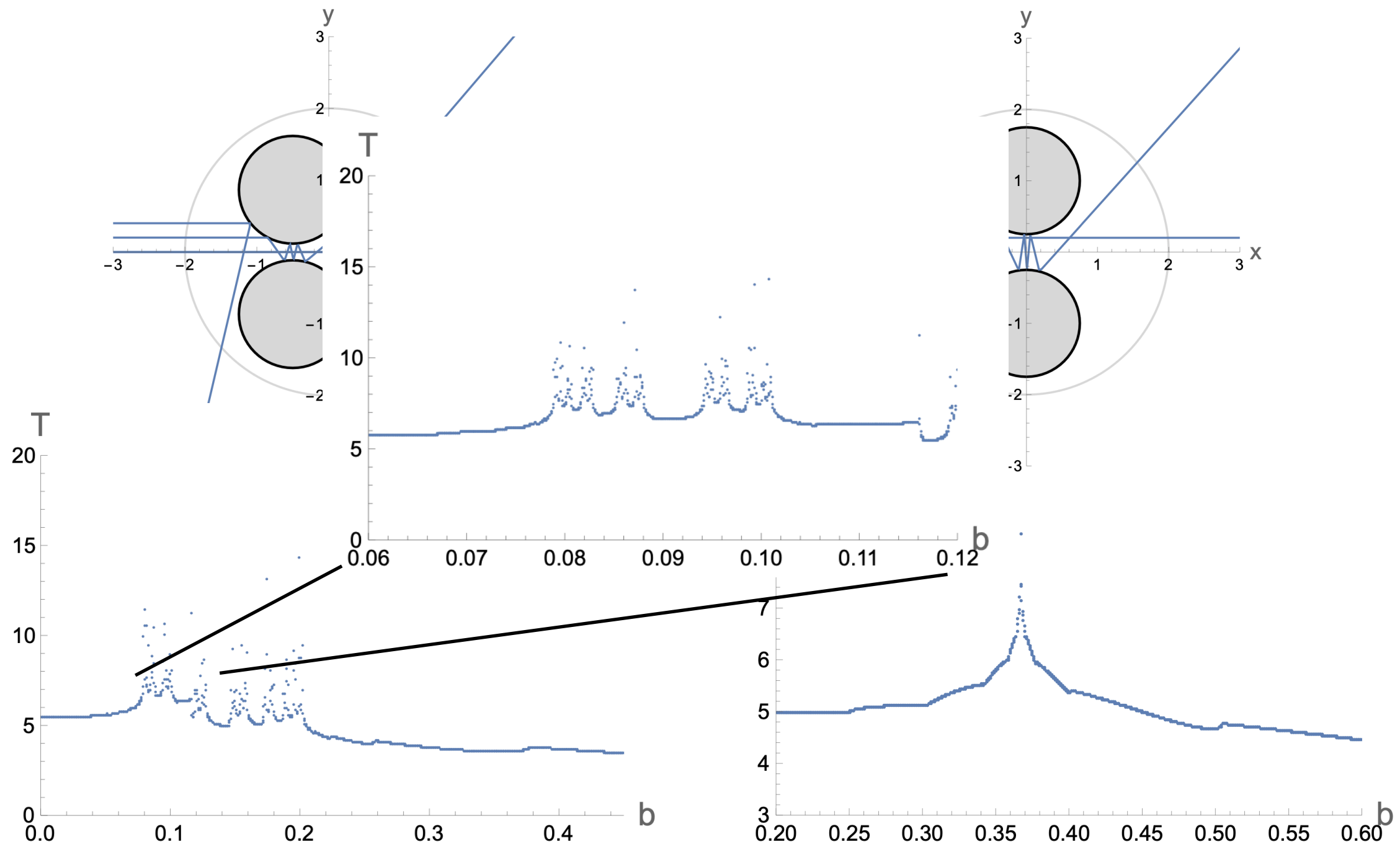
Chaos can be detected by fractal structure of scattering data



In fact, the Lyapunov exponent is determined by fractal dimension

Chaos in classical scatterings

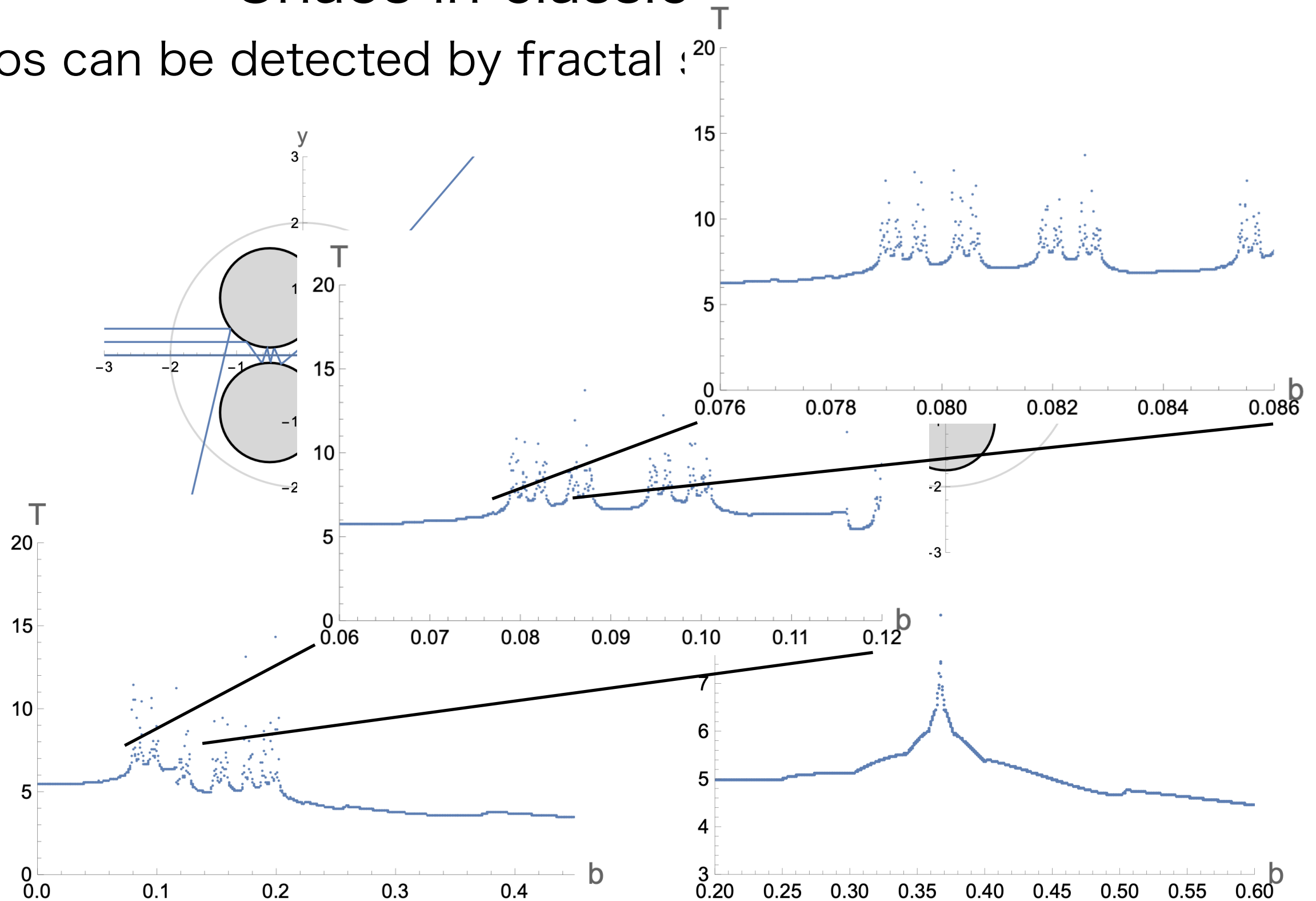
Chaos can be detected by fractal structure of scattering data



In fact, the Lyapunov exponent is determined by fractal dimension

Chaos in classical scatterings

Chaos can be detected by fractal



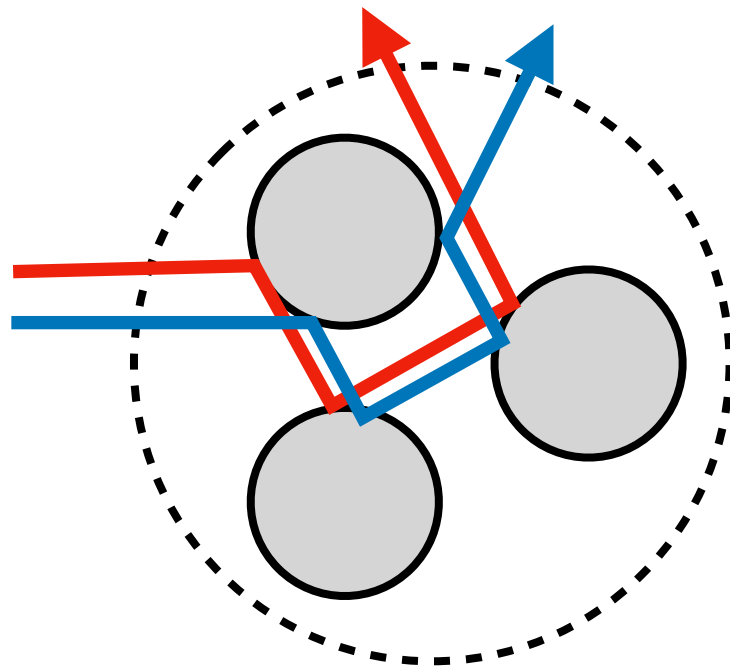
In fact, the Lyapunov exponent is determined by fractal dimension

Key mechanisms

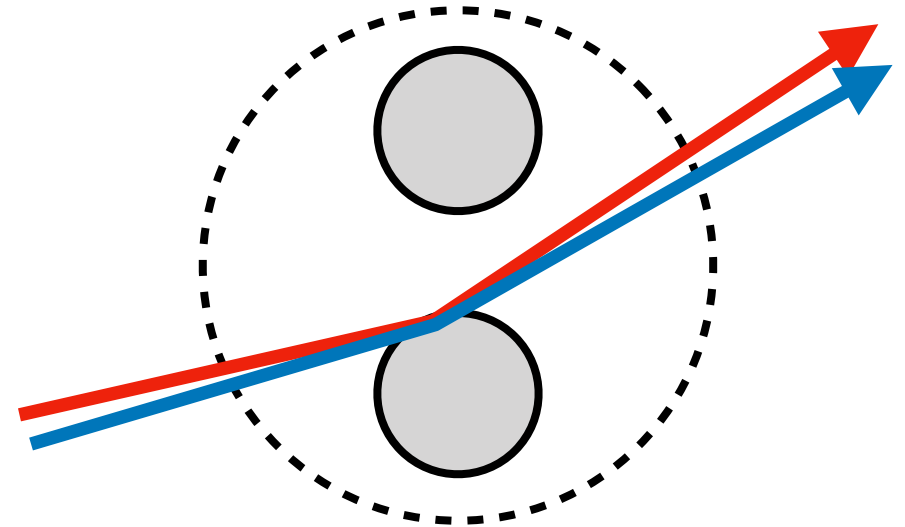
Chaotic scatterings are associated with three key mechanisms:

- **Stretching**
- **Folding**
- **Escape**

Chaotic



Not chaotic

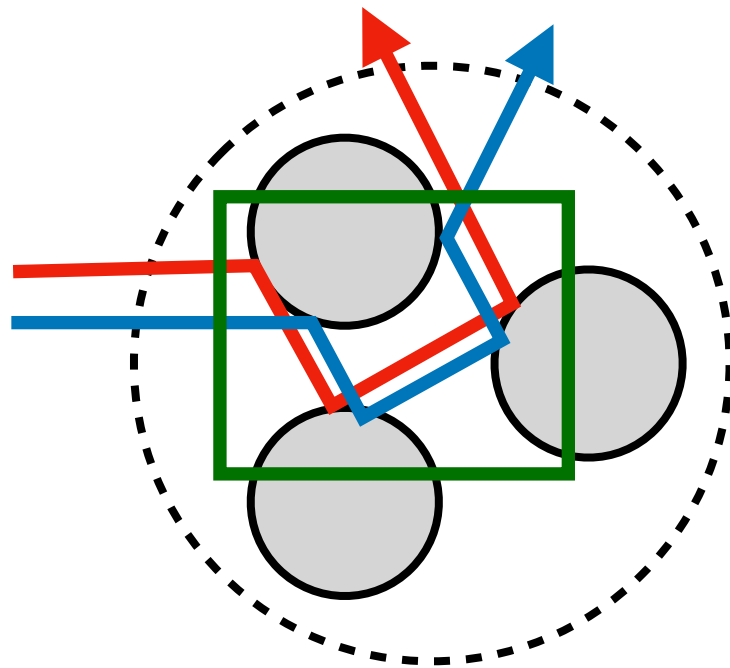


Key mechanisms

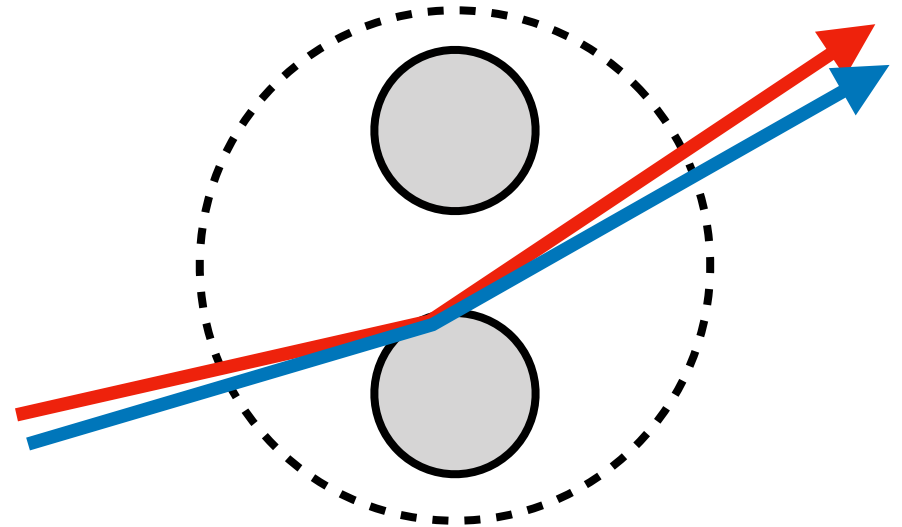
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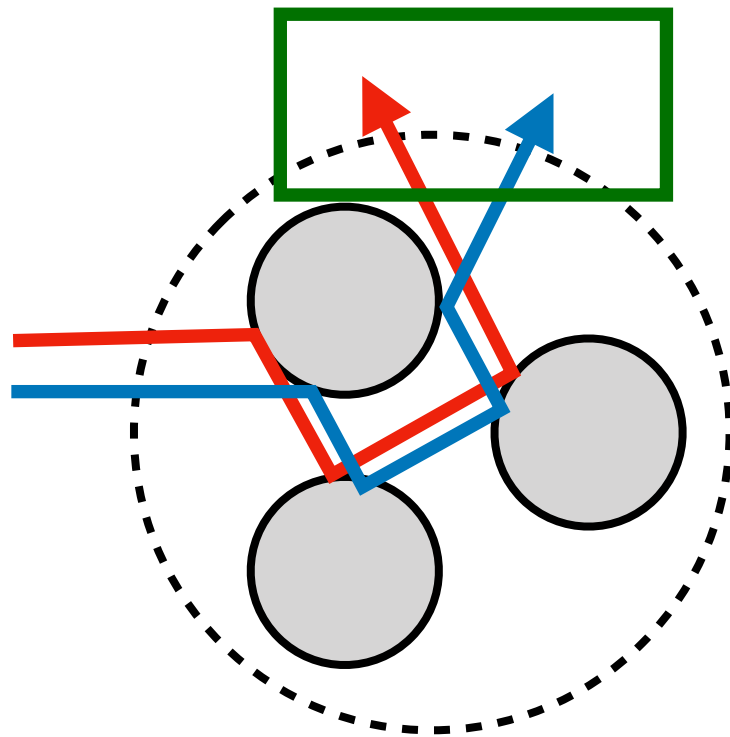


Key mechanisms

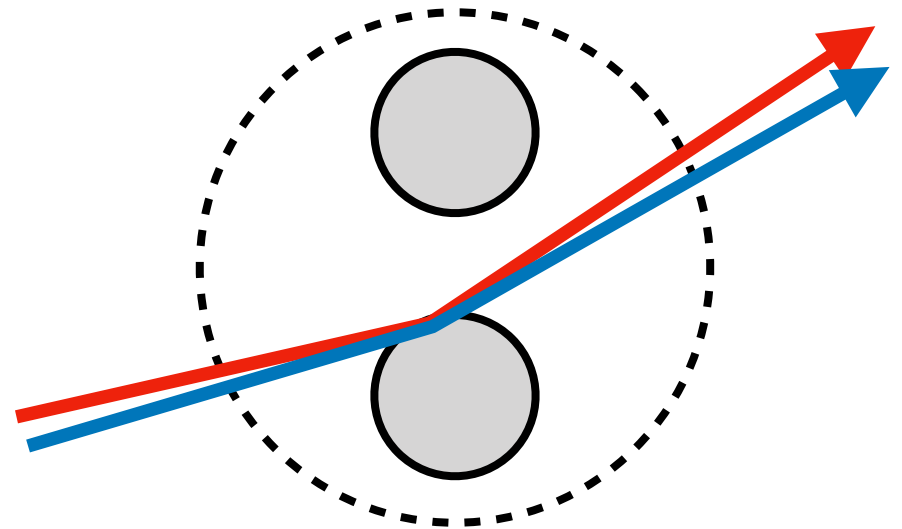
Chaotic scatterings are associated with three key mechanisms:

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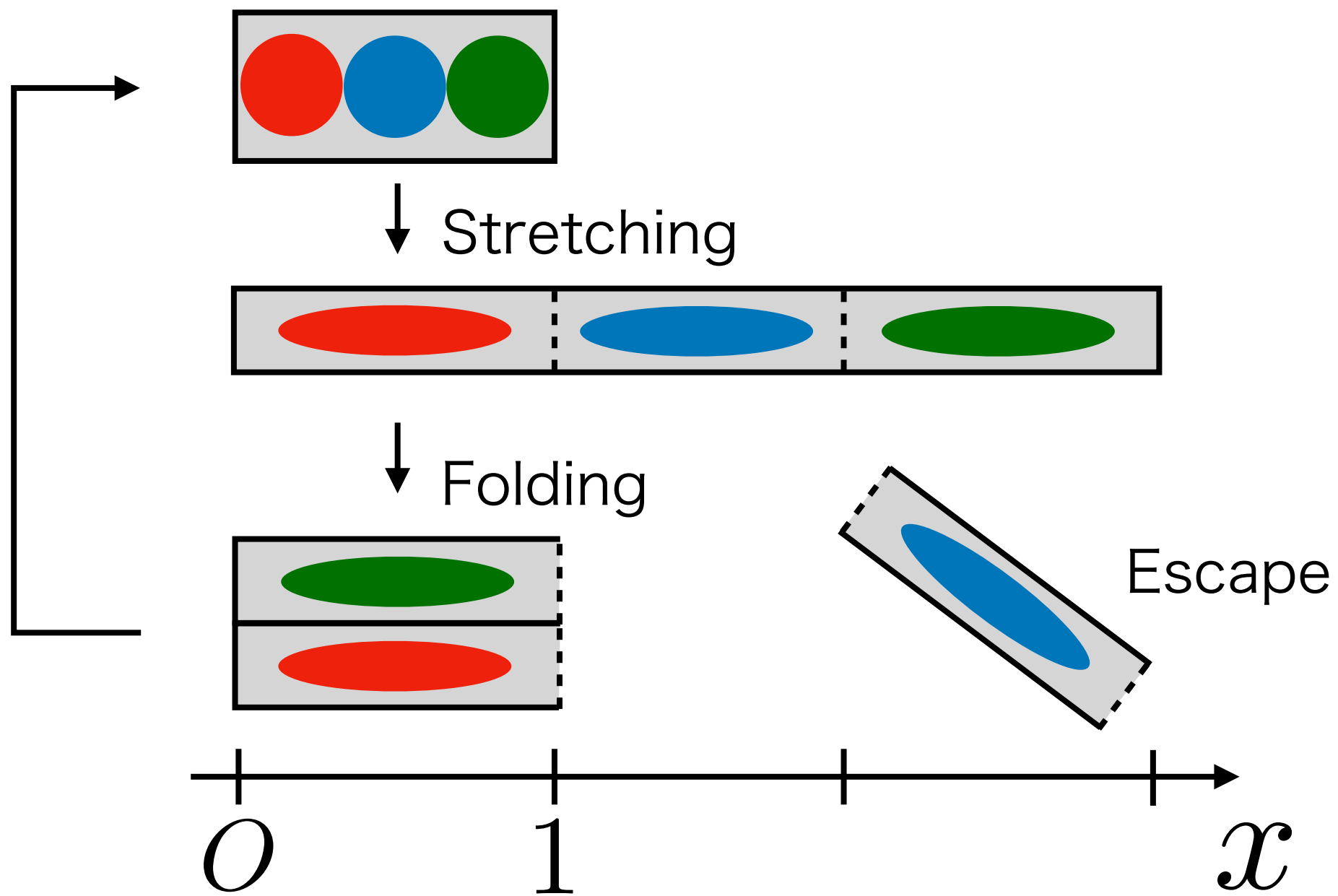
Chaotic



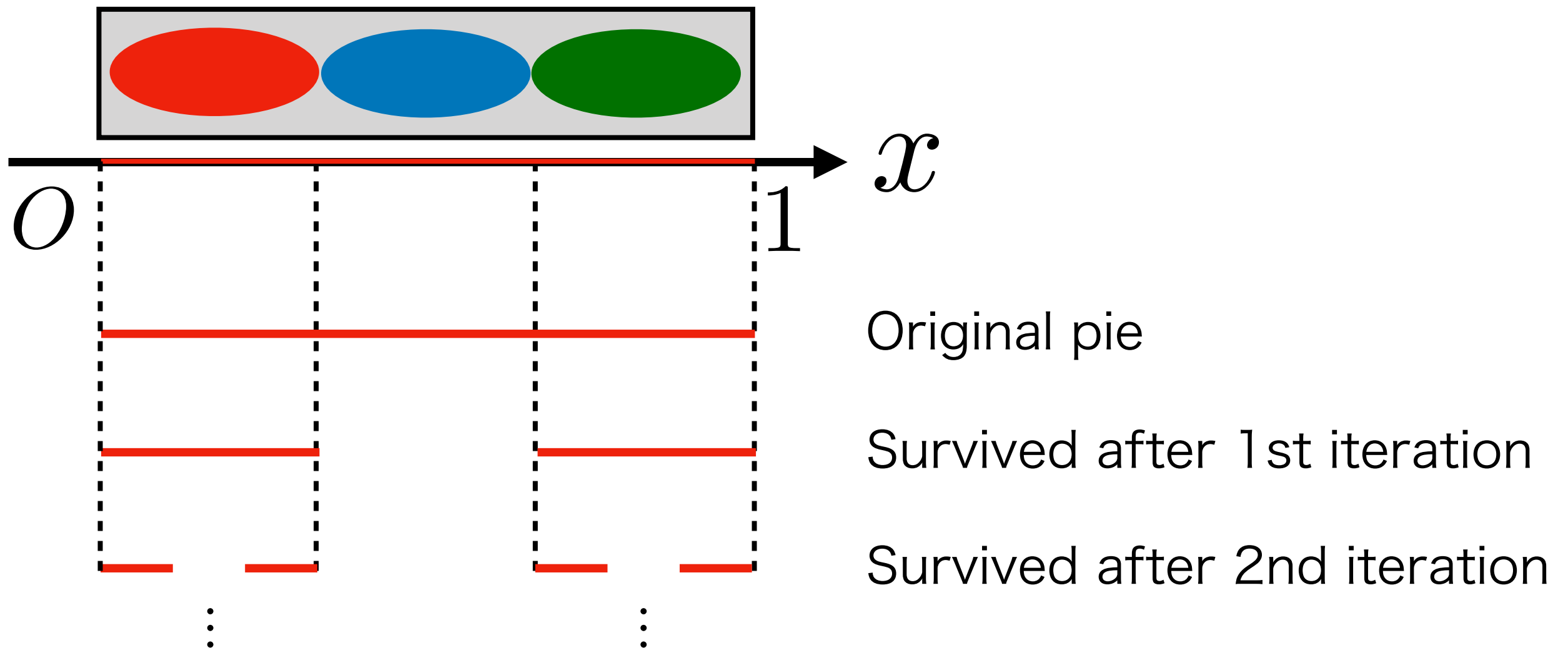
Not chaotic



Toy model for key mechanisms



Toy model for key mechanisms



Fractal dimension & Lyapunov exponent

- Stretching
- Folding
- Escape

Fractal dimension & Lyapunov exponent

- Stretching
- Folding
- Escape

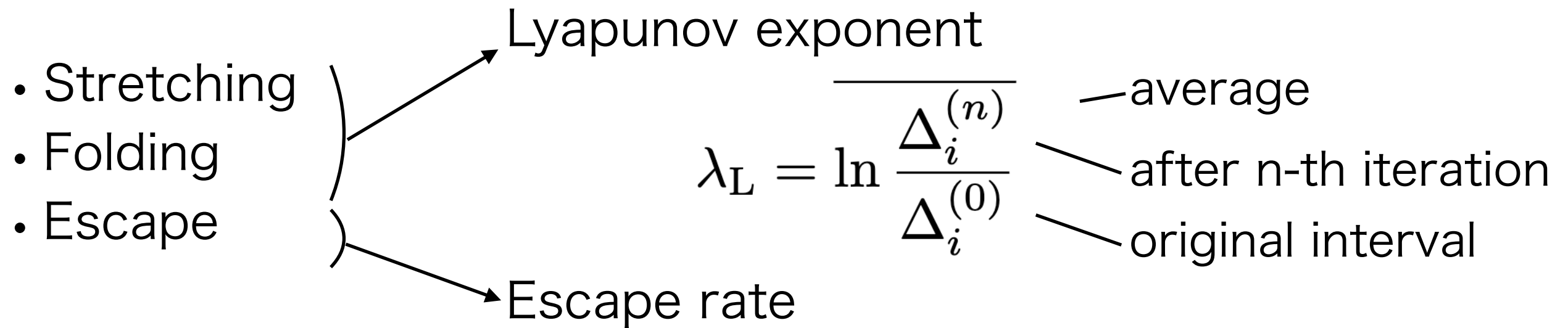
Lyapunov exponent

$$\lambda_L = \ln \frac{\overline{\Delta_i^{(n)}}}{\Delta_i^{(0)}}$$

— average
— after n-th iteration
— original interval

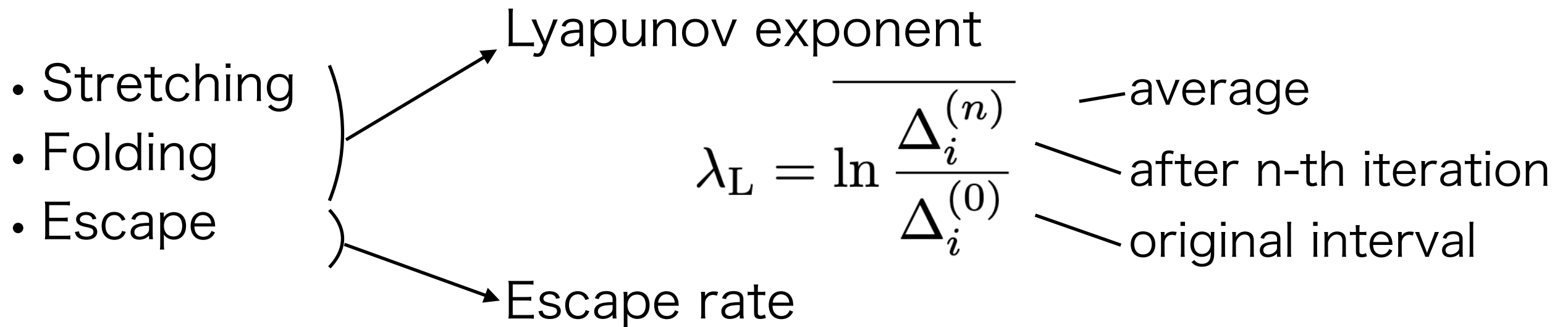
The diagram illustrates the components of the Lyapunov exponent. On the left, a list of three items: 'Stretching', 'Folding', and 'Escape' is enclosed in a large right-facing curly bracket. An arrow points from the middle of this bracket to the text 'Lyapunov exponent'. To the right of this text is the mathematical formula for the Lyapunov exponent: $\lambda_L = \ln \frac{\overline{\Delta_i^{(n)}}}{\Delta_i^{(0)}}$. Three lines with labels point to parts of the formula: a horizontal line from 'average' to the overline, a diagonal line from 'after n-th iteration' to the $\Delta_i^{(n)}$ term, and another diagonal line from 'original interval' to the $\Delta_i^{(0)}$ term.

Fractal dimension & Lyapunov exponent



$$(\text{survived sets}) \propto e^{-\kappa n}$$

Fractal dimension & Lyapunov exponent



$$(\text{survived sets}) \propto e^{-\kappa n}$$

Then we find

$$\lambda_L = \frac{\kappa}{1 - D}$$

Lyapunov exponent
harder to compute

escape rate & fractal dim
easier to compute

Complex saddles & possible chaos in string scatterings

1. Complex saddles in string(y) amplitudes
 - 1.1. Veneziano amplitude (tree) (9)
 - 1.2. Stringy toy model (1-loop, multi-loop) (6)
 - 1.3. Higher-genus corrections (2)
- 2. Application: finding chaos in string scatterings**
 - 2.1. BH chaos & BH-string correspondence (3)
 - 2.2. Chaos in classical scatterings (2)
 - 2.3. Possible chaos in string scatterings** (2)

Complex saddle & semi-classical picture

Complex saddle approx gives us a semi-classical picture

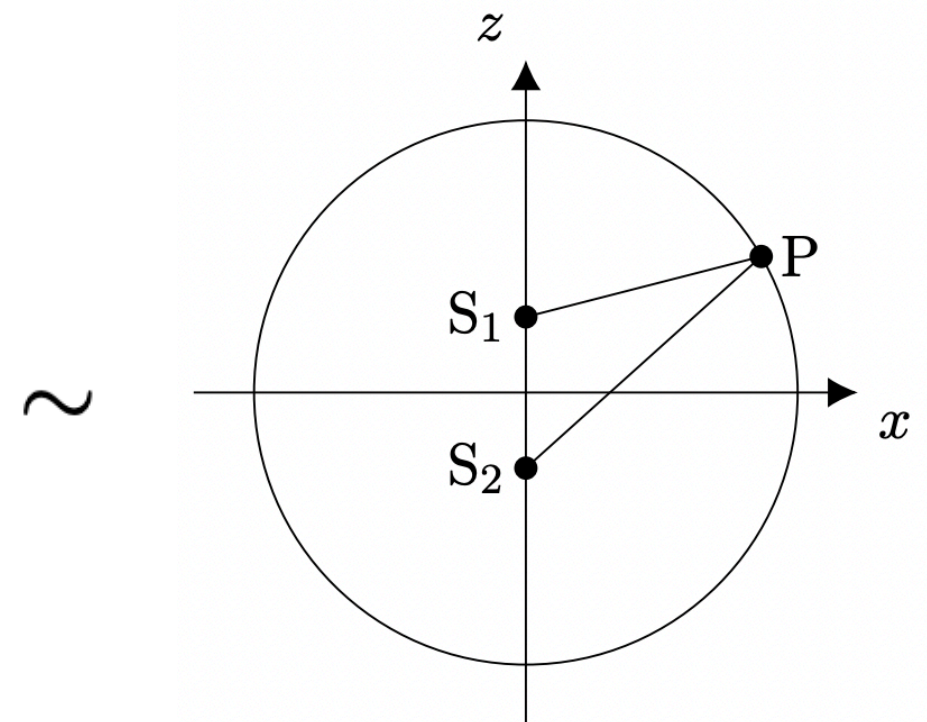
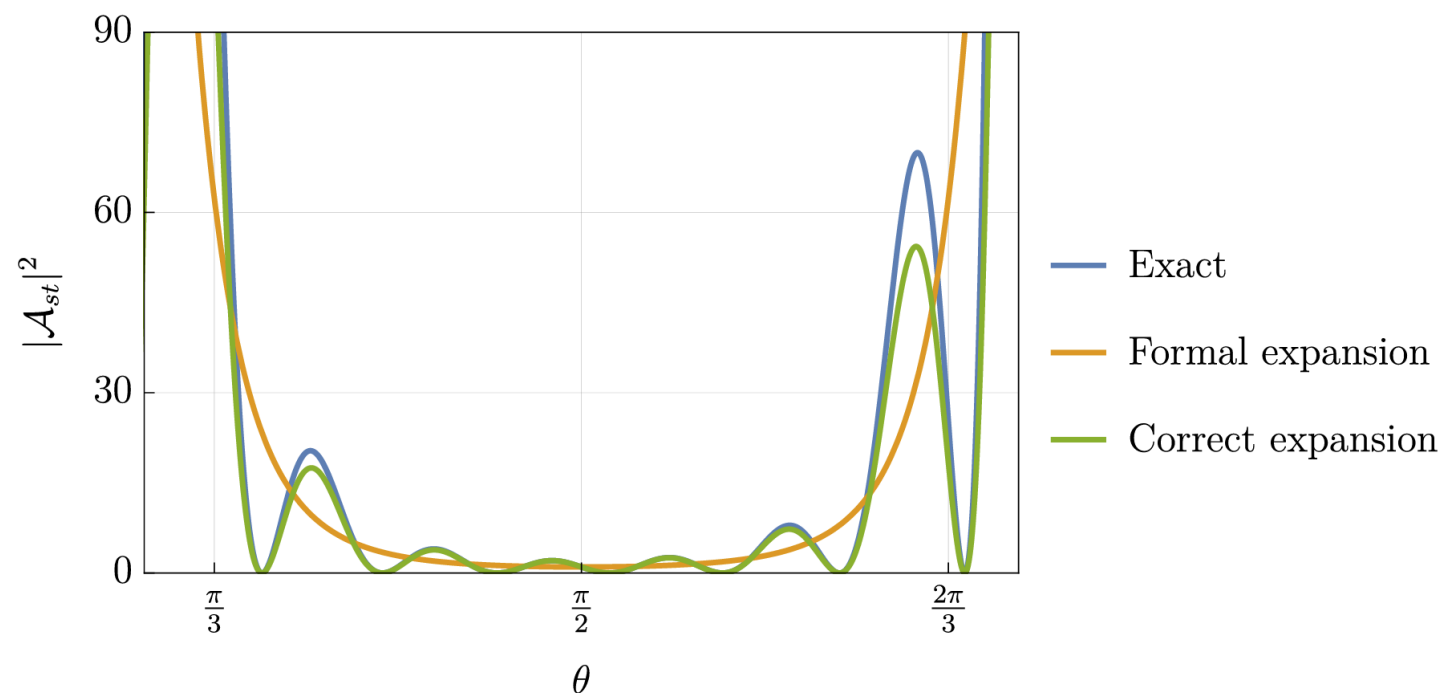
[TY, 24]

$$X(z_{n,m}) - X(z_{0,0}) = p_2(2\pi n) + p_3(2\pi m)$$

that zeros are interpreted as interference of optical paths

Tree-level scatterings are analogous to scatterings by slits,
which imply that string scatterings cannot be chaotic at tree-level

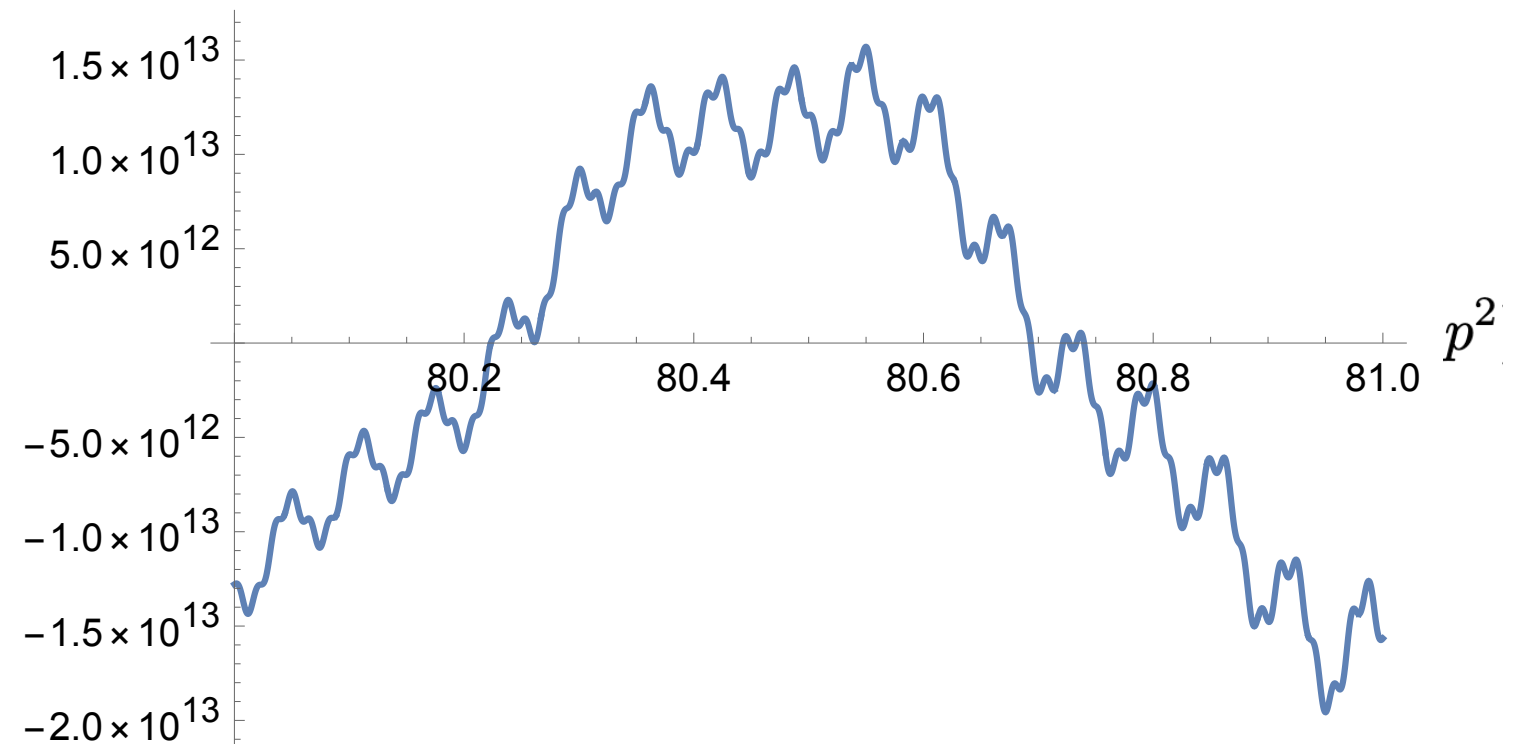
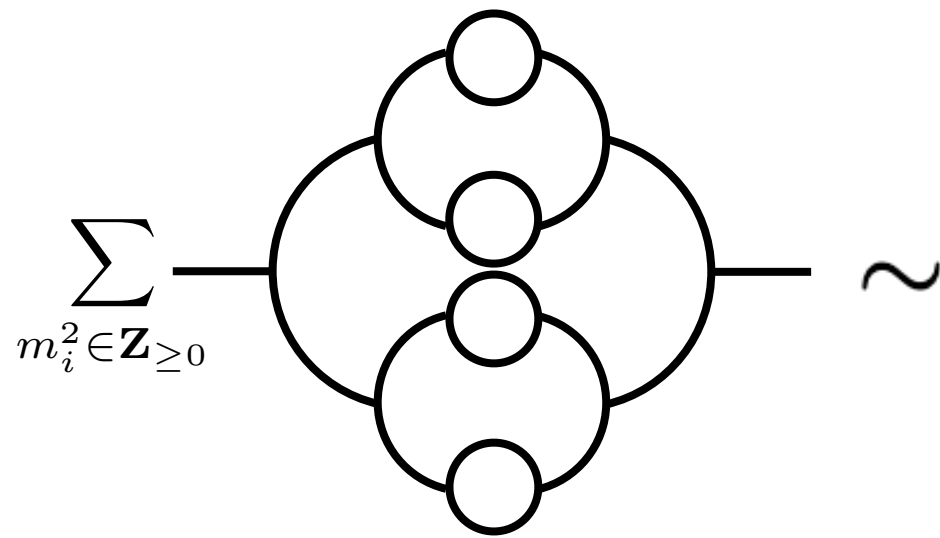
[Hashimoto, Matsuo, TY, 22, 23]



Complex saddle & semi-classical picture

Complex saddle analysis imply that higher genus corrections may generate highly oscillating structure

If the spacetime dim approaches classically conformal point



amplitude looks self-similar

[TY, in preparation]

At the limit of infinitely many genera, string scatterings may approach chaotic region

If so, **the Lyapunov exponent is determined by fractal dim**,
it will be nice if it agrees with the BH surface gravity

Complex saddles & possible chaos in string scatterings

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Complex saddles & possible chaos in string scatterings

- Complex saddles contribute to string amplitudes
- Complex saddles have physical meanings:
 - Semi-classical string path in Lorentzian signature
 - Stringy mass tower
 - Partial wave unitarity
- Complex saddle analysis can be applied to finding chaos in string scatterings

Backups

1. Is single string a chaotic scatterer?

1-1. Chaos in general

1-2. Black hole chaos

1-3. String (basics)

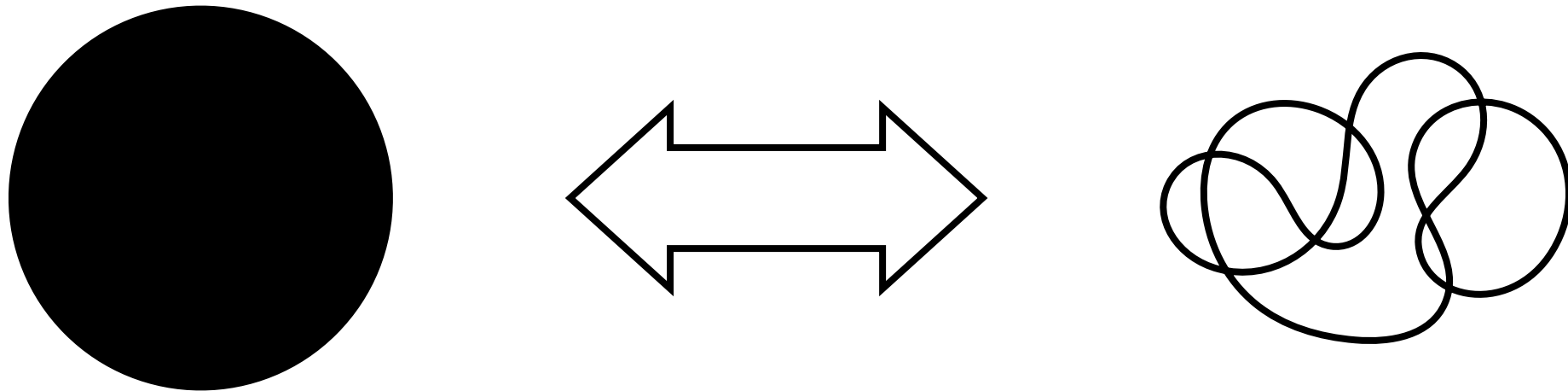
1-4. String (characteristic properties)

1-5. Black hole-string correspondence

1-6. Comparison of them as scatterers

Motivation & Main question

Black hole is chaotic

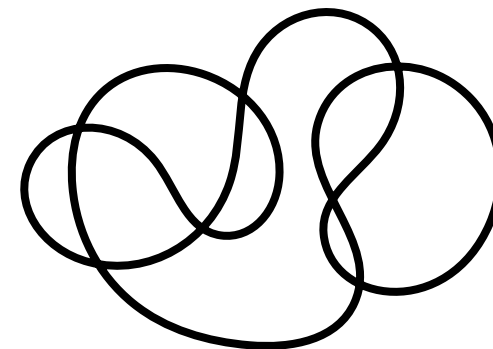
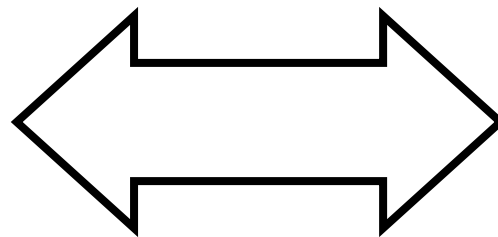
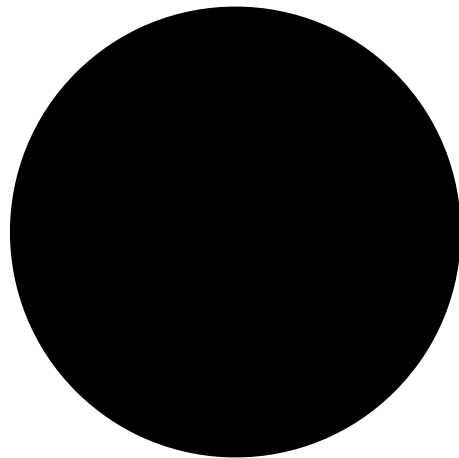


Black hole-string correspondence

Motivation & Main question

Black hole is chaotic

**Then, is single string
also chaotic?**

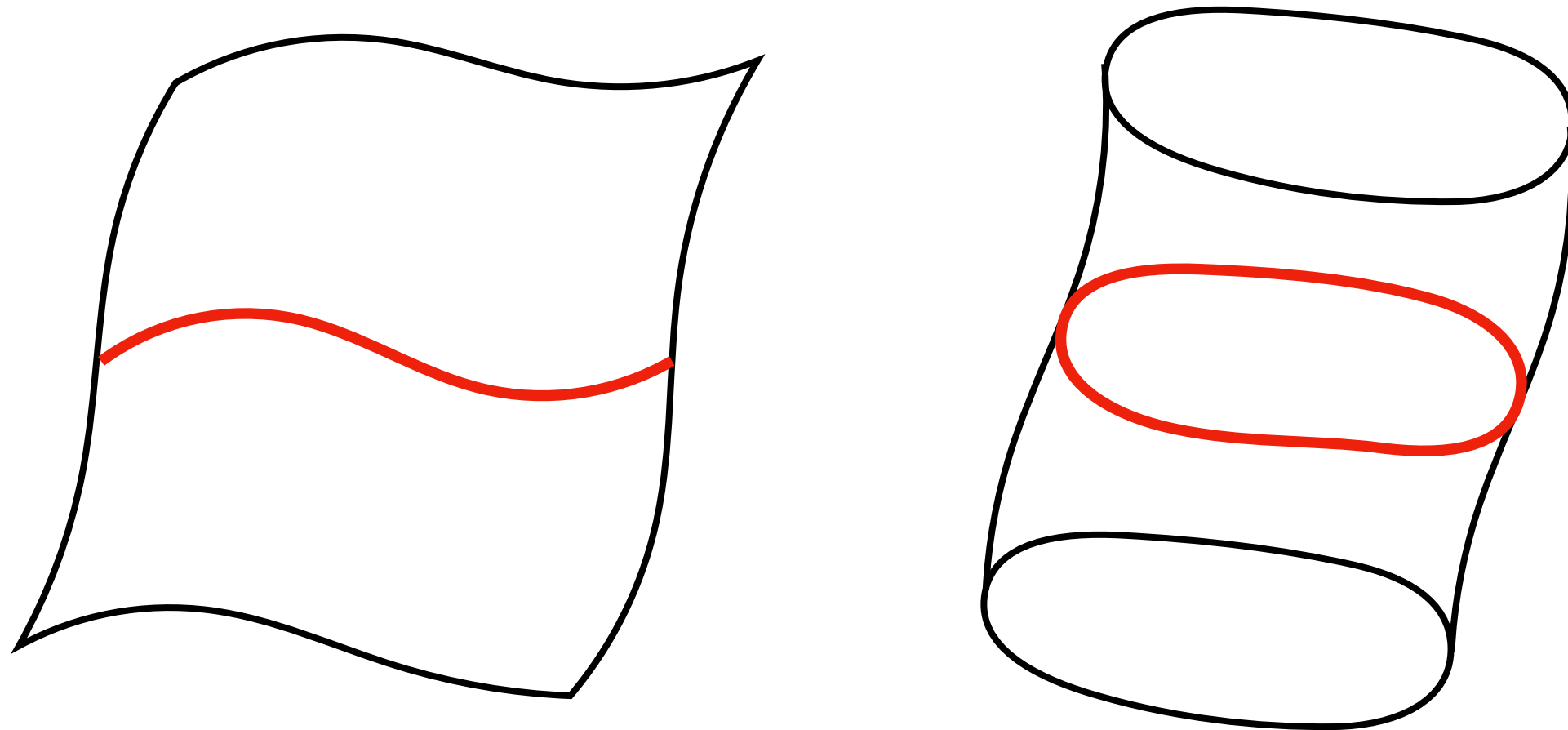


Black hole-string correspondence

Toward chaos in string scatterings

1. Is single string a chaotic scatterer?
2. Chaotic scatterings & fractal
3. Analysis in string scatterings
 - 3.1. Tree-level
 - 3.2. Strategy for higher-genus corrections
 - 3.3. String-like melon diagrams

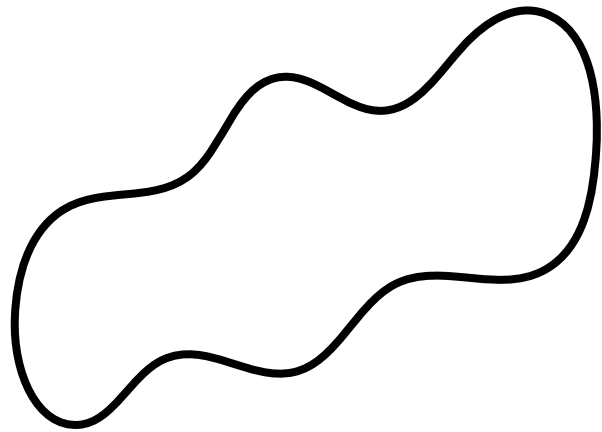
1-3. String (basics)



- Finite sized 1-dim object
- Constant tension
- Infinite tower of excitations

1-4. String (characteristic properties)

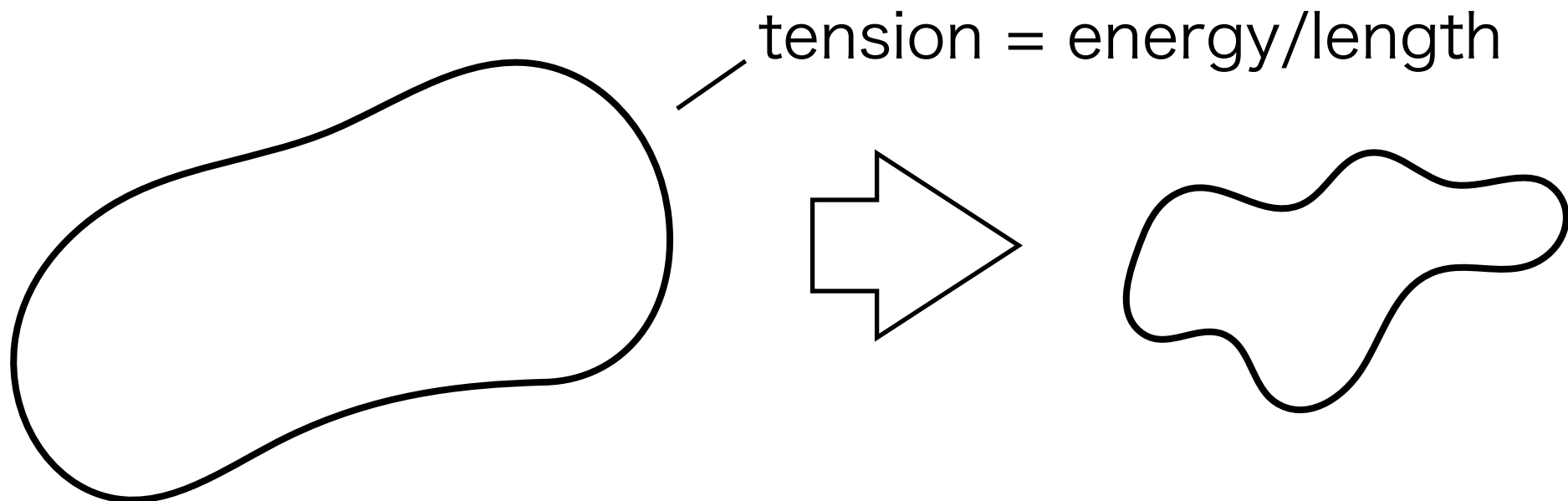
- Finite sized -> Single string has entropy



$$M^2 \propto N$$

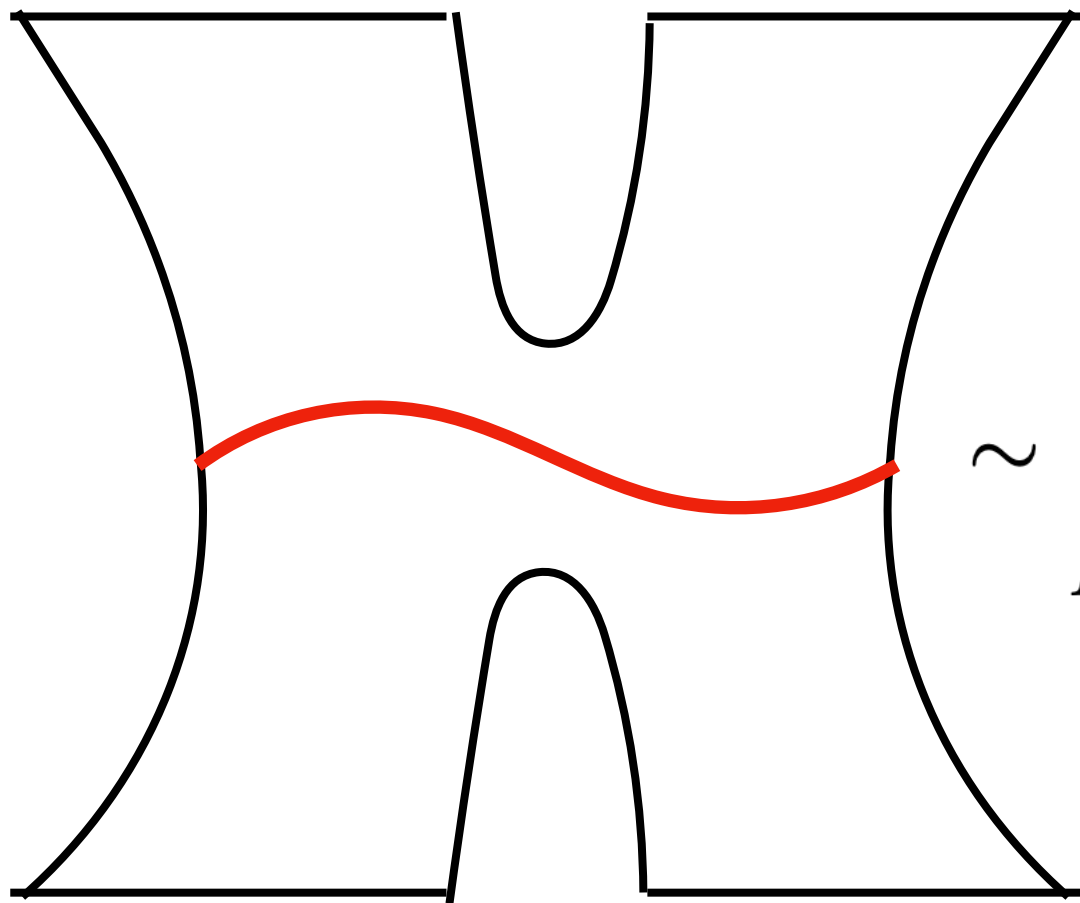
(# states) $\sim$ (# integer partition of N)

- Constant tension -> Self-gravitating

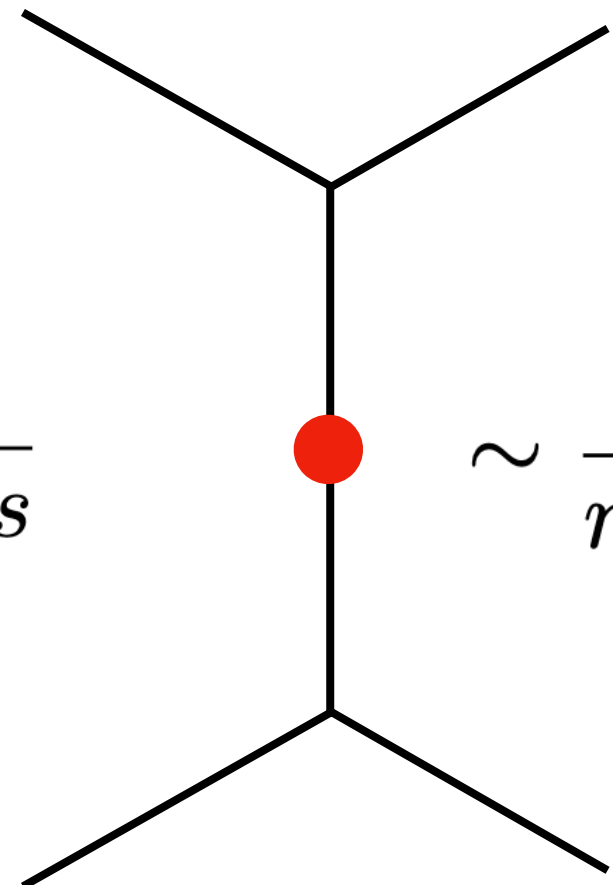


1-4. String (characteristic properties)

- Infinite tower of excitations $\rightarrow$ multiple poles/zeros



$$\sim \sum_{M^2 \propto N} \frac{1}{M^2 + s}$$



$$\sim \frac{1}{m^2 + s}$$

1. Is single string a chaotic scatterer?

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1-2. Black hole chaos

1-3. String (basics)

1-4. String (characteristic properties)

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2. Chaotic scatterings & fractal

[Textbooks e.g. Lai, Tél, 11. Seoane, Sanjuán, 12]

- 2-1. Requirements for chaotic scatterings
- 2-2. Fractal structure in phase space
- 2-3. Toy model: the gluttonous baker's map
- 2-4. Toy model: the map generates Cantor set
- 2-5. Fractal dimension & Lyapunov exponent

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3-1. Tree-level analysis

[K. Hashimoto, Y. Matsuo, TY, 22, 23]

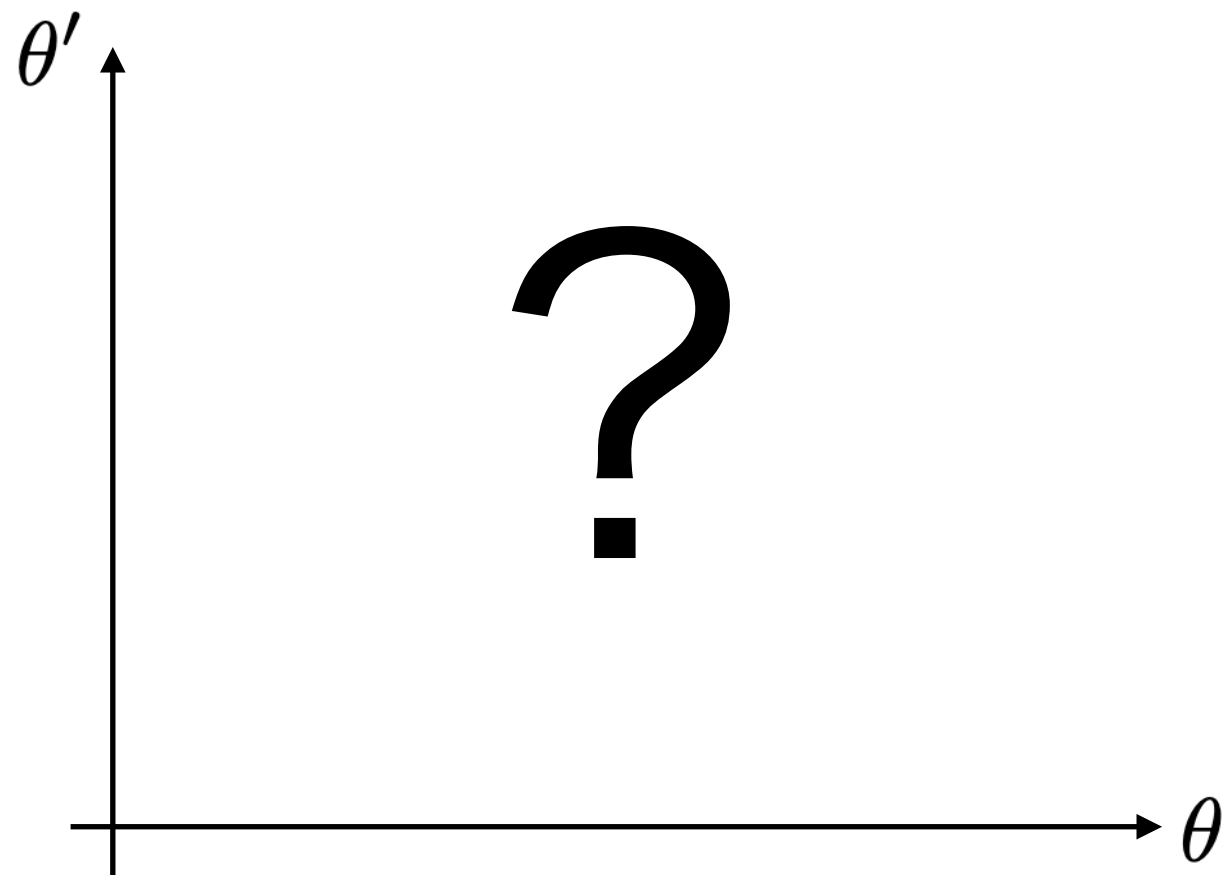
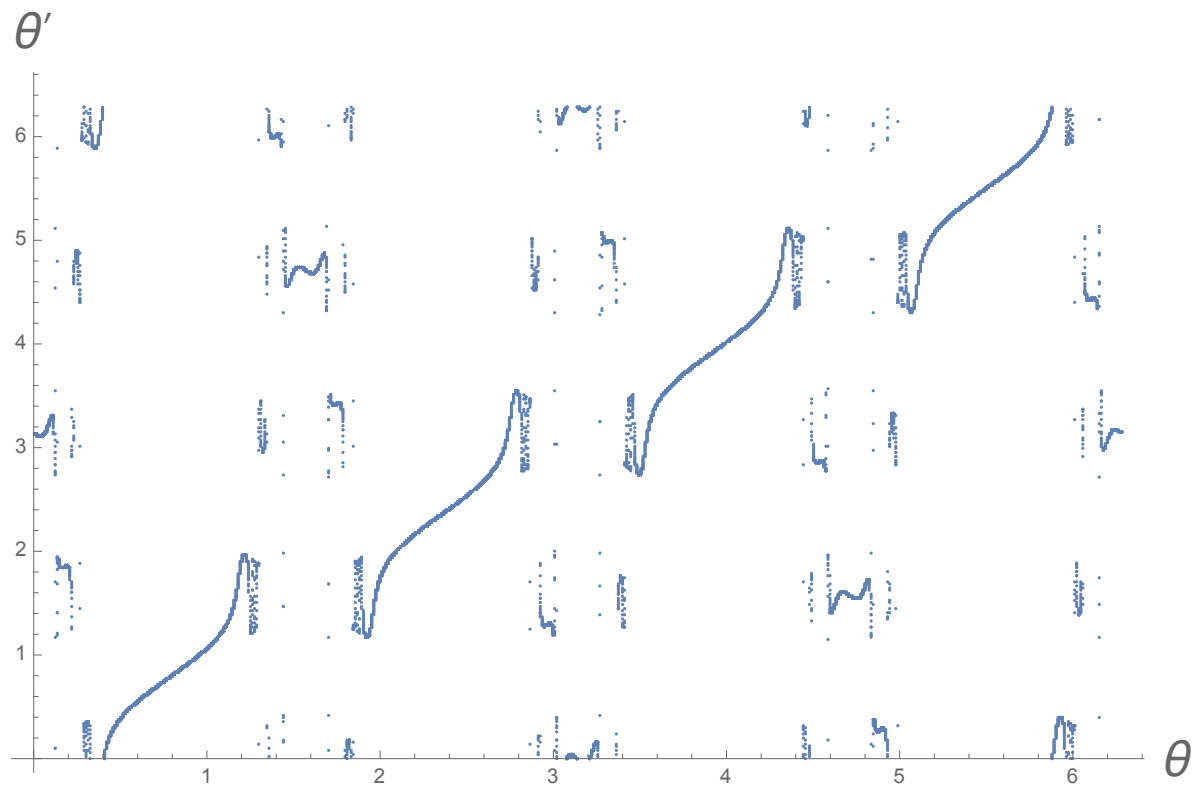
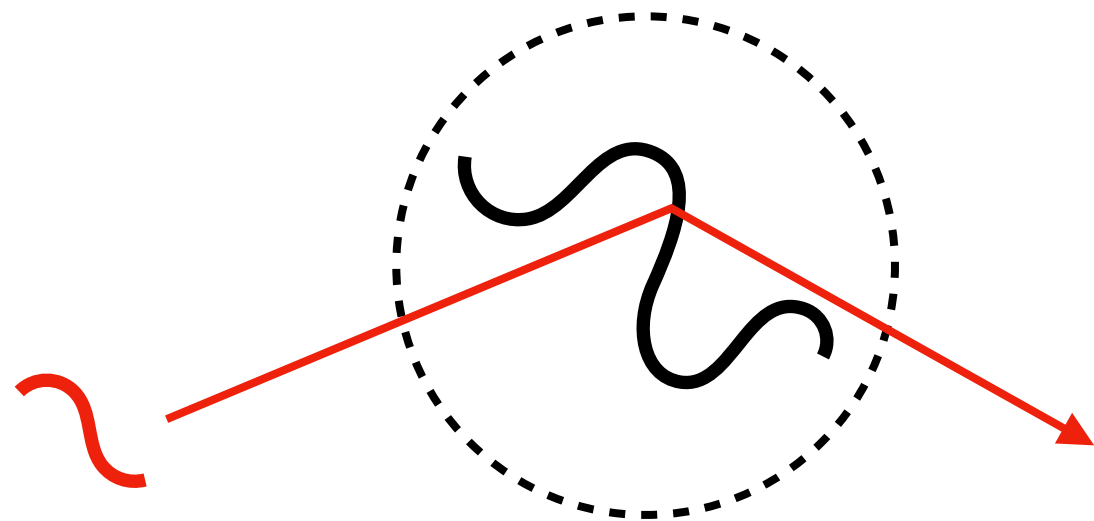
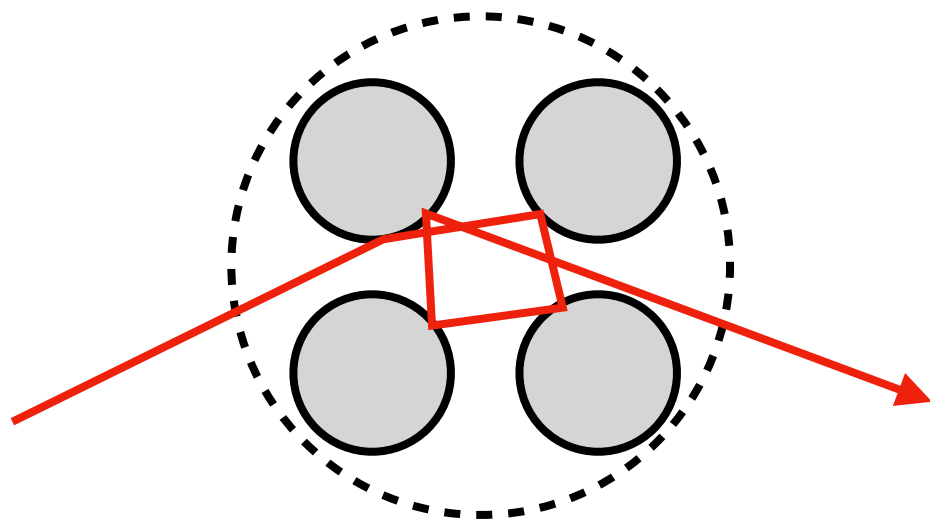
Tree-level scattering is NOT chaotic

3-1-1. Classical to quantum

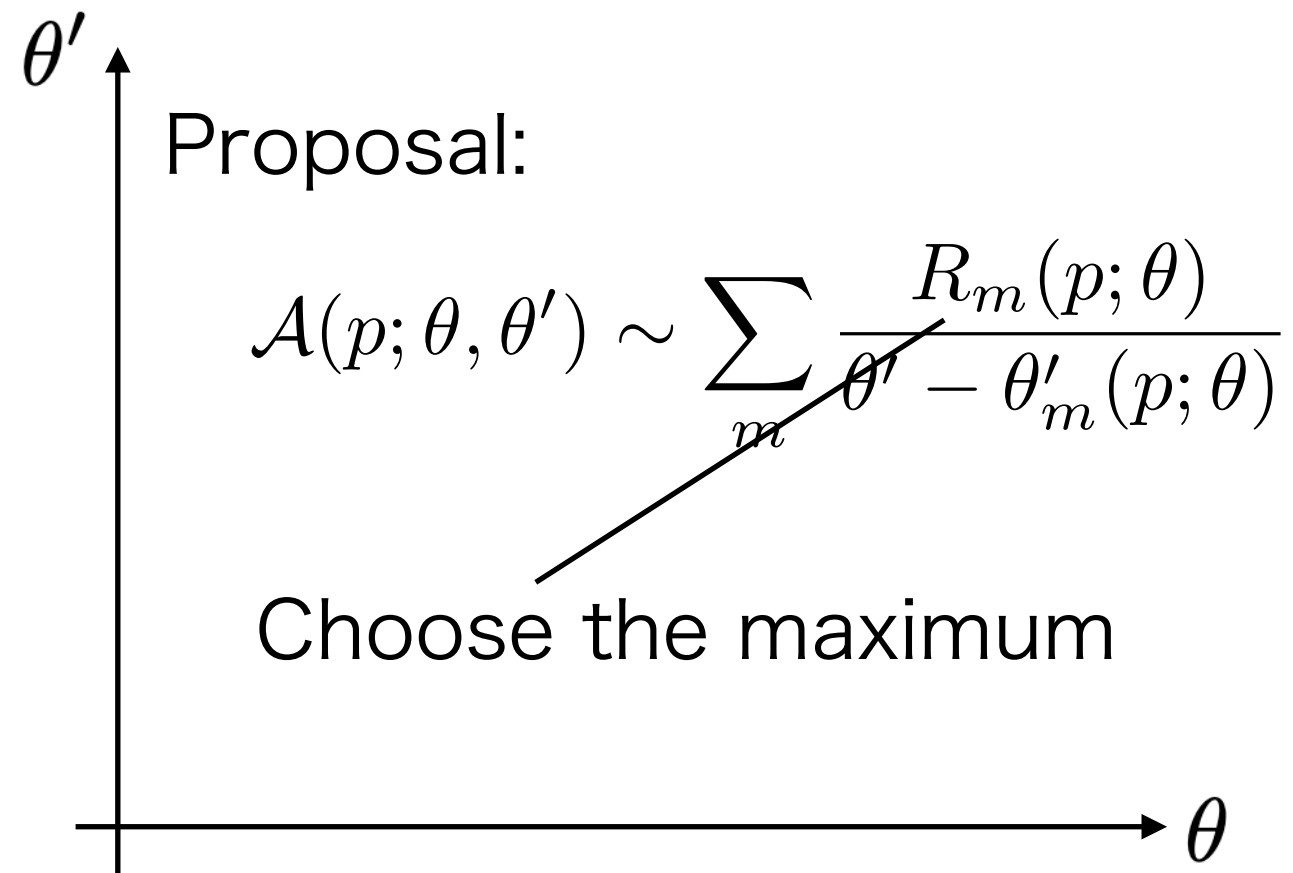
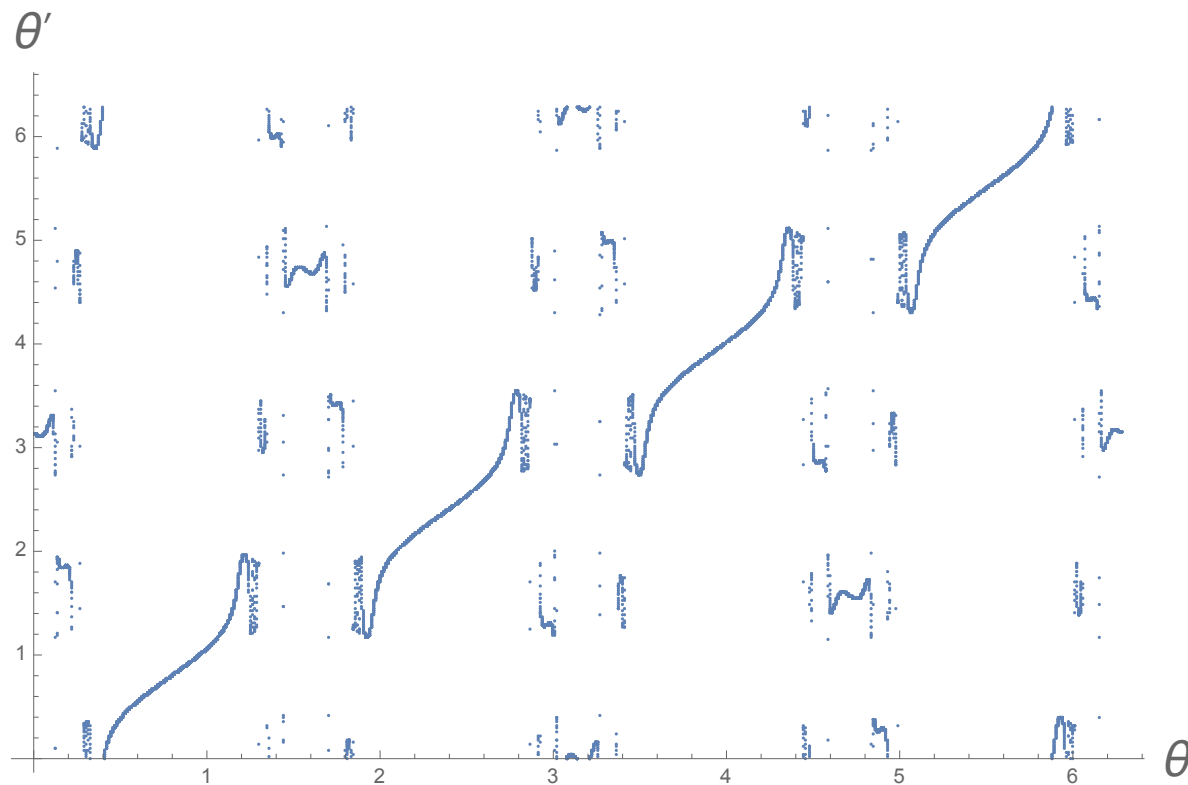
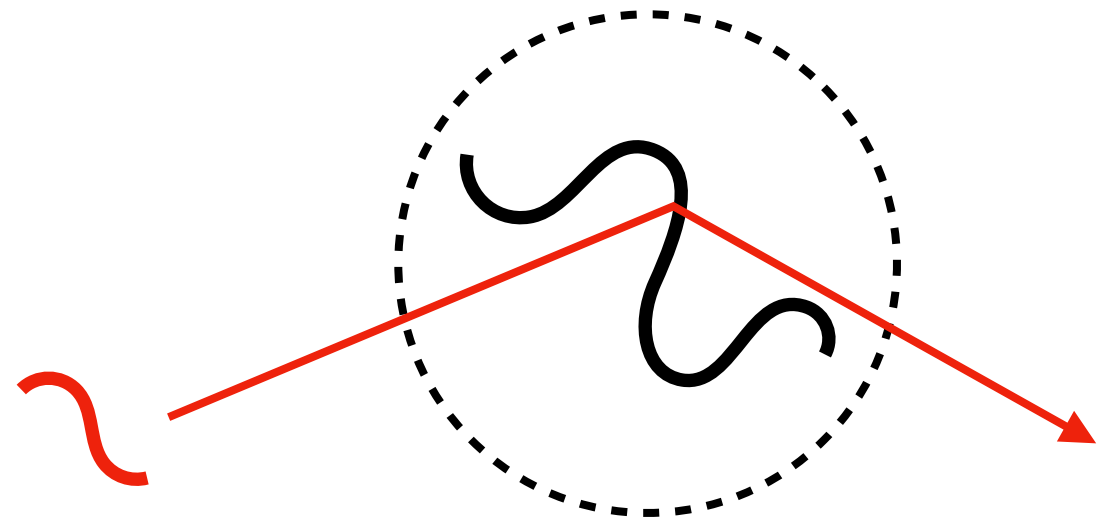
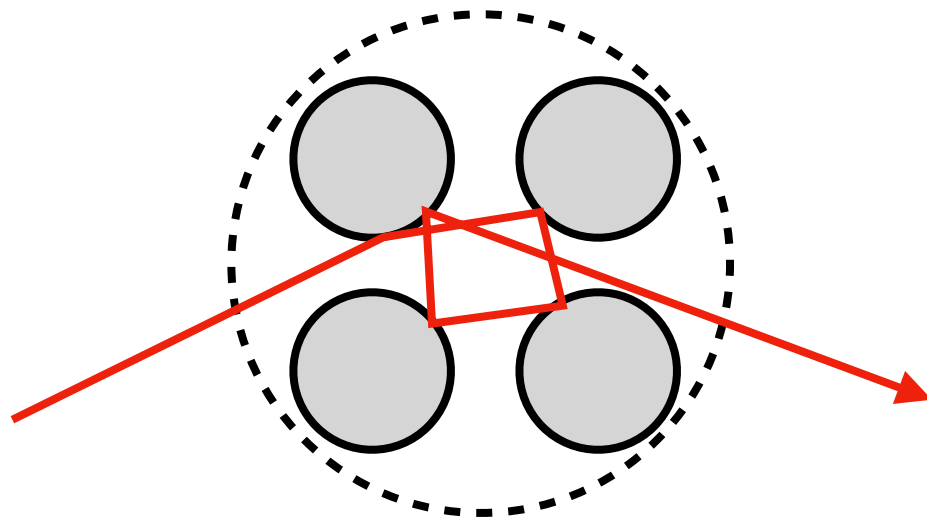
3-1-2. Setups for highly excited string

3-1-3. Result: no self-similar structure

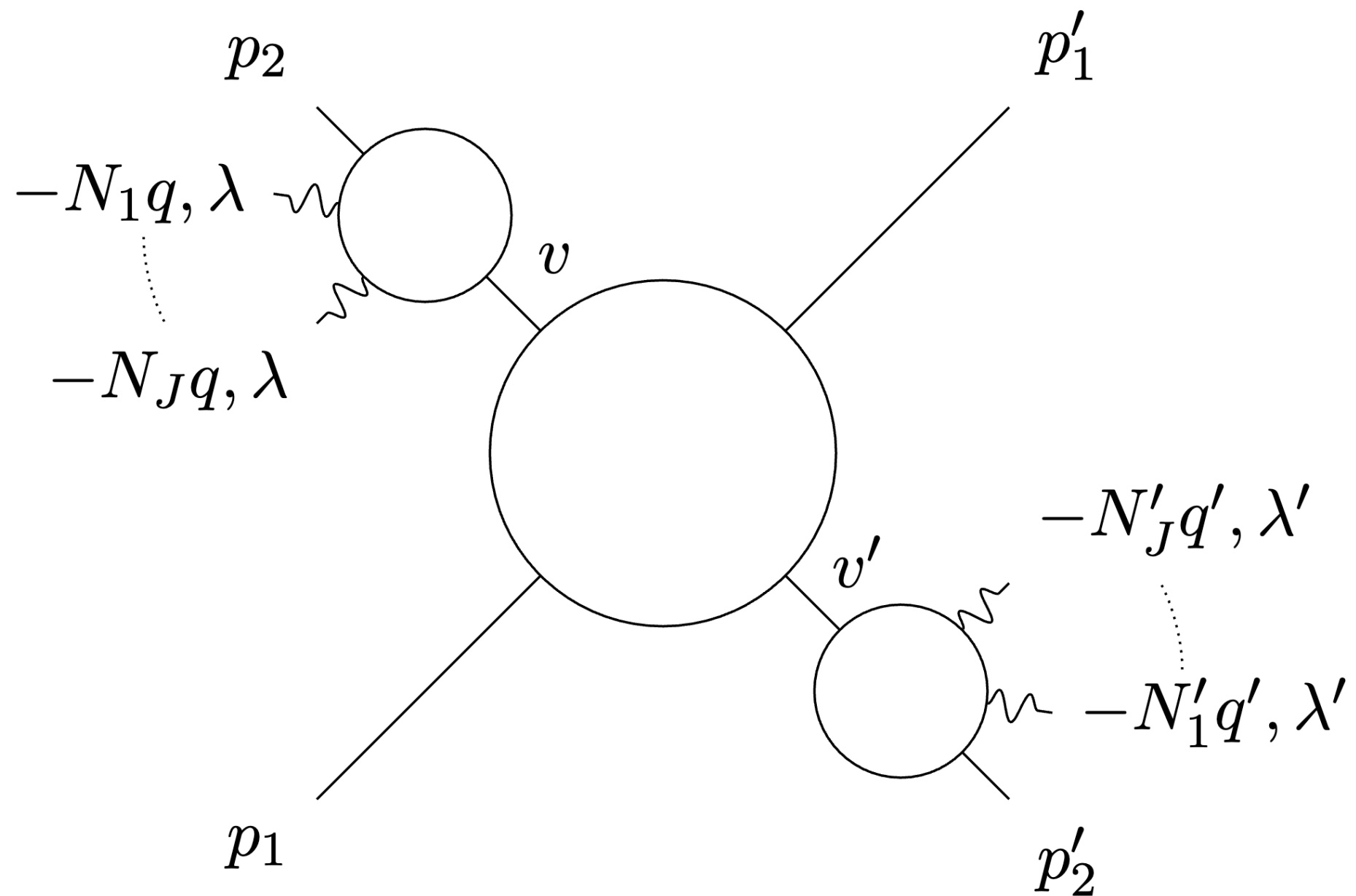
3-1-1. Classical to quantum



3-1-1. Classical to quantum



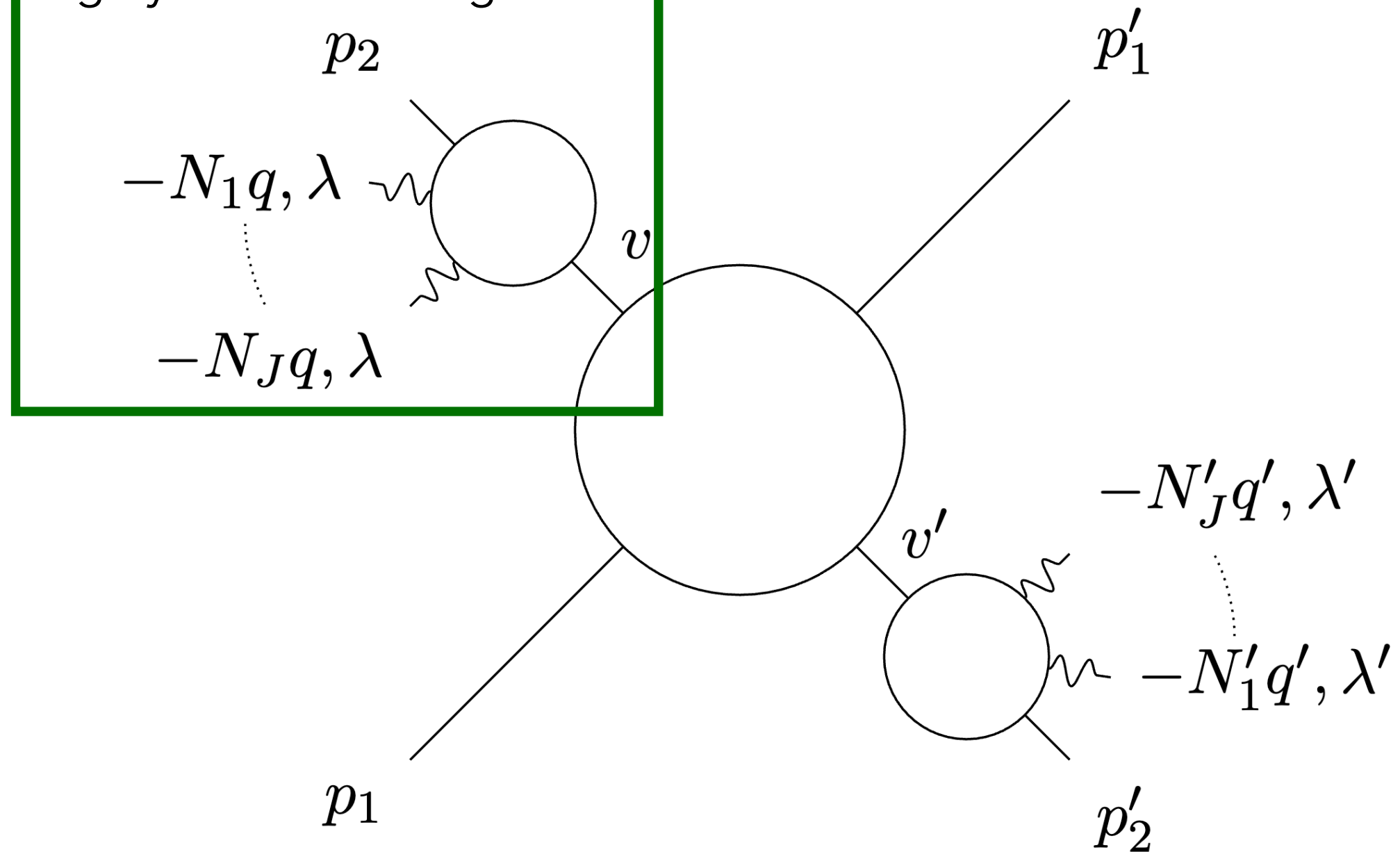
3-1-2. Setups for highly excited string



3-1-2. Setups for highly excited string

Target:

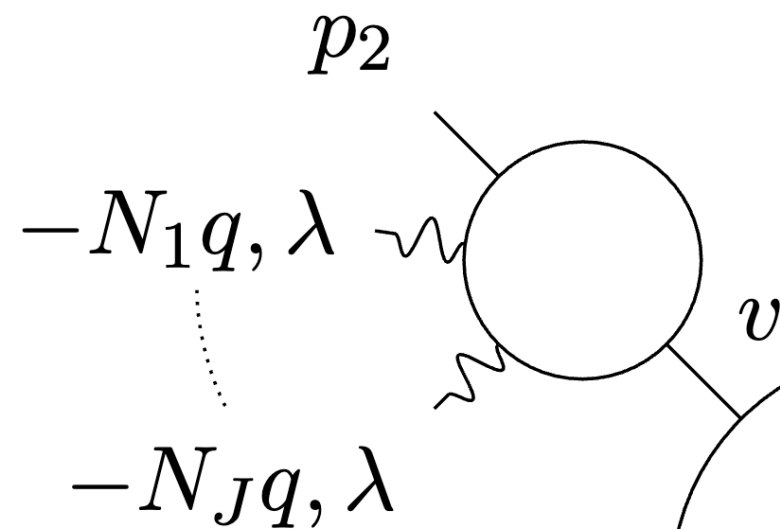
highly excited string



3-1-2. Setups for highly excited string

Target:

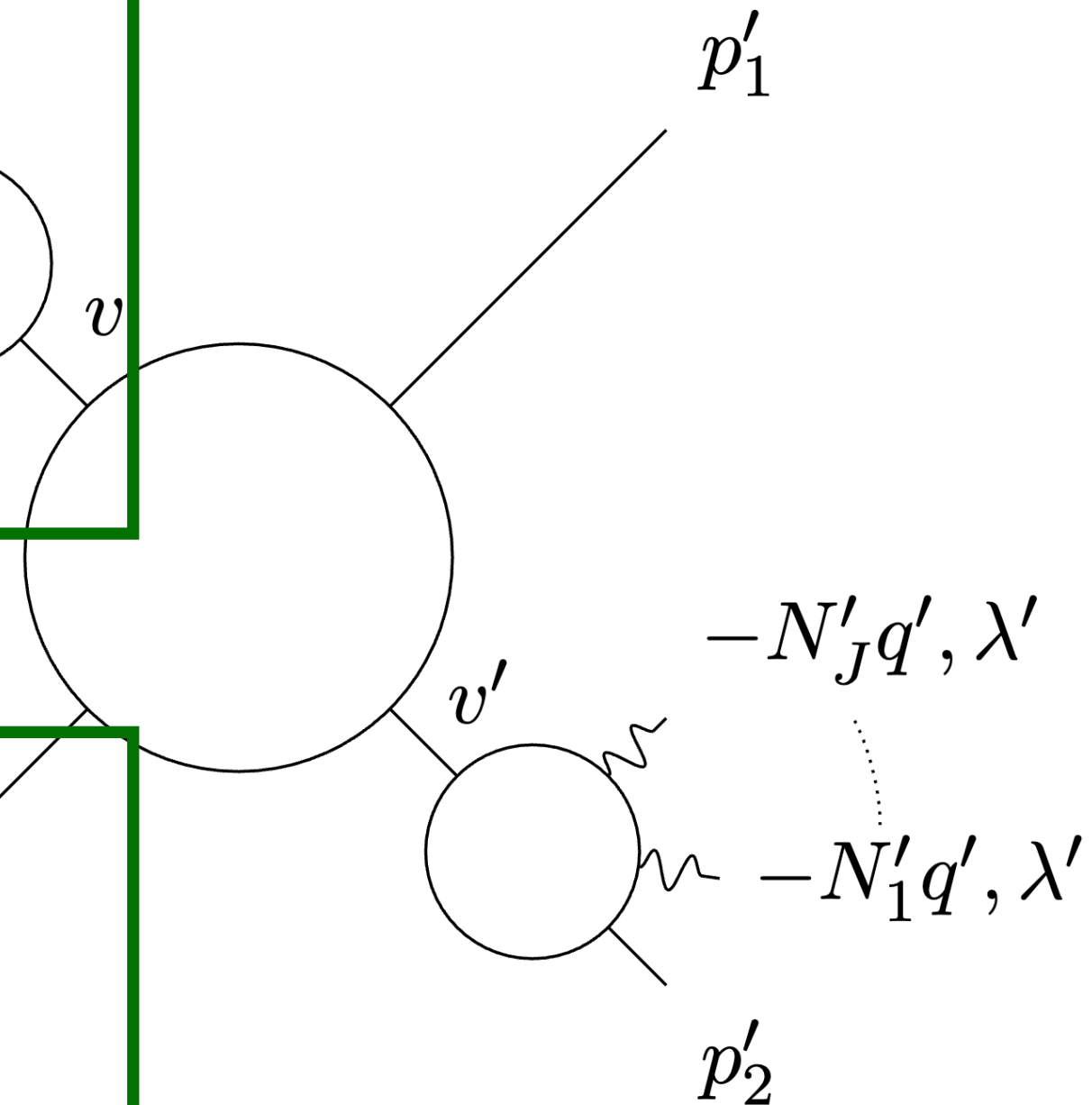
highly excited string



Probe:

light tachyon

p_1

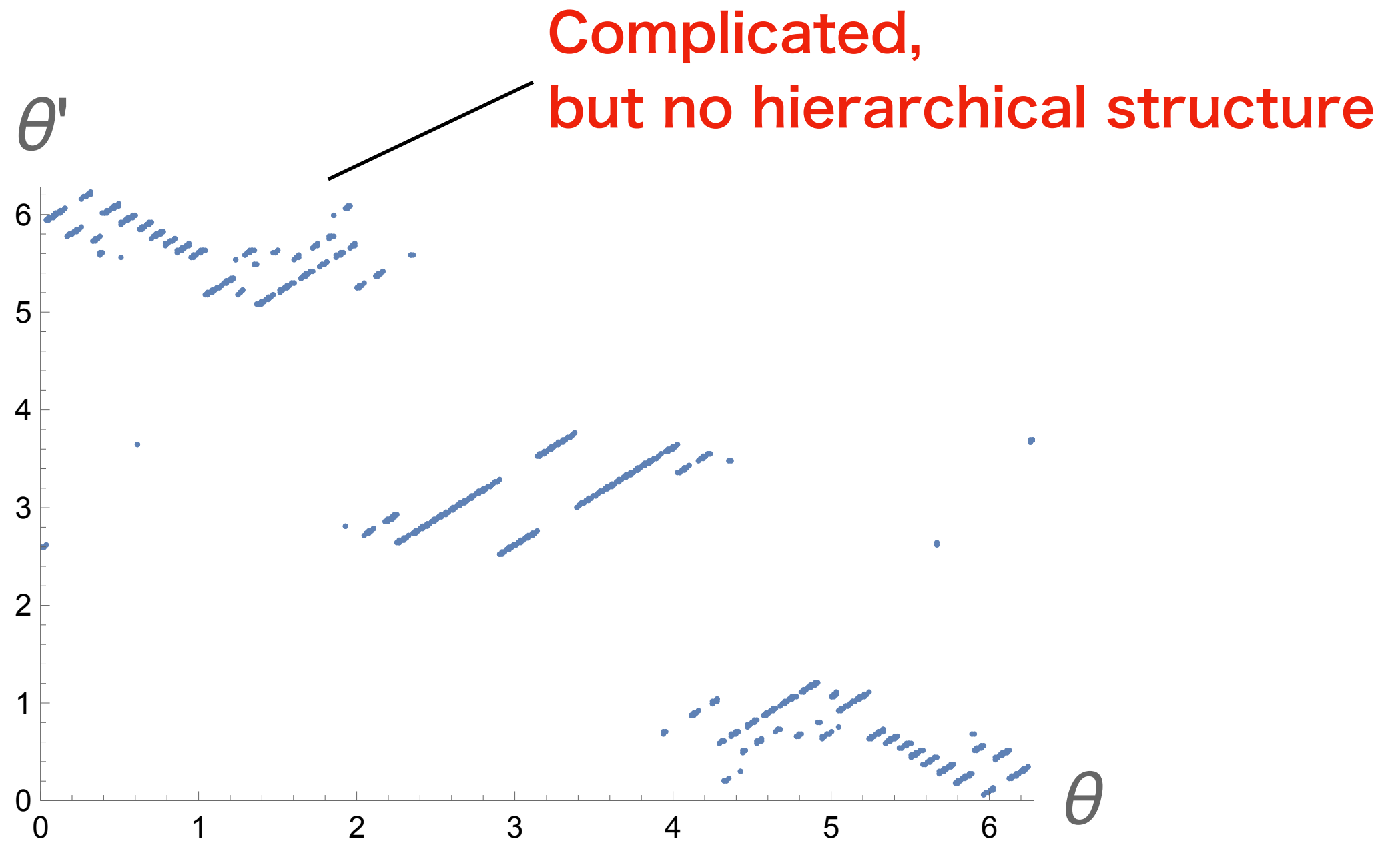


3-1-2. Setups for highly excited string

$$\begin{aligned}
 \mathcal{A} = & \frac{1}{\text{vol.}} \int_{-\infty}^{\infty} \prod_{i=1,2,2',1'} dw_i \prod_{i < j} |w_{ij}|^{p_i \cdot p_j} \\
 & \times \int_{-\infty}^{\infty} \prod_{a=1}^J dz_a \prod_{a,i} |z_a - w_i|^{-\alpha_i} \sum_i \frac{-p_i \cdot \lambda}{w_i - z_a} \\
 & \times \int_{-\infty}^{\infty} \prod_{b=1}^J dz'_b \prod_{b,j} |z'_b - w_j|^{-\beta_j} \sum_j \frac{-p_j \cdot \lambda'}{w_j - z'_b}
 \end{aligned}$$

For simplicity, $q \propto q'$ to drop $|z_a - z'_b|^{N_a q \cdot N'_b q'}$

3-1-3. Result: no self-similar structure



numerical plot $N = 27$

3-1. Tree-level analysis

Tree-level scattering is NOT chaotic

3-1-1. Classical to quantum

3-1-2. Setups for highly excited string

3-1-3. Result: no self-similar structure

Toward chaos in string scatterings

1. Is single string a chaotic scatterer?
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3-2. Strategy for higher-genus corrections

[TY, in progress]

Complex saddles tell us poles/zeros

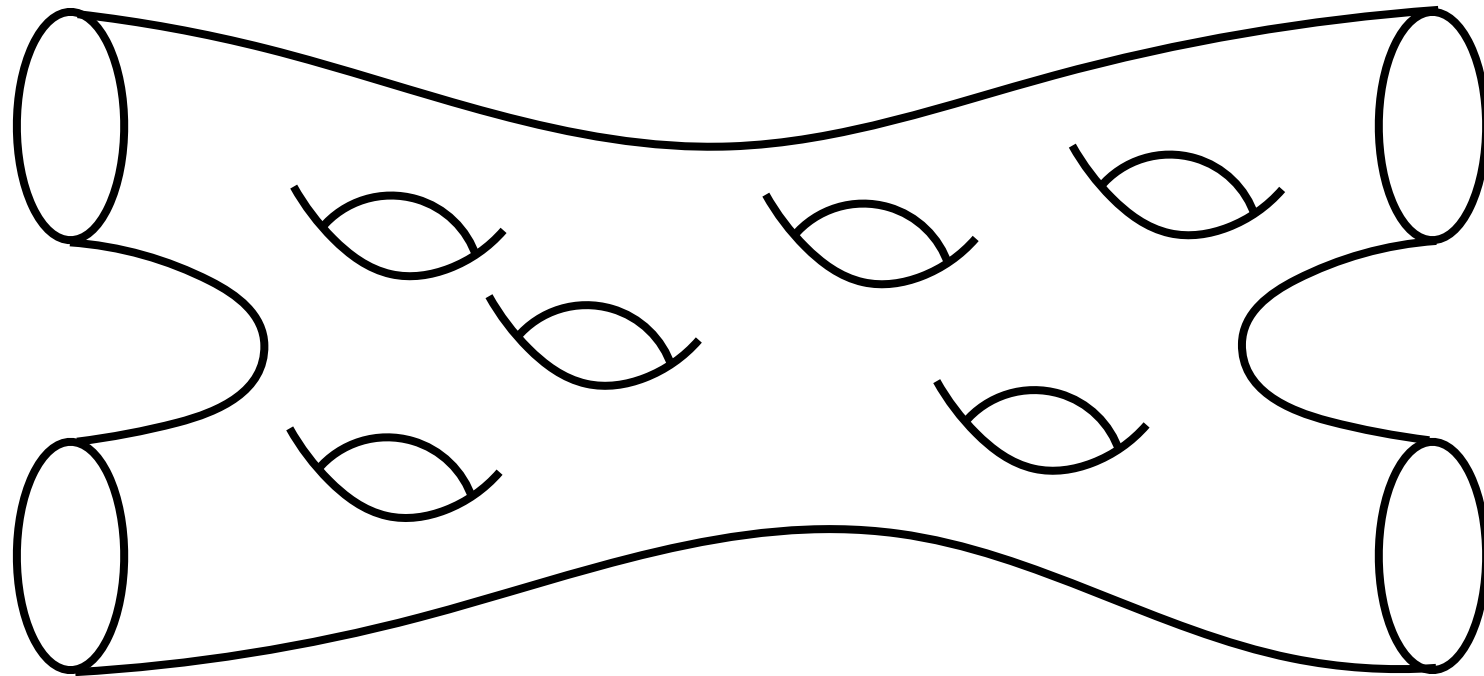
3-2-1. Corrections are notoriously hard to compute

3-2-2. Saddle approximation of string amplitudes

3-2-3. Real saddle is insufficient to detect poles/zeros

3-2-4. Complex saddles detect poles/zeros

3-2-1. Notoriously hard to compute



$$\mathcal{A} = \sum_{\text{topo.}} \int \mathcal{D}g_{ab} \int \mathcal{D}X^\mu \prod_i V_i e^{-S_{\text{poly.}}}$$

No systematical method to compute
full higher genus corrections

see e.g.

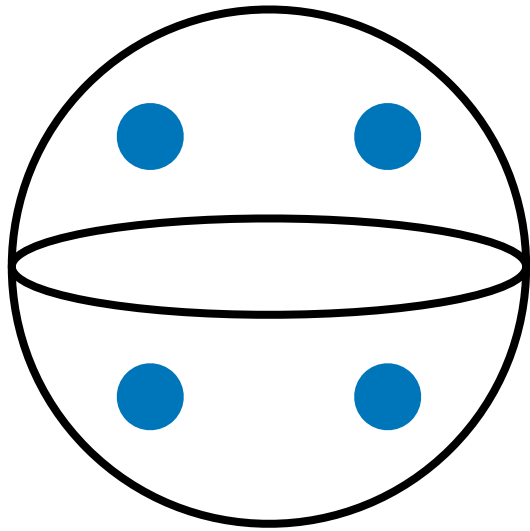
[Kaku, Introduction to Superstrings]

[D'Hoker, Phong, The geometry of string

[Gopakumar, Perlmutter, Pufu, Yin, Snowmass White Paper]

3-2-2. Saddle approximation of string amplitudes

[Gross, Mende, 88; Gross Manes, 8



$$= g^2 \int \frac{\prod_{i=1}^4 d^2 z_i}{\text{Vol.}} \exp \left[\frac{1}{2} \sum_{i < j} p_i \cdot p_j \ln |z_i - z_j| \right]$$

$$= g^2 \int d^2 z \exp \left[\frac{1}{2} p_1 \cdot p_3 \ln |z| + \frac{1}{2} p_1 \cdot p_4 \ln |1 - z| \right]$$

$$z = \frac{p_1 \cdot p_3}{p_1 \cdot p_3 + p_1 \cdot p_4}$$

$$\simeq (\text{fluc.}) \exp \left[-\frac{1}{4} (s \ln s + t \ln t + u \ln u) \right]$$

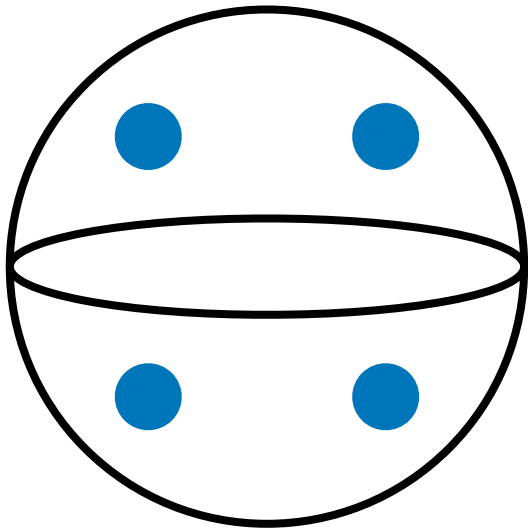
$$s = -(p_1 + p_2)^2 = +4(p^2 - 2)$$

$$t = -(p_1 + p_3)^2 = -4p^2 \cos^2 \frac{\theta}{2}$$

$$u = -(p_1 + p_4)^2 = -4p^2 \sin^2 \frac{\theta}{2}$$

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Saddle

$$z = \frac{p_1 \cdot p_3}{p_1 \cdot p_3 + p_1 \cdot p_4}$$

Approx.

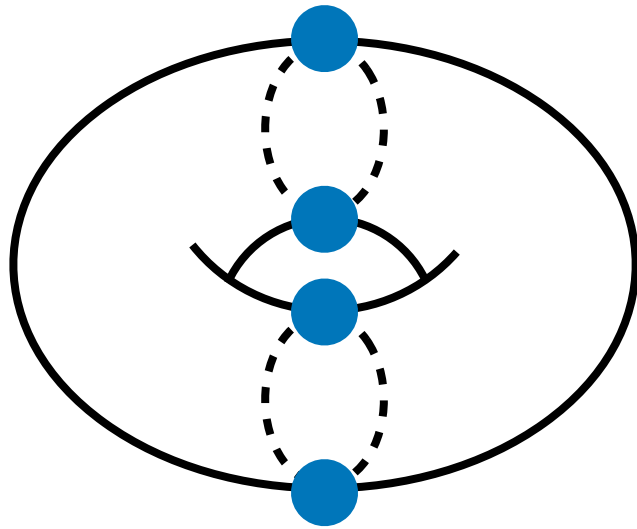
$$\simeq (\text{fluc.}) \exp \left[-\frac{1}{4} (s \ln s + t \ln t + u \ln u) \right]$$

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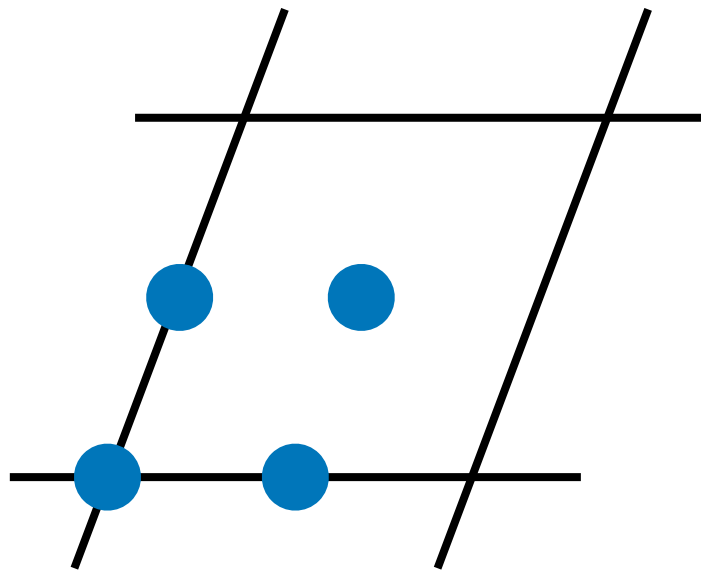
3-2-2. Saddle approximation of string amplitudes



[Gross, Mende, 88; Gross Manes, 88]

$$= g^4 \int d\tau \prod_{i=1}^{4-1} d^2 \nu_i \Omega(\tau) \exp \left[\frac{1}{2} \sum_{i < j} p_i \cdot p_j \ln \chi_{ij} \right]$$

$$\ln \chi_{ij} = \frac{-\pi \operatorname{Im}^2(\nu_i - \nu_j)}{\operatorname{Im} \tau} + \ln \left| \frac{\theta_1(\nu_i - \nu_j | \tau)}{\theta'_1(0 | \tau)} \right|$$



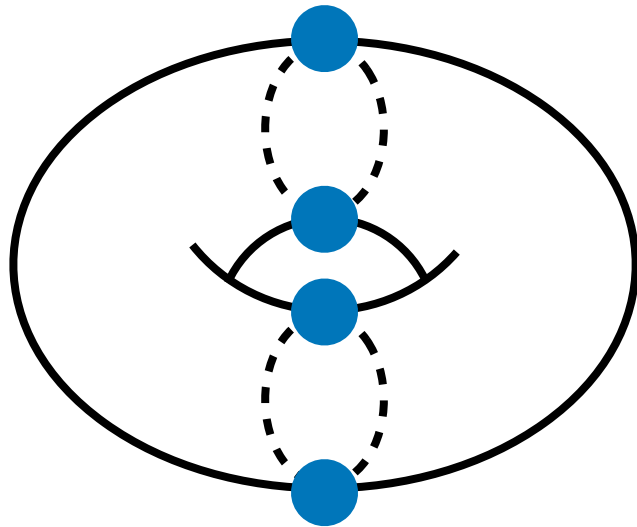
$$0 = \frac{d}{d\tau} \ln \chi_{ij}$$

$$\nu_1 = 1/2, \quad \nu_2 = \tau/2, \quad \nu_3 = (1 + \tau)/2, \quad \nu_4 = 0,$$

$$\simeq (\text{fluc.}) \exp \left[-\frac{1}{8} (s \ln s + t \ln t + u \ln u) \right]$$

*Other 63 saddles are subdominant

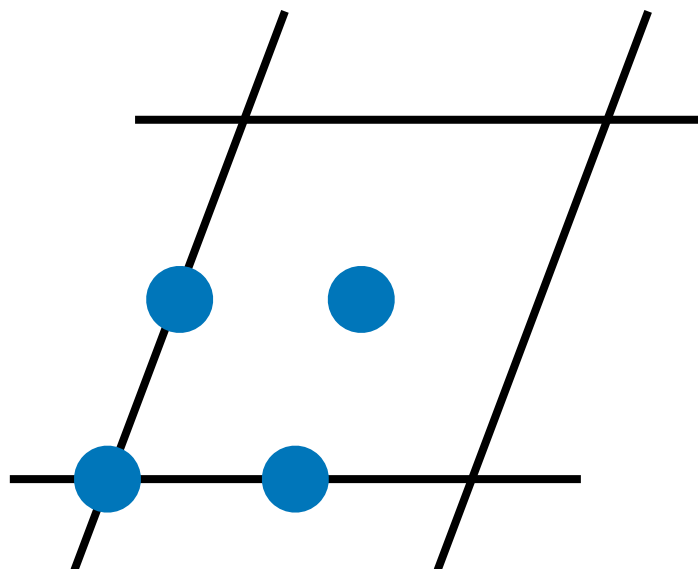
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Saddle

$$0 = \frac{d}{d\tau} \ln \chi_{ij}$$

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Approx.

$$\simeq (\text{fluc.}) \exp \left[-\frac{1}{8} (s \ln s + t \ln t + u \ln u) \right]$$

*Other 63 saddles are subdominant

3-2-2. Saddle approximation of string amplitudes

[Gross, Mende, 88; Gross Manes, 8

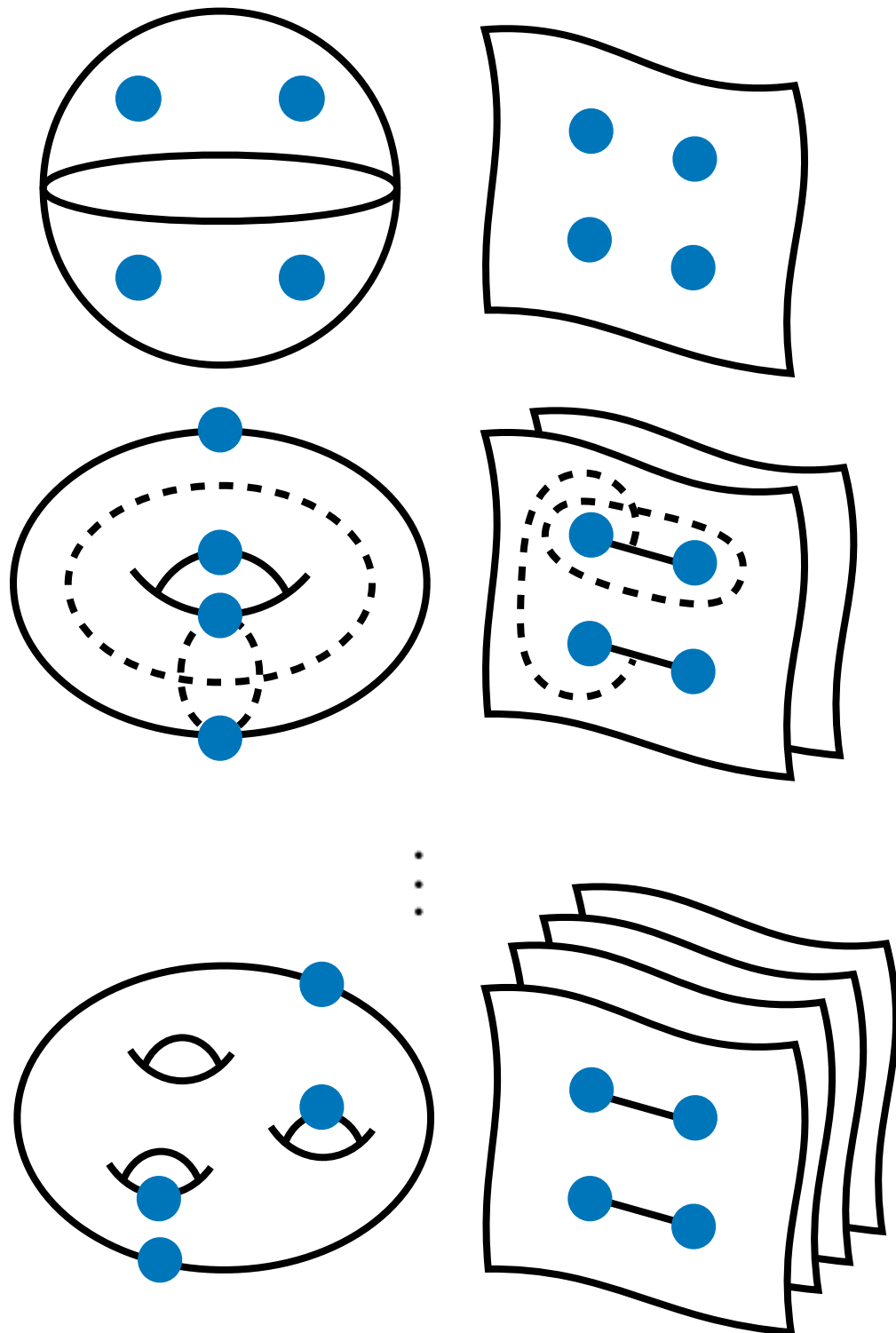
$$\mathcal{A}_{(N-1)\text{-loop}} \simeq (\text{fluc.}) e^{\mathcal{E}_N}, \quad \mathcal{E}_N = -\frac{1}{4N} (s \ln s + t \ln t + u \ln u)$$

*Other saddles are expected to be subdominant

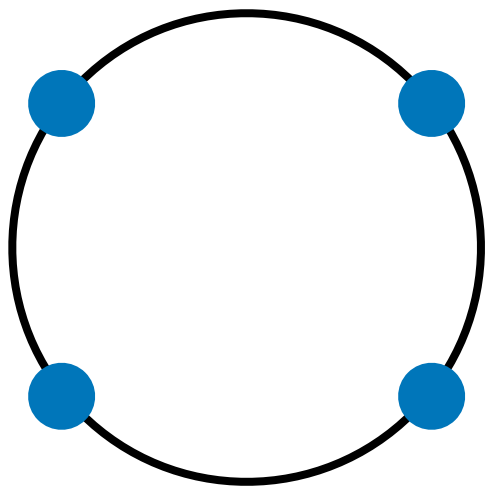
$$\mathcal{E}_1 = \int_{\text{sphere}} |\mathbf{E}_1|^2$$

$$\mathcal{E}_2 = \int_{\Sigma_2} |\mathbf{E}_2|^2 = 2 \int_{\text{sphere}} \left| \frac{1}{2} \mathbf{E}_1 \right|^2 = \frac{1}{2} \mathcal{E}_1$$

$$\mathcal{E}_N = \int_{\Sigma_N} |\mathbf{E}_N|^2 = N \int_{\text{sphere}} \left| \frac{1}{N} \mathbf{E}_1 \right|^2 = \frac{1}{N} \mathcal{E}_1$$

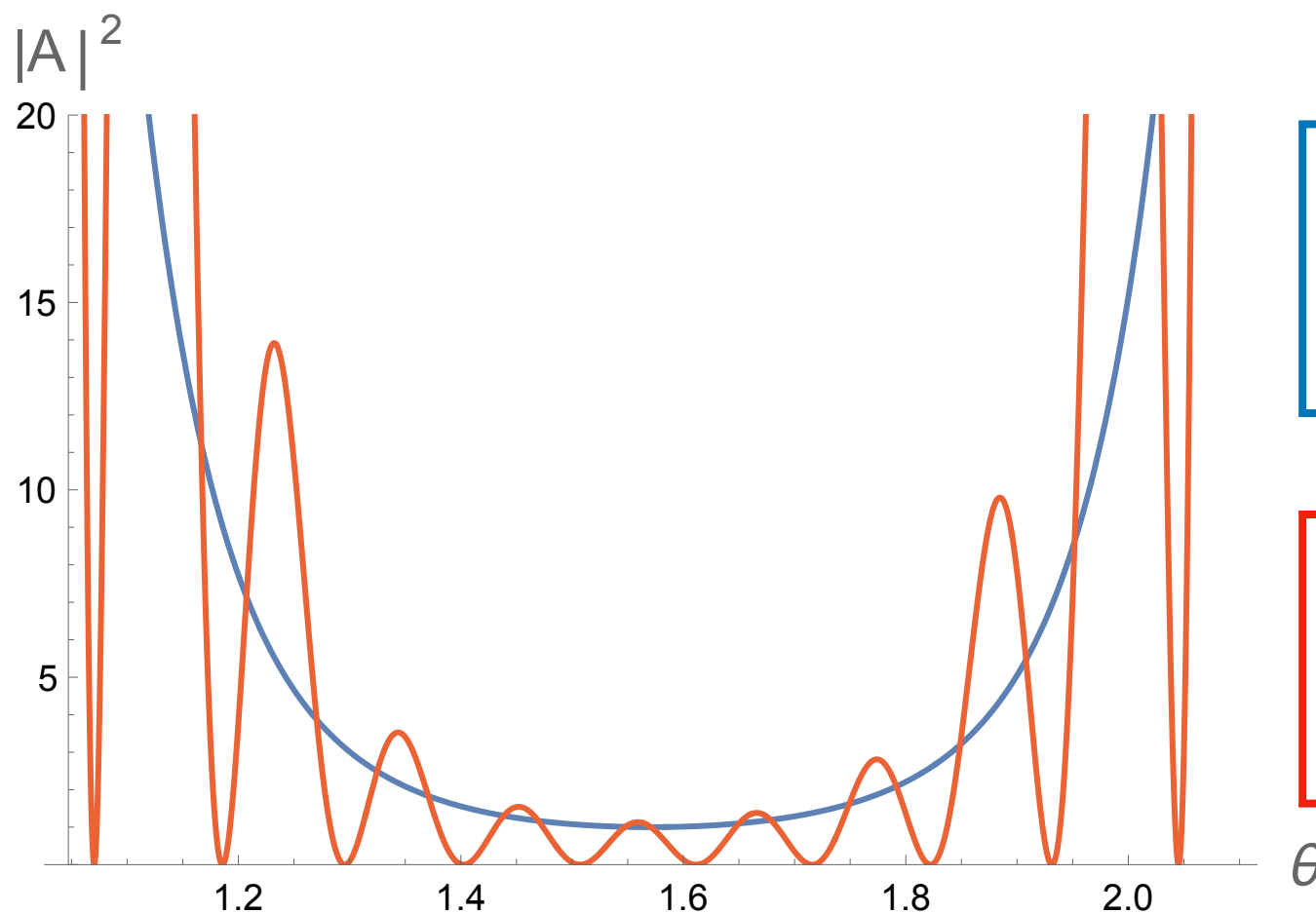


3-2-3. Real saddle is insufficient to detect poles/zeros



$$= g \int_0^1 dz \exp [p_1 \cdot p_2 \ln |z| + p_1 \cdot p_3 \ln |1 - z|]$$

*For simplicity, open & bosonic



Real saddle approx

$$(\text{fluc.}) \exp \left[-\frac{1}{2} (s \ln s + t \ln t + u \ln u) \right]$$

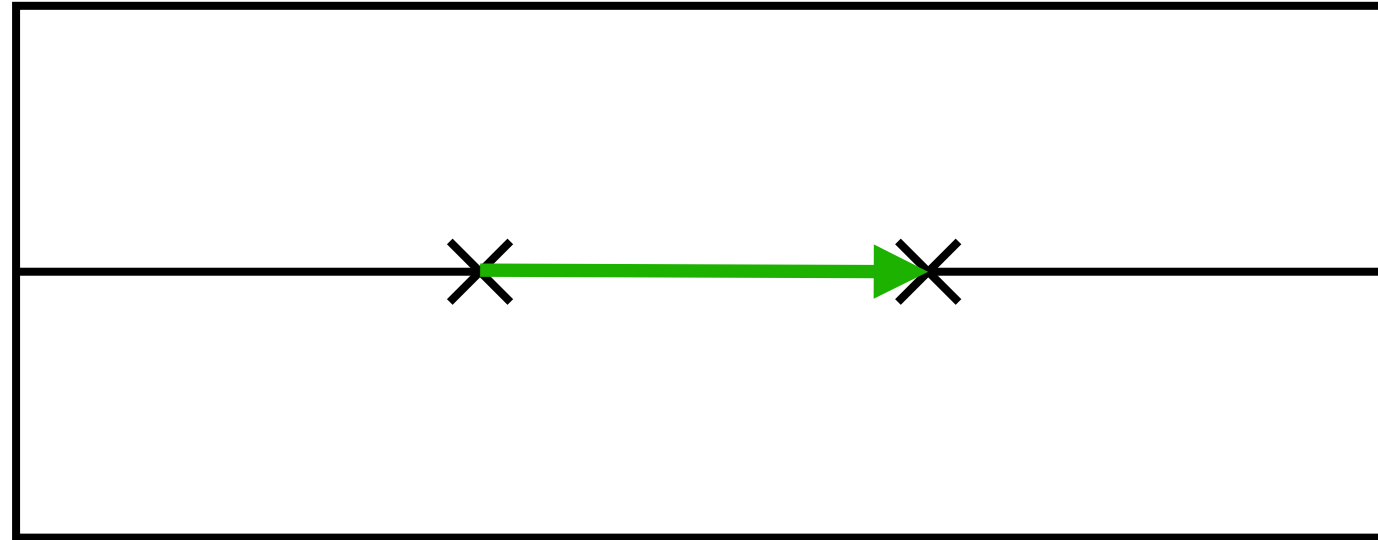
Exact

$$\frac{\Gamma(-1 - s/2) \Gamma(-1 - t/2)}{\Gamma(-2 - s/2 - t/2)}$$

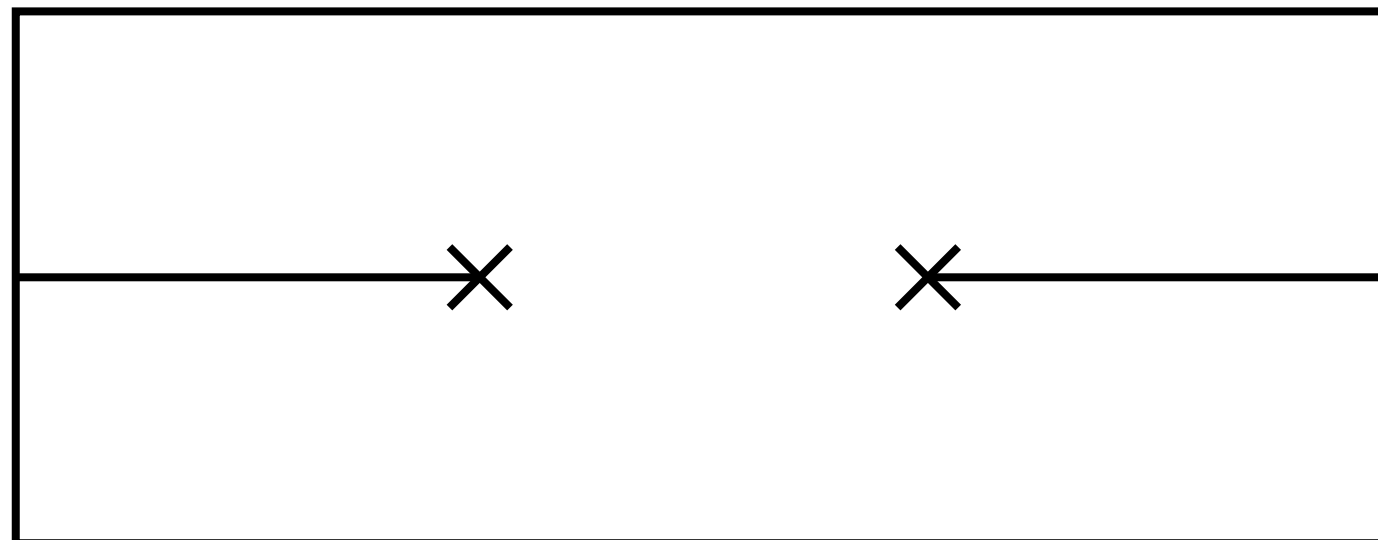
3-2-4. Complex saddles detect poles/zeros

[Witten]

$$g \int_0^1 dz \exp [p_1 \cdot p_2 \ln |z| + p_1 \cdot p_3 \ln |1 - z|]$$



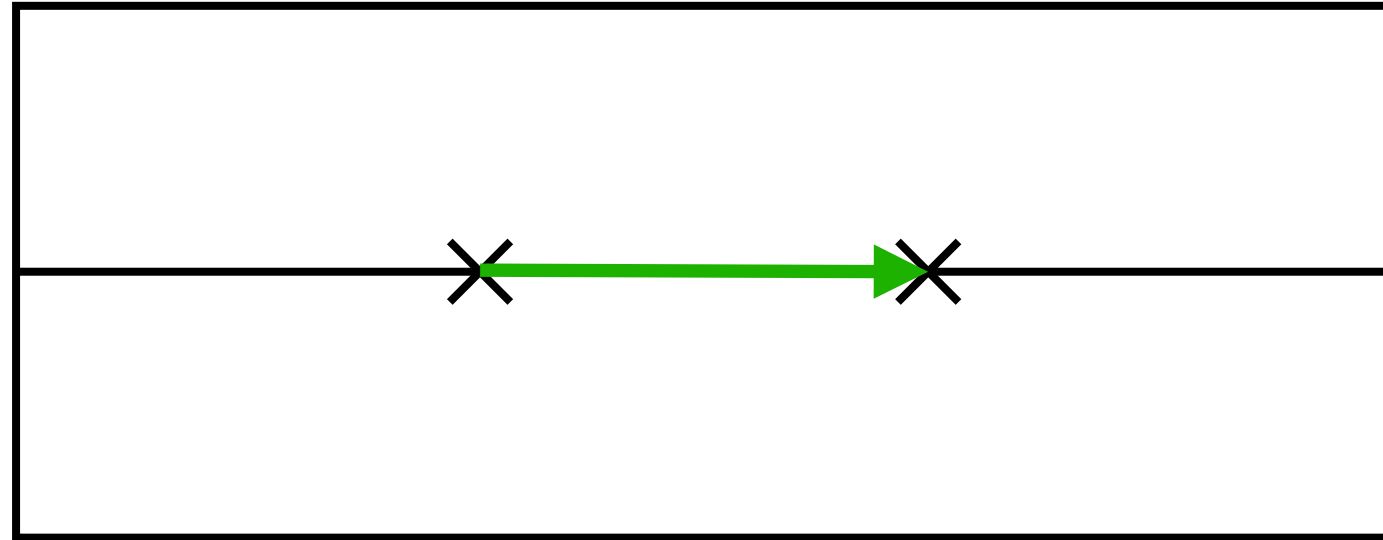
$$g \int_{C_\epsilon} dz \exp [(p_1 \cdot p_2 - i\epsilon) \ln |z| + (p_1 \cdot p_3 - i\epsilon) \ln |1 - z|]$$



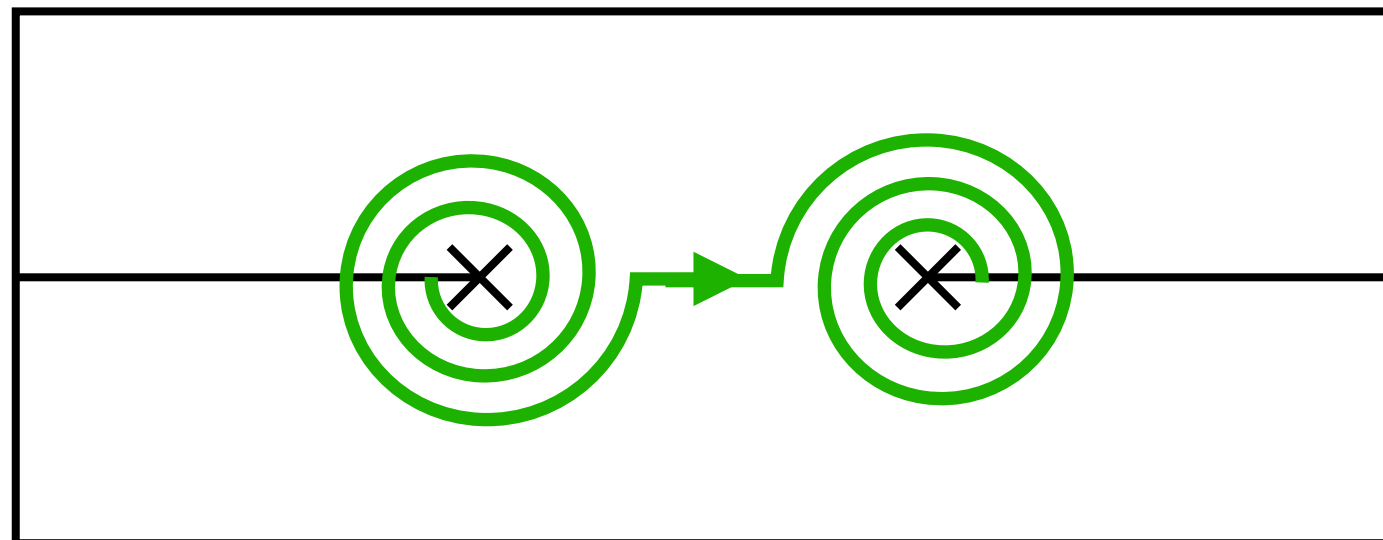
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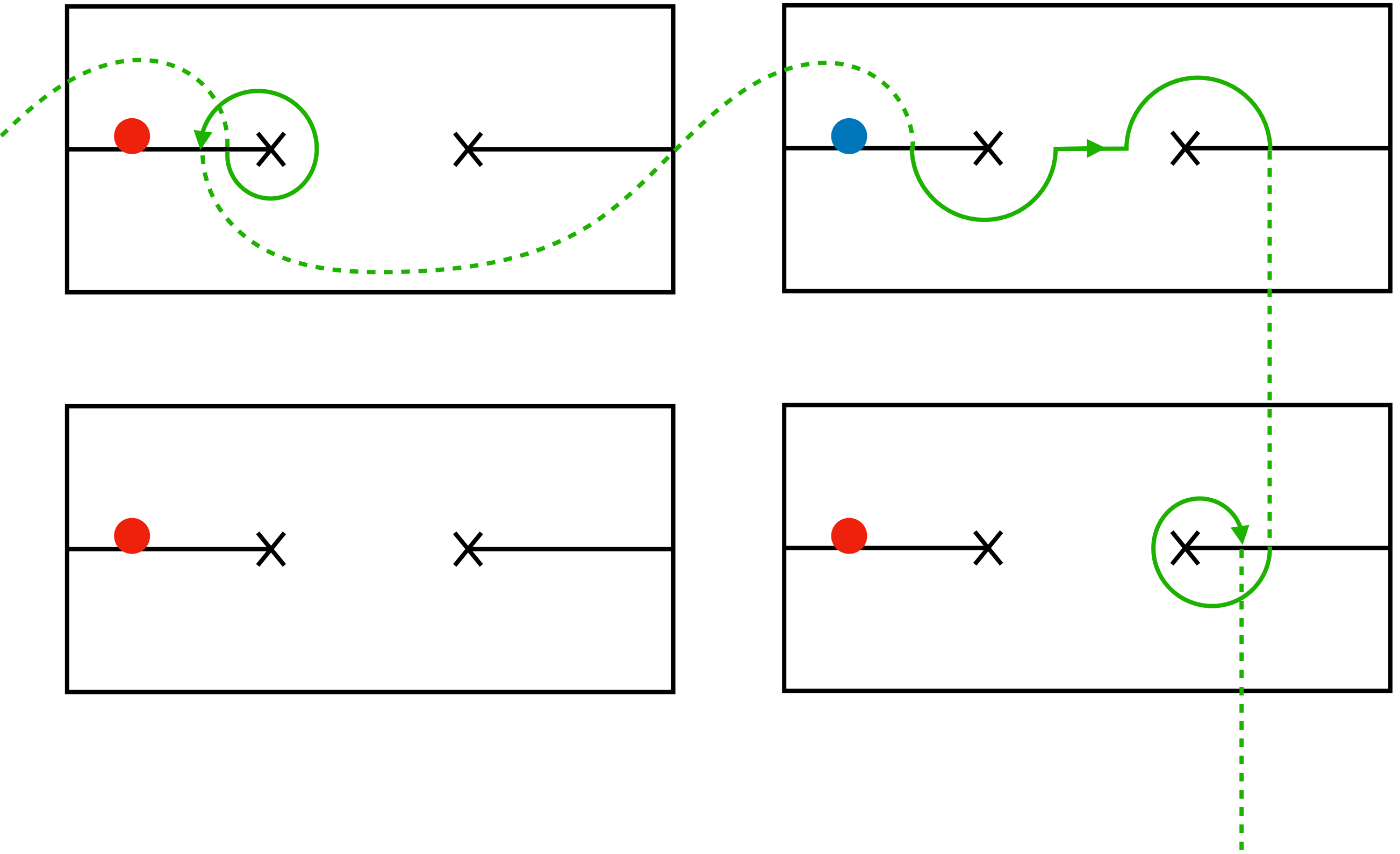


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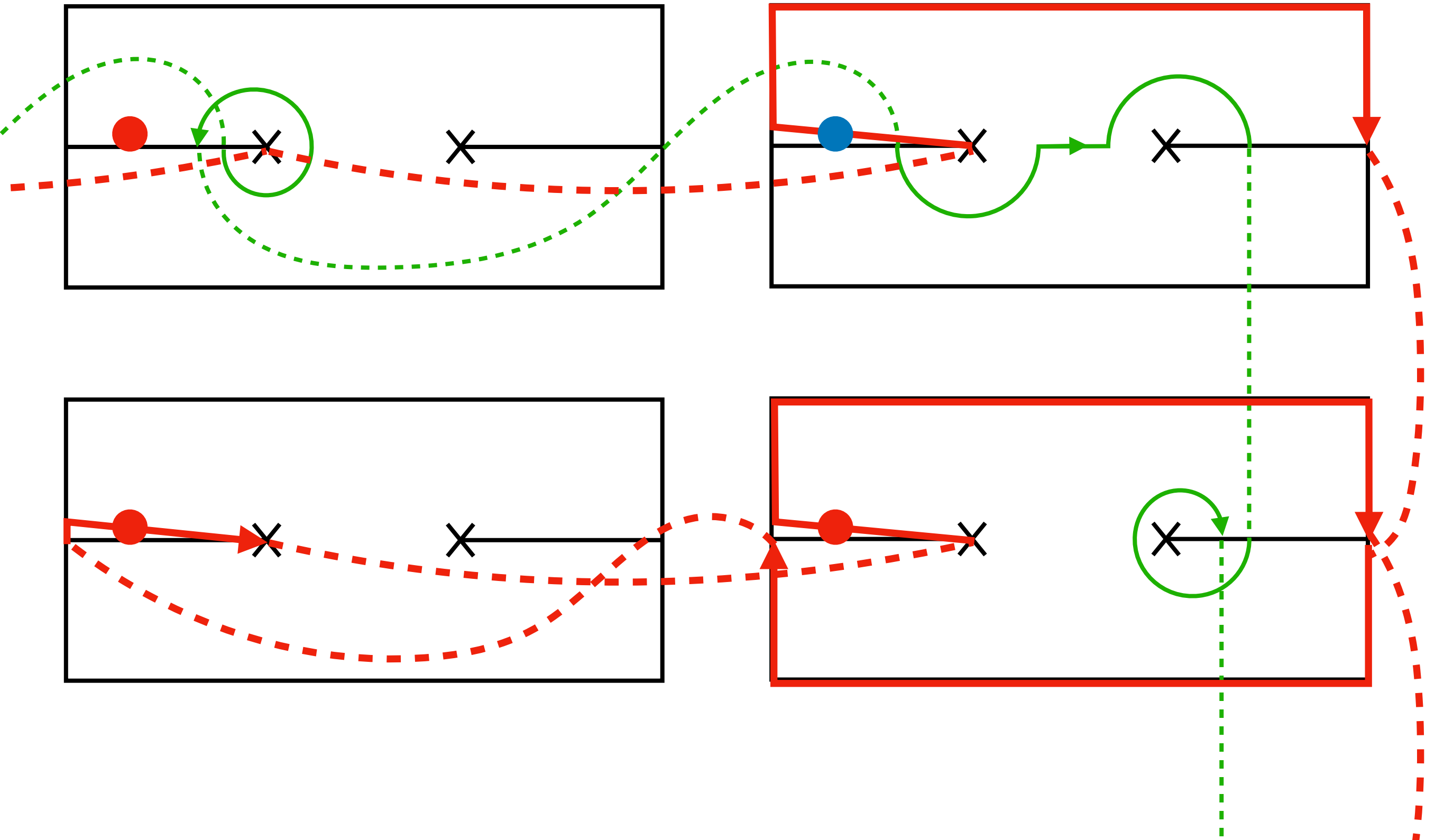
3-2-4. Complex saddles detect poles/zeros

[TY, in pr



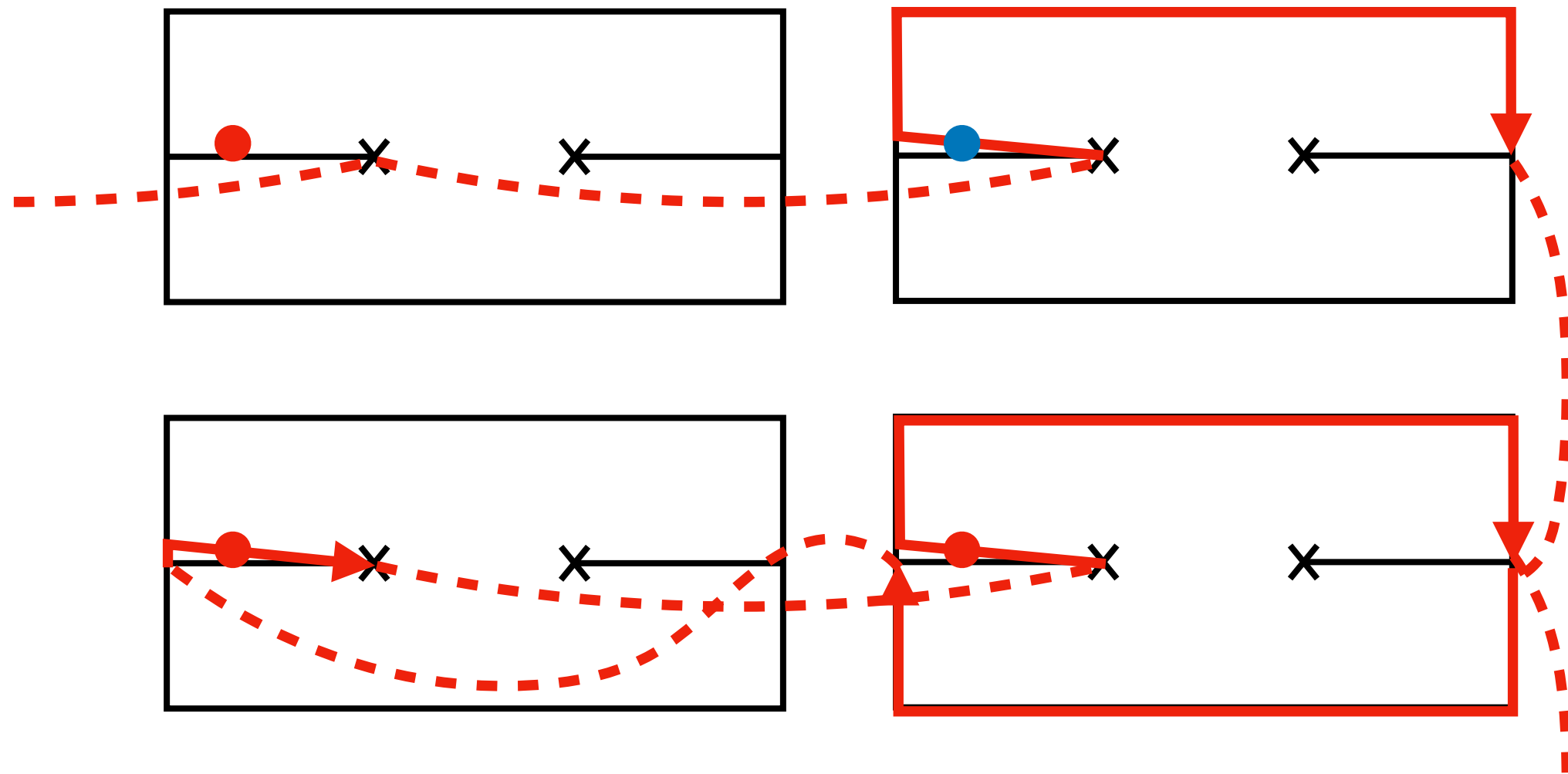
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[TY, in pr



3-2-4. Complex saddles detect poles/zeros

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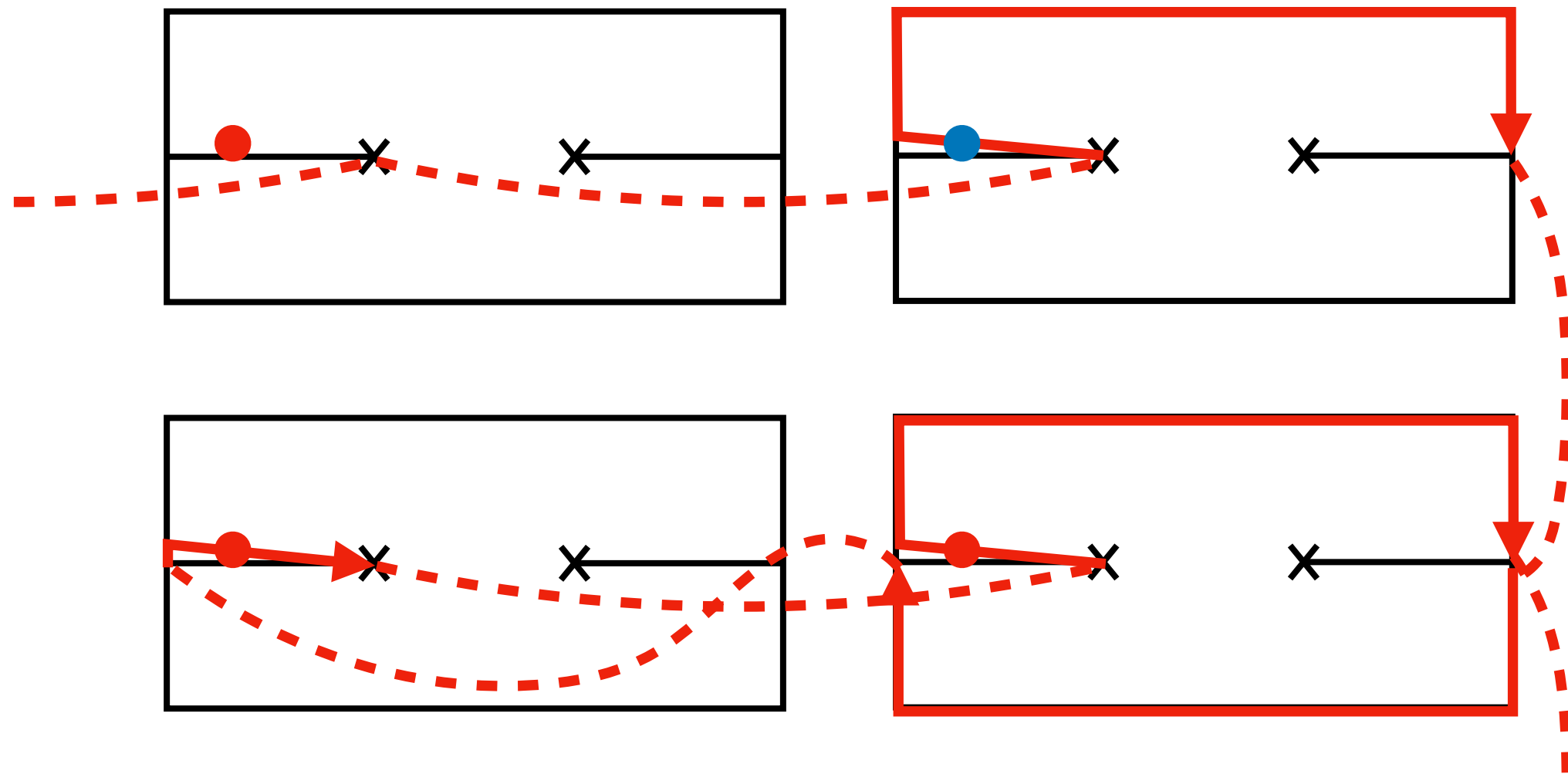


$$\mathcal{A}_{\text{tree}}^{(st)} \simeq (\text{fluc.}) \exp \left[-\frac{1}{2}(s \ln s + t \ln t + u \ln u) \right] \left[1 + \sum_{n \leq -1} e^{2\pi i n(-2-s/2)} - \sum_{m \leq -1} e^{-2\pi i(-2-t/2)+2\pi i m(-2-s/2)} \right]$$

$$= (\text{fluc.}) \frac{1 - e^{2\pi i(4+s/2+t/2)}}{1 - e^{2\pi i(2+s/2)}} \exp \left[-\frac{1}{2}(s \ln s + t \ln t + u \ln u) \right]$$

3-2-4. Complex saddles detect poles/zeros

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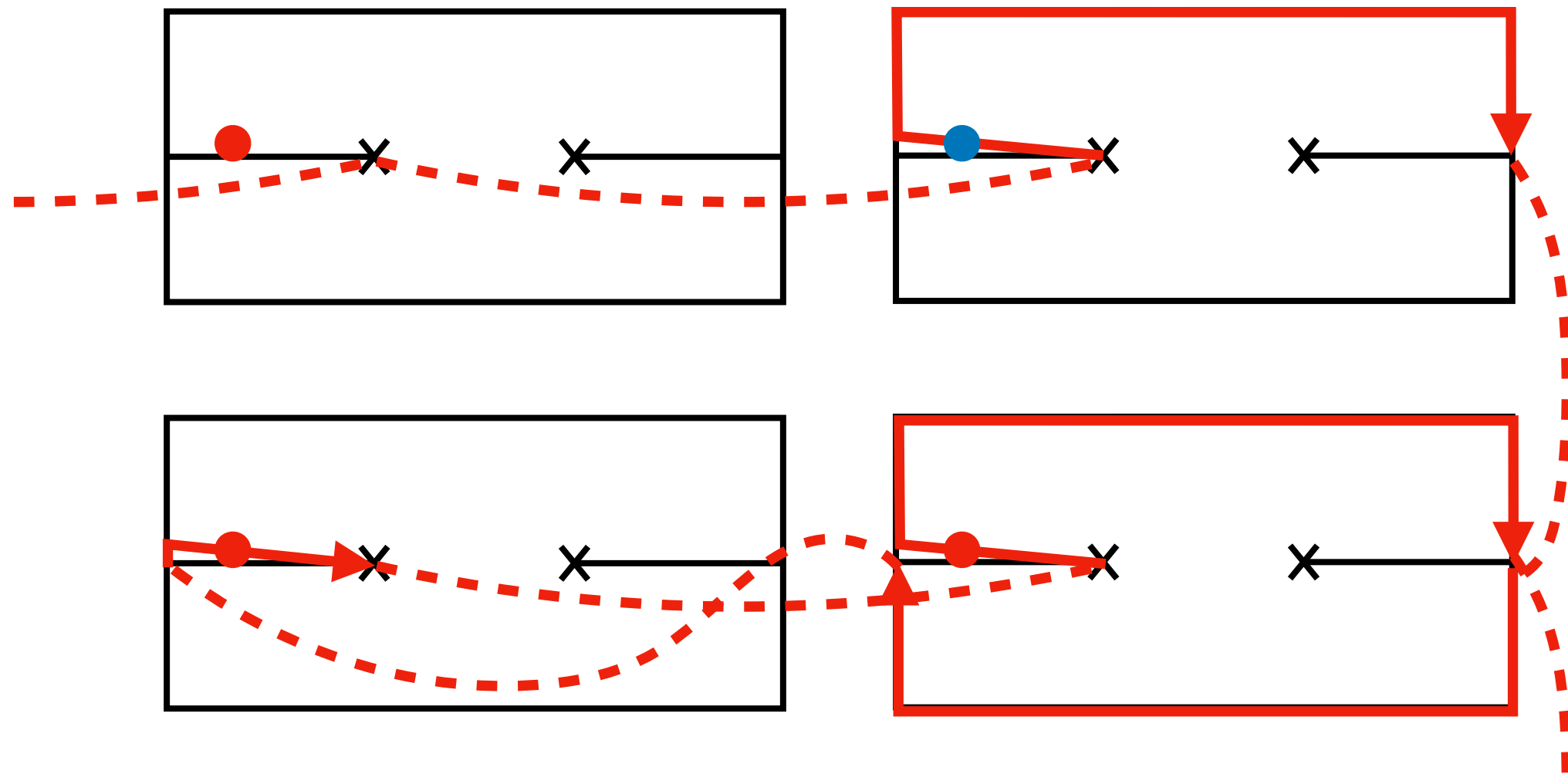


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poles/zeros

3-2. Strategy for higher-genus corrections

[TY, in progress]

Complex saddles tell us poles/zeros

3-2-1. Corrections are notoriously hard to compute

3-2-2. Saddle approximation of string amplitudes

3-2-3. Real saddle is insufficient to detect poles/zeros

3-2-4. Complex saddles detect poles/zeros

Toward chaos in string scatterings

1. Is single string a chaotic scatterer?
2. Chaotic scatterings & fractal
- 3. Analysis in string scatterings**
 - 3.1. Tree-level
 - 3.2. Strategy for higher-genus corrections
 - 3.3. String-like melon diagrams**

3-3. String-like melon diagrams

[TY, in preparation]

String-like melon diagrams exhibit fractal behavior

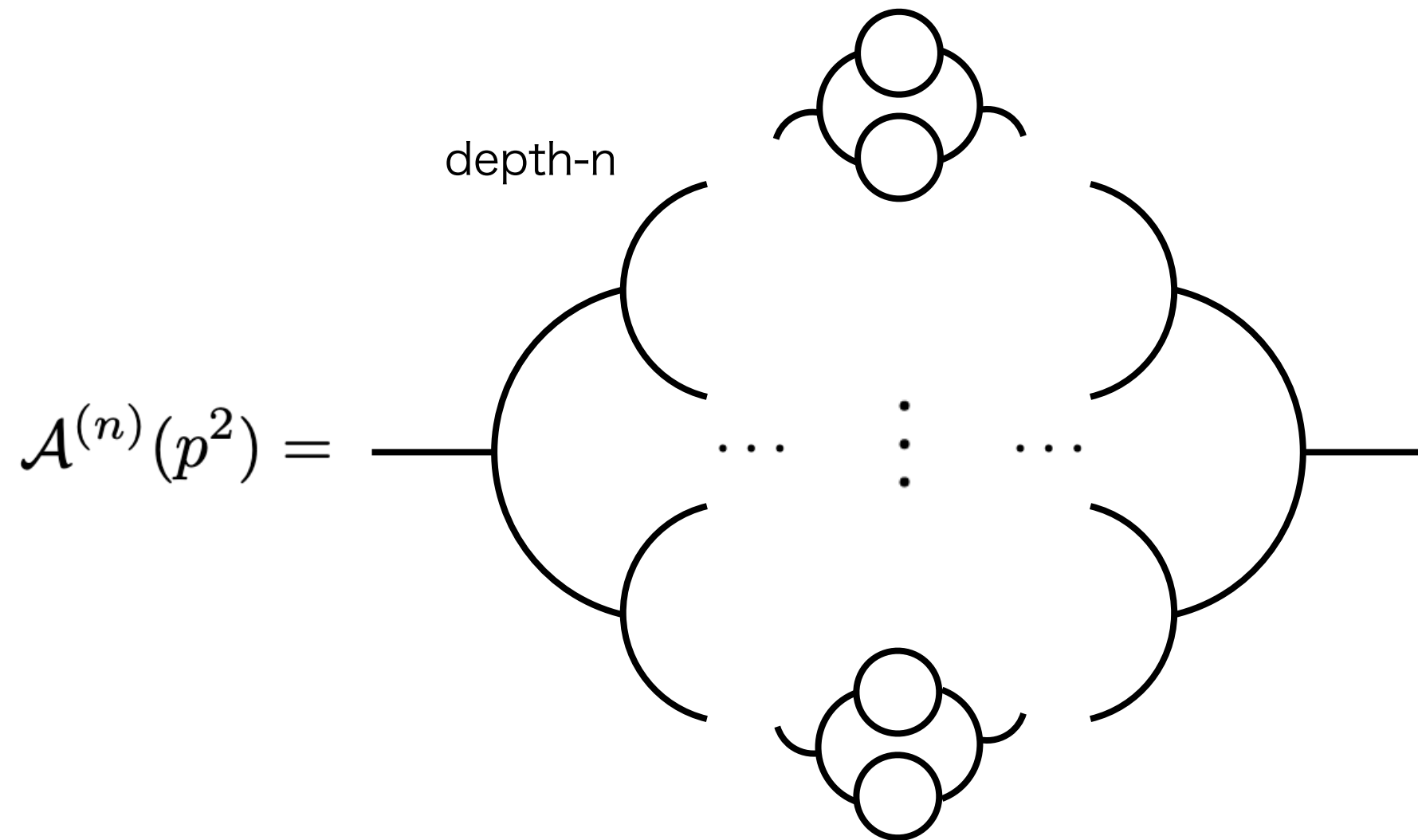
3-3-1. A model to mimic stringy mass tower

3-3-2. Mass tower makes amplitude periodic

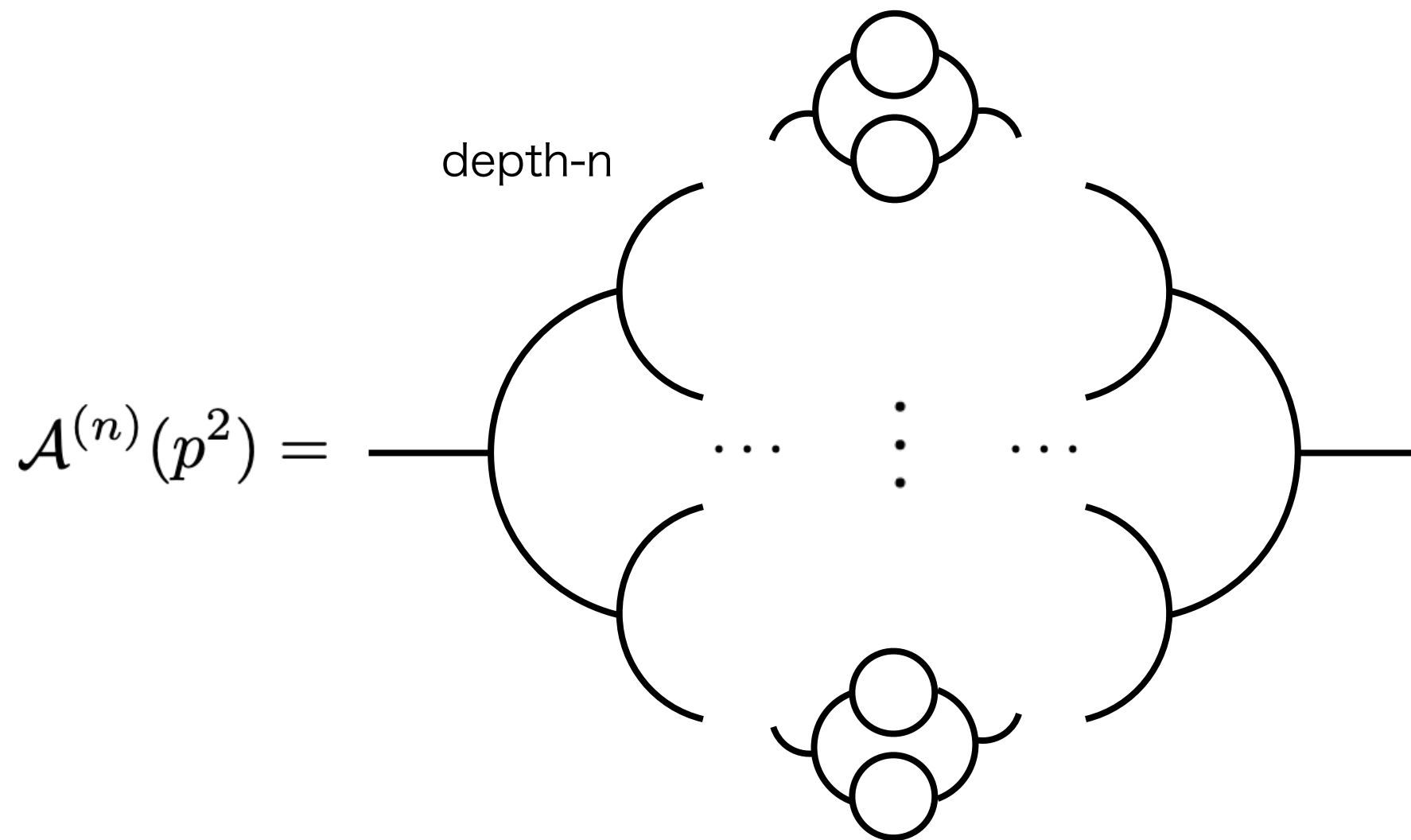
3-3-3. Nested melon diagrams exhibit fractal behavior

3-3-4. Fractal when classically conformal!?

3-3-3. Nested melon diagram exhibits fractal behavior



3-3-3. Nested melon diagram exhibits fractal behavior

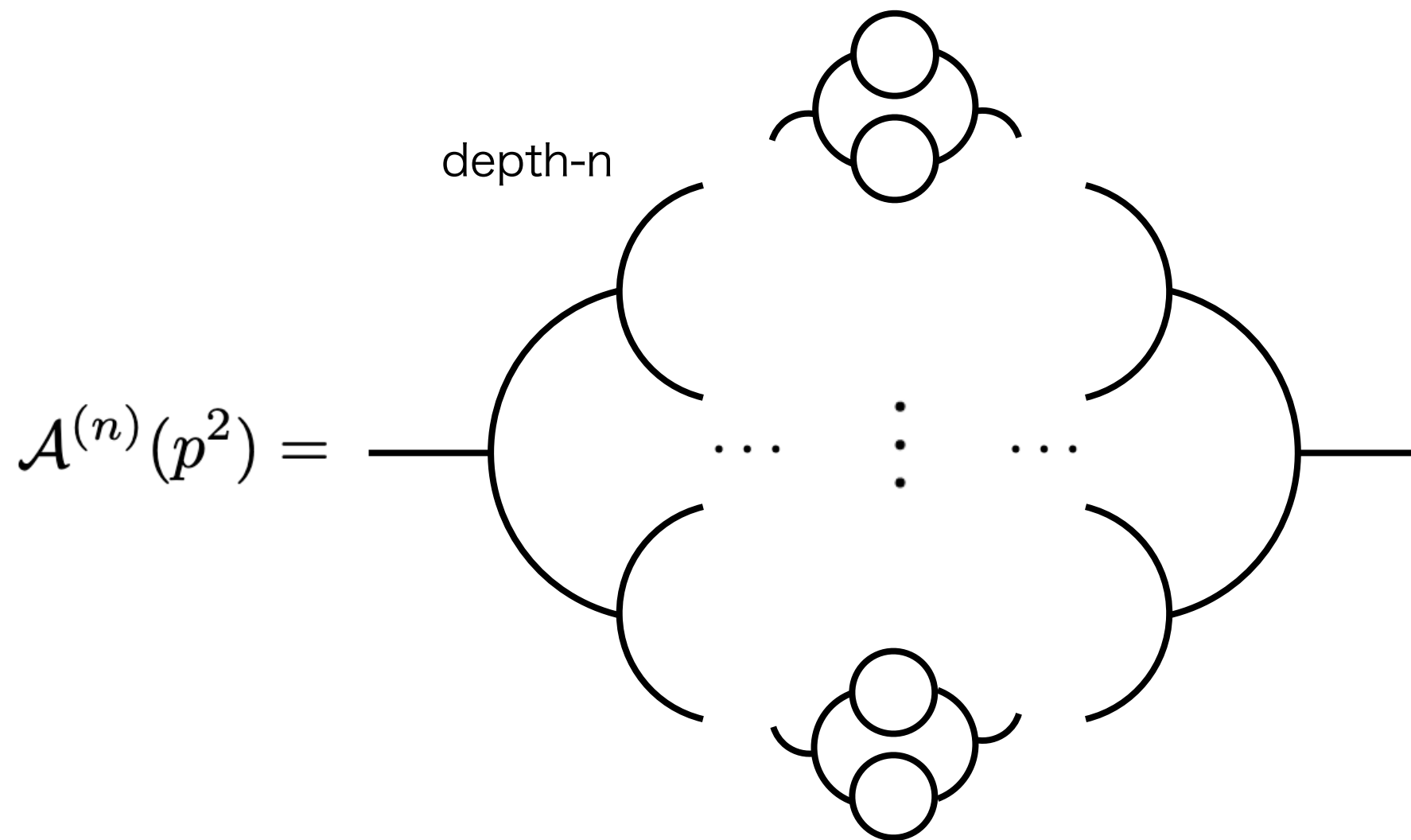


saddle pt. approx. and assuming large p, large d

$$\sim \Gamma(-k) k! \sum_{m \notin 4\mathbb{Z}} \sum_{i=1}^{n+1} c_{4^{n-i+1}m}(\xi, k) \left(\frac{1/4^{n-i+1}}{2\pi i m} \right)^{k+1} e^{-2\pi i m \cdot 4^{n-i+1} \xi} + \dots$$

$$\xi = \frac{3 - 2^{-n+1}}{2^{n+1}} \alpha' p^2 \quad k = (2^{n+1} - 1) \left(\frac{d}{2} - \frac{3 \cdot 2^{n+1} - 4}{2^{n+1} - 1} \right)$$

3-3-3. Nested melon diagram exhibits fractal behavior



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3-3-4. Estimation of fractal dimension

Neat model: Weierstrass function

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(2\pi b^n x) \quad b > 1, \frac{1}{b} < a < 1$$

fractal dimension $D = 2 + \frac{\ln a}{\ln b}$

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Nested melon diagram

$$\sum_{i=1}^n \left(\frac{1}{2\pi i m \cdot 4^{n-i+1}} \right)^{k+1} e^{-2\pi i m \cdot 4^{n-i+1} \xi}$$

estimated
fractal dimension $D \sim 2 + \frac{4^{-k}}{4} = 2 - k$

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3-3-4. Fractal when classically conformal!?

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Nested melon diagram

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estimated
fractal dimension

$$D \sim 2 + \frac{4^{-k}}{4} = 2 - k$$

$1 < D < 2$ when the
theory is classically
conformal

$$d = 6 + \mathcal{O}(2^{-n-1})$$

$$k = (2^{n+1} - 1) \left(\frac{d}{2} - \frac{3 \cdot 2^{n+1} - 4}{2^{n+1} - 1} \right)$$

3-3. String-like melon diagrams

[TY, in preparation]

String-like melon diagrams exhibit fractal behavior

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Toward chaos in string scatterings

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Toward chaos in string scatterings

Take-home messages

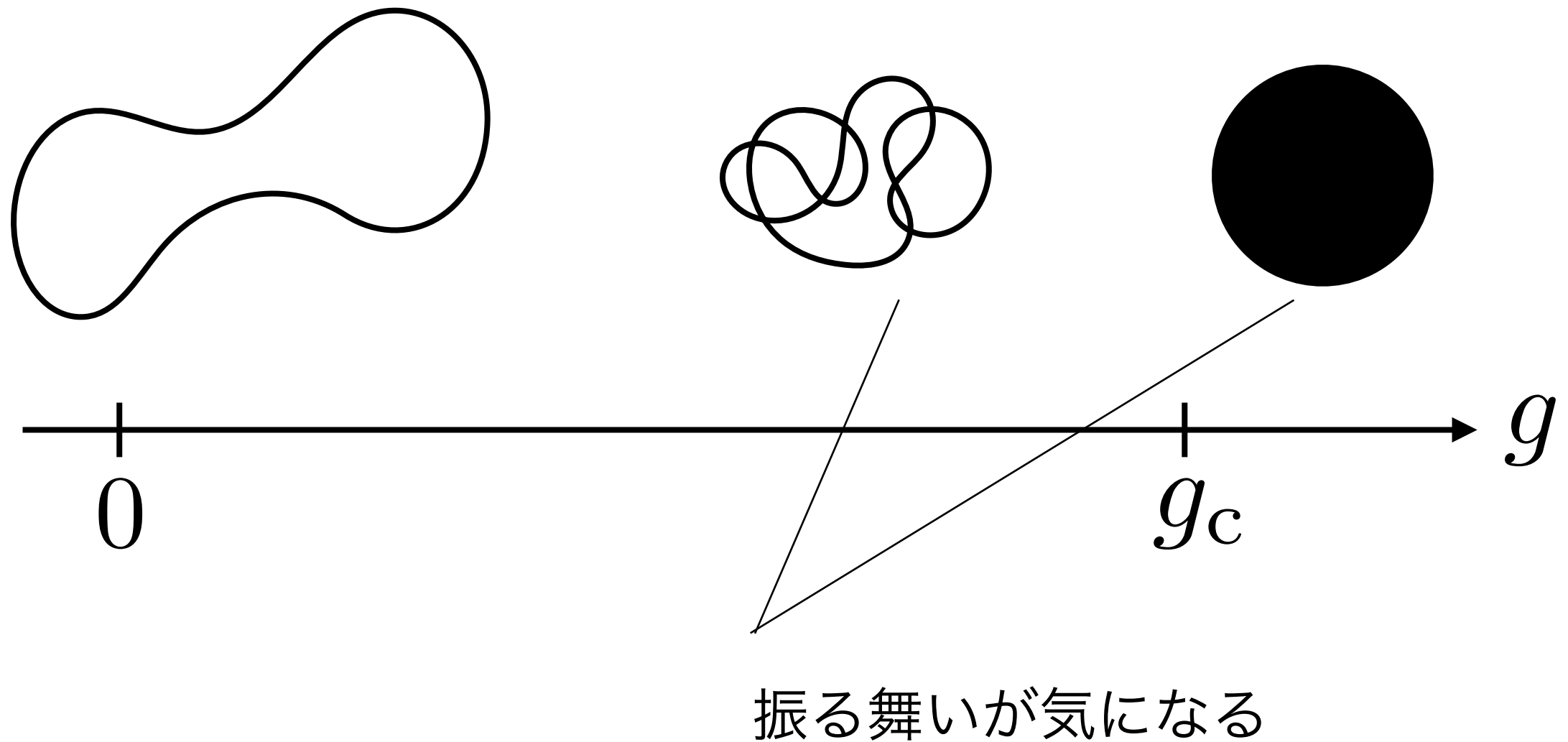
- Tree-level scattering is not chaotic
- Complex saddles tell us poles/zeros
- String mass tower may cause fractal

Backups

[動機] 弦とブラックホールの対応

弦は自重で潰れてブラックホールに転移する

[Horowitz, Polchinski, 97, 98. Amati, Russo, 99. Chen, Maldacena, 99.]



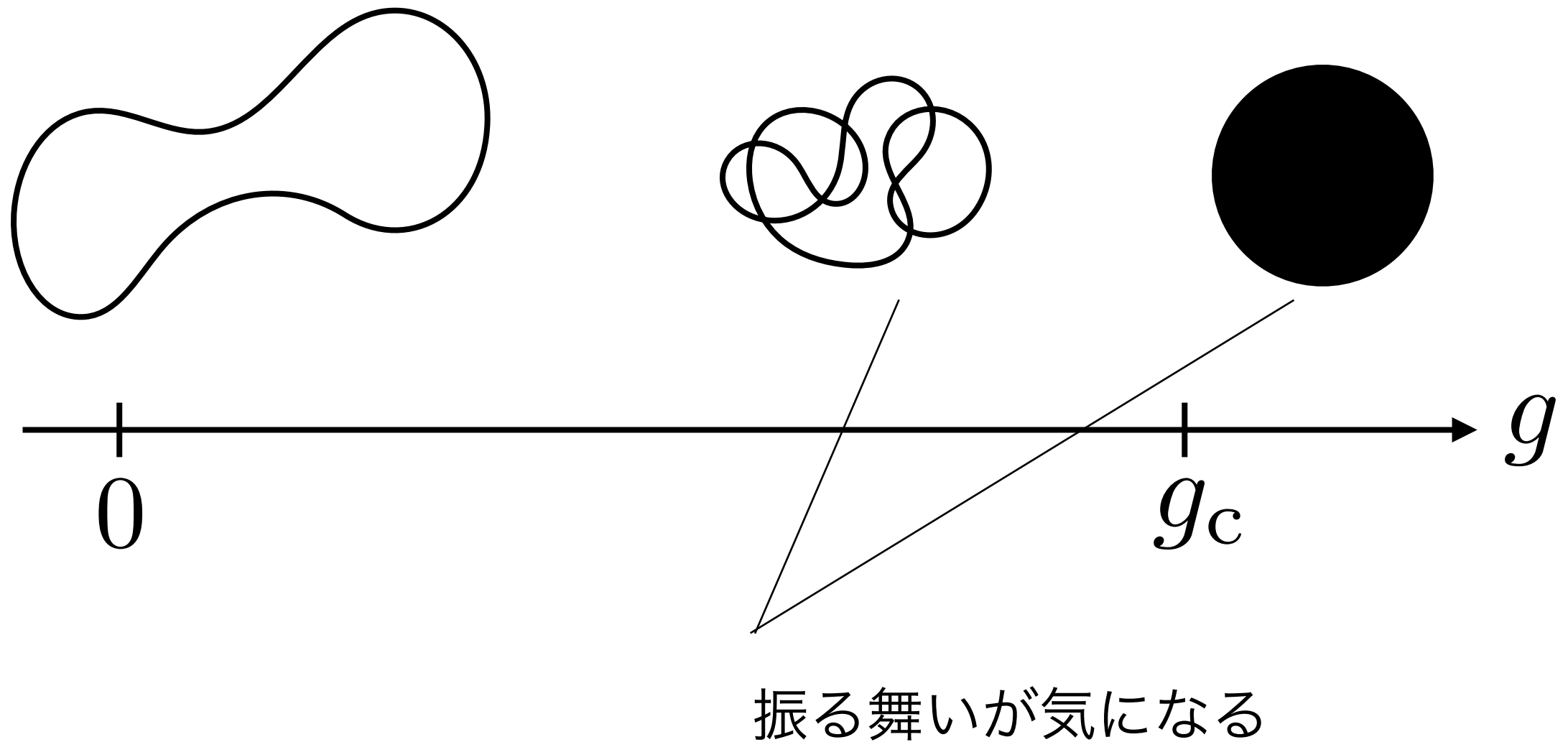
弦の散乱は
カオスか？

概観 [5枚]

[動機] 弦とブラックホールの対応

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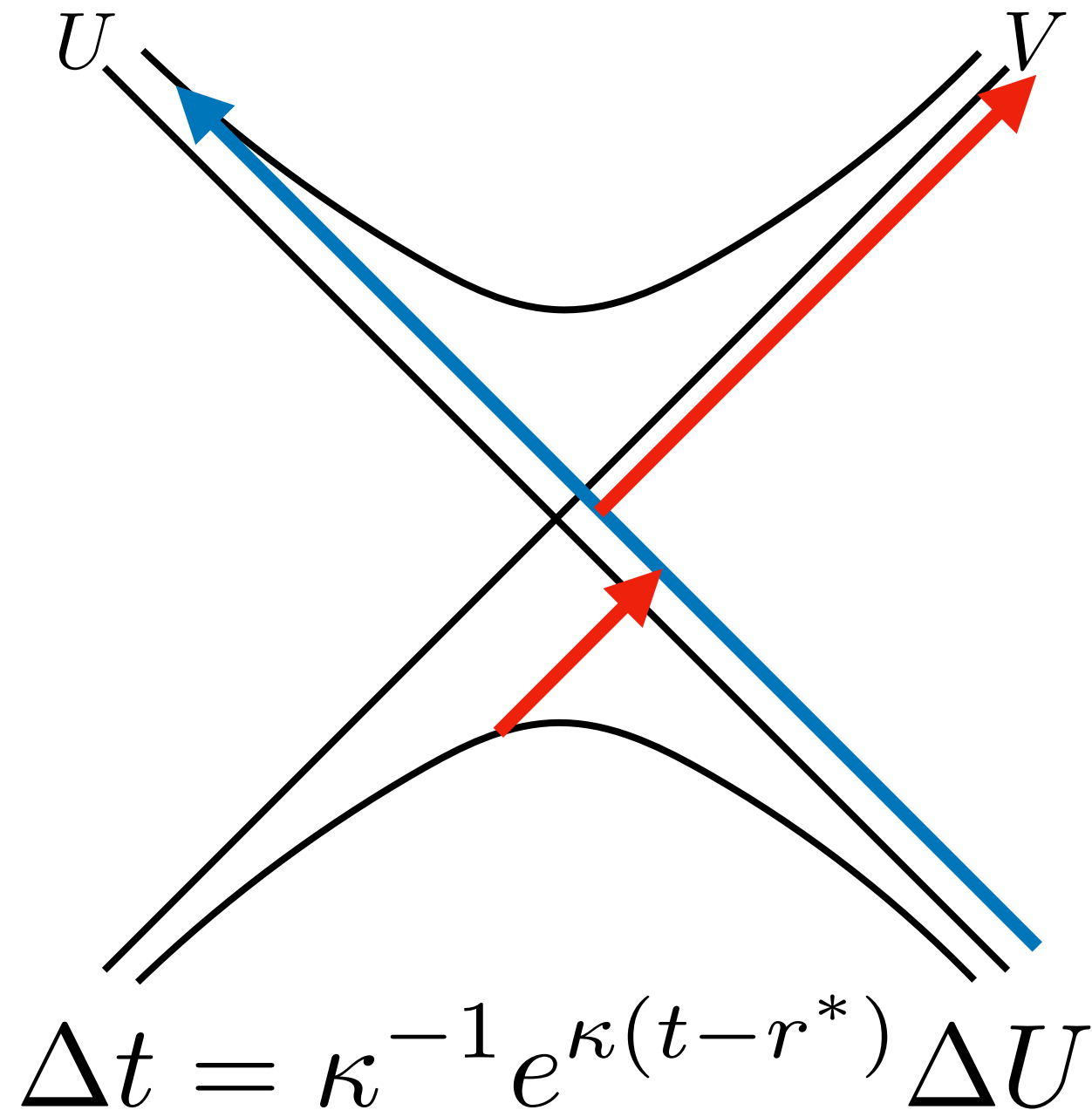
[Horowitz, Polchinski, 97, 98. Amati, Russo, 99. Chen, Maldacena, 99.]



[動機] 地平面近傍の赤方偏移=カオス？

[Polchinski 15. See also Polchinski 16

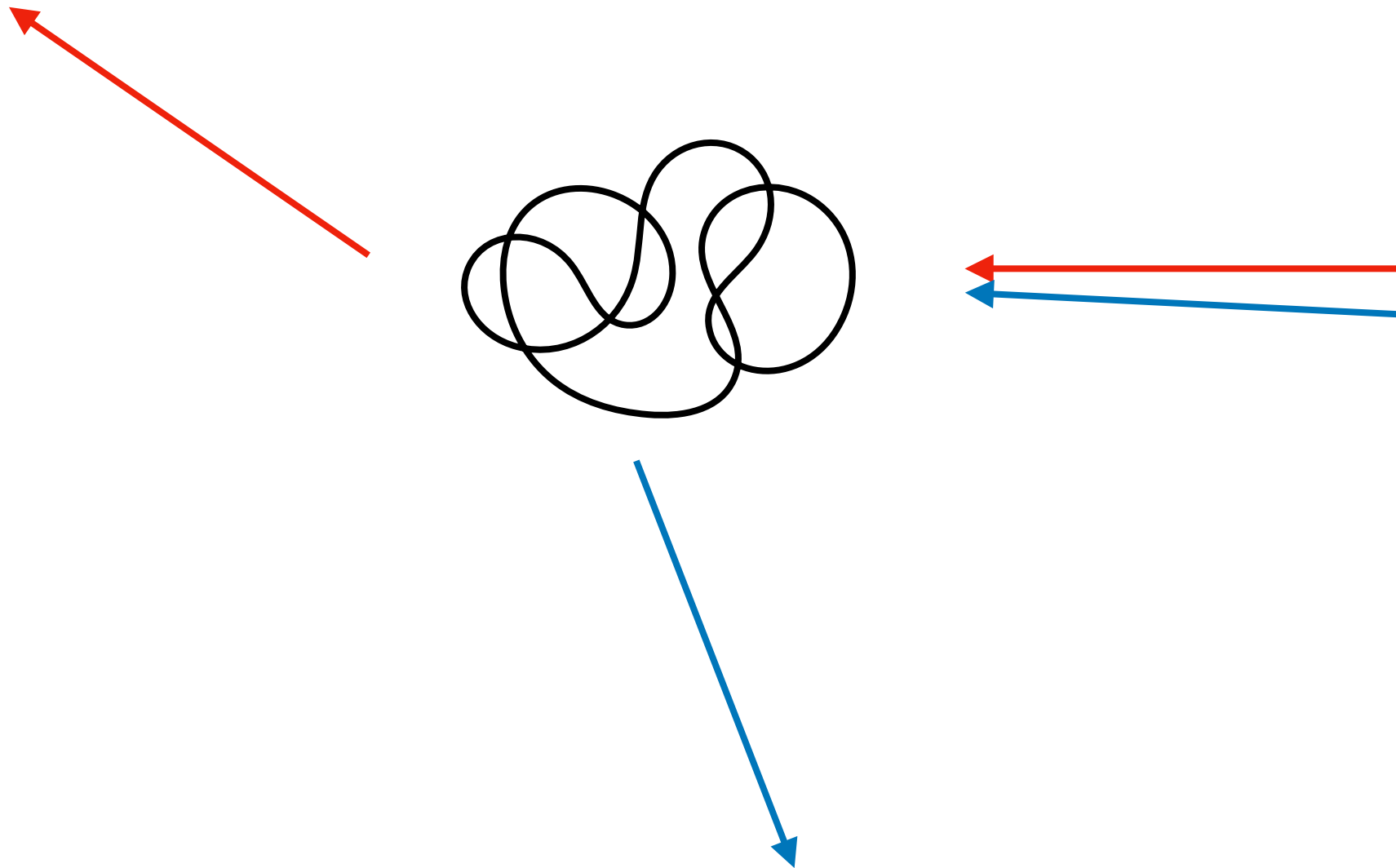
[Shenker, Stanford 14, 15. Kit



ホログラフィック双対における量子カ

[問] 弦の散乱はカオス？

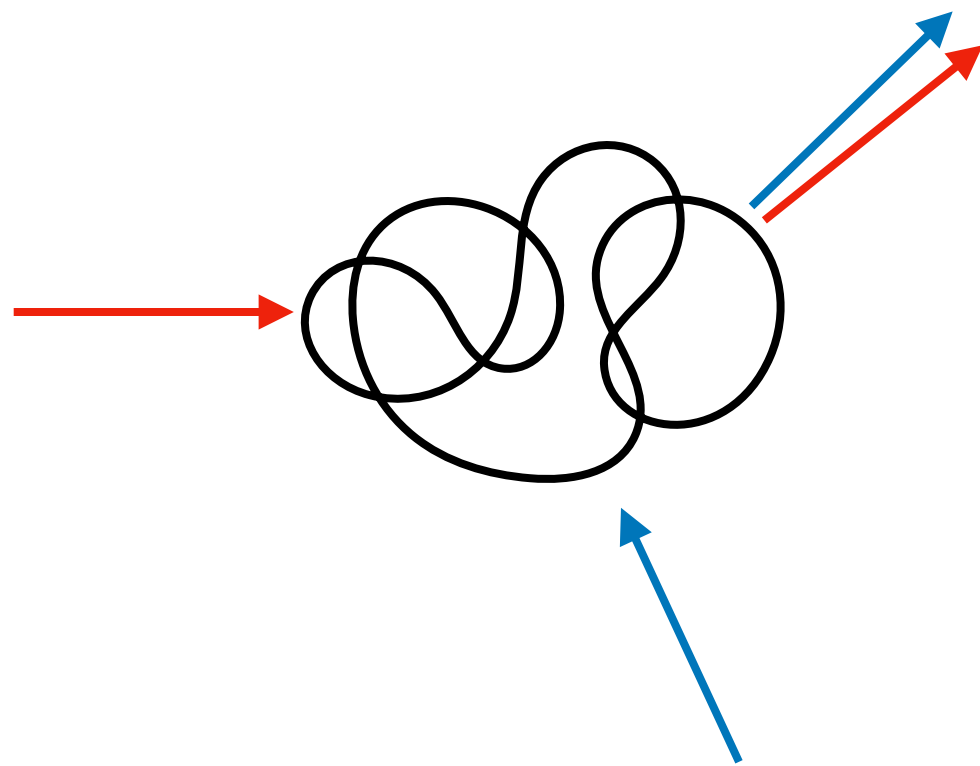
潰れた弦の散乱は初期値鋭敏？



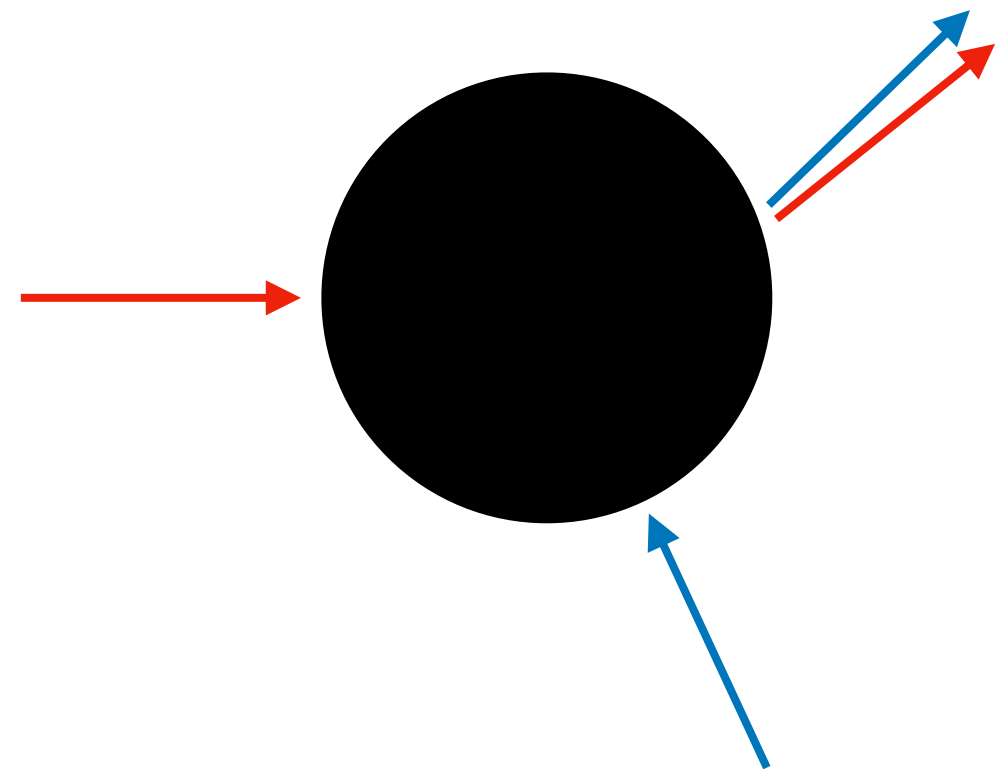
実際、高励起な弦の崩壊は極めて不規則

[意義] ブラックホールの熱的性質を説明!?

熱化や赤方偏移に関わる!?



- 入射角の**解説が困難**
- **長い**遅延時間



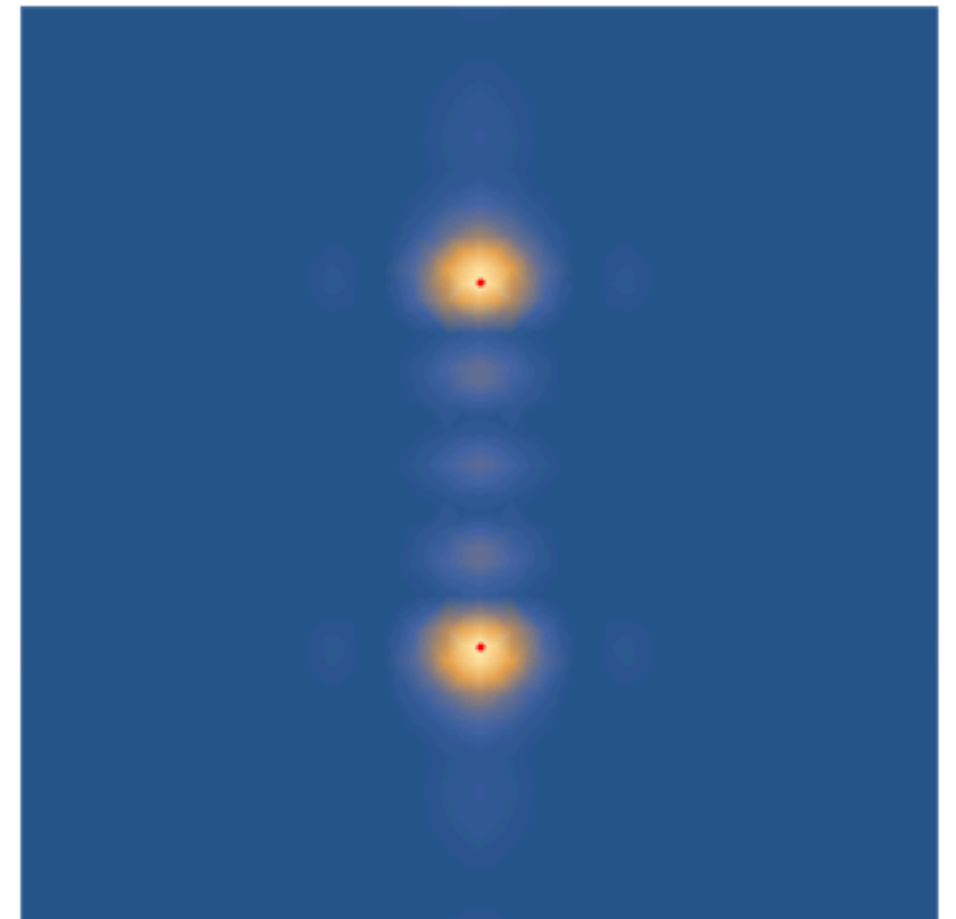
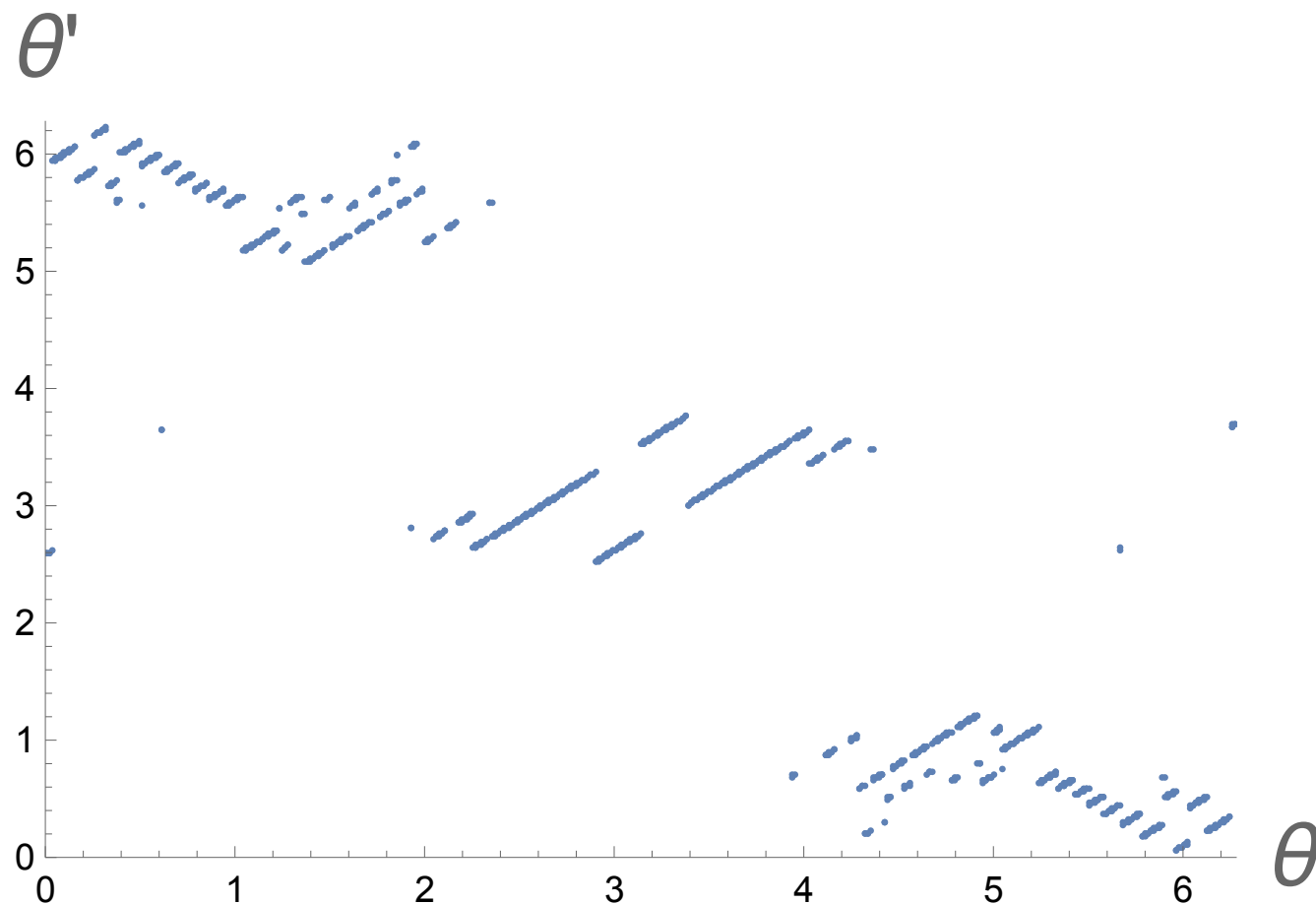
- 初期状態の**解説が困難**
- **長い**蒸発時間

[研究] 手始めにボソン開弦のツリーレベル

[Hashimoto, Matsuo, TY, 22, 22]

- 少なくともボソン開弦のツリーレベルでカオスは無い
- 高次補正や閉弦の多点散乱を考えるとカオスだと予想

1. 散乱データのフラクタル構造を探した 2. 散乱領域の幾何学的構造を見た



目次

序 章	概観	[5枚]
第1章	過渡的カオス	[8枚]
第2章	弦の散乱データとフラクタル構造	[12枚]
第3章	弦の散乱領域と幾何学的構造	[14枚]
終 章	まとめ	[3枚]
		[42枚]

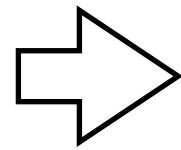
1/3章 過渡的カオス まとめ

- 過渡的カオス = 有限時間の間だけカオス
- 古典散乱におけるカオスの必要条件
 - 散乱データのフラクタル構造
 - 散乱領域の幾何学的構造

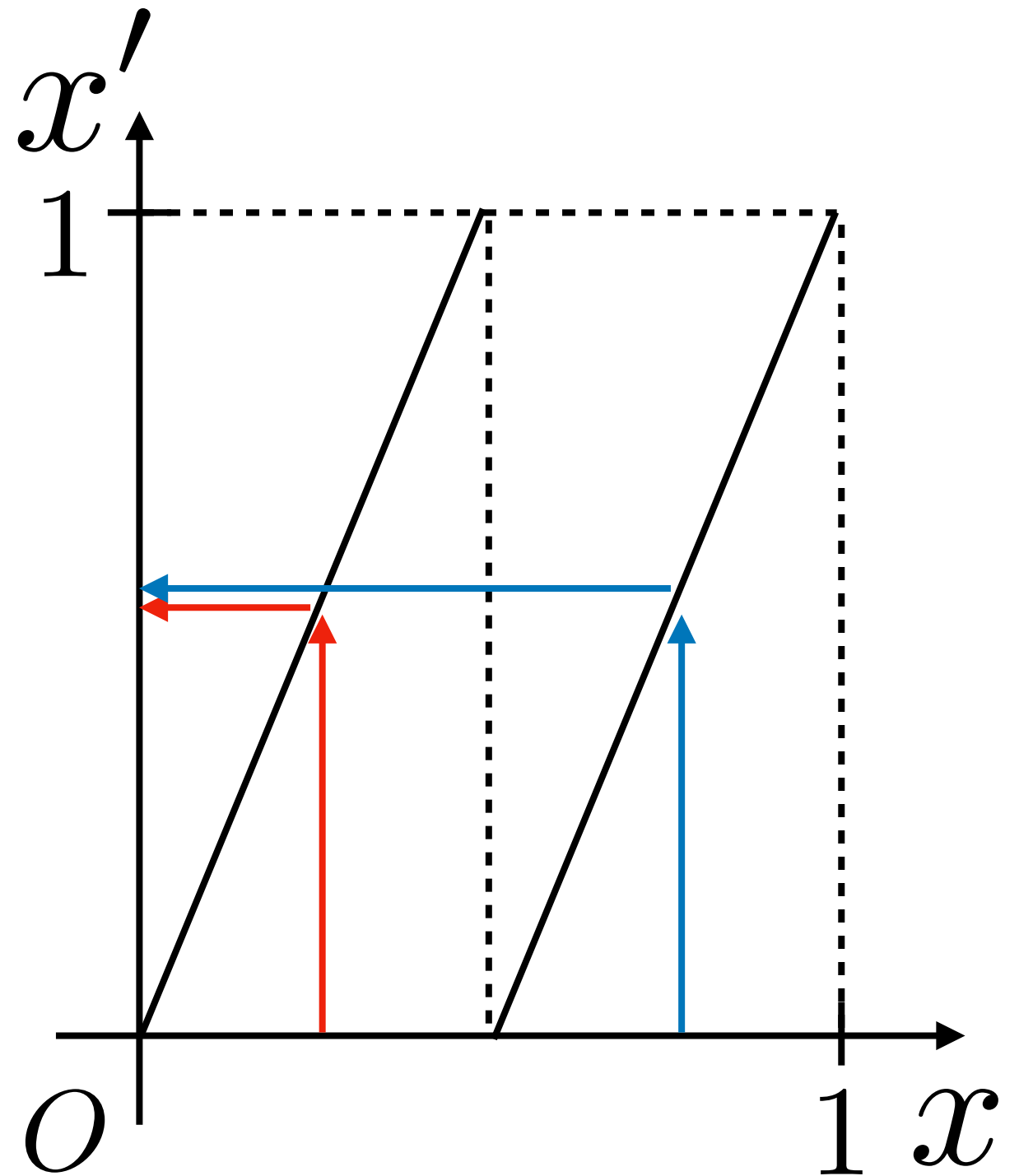
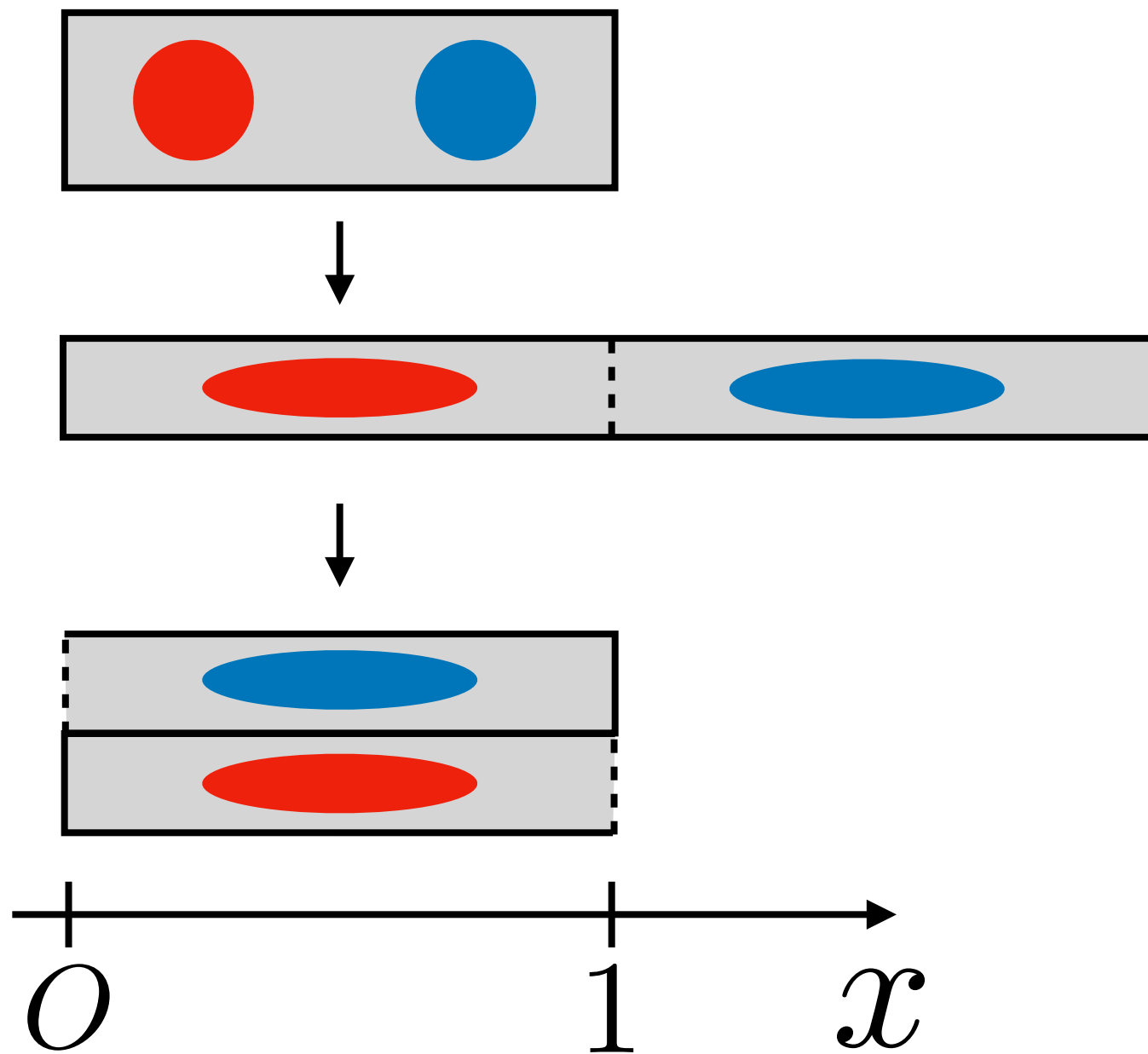
この章は[Lai, Tél, 11. Seoane, Sanjuán, 12]等に基づくレビュー

パン屋の写像とカオス

引き伸ばし
折りたたみ

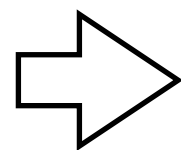


非周期性
初期値鋭敏性



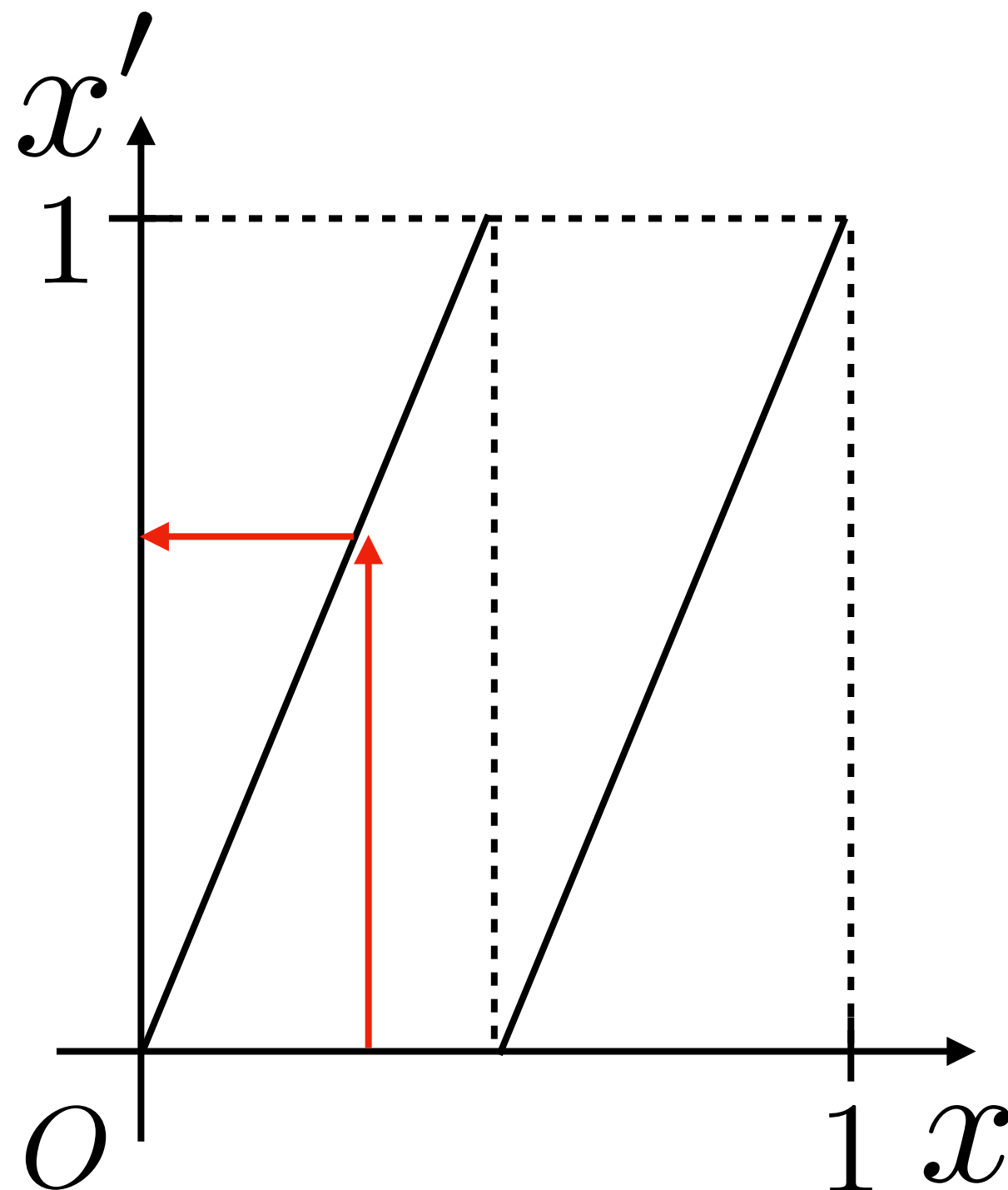
パン屋の写像とカオス

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折りたたみ



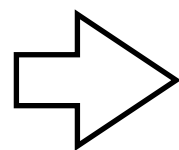
非周期性
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$$\begin{aligned}x &= (0.0101 \cdots x_{n+1} \cdots)_2 \\x' &= (0.101 \cdots x_{n+1} \cdots)_2 \\x'' &= (0.01 \cdots x_{n+1} \cdots)_2 \\&\vdots \\x^{(n)} &= (0.x_{n+1} \cdots)_2\end{aligned}$$



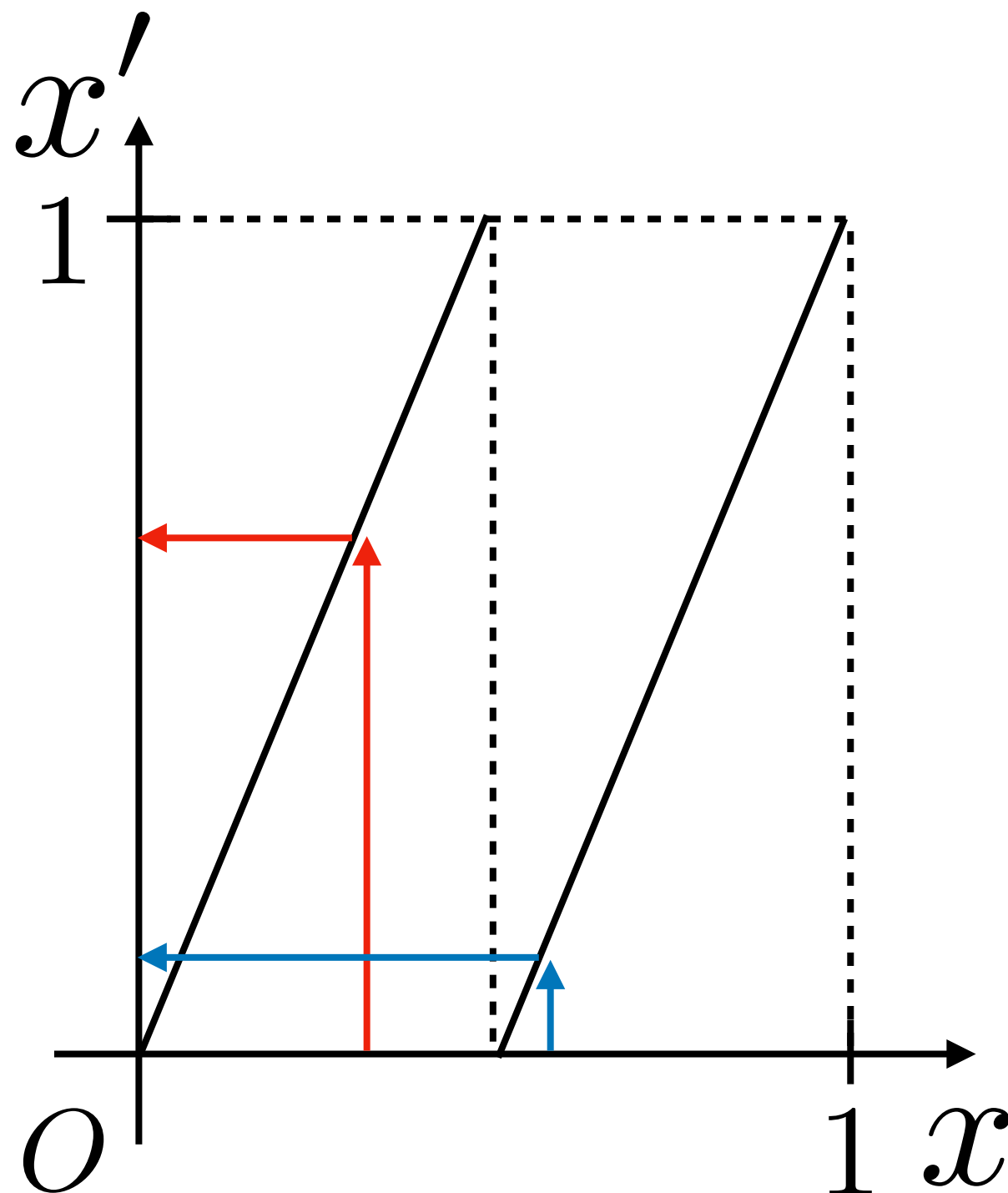
パン屋の写像とカオス

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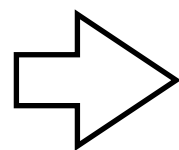
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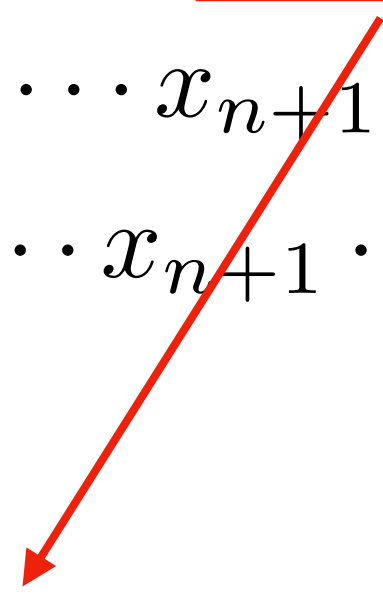


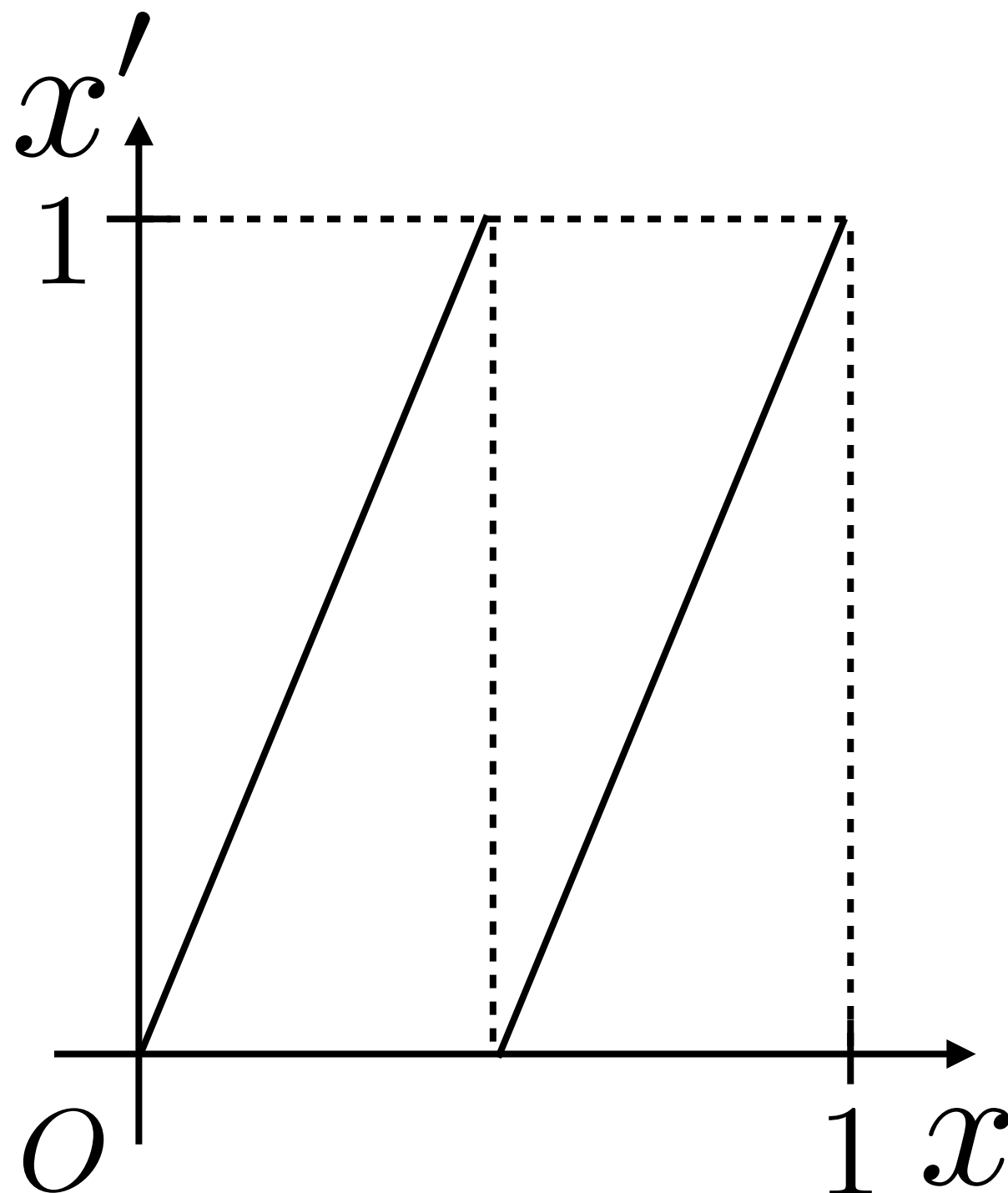
パン屋の写像とカオス

引き伸ばし
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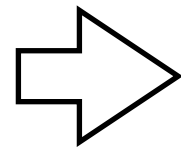
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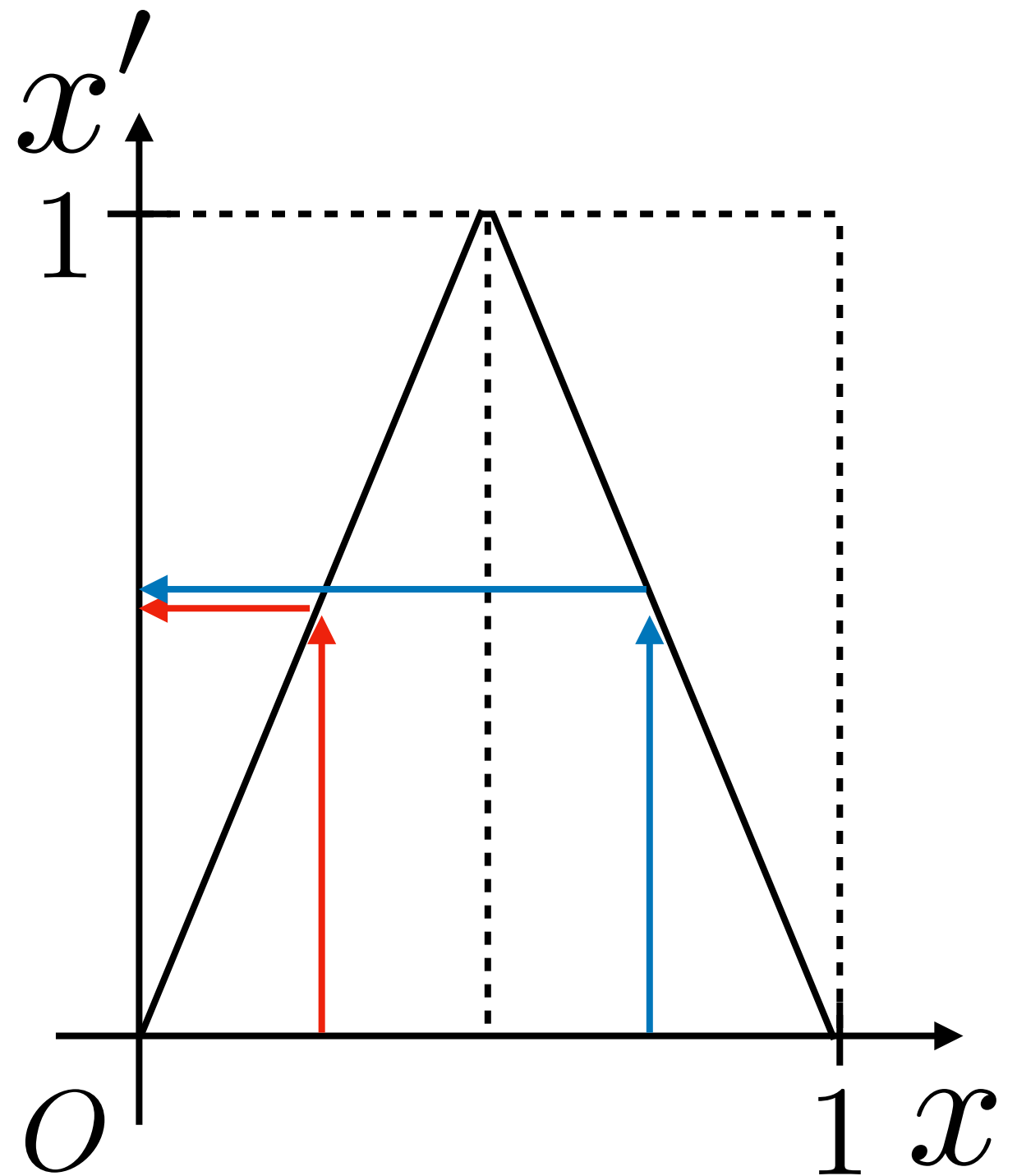
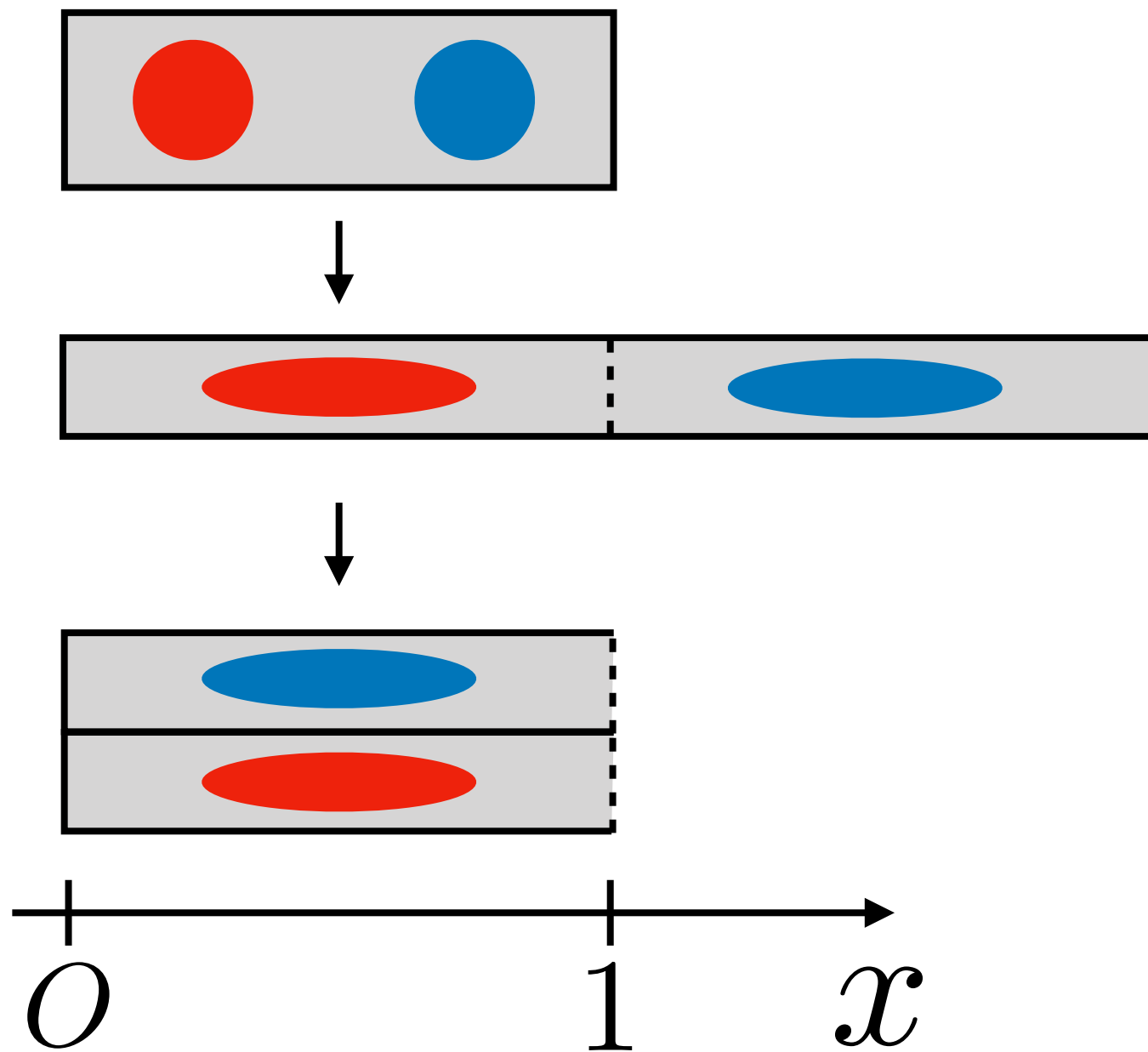


テント型の写像もカオス

引き伸ばし
折りたたみ

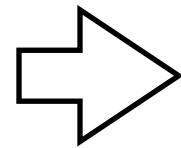


非周期性
初期値鋭敏性

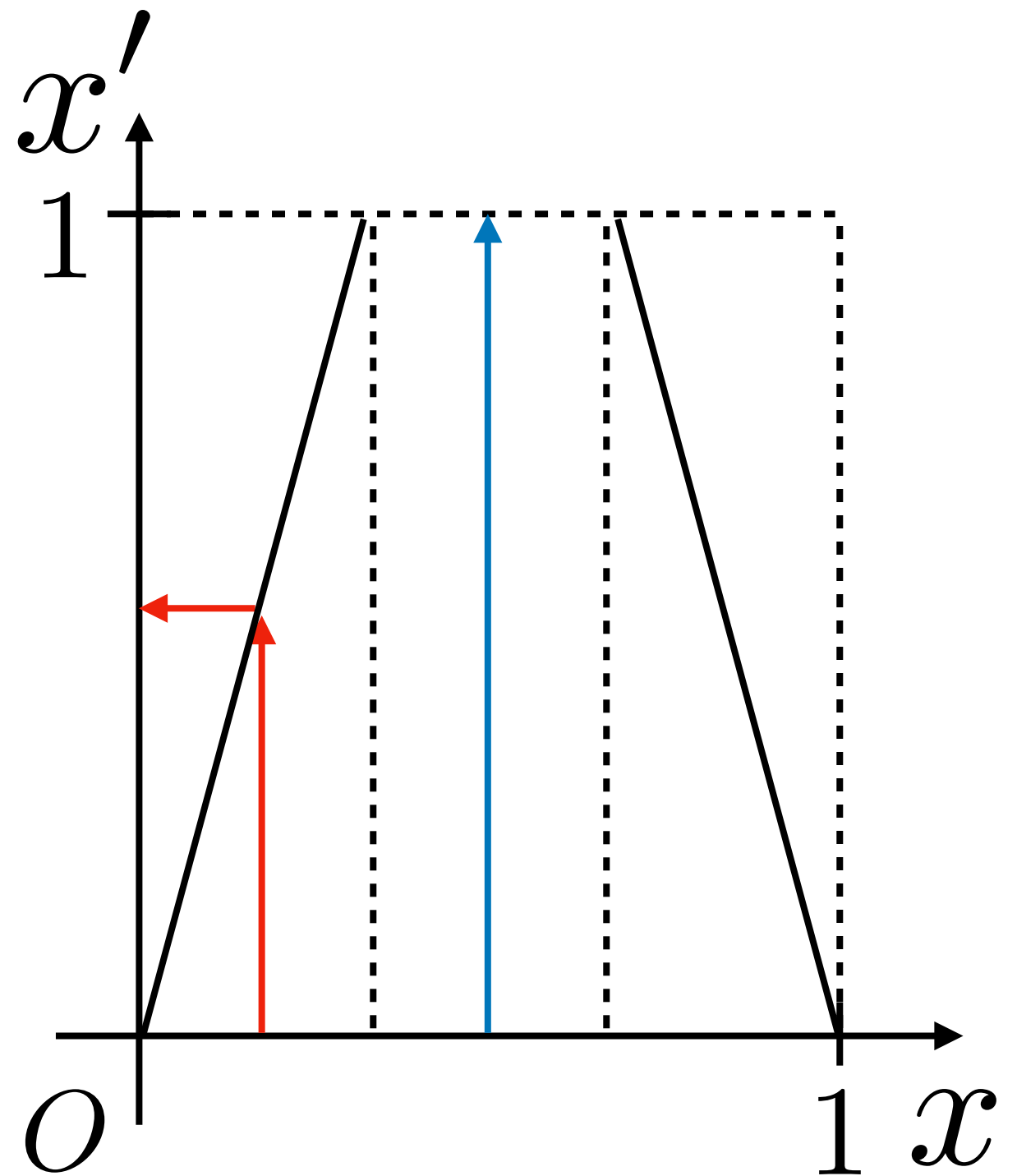
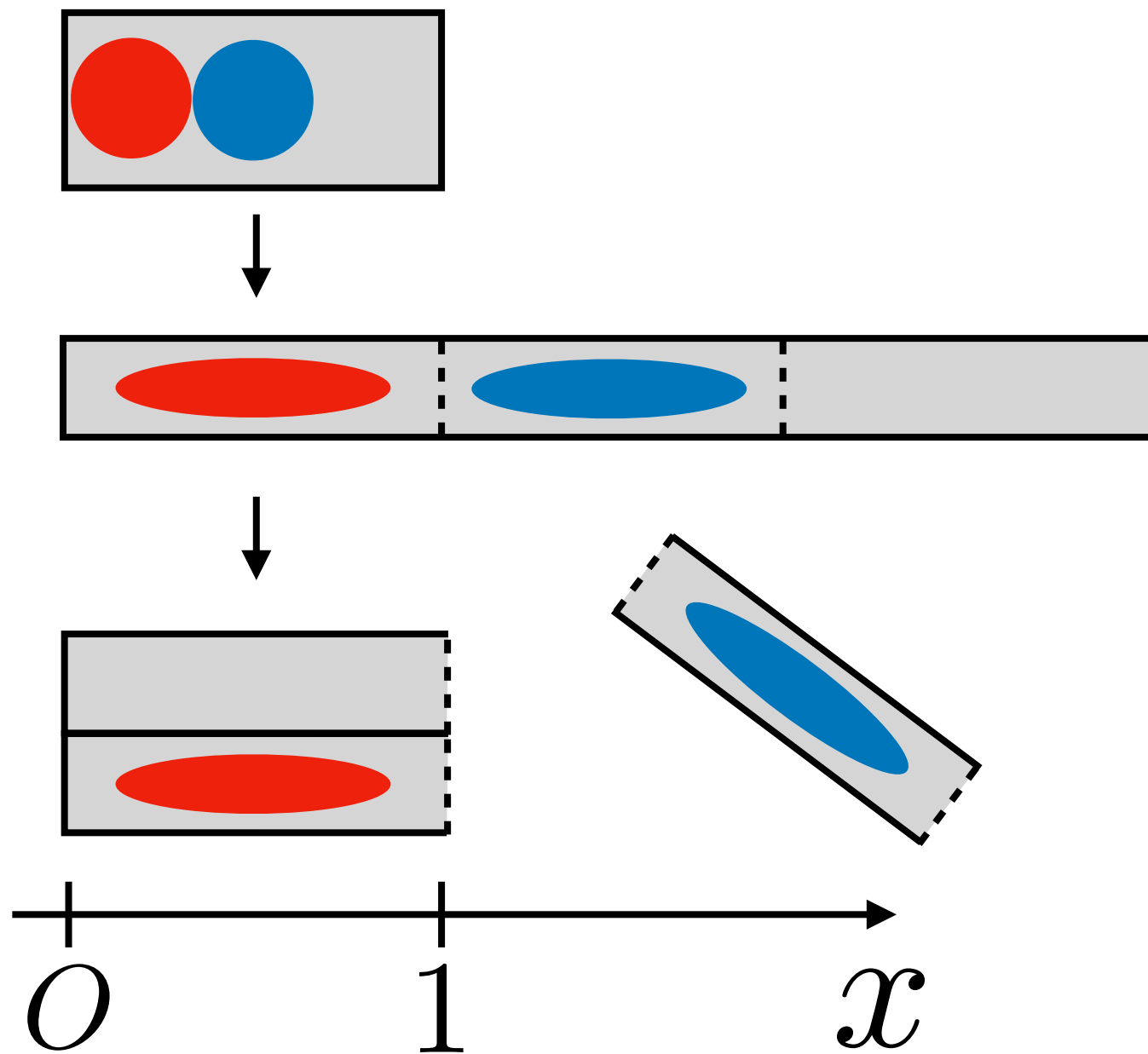


食いしん坊なパン屋の写像と過渡的カオス

離脱

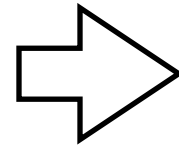


フラクタル性

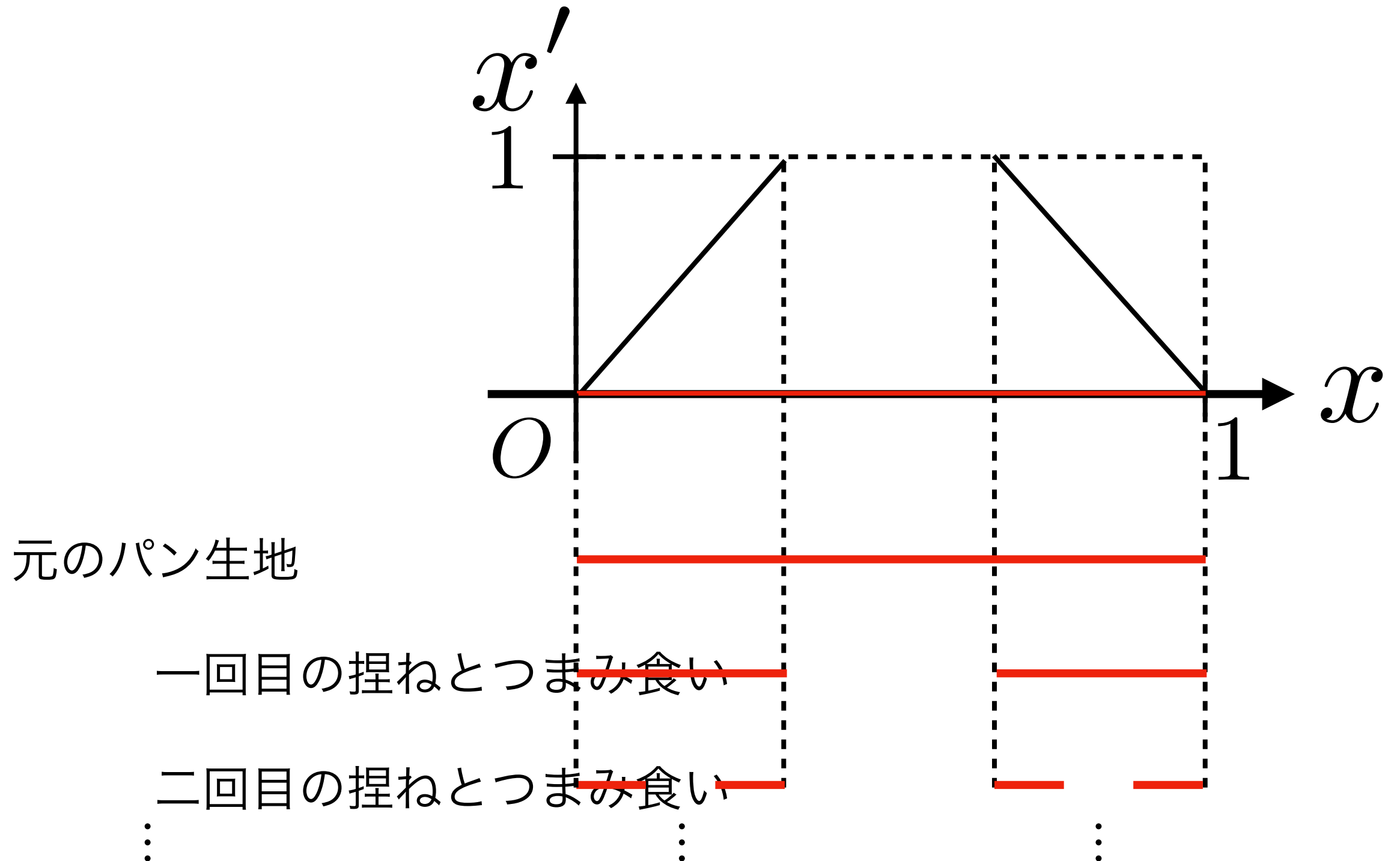


食いしん坊なパン屋の写像と過渡的カオス

離脱



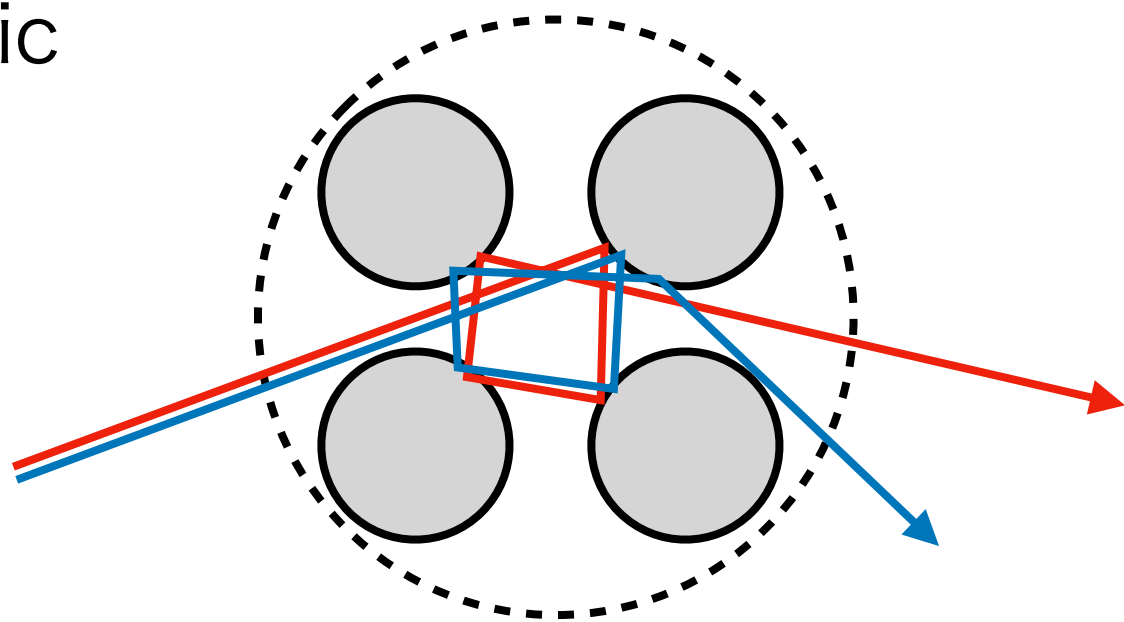
フラクタル性



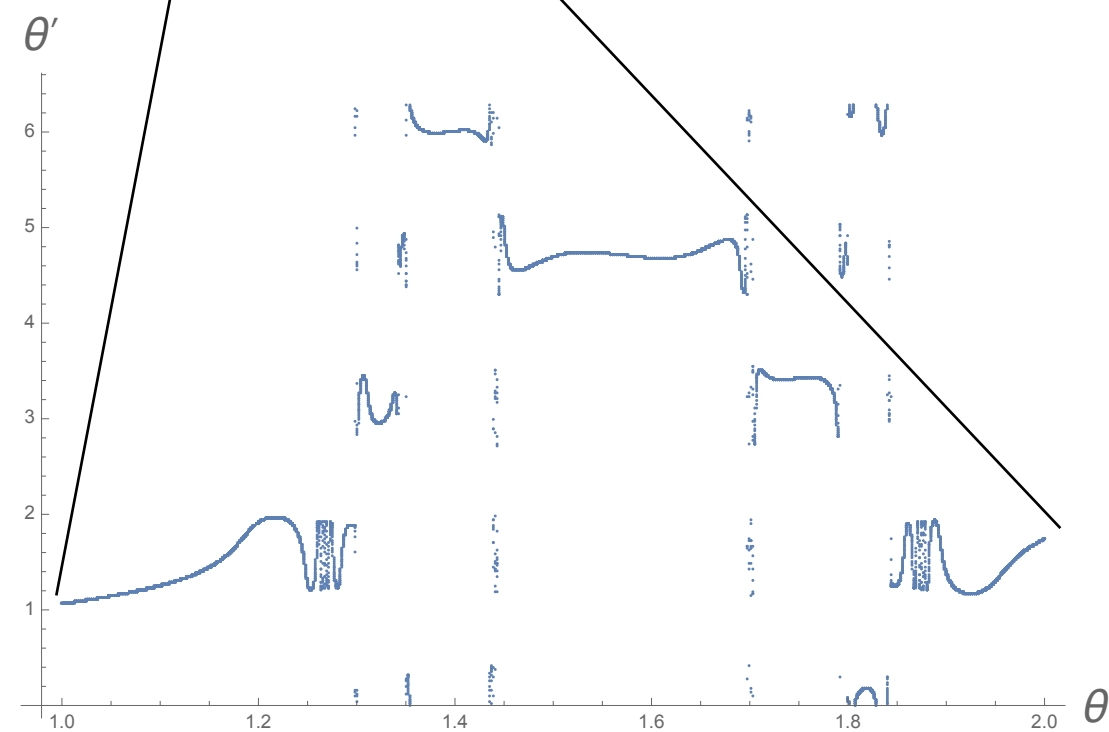
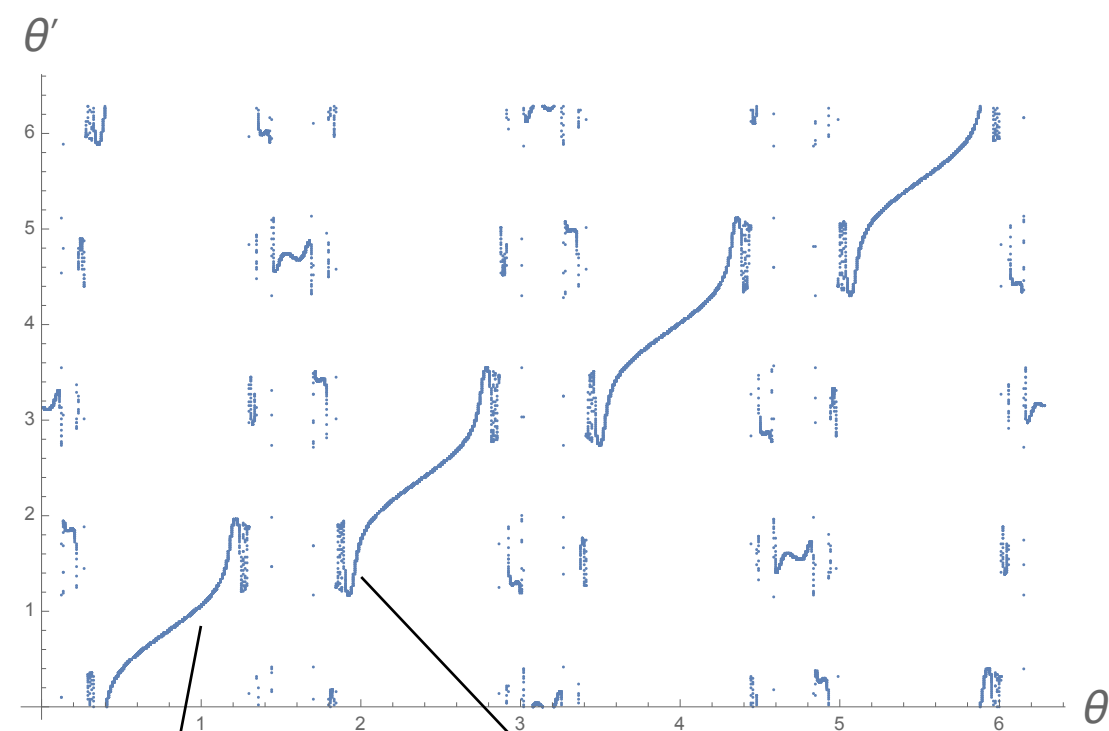
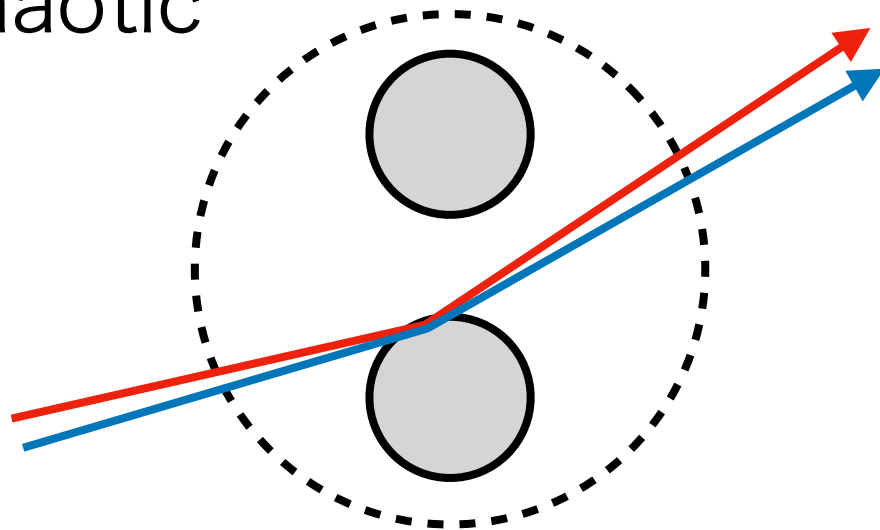
古典散乱の過渡的カオス

引き伸ばし, 折りたたみ, 離脱

chaotic



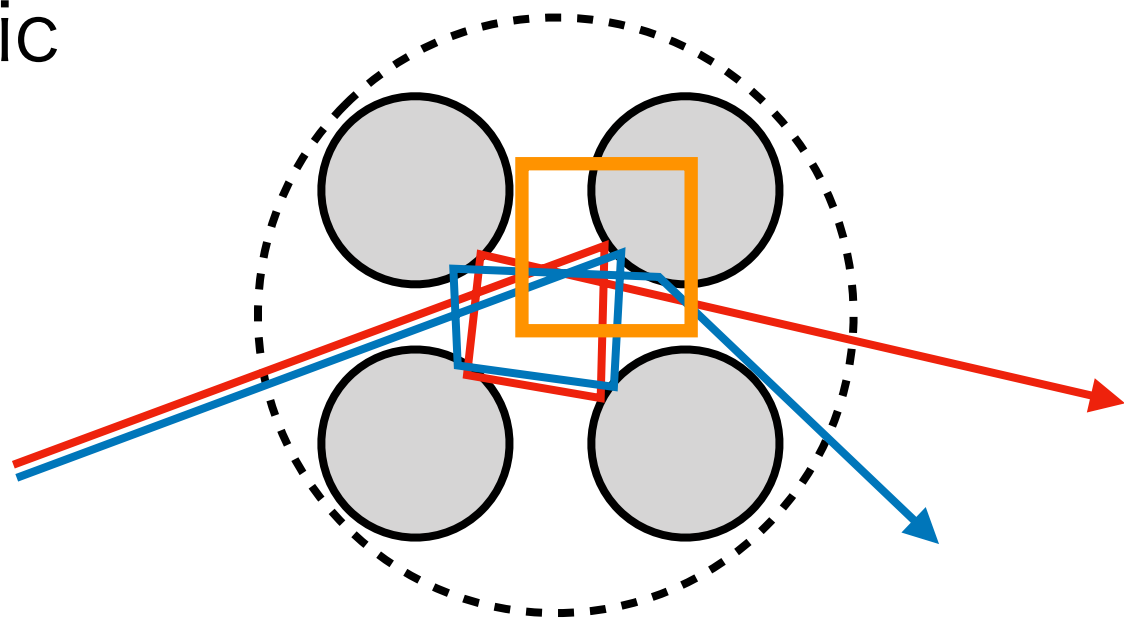
Not chaotic



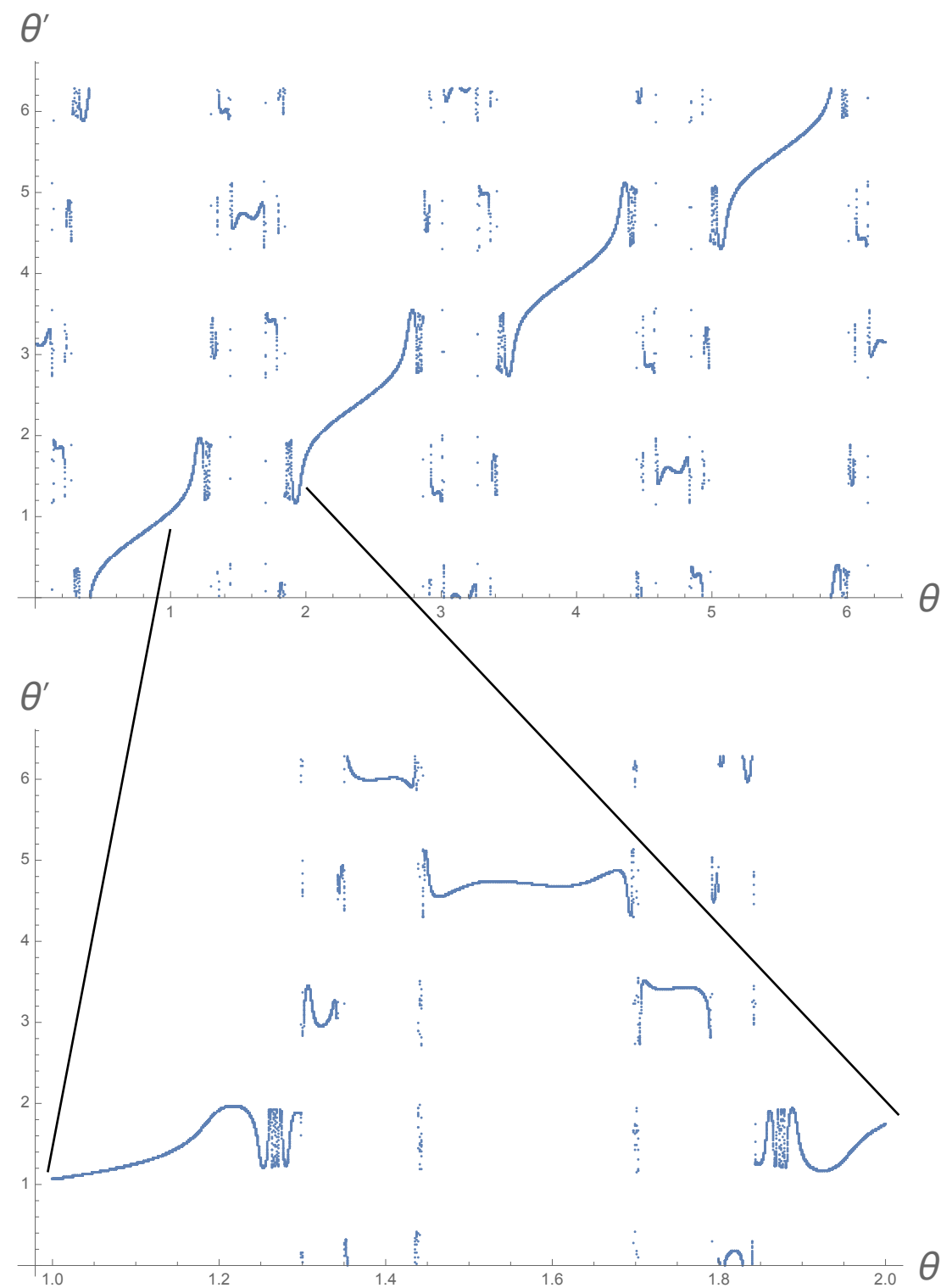
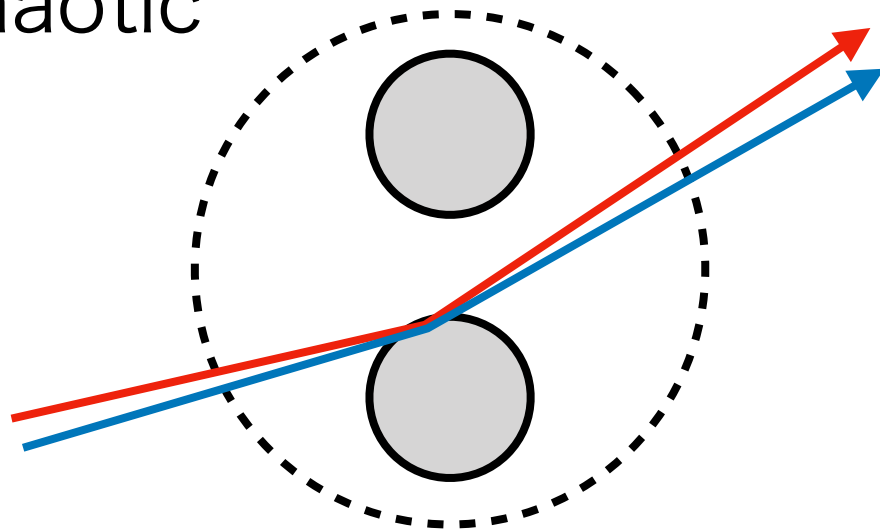
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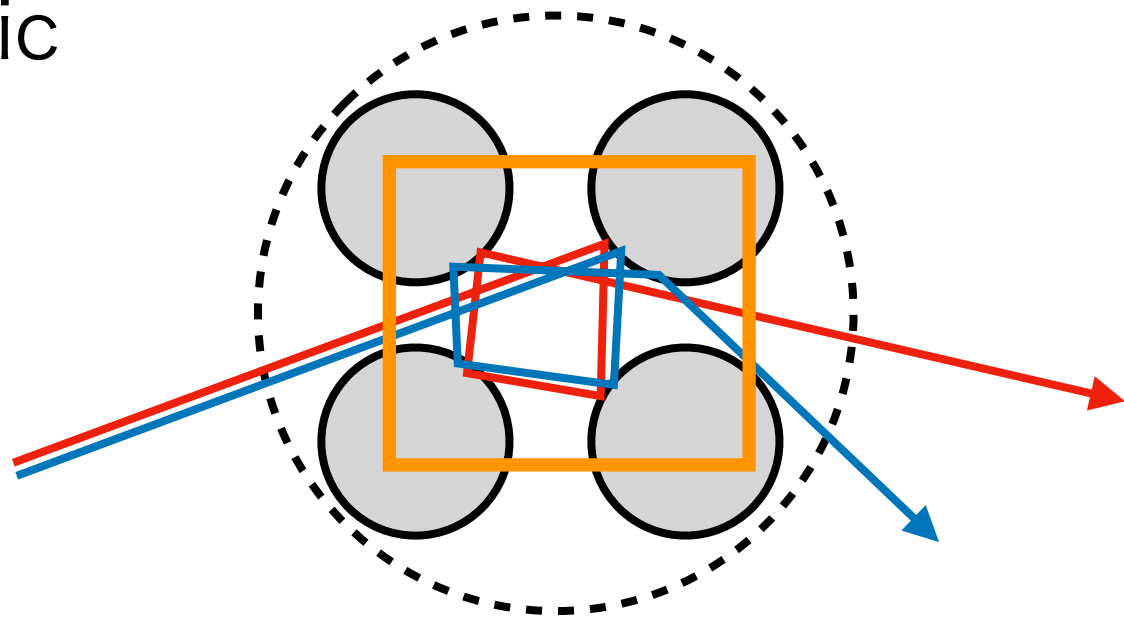
Not chaotic



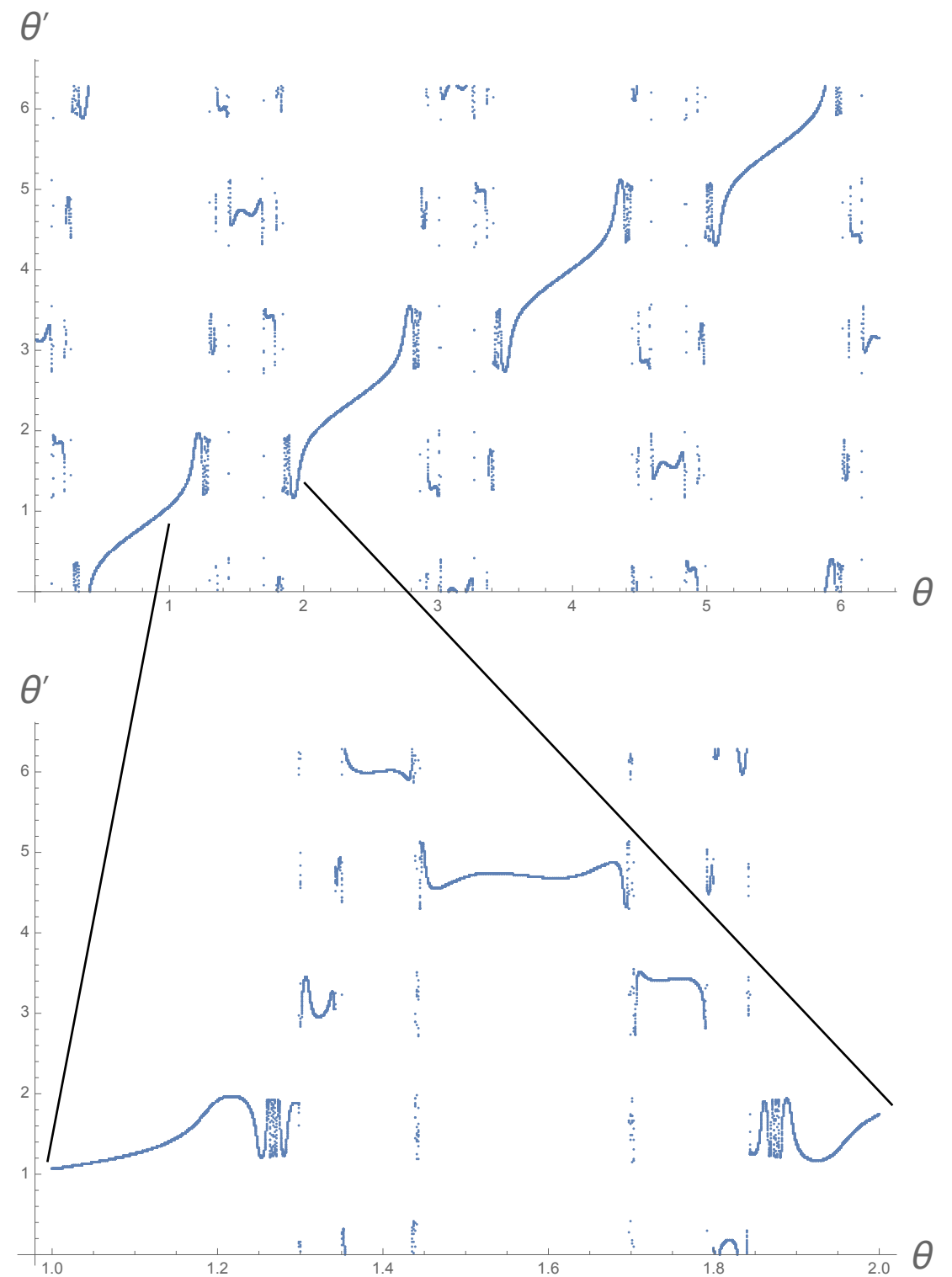
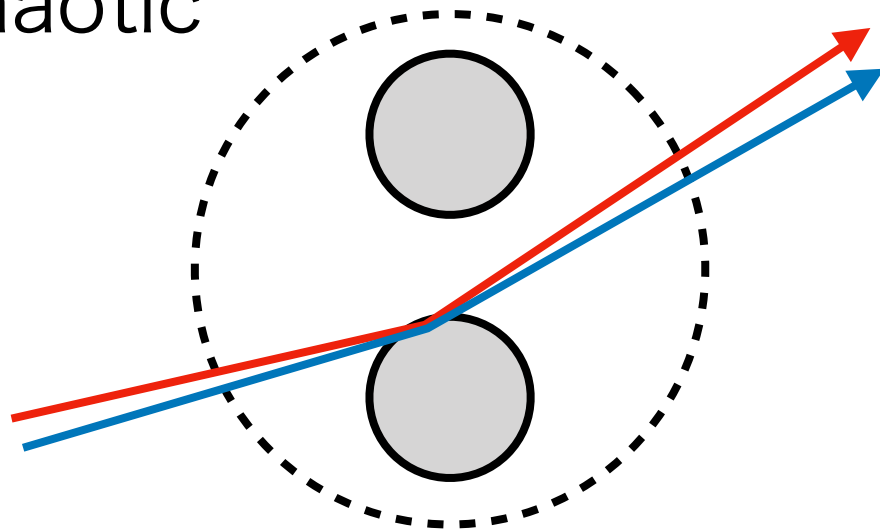
古典散乱の過渡的カオス

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chaotic



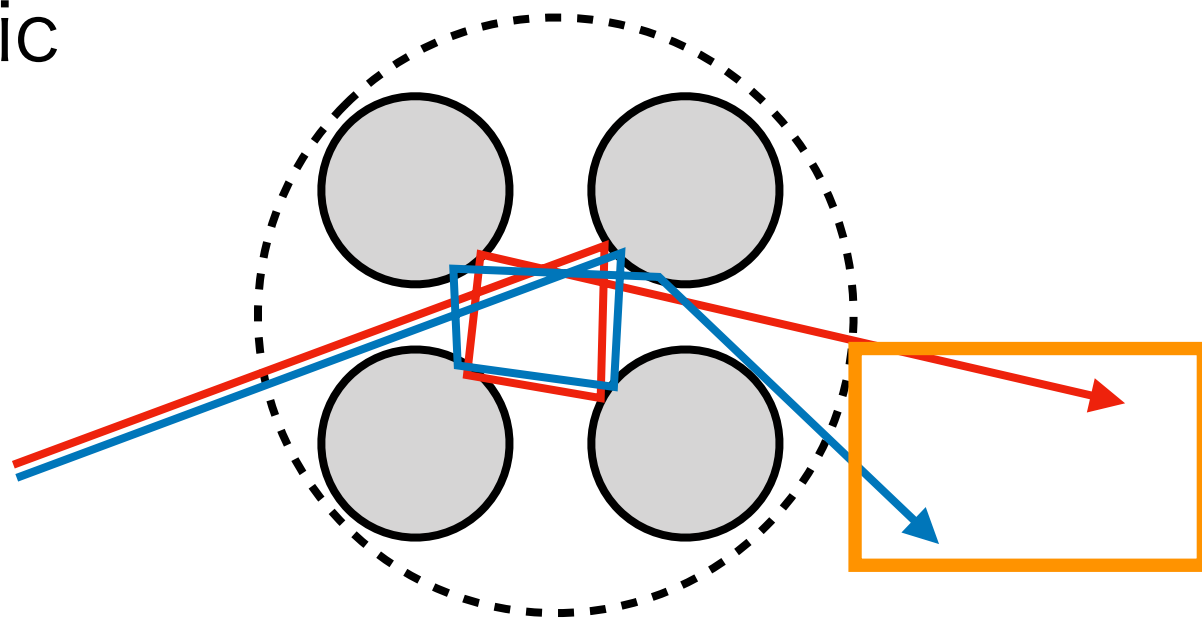
Not chaotic



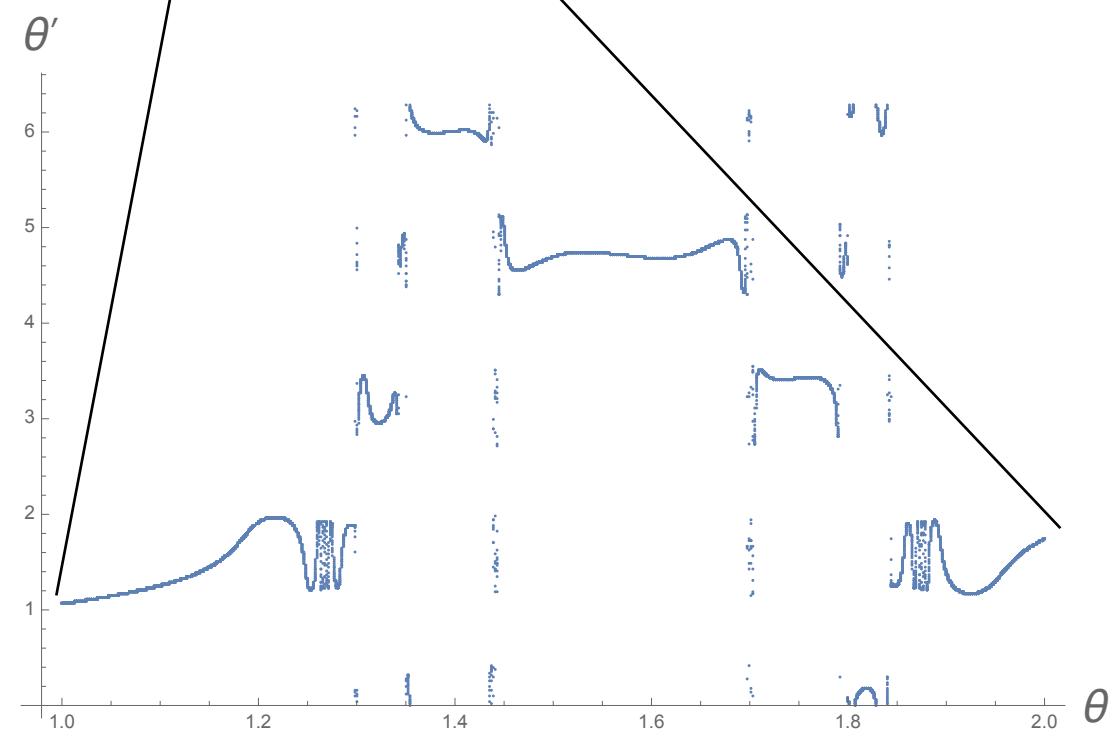
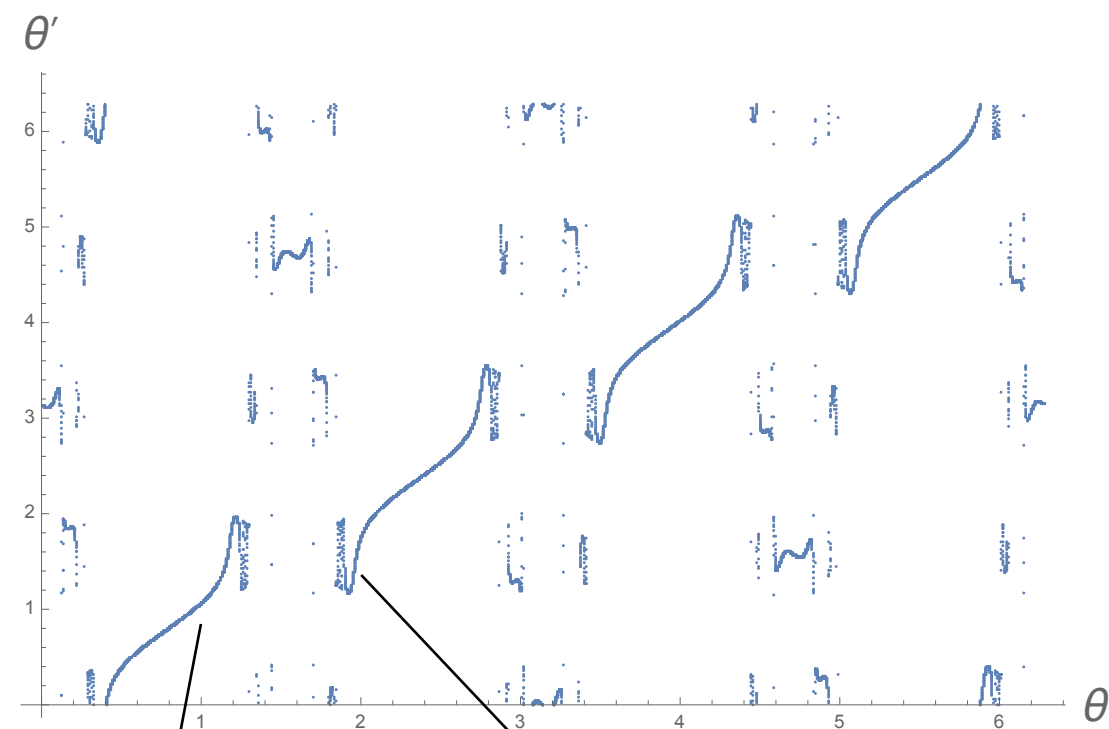
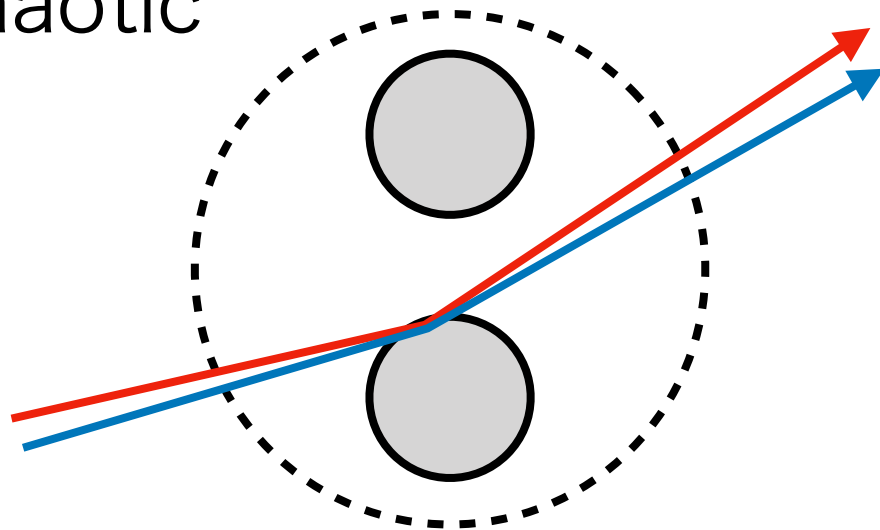
古典散乱の過渡的カオス

引き伸ばし, 折りたたみ, 離脱

chaotic



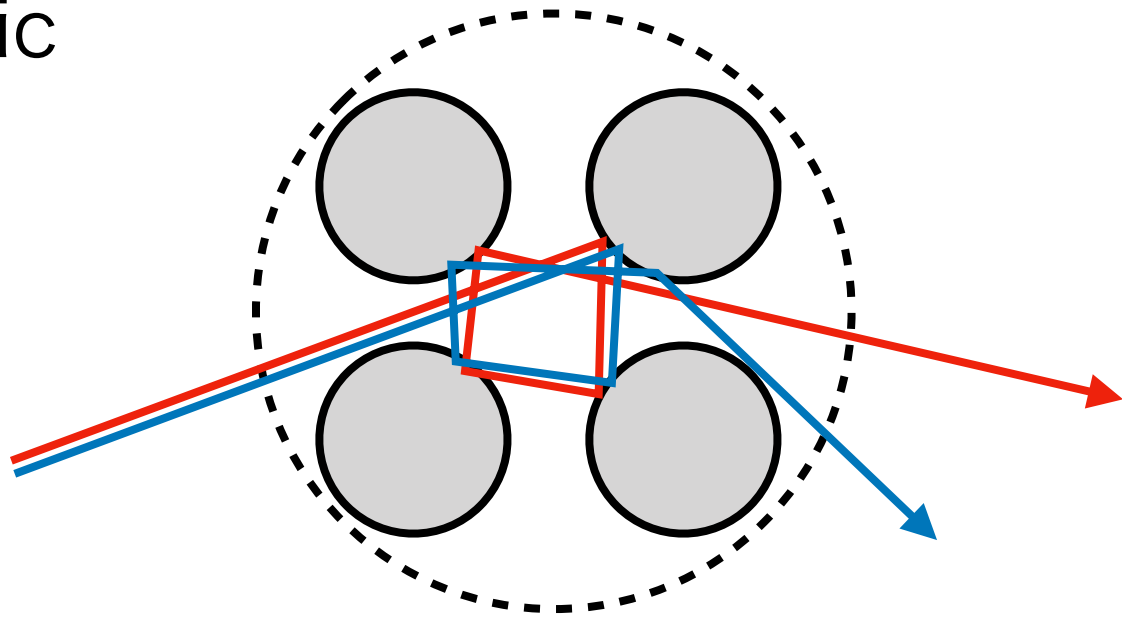
Not chaotic



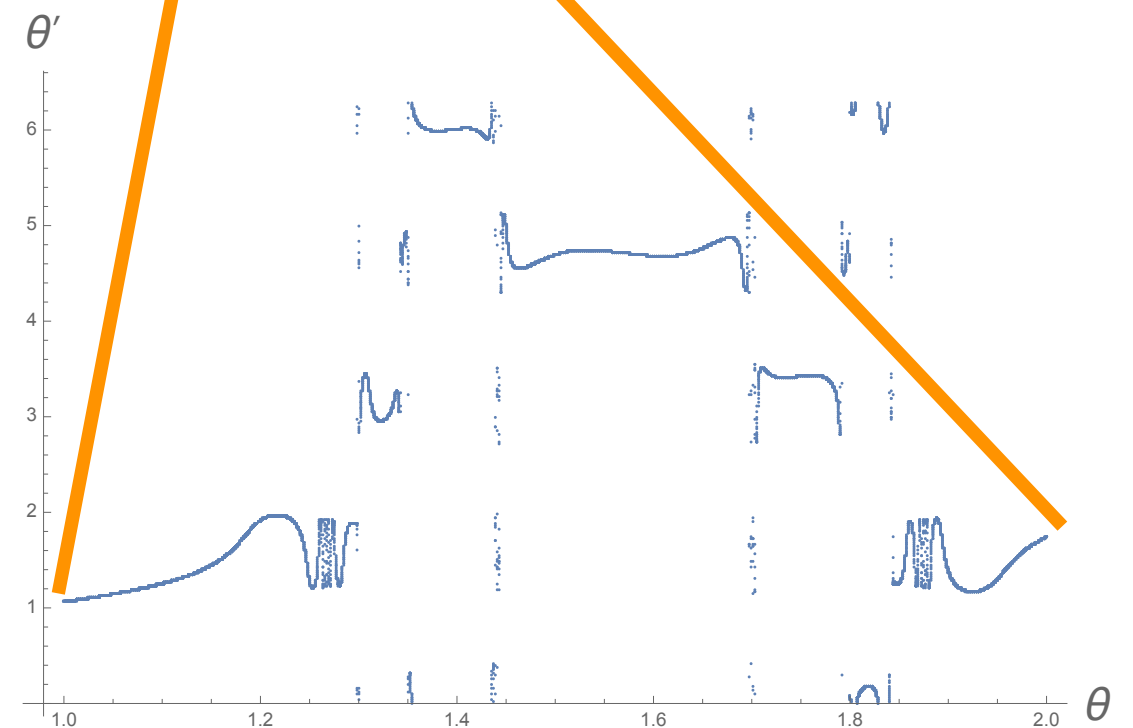
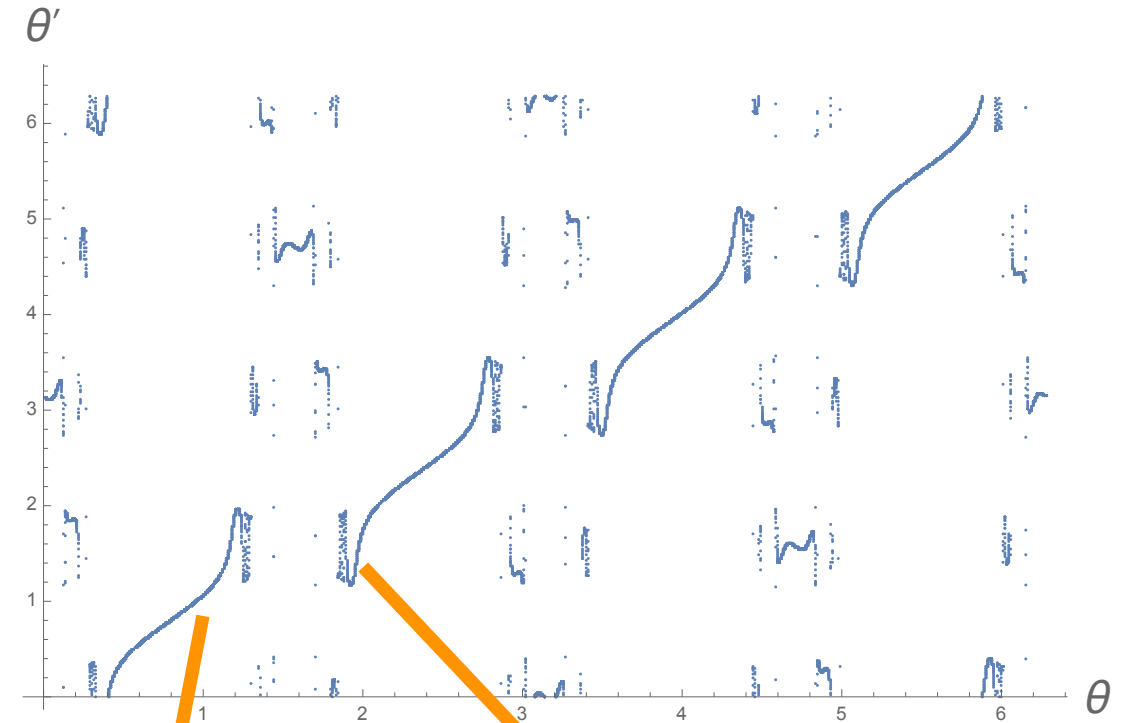
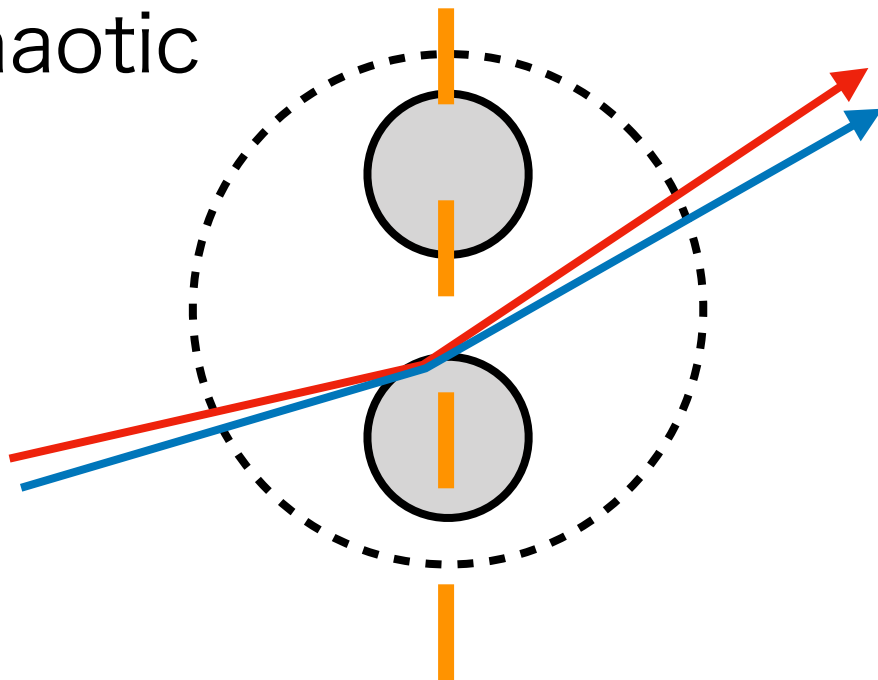
古典散乱の過渡的カオス

- 散乱データのフラクタル構造
- 散乱領域の非自明な(一直線上に無い)幾何学的構造

chaotic



Not chaotic



1/3章 過渡的カオス まとめ

- 過渡的カオス = 有限時間の間だけカオス
- 古典散乱におけるカオスの必要条件
 - **散乱データのフラクタル構造**
 - **散乱領域の幾何学的構造**

この章は[Lai, Tél, 11. Seoane, Sanjuán, 12]等に基づくレビュー

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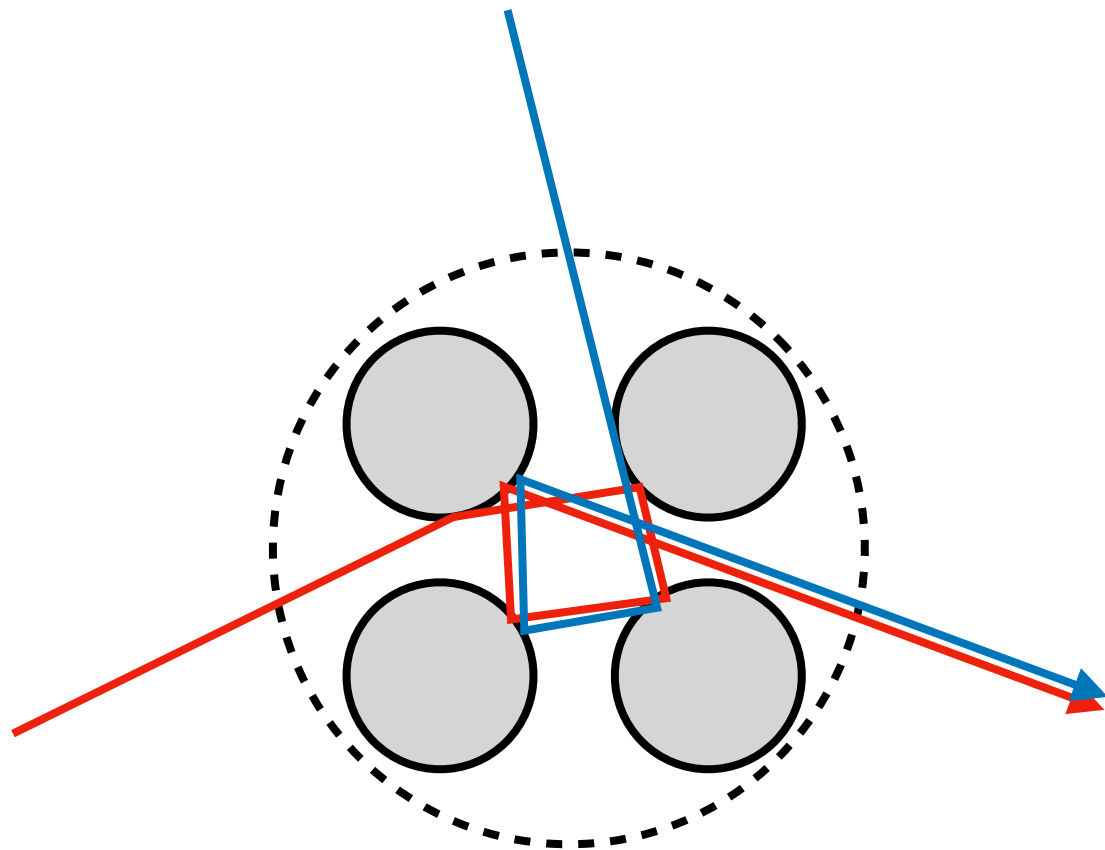
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終 章	まとめ	[3枚]
		[42枚]

2/3章 散乱データとフラクタル構造 まとめ

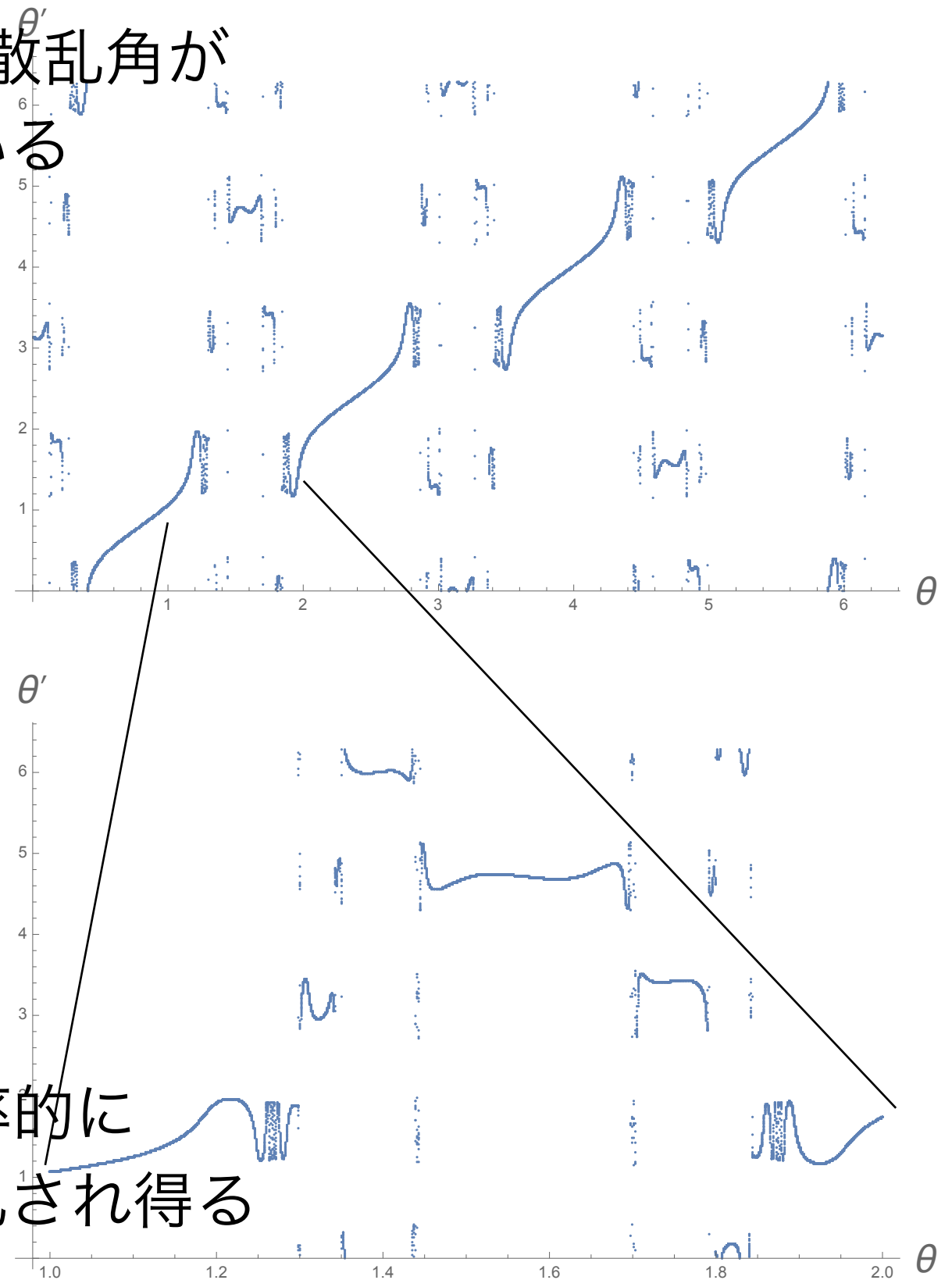
- 古典散乱のアイデアを量子に拡張
- 高励起弦+タキオン $\rightarrow$ 高励起弦+タキオン
 - 散乱データが微細に分裂するがフラクタルでは無い
 - 高次補正でカオスが出る可能性

入射-散乱角の図を得たい

古典論では入射/散乱角が
良く定義されている



量子論では、確率的に
どの角度へも散乱され得る

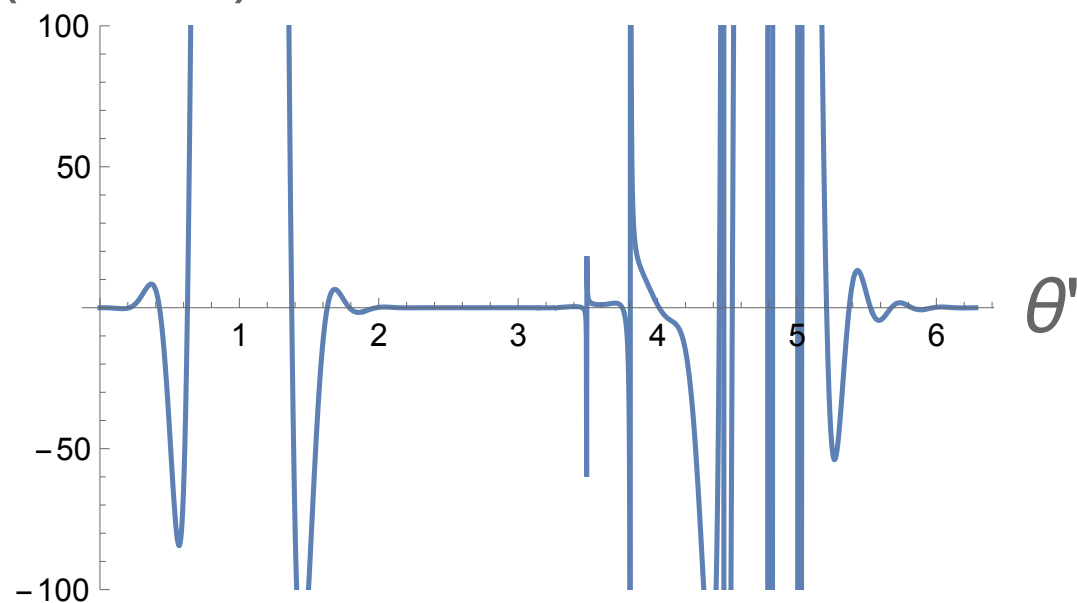


古典のアイデアを量子に拡張

最大の極に対応する角度を散乱角とする

弦の散乱振幅は多数の極を持つ

$A(\theta=1, \theta')$



θ'

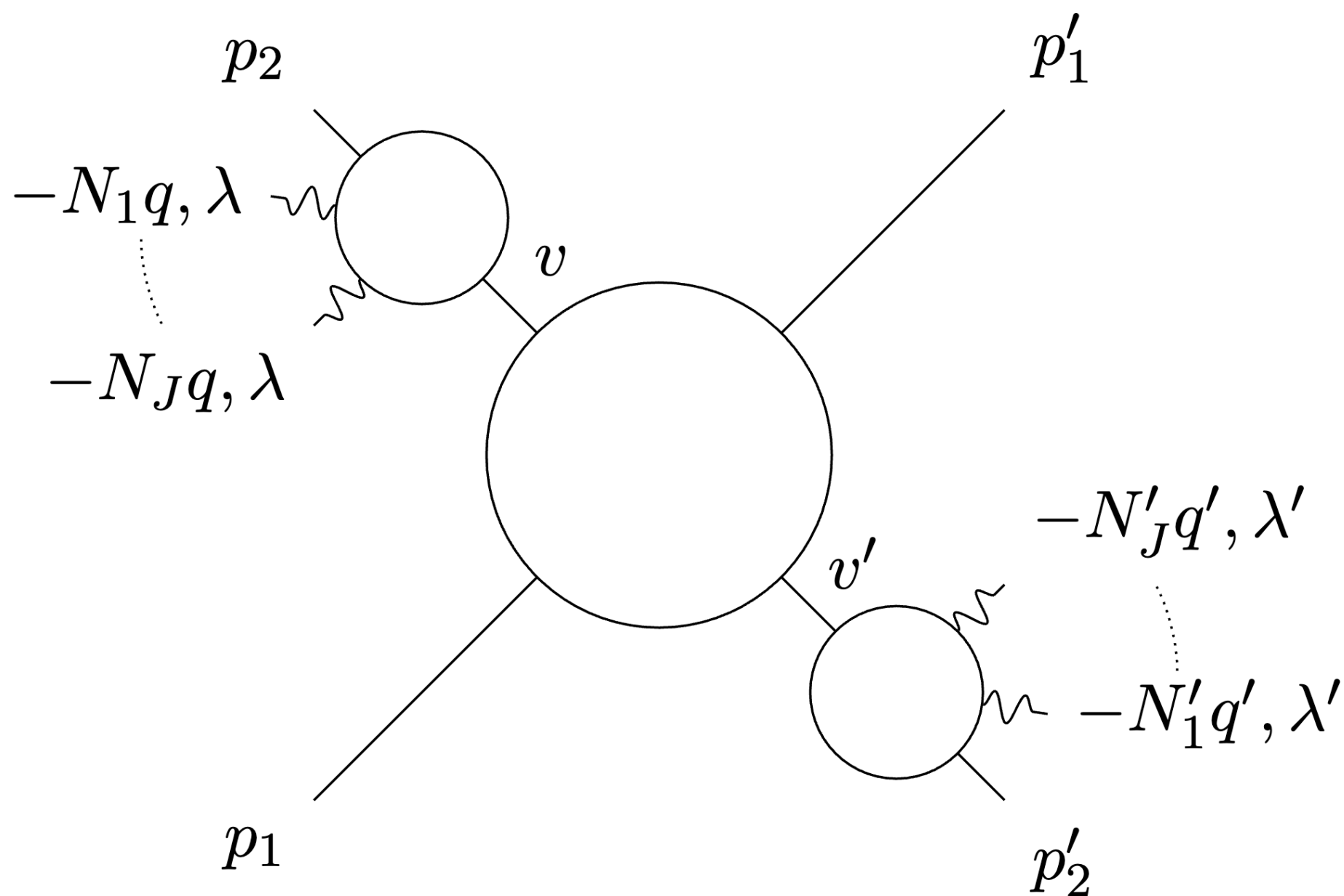
$$\mathcal{A}(p; \theta, \theta') \sim \sum_m \frac{R_m(p; \theta)}{\theta' - \theta'_m(p; \theta)}$$

最大となる角度を

高励起弦の散乱振幅

まず、ボソン開弦の散乱振幅を考える

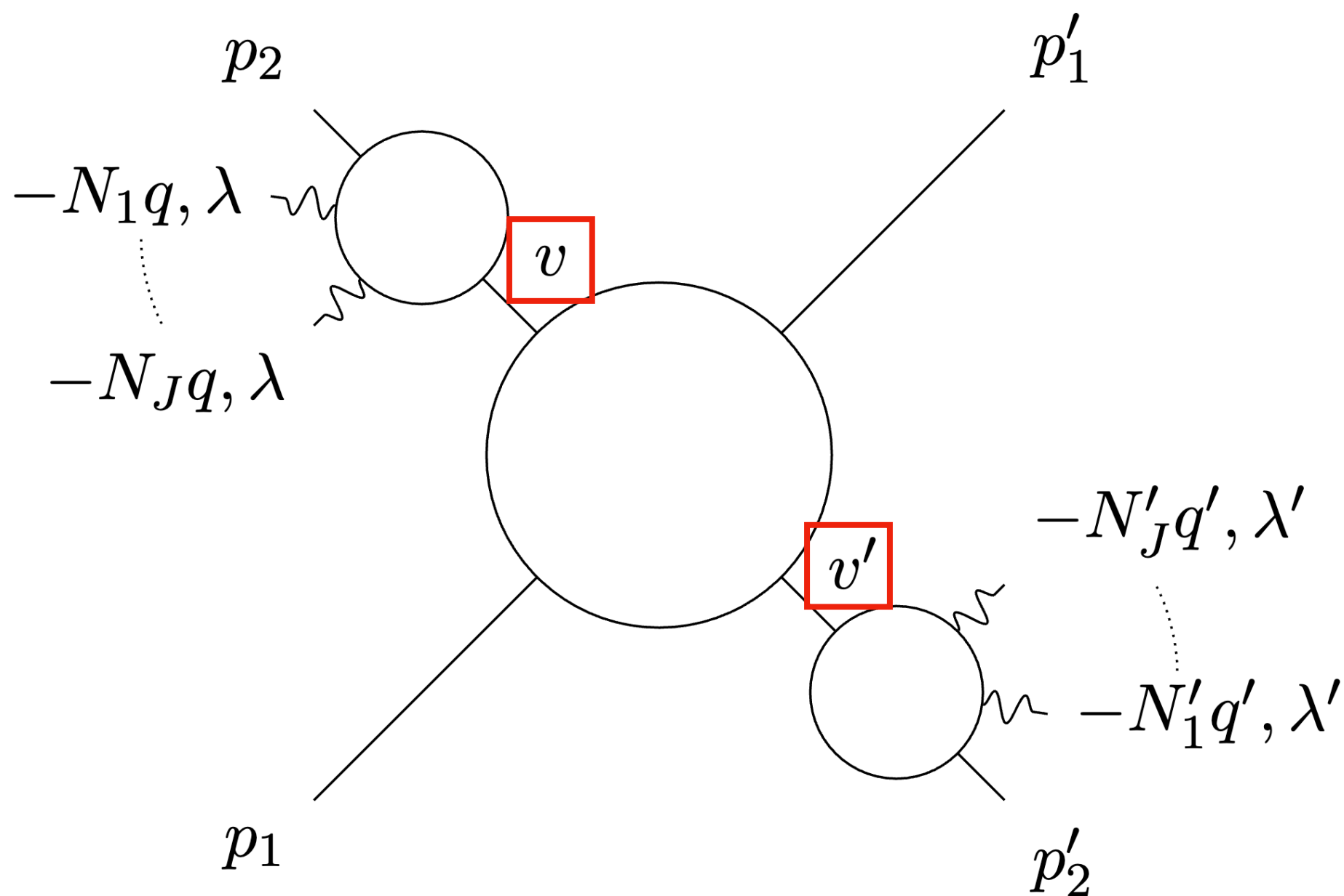
(タキオン+光子)+タキオン $\rightarrow$ (タキオン+光子)+タキオン



高励起弦の散乱振幅

極を拾い、高励起弦の寄与を見る

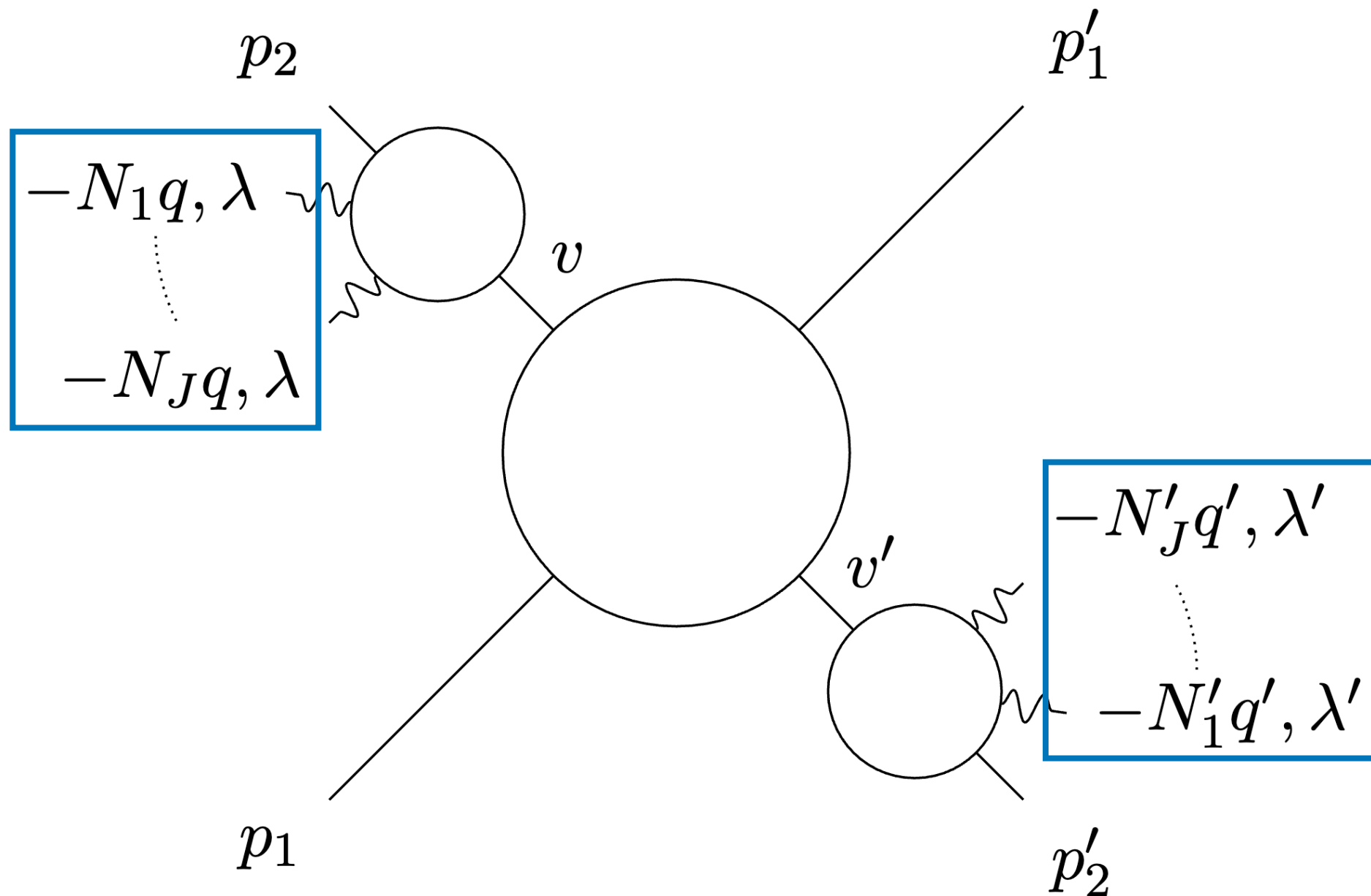
(タキオン+光子)+タキオン $\rightarrow$ (タキオン+光子)+タキオン



高励起弦の散乱振幅

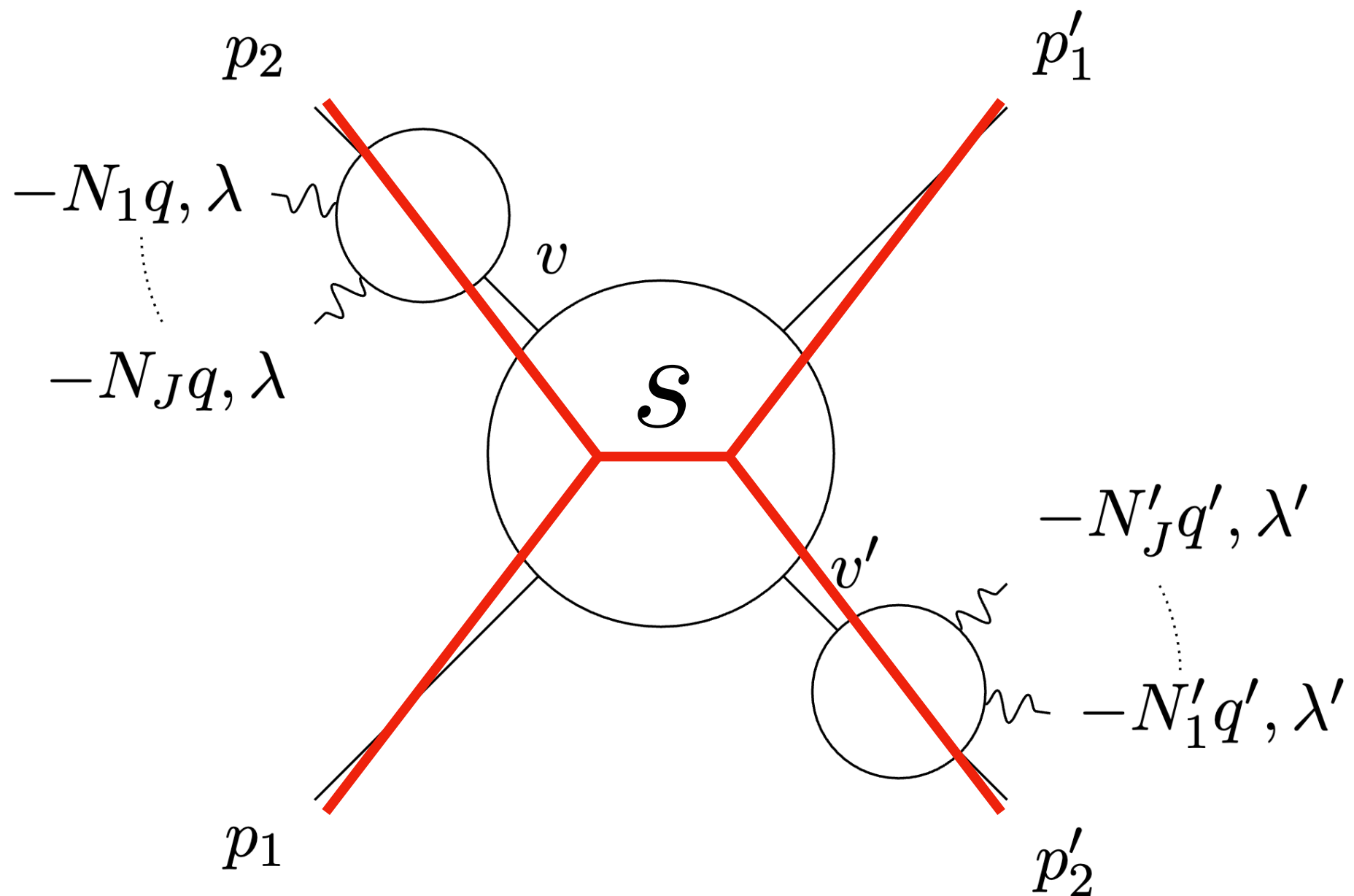
高励起弦の状態は、光子で制御する

$$\begin{array}{l} \text{質量} \quad : M^2 = 2(N - 1), N = \sum_a^J N_a \\ \text{角運動量} : J \end{array}$$



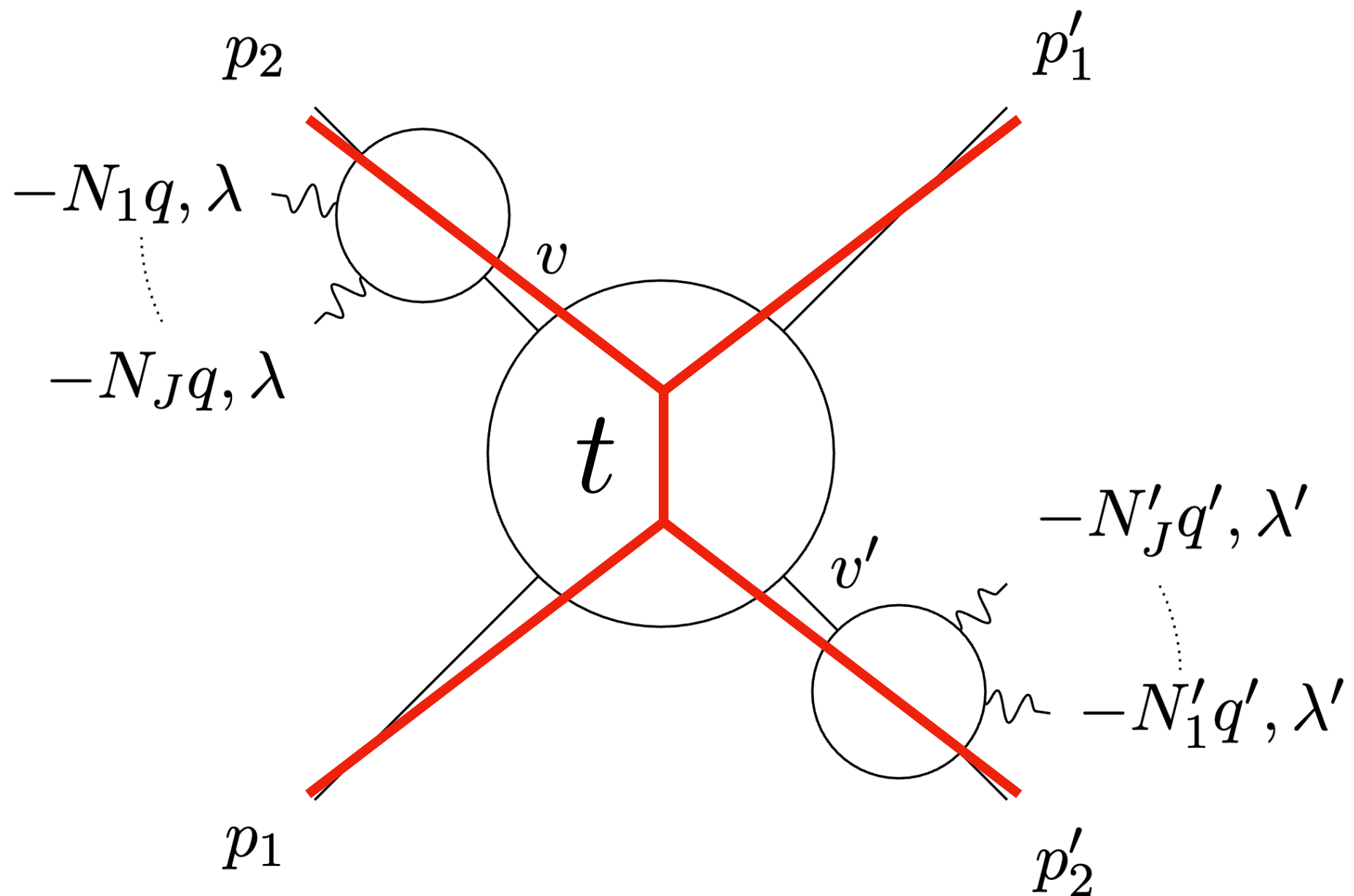
高励起弦の散乱振幅

高励起弦の運動量を使って、チャンネルを定義する



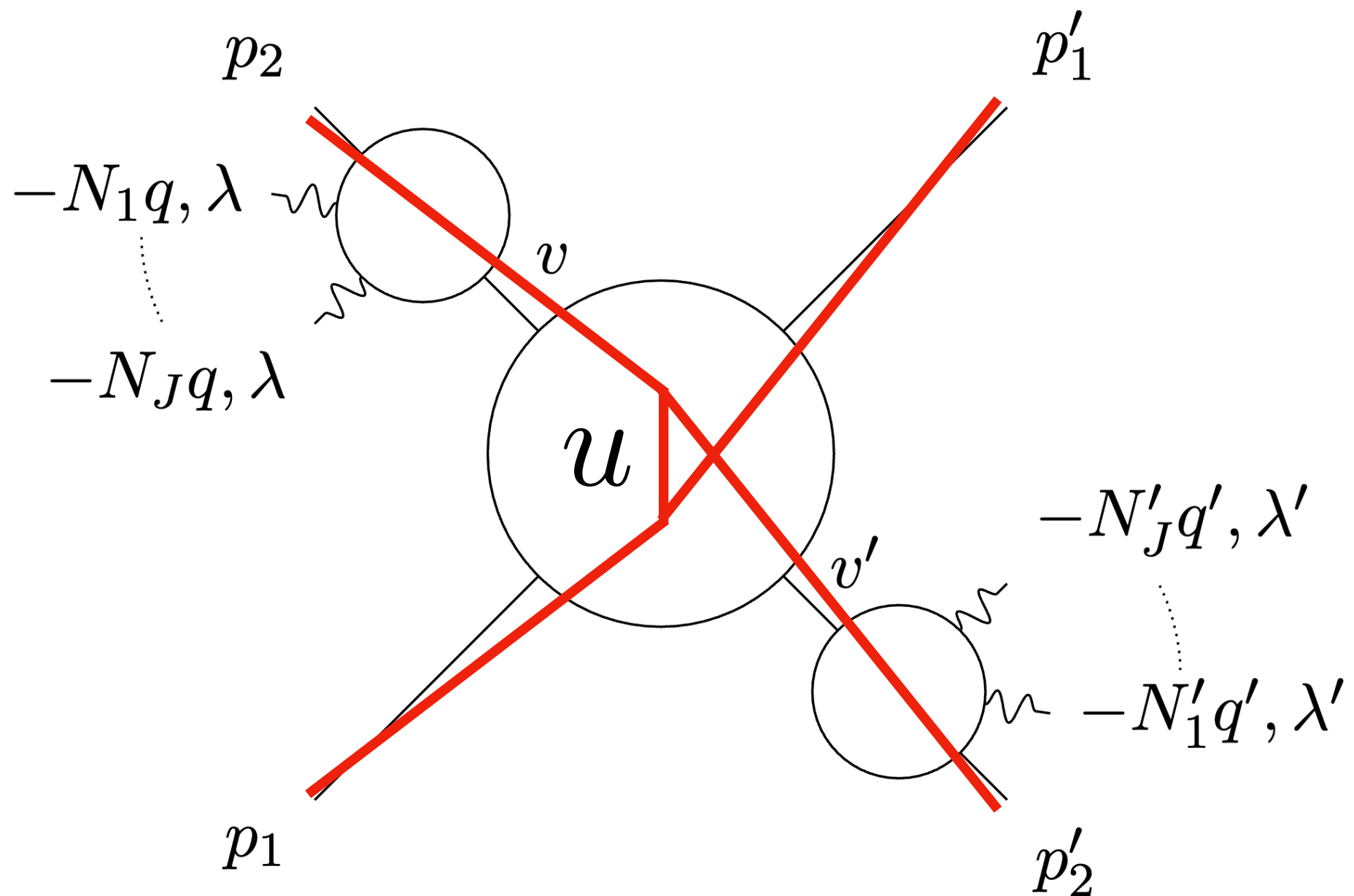
高励起弦の散乱振幅

高励起弦の運動量を使って、チャンネルを定義する



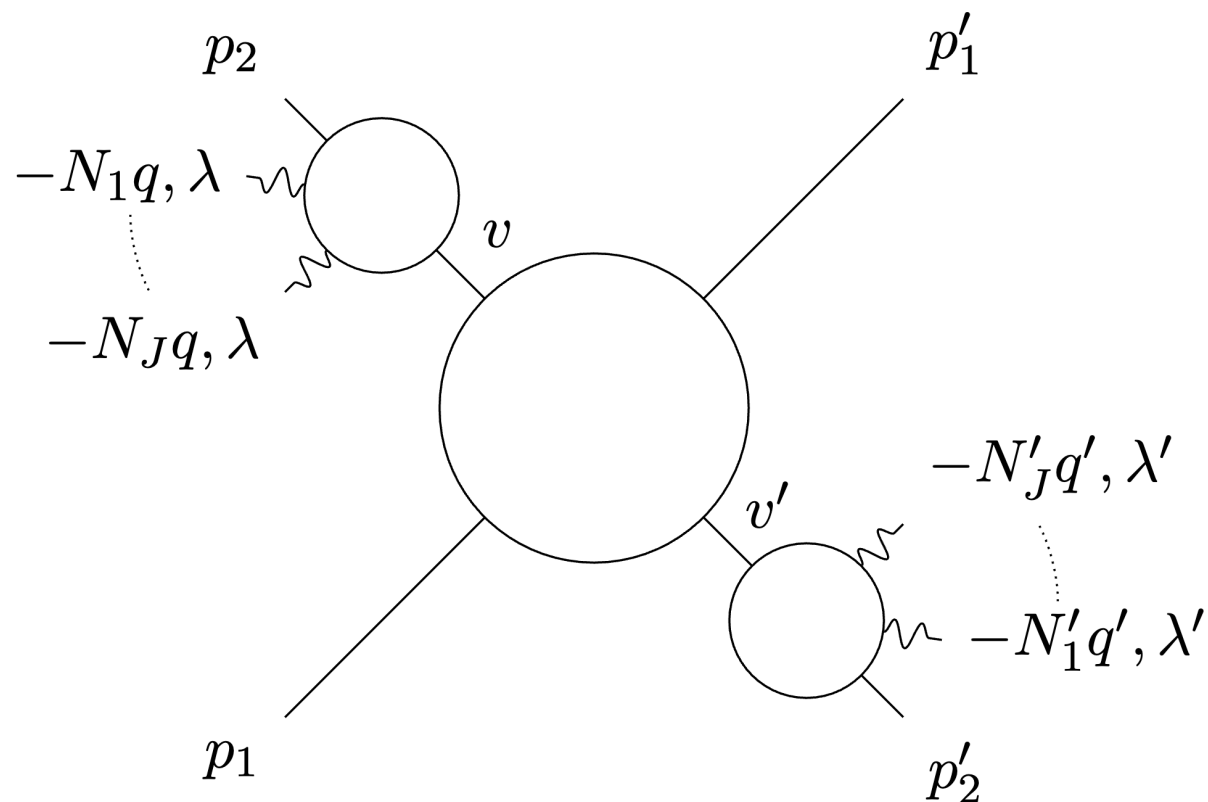
高励起弦の散乱振幅

高励起弦の運動量を使って、チャンネルを定義する



振幅の計算

$$\begin{aligned} \mathcal{A} = & \frac{1}{\text{vol.}} \int_{-\infty}^{\infty} \prod_{i=1,2,2',1'} dw_i \prod_{i < j} |w_{ij}|^{p_i \cdot p_j} \\ & \times \int_{-\infty}^{\infty} \prod_{a=1}^J dz_a \prod_{a,i} |z_a - w_i|^{-\alpha_i} \sum_i \frac{-p_i \cdot \lambda}{w_i - z_a} \\ & \times \int_{-\infty}^{\infty} \prod_{b=1}^J dz'_b \prod_{b,j} |z'_b - w_j|^{-\beta_j} \sum_j \frac{-p_j \cdot \lambda'}{w_j - z'_b} \end{aligned}$$



簡単のため、

$$q \propto q'$$

を仮定し交差項を落とした

$$|z_a - z'_b|^{N_a q \cdot N'_b q'}$$

計算結果

$$\mathcal{A}_{st} \sim \sum_{\{i_a=2,2'\}} \sum_{\{j_b=2,2'\}} \sum_{\{N_a\}}^{\{N_a\}} \sum_{\{N'_b\}}^{\{N'_b\}} \left(\prod_{a=1}^J (p_{i_a} \cdot \lambda) c_{k_a}^{(i_a)} \right) \left(\prod_{b=1}^J (p_{j_b} \cdot \lambda') d_{l_b}^{(j_b)} \right) B(-\alpha(s) + k + l, -\alpha(t))$$

角度に対して激しく振動する係数

Veneziano振幅と似た項

$$c_k^{(2)} = +c_k(\alpha_2 + 1, \alpha_1 + 1, \alpha'_2),$$

$$c_k^{(2')} = -c_{k-1}(\alpha_2, \alpha_1, \alpha'_2 + 1),$$

$$d_l^{(2)} = -c_{l-1}(\beta'_2, \beta'_1, \beta_2 + 1),$$

$$d_l^{(2')} = +c_l(\beta'_2 + 1, \beta'_1 + 1, \beta_2),$$

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

$$\alpha(x) = 1 + x/2,$$

$$k = \sum_{a=1}^J k_a,$$

$$l = \sum_{b=1}^J l_b$$

$$c_k(\alpha_2, \alpha_1, \alpha'_2)$$

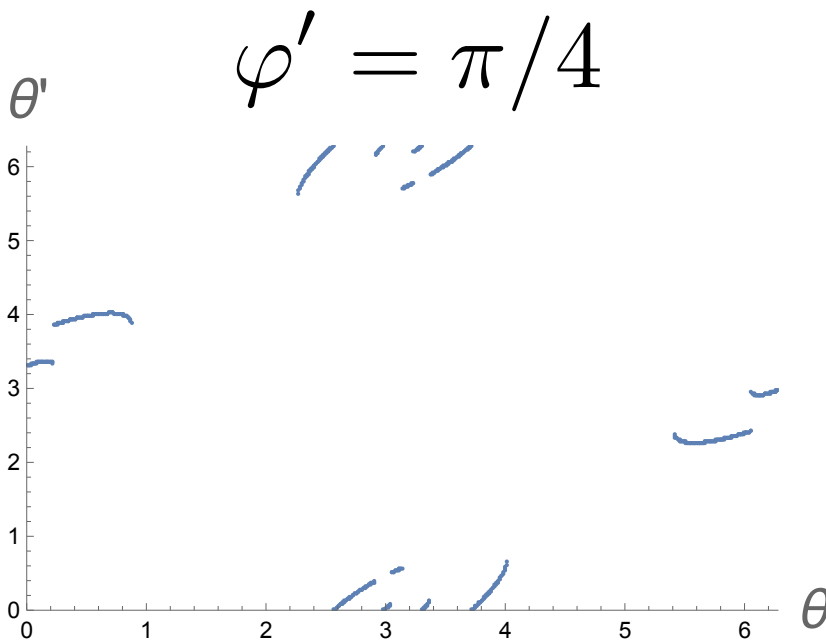
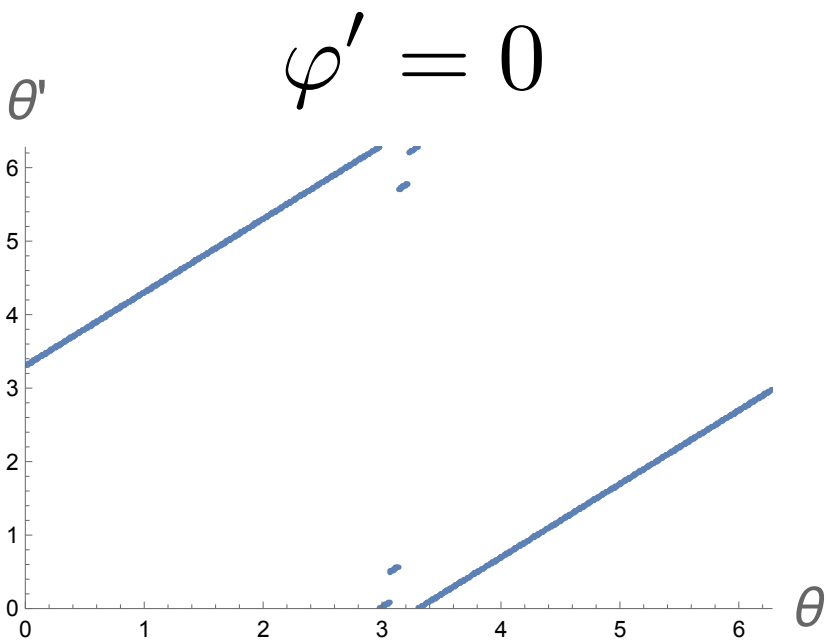
$$= \frac{\Gamma(-\alpha_2 + 1)\Gamma(-\alpha_1 + 1)}{\Gamma(-\alpha_2 - \alpha_1 + 2)} \frac{\Gamma(\alpha'_2 + k)}{\Gamma(\alpha'_2)} \frac{\Gamma(-\alpha_2 + 1 + k)}{\Gamma(-\alpha_2 + 1)} \frac{\Gamma(-\alpha_2 - \alpha_1 + 2)}{\Gamma(-\alpha_2 - \alpha_1 + 2 + k)} \frac{1}{\Gamma(k + 1)}$$

$$\alpha_i = -(-N_a q) \cdot p_i, \quad \beta_j = -(-N'_b q') \cdot p_j$$

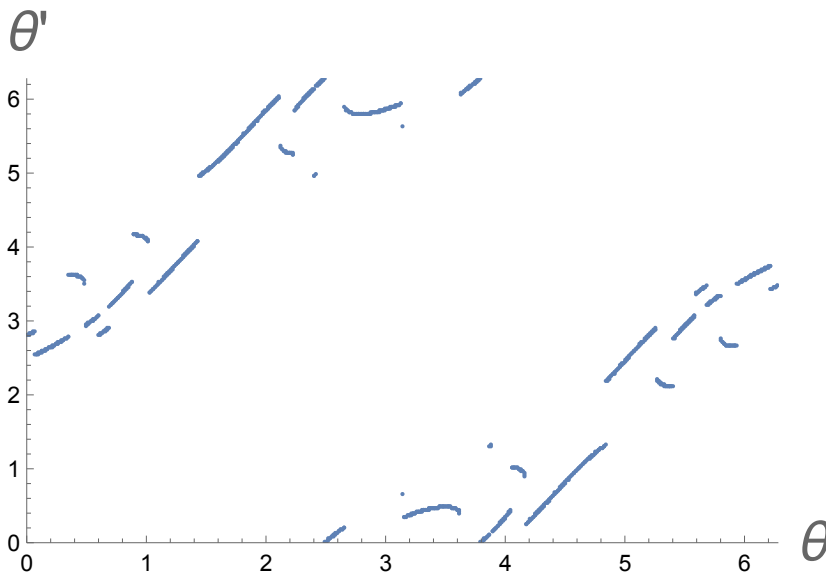
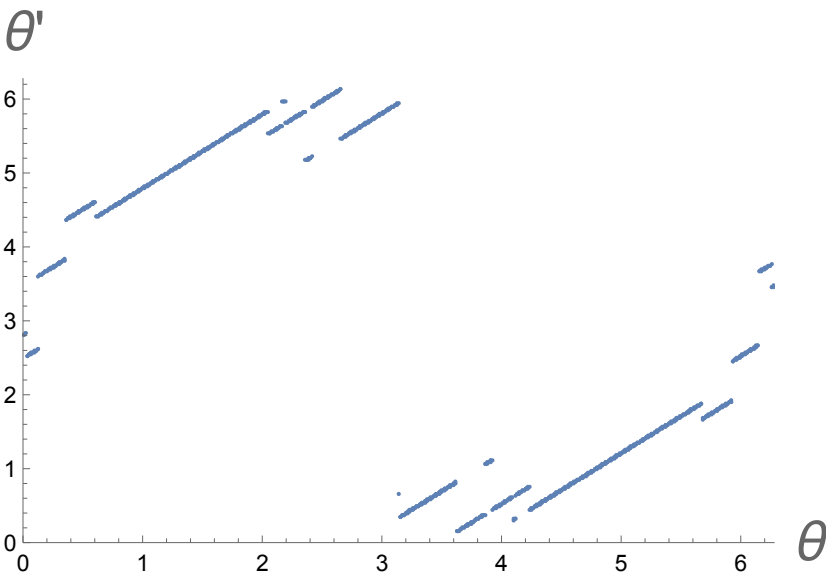
入射角-散乱角($J = 1$)

重心系
光子を基準とした
極座標表示

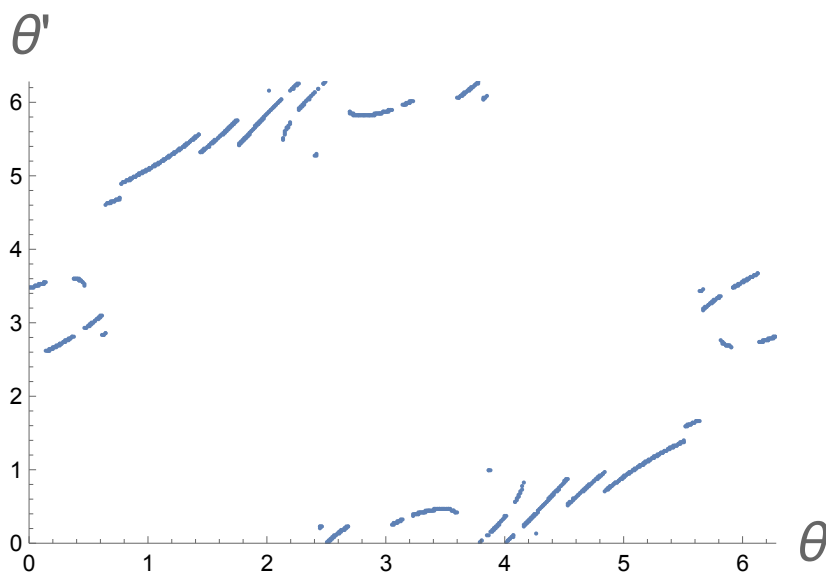
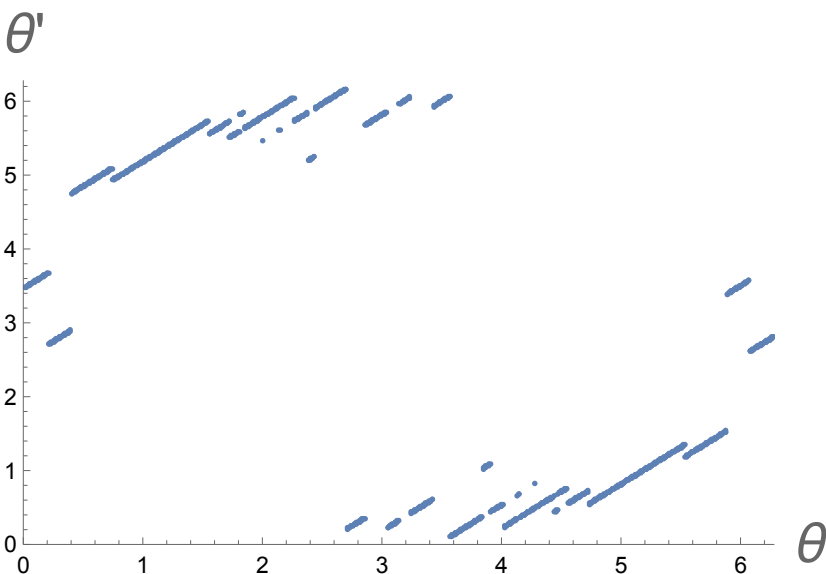
$N = 1$



$N = 5$



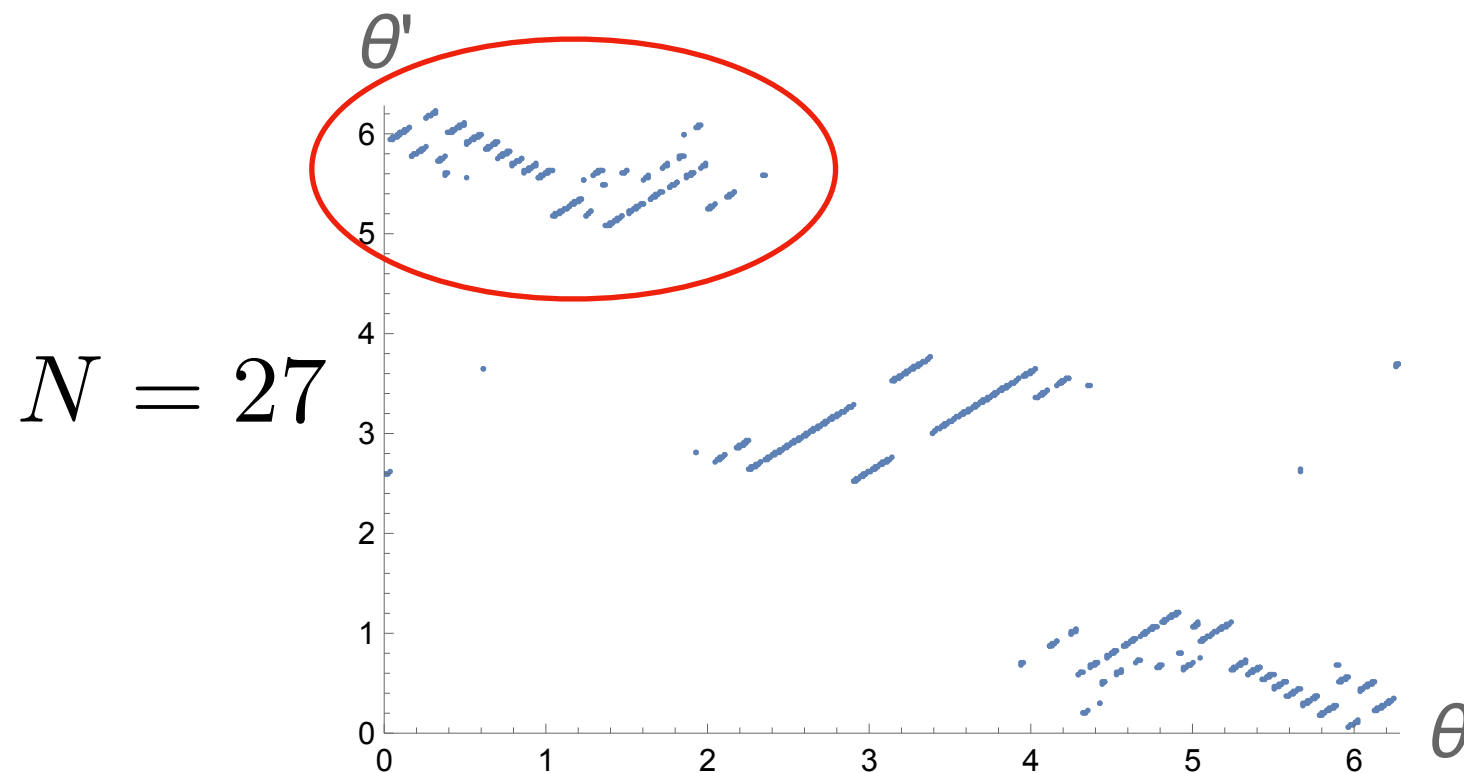
$N = 9$



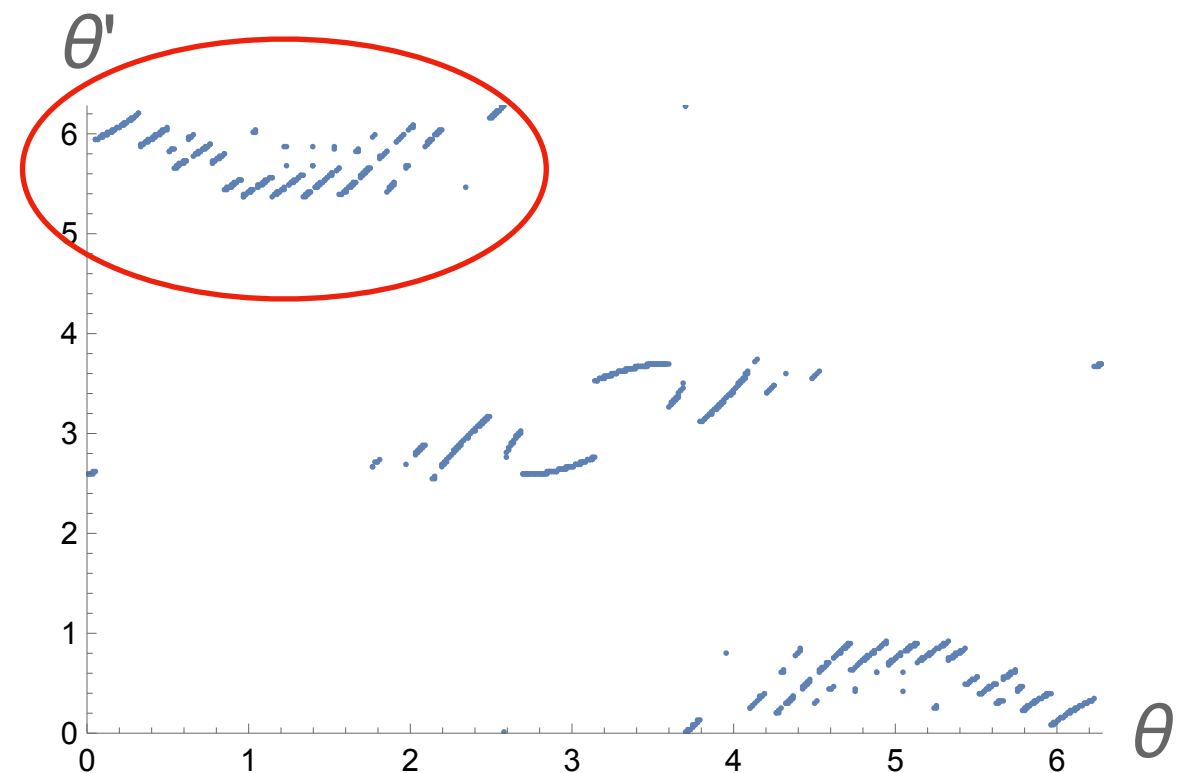
複雑だがカオスとは言えない

- ・散乱図が微小な断片に分裂
- ・断片の数が指数的には増加しない → フラクタルでは無い

$$\varphi' = 0$$



$$\varphi' = \pi/4$$



重心系、高励起弦の向きを基準とした極座標表示

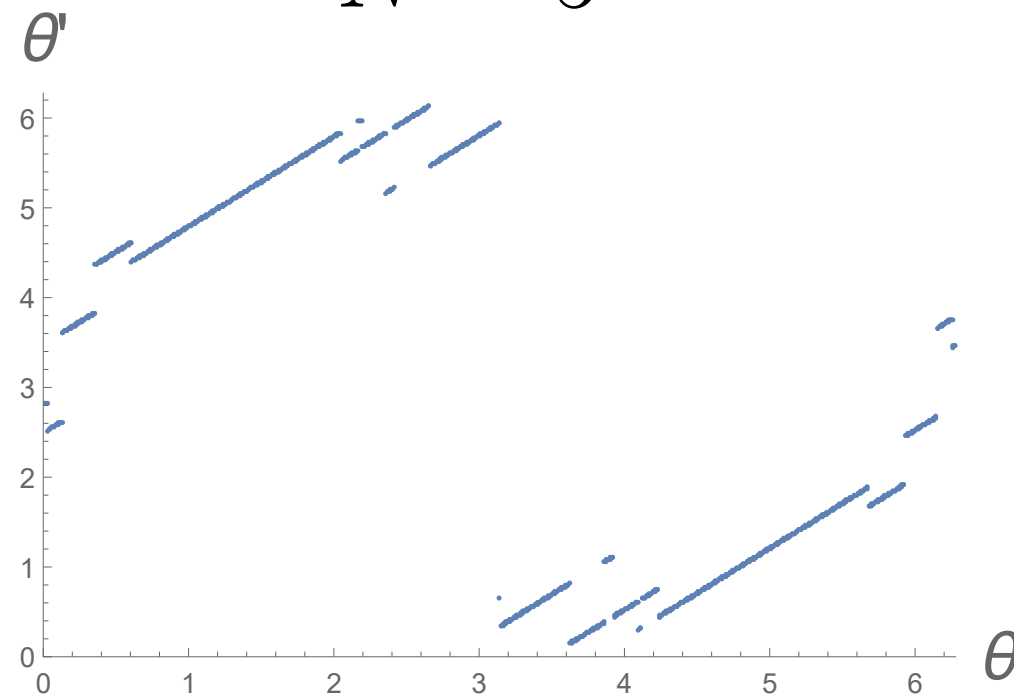
入射角-散乱角($J \neq 1$)

励起数を分割してもフラクタルでは無い

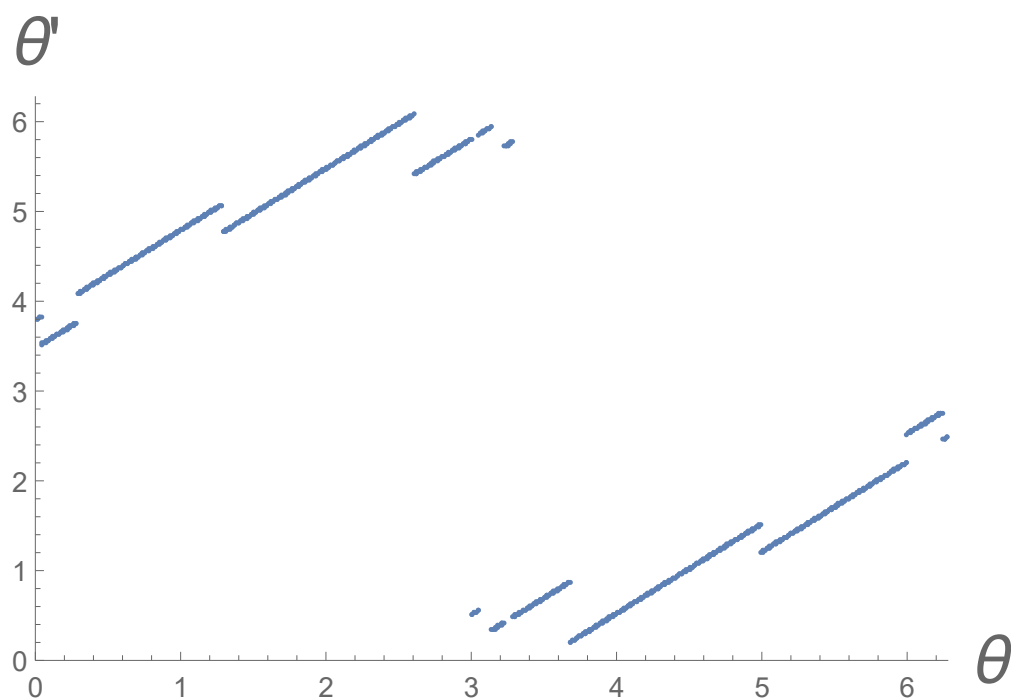
重心系
光子を基準とした
極座標表示

$$\varphi' = 0$$

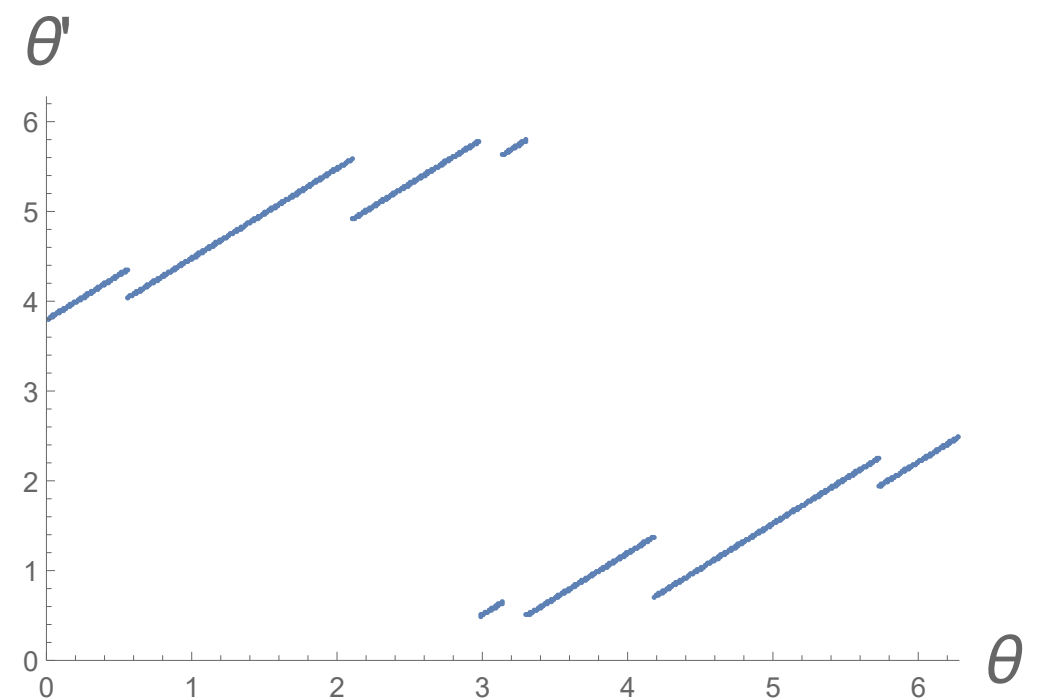
$$N = 5$$



$$N = 3 + 1 + 1$$

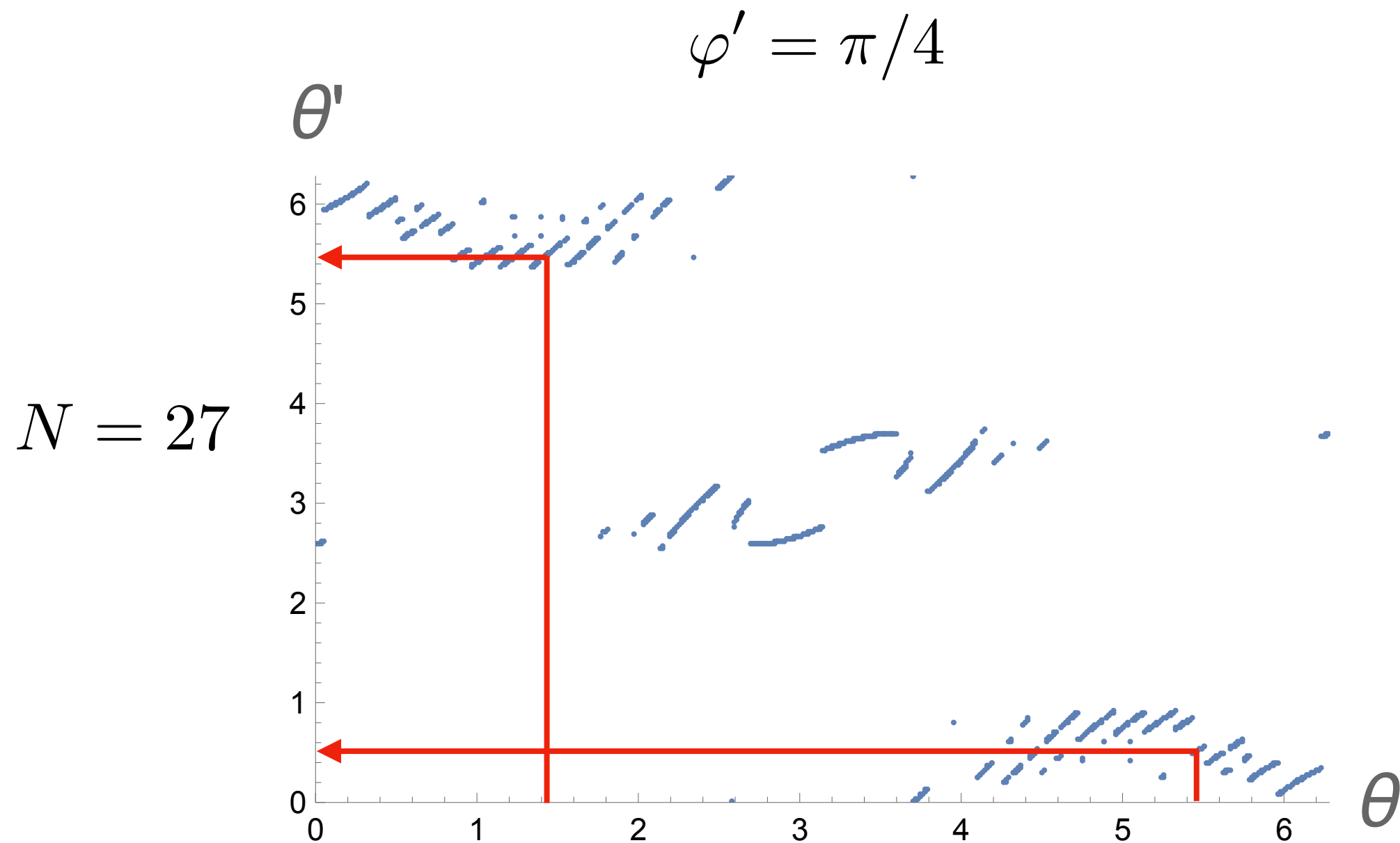


$$N = 1 + 1 + 1 + 1 + 1$$



高次補正を考えるとカオス？

- n次補正 ~ n回のTree-level散乱 → パン屋の写像？



重心系、高励起弦の向きを基準とした極座標表示

2/3章 散乱データとフラクタル構造 まとめ

- 古典散乱のアイデアを量子に拡張
- 高励起弦+タキオン $\rightarrow$ 高励起弦+タキオン
 - 散乱データが微細に分裂するがフラクタルでは無い
 - 高次補正でカオスが出る可能性

目次

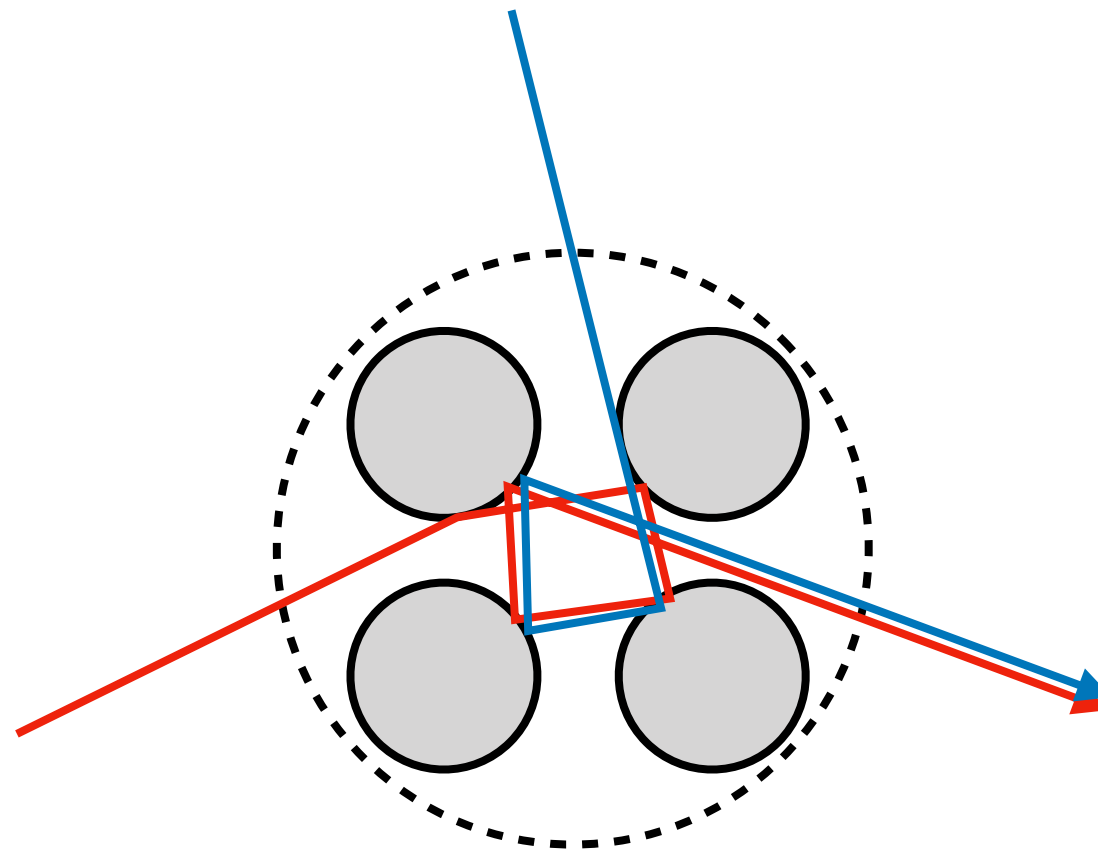
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終 章	まとめ	[3枚]
		[42枚]

3/3章 散乱領域と幾何学的構造 まとめ

- 散乱領域の写真を撮る
- タキオン+タキオン $\rightarrow$ タキオン+タキオン
高励起弦 $\rightarrow$ タキオン+タキオン
 - 散乱領域は二重/多重スリットのような一直線の構造
 - 閉弦の散乱や多点散乱でカオスが出る可能性

散乱領域内の構造を見たい

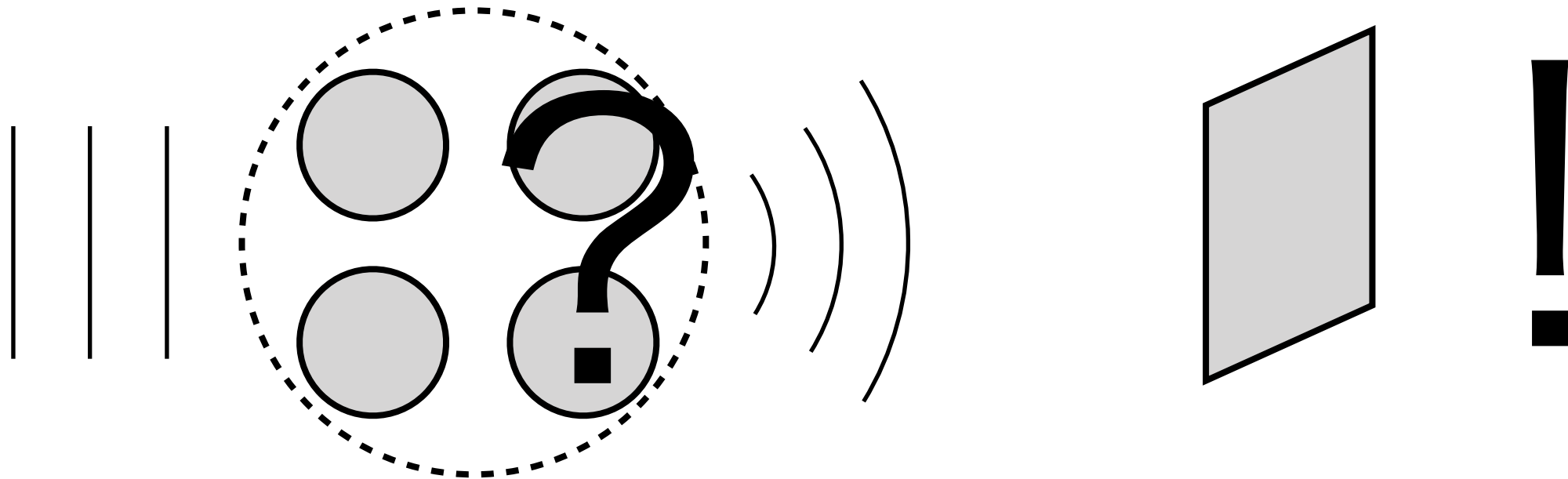
古典では構造が先に与えられる



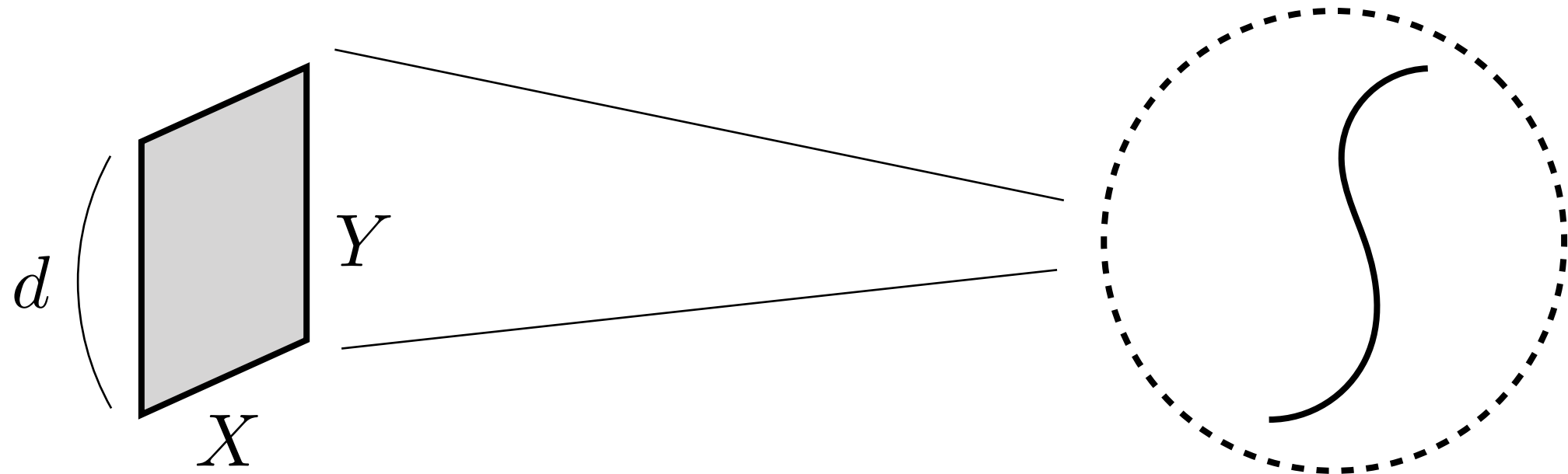
構造が不明な場合はどうする？

波を当てて像を撮れば良い

スクリーンで散乱波を受け取り、角度方向にFourier変換



弦の写真を撮る



$$I(X, Y) = \frac{1}{(2d)^2 \sin \theta'_0} \int_{\theta'_0 - d}^{\theta'_0 + d} d\theta' \int_{\varphi'_0 - d/\sin \theta'_0}^{\varphi'_0 + d/\sin \theta'_0} d\varphi' e^{i\omega((\theta' - \theta'_0)X + (\varphi' - \varphi'_0)Y)} \mathcal{A}(\theta', \varphi')$$

|

像

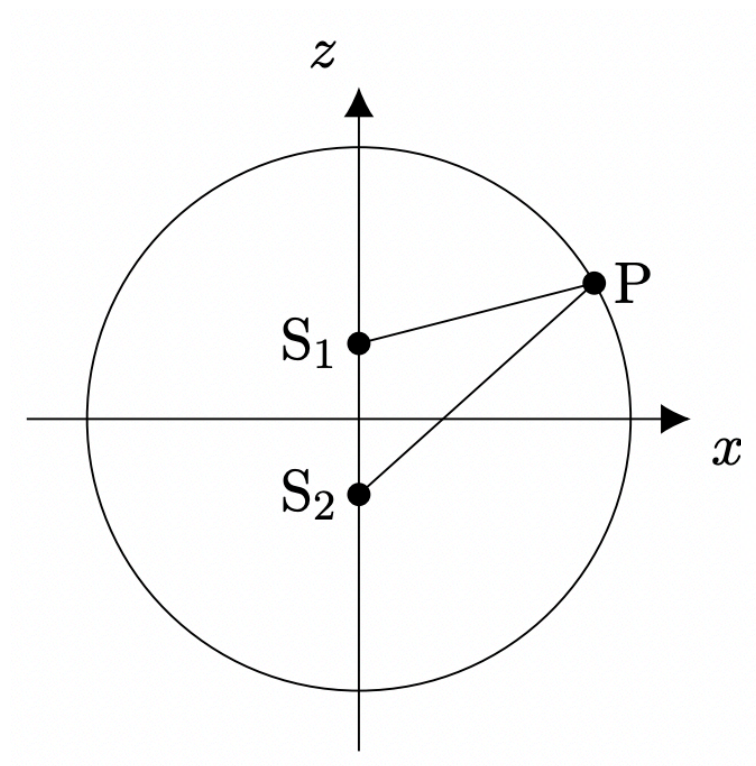
|

角度方向のFourier変換

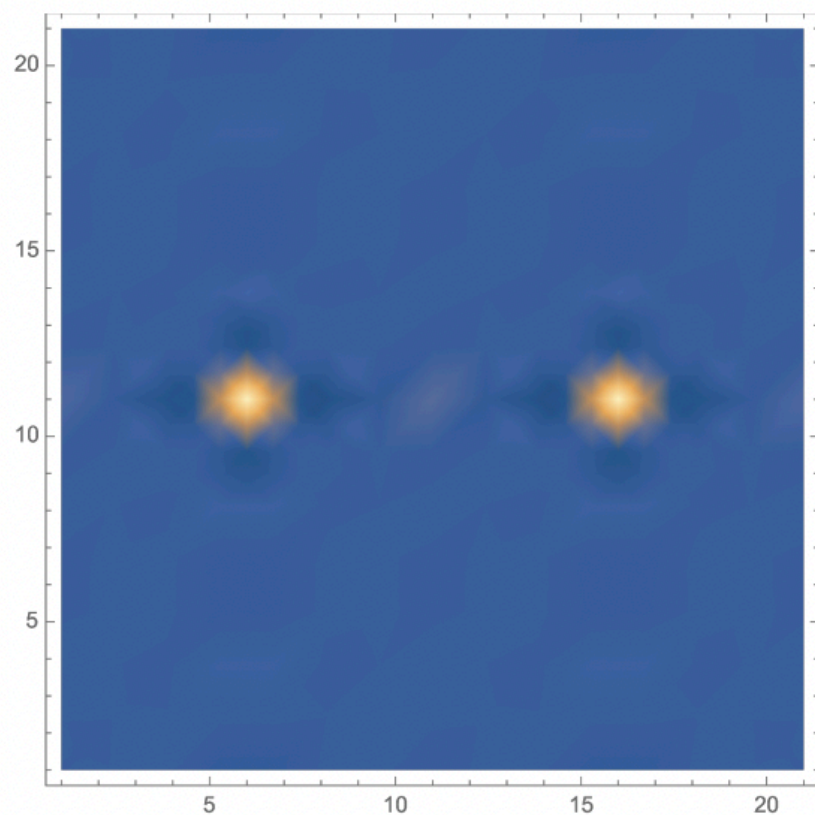
|

振幅

通常の実二重スリット実験で確認

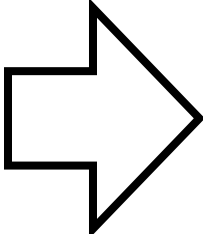


$$\mathcal{A}(\theta', \varphi') \simeq 2A \cos \left(\frac{\pi l}{\lambda_{\text{ds}}} \cos \theta' \right)$$

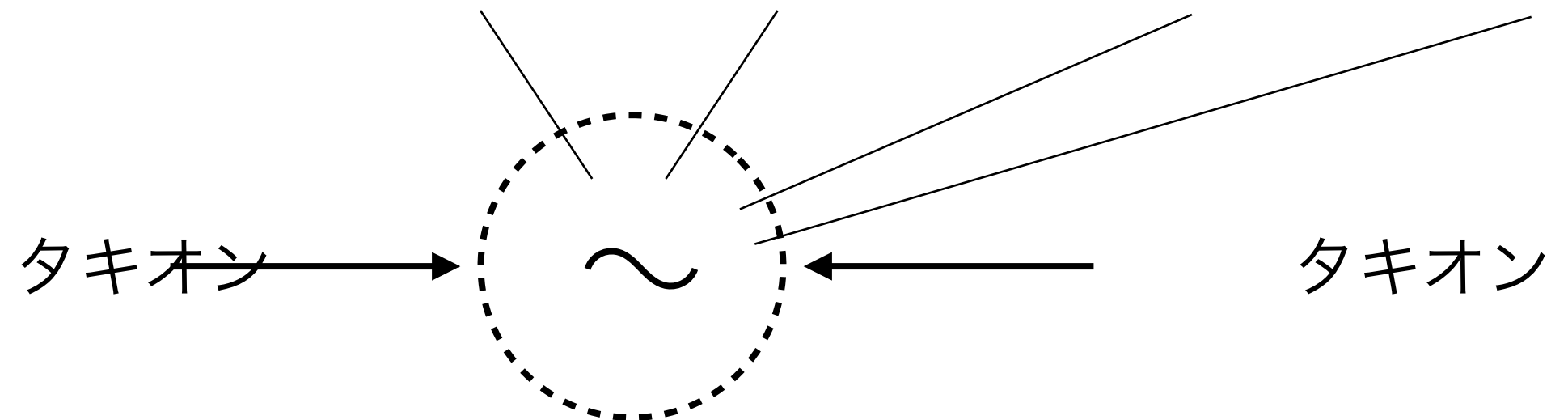


$$I(X, Y) = \frac{1}{(2d)^2 \sin \theta'_0} \int_{\theta'_0 - d}^{\theta'_0 + d} d\theta' \int_{\varphi'_0 - d / \sin \theta'_0}^{\varphi'_0 + d / \sin \theta'_0} d\varphi' e^{i\omega((\theta' - \theta'_0)X + (\varphi' - \varphi'_0)Y)} \mathcal{A}(\theta', \varphi')$$

弦の散乱で調べてみる

1. タキオン+タキオン $\rightarrow$ タキオン+タキオン
 2. 高励起弦 $\rightarrow$ タキオン+タキオン
- 
- 二重スリット
多重スリット

Veneziano振幅の像は二重スリット



振幅の零が一致している

二重スリット実験(スリットの間隔が小さい極限)

$$\mathcal{A}(\theta', \varphi') \simeq 2A \cos \left(\frac{\pi l}{\lambda_{\text{ds}}} \cos \theta' \right)$$

$$\cos \theta' \sim 2\mathbb{Z} \pm 1$$

が振幅の

Veneziano振幅(散乱のエネルギーが大きい極限)

$$\begin{aligned} \tilde{\mathcal{A}}_s^{\text{Ven}} &= \lim_{s \rightarrow 2n} \frac{\sin \pi(-1 - s/2)}{\pi} (\mathcal{A}_{st}^{\text{Ven}} + \mathcal{A}_{us}^{\text{Ven}}) \\ &= \frac{1}{(n+1)!} \left[\frac{\Gamma\left(\frac{n}{2} + 1 - \frac{n+4}{2} \cos \theta'\right)}{\Gamma\left(-\frac{n}{2} - \frac{n+4}{2} \cos \theta'\right)} + \frac{\Gamma\left(\frac{n}{2} + 1 + \frac{n+4}{2} \cos \theta'\right)}{\Gamma\left(-\frac{n}{2} + \frac{n+4}{2} \cos \theta'\right)} \right] \end{aligned}$$

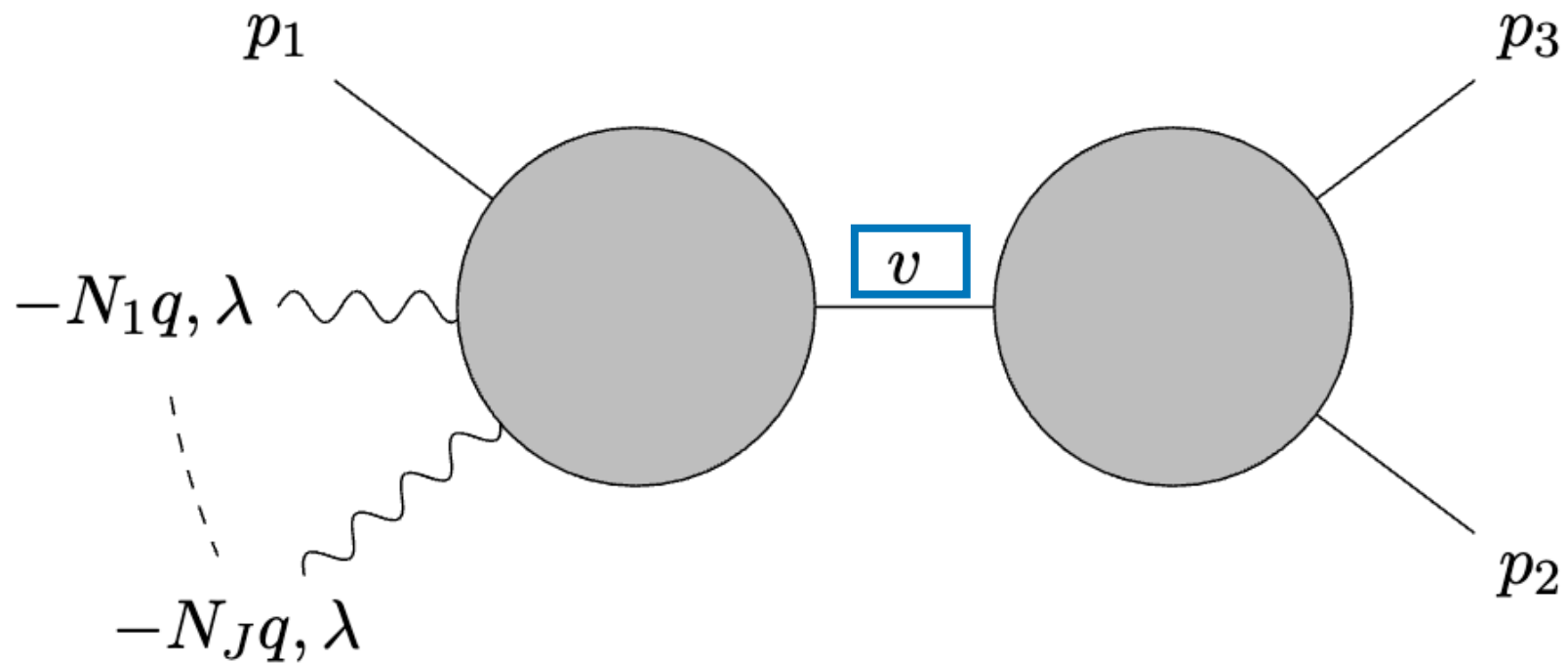
$$s = 4(p^2 - 2) = 2n$$

$$t = -2p^2(1 + \cos \theta')$$

$$u = -2p^2(1 - \cos \theta')$$

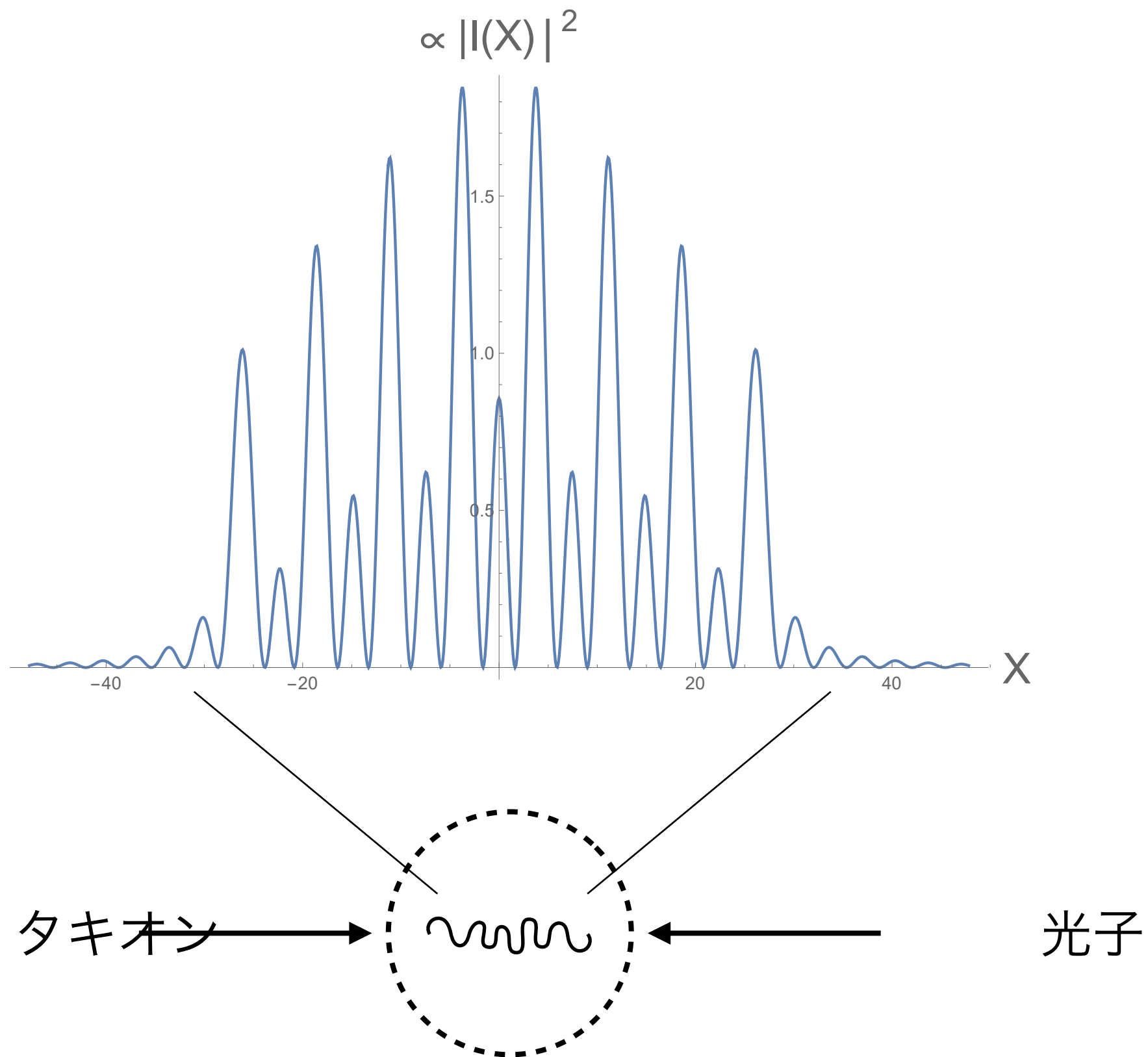
高励起弦の散乱振幅

高励起弦の質量 : $M^2 = 2(N - 1), N = \sum_a^J N_a$
角運動量 : J



(タキオン+光子) $\rightarrow$ タキオン+タキオン

高励起弦の像は多重スリット



振幅がVeneziano振幅の積になってる

高励起弦の振幅(散乱のエネルギーが大きい極限)

$$\begin{aligned}\tilde{A}_v^{\text{HES}} &\simeq \prod_{a=1}^J \sin \theta' \left[\frac{\Gamma\left(+\frac{N_a}{2}(1 + \cos \theta')\right)}{\Gamma\left(-\frac{N_a}{2}(1 - \cos \theta')\right)} + \frac{\Gamma\left(+\frac{N_a}{2}(1 - \cos \theta')\right)}{\Gamma\left(-\frac{N_a}{2}(1 + \cos \theta')\right)} \right] \\ &\simeq (\sin \theta')^J \prod_{a=1}^J \tilde{\mathcal{A}}_s^{\text{Ven}} \Big|_{n \rightarrow N_a}\end{aligned}$$

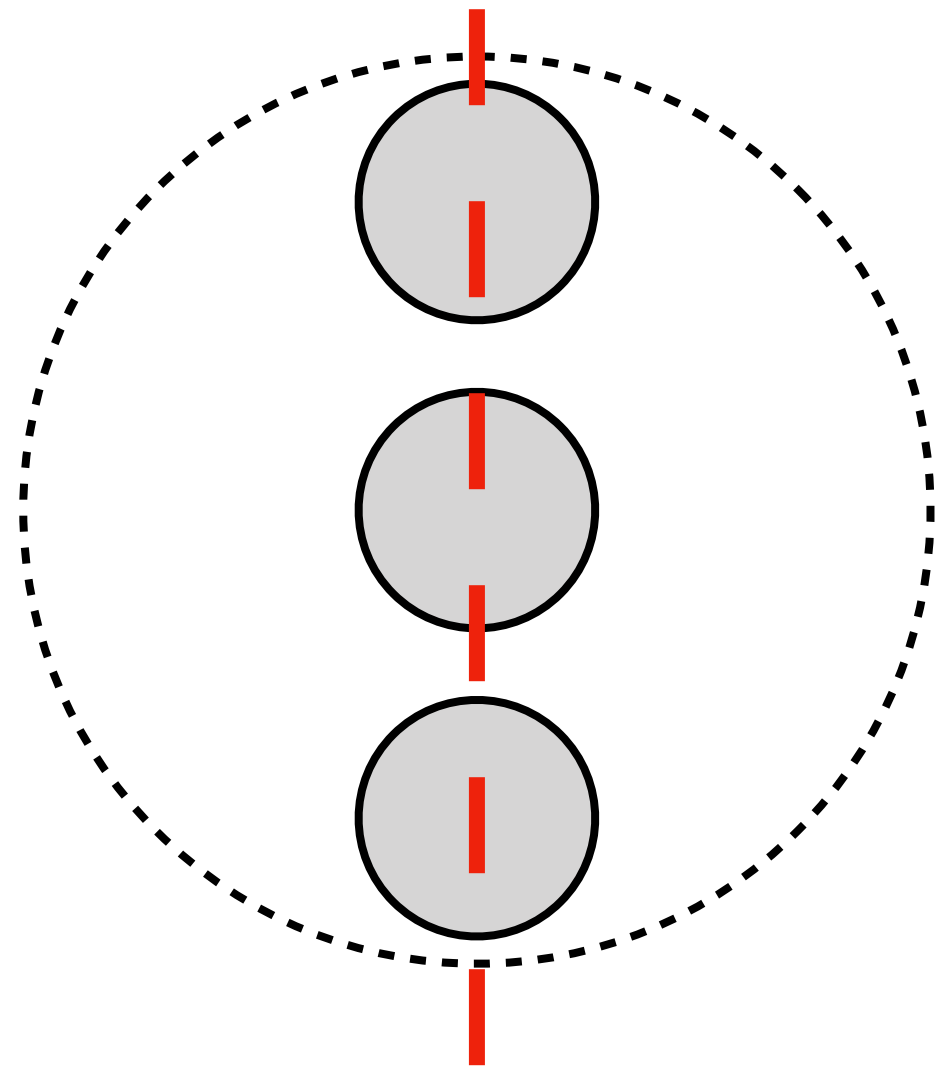
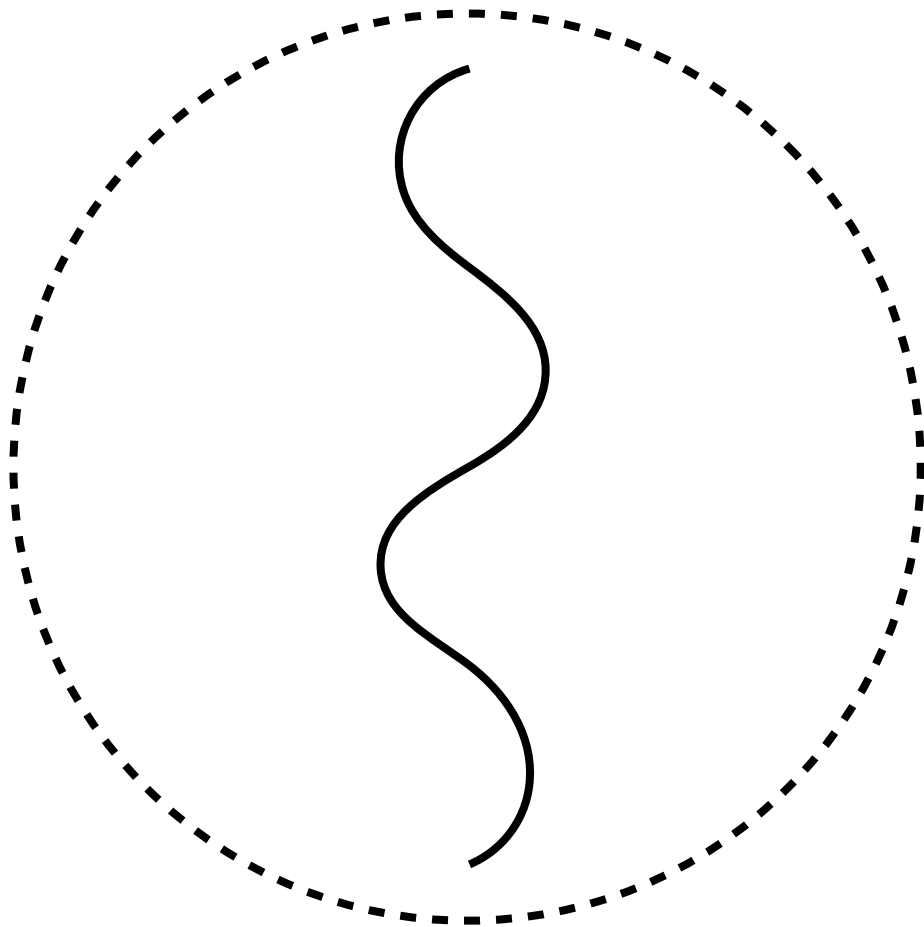
$$I^{\text{HES}} \simeq (\sin \theta')^J * I^{\text{Ven}} \Big|_{n \rightarrow N_1} * \dots * I^{\text{Ven}} \Big|_{n \rightarrow N_J}$$

高励起弦の像は、

二重スリットのコンボリューション → 多重スリット

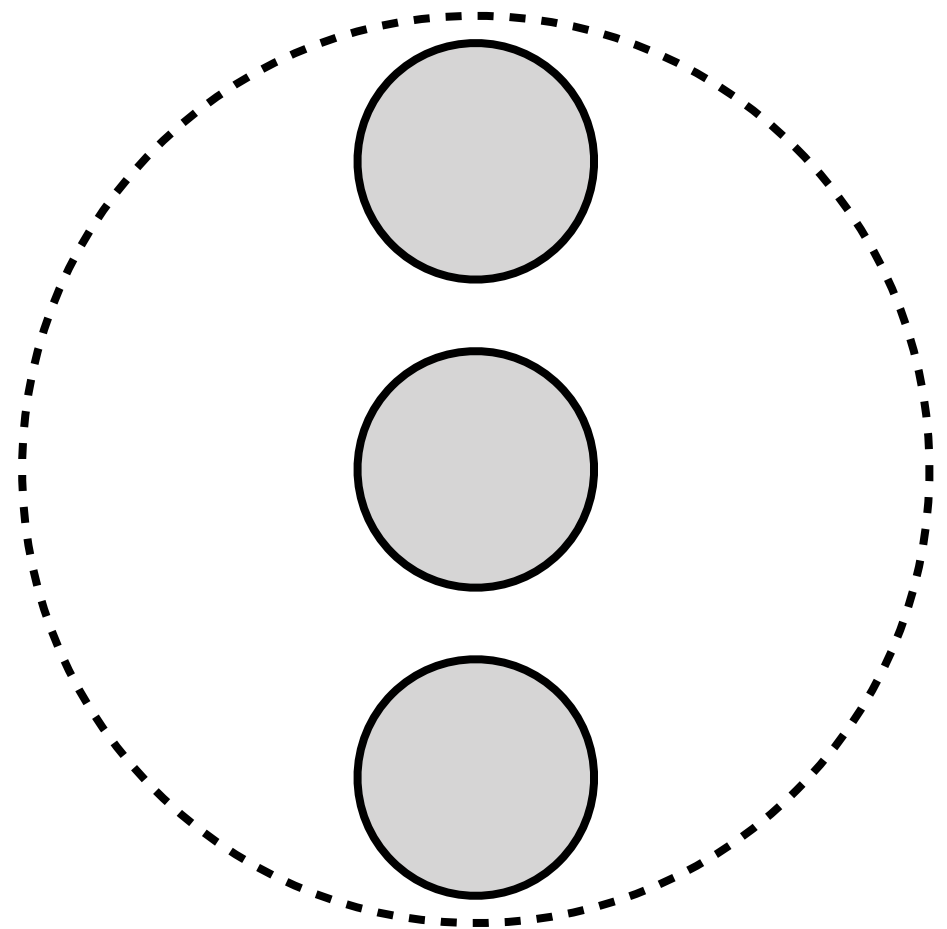
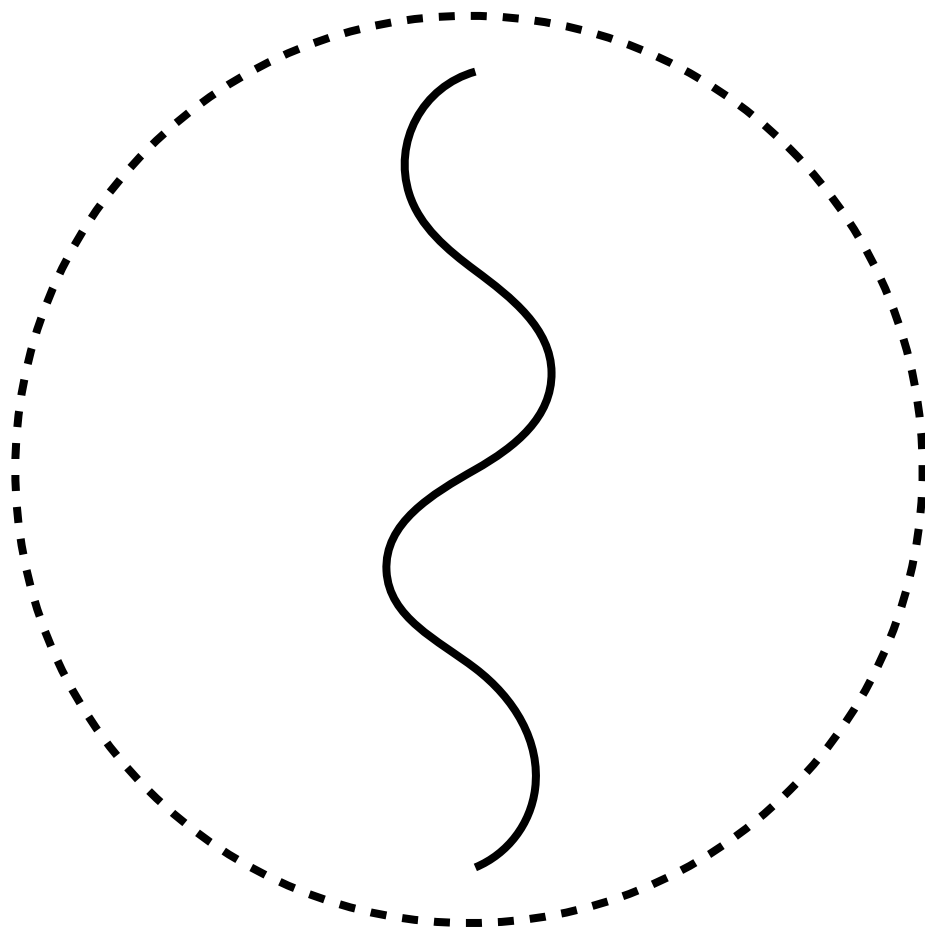
散乱領域の幾何学的構造

- 散乱領域は二重/多重スリットに見える
- 構造が一直線上 $\rightarrow$ カオスでは無い



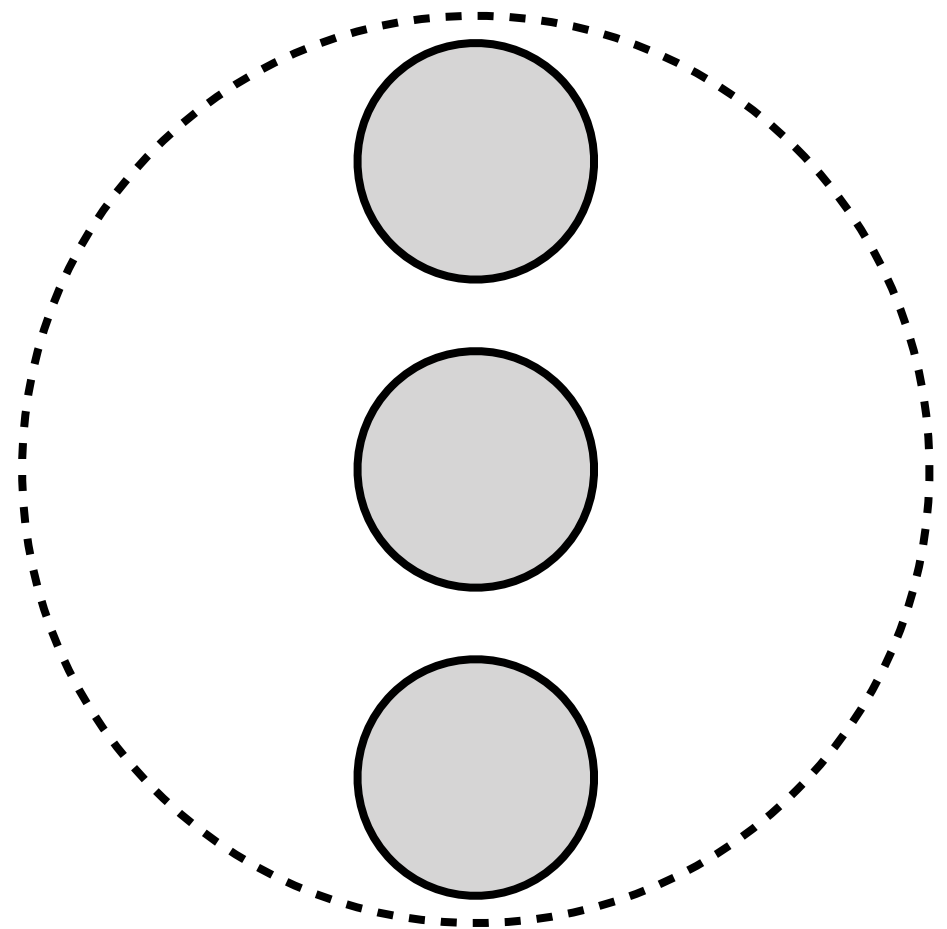
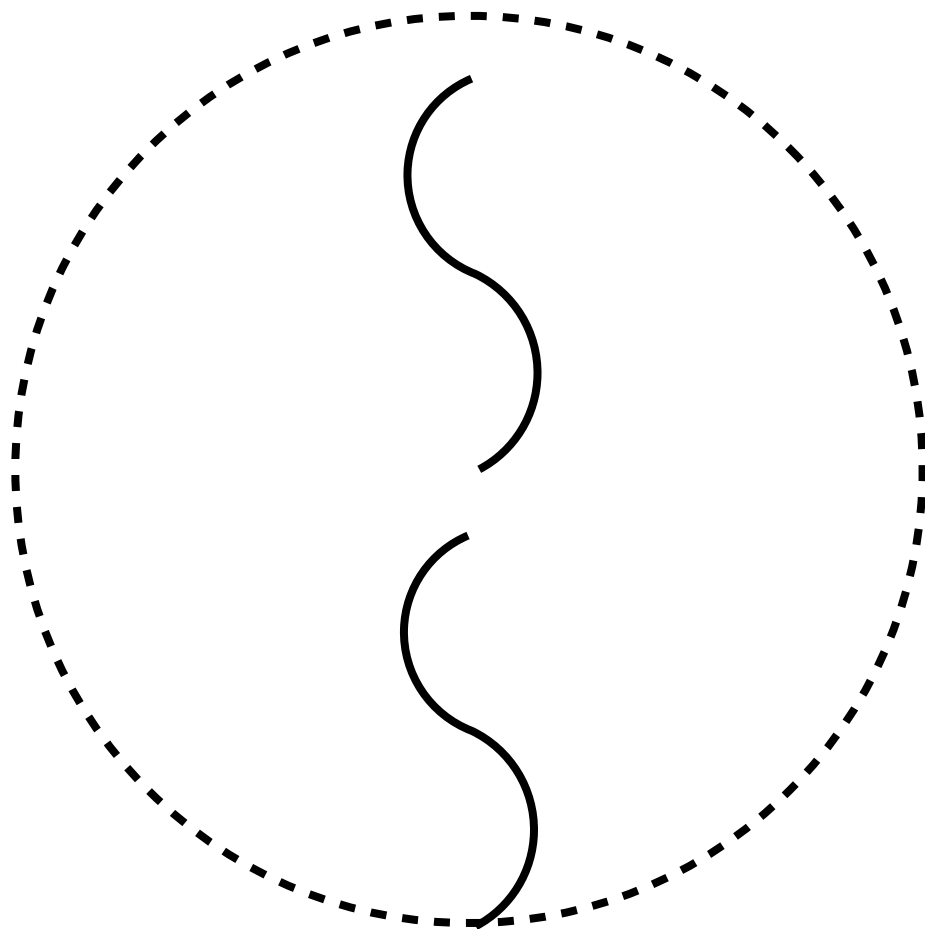
散乱領域の幾何学的構造

- 弦がどこで切れるかの重ね合わせと解釈できる
- 閉弦や多点散乱 → カオスが出る可能性



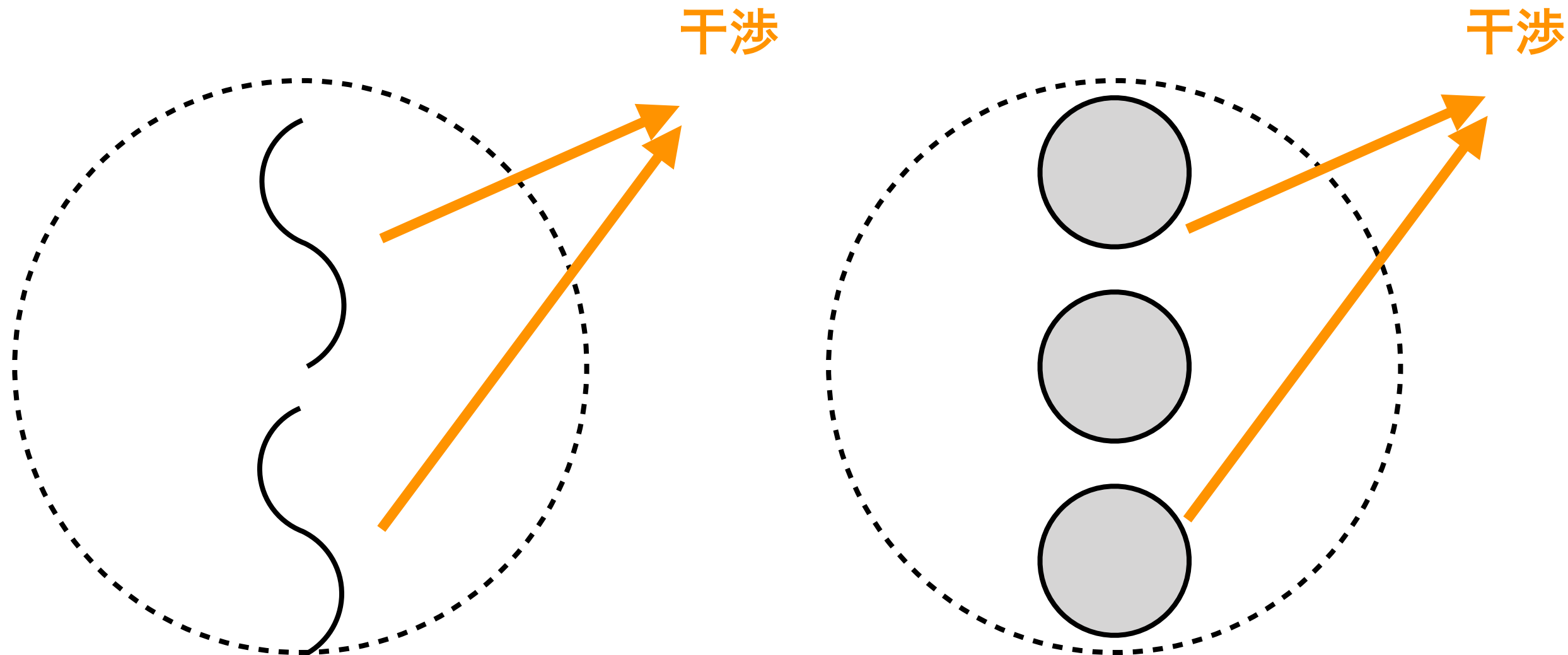
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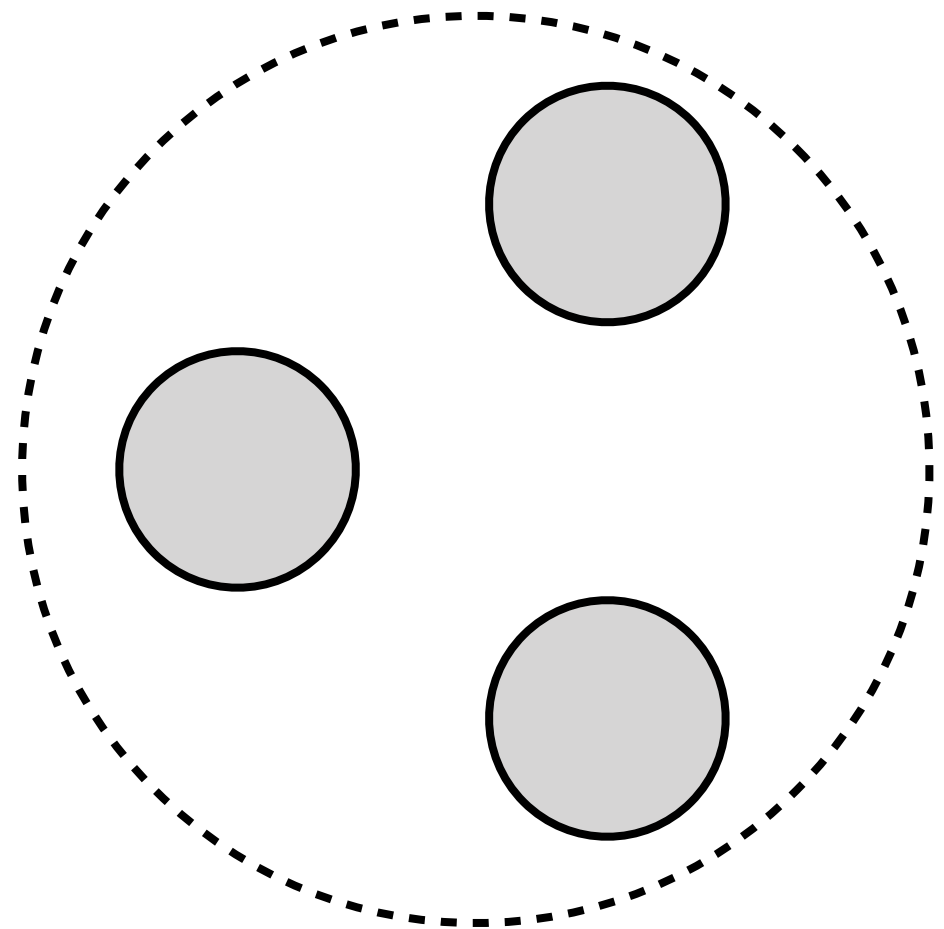
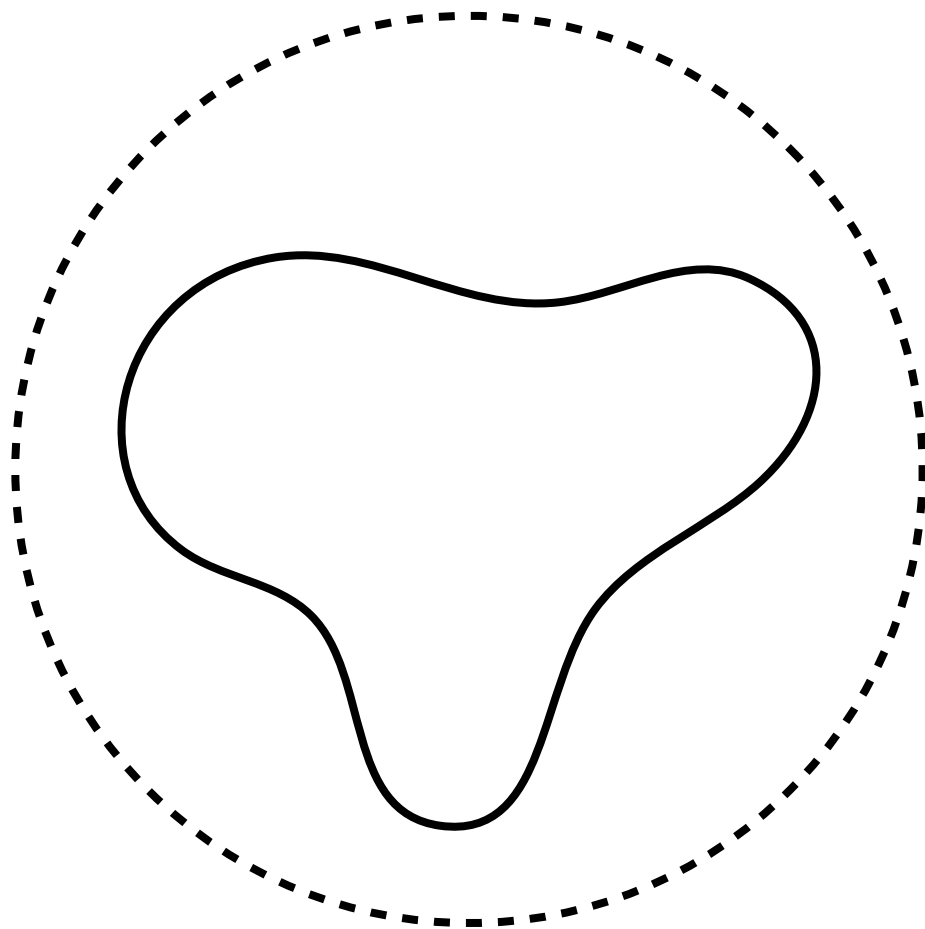
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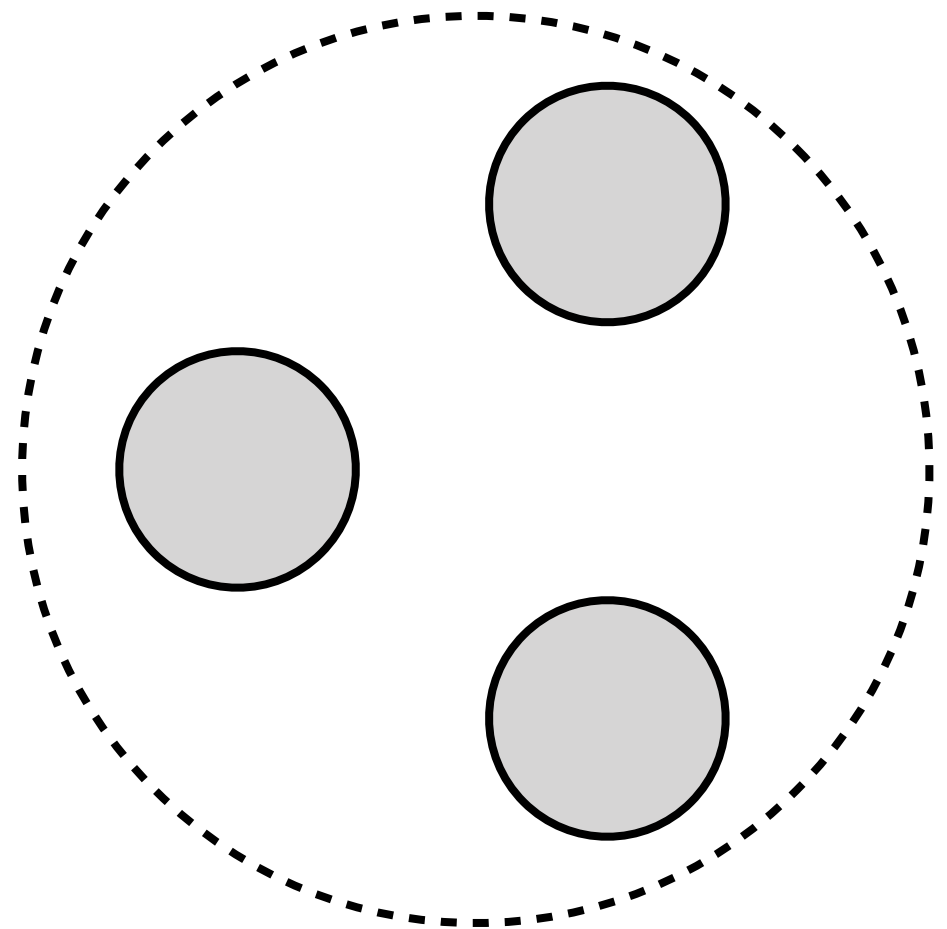
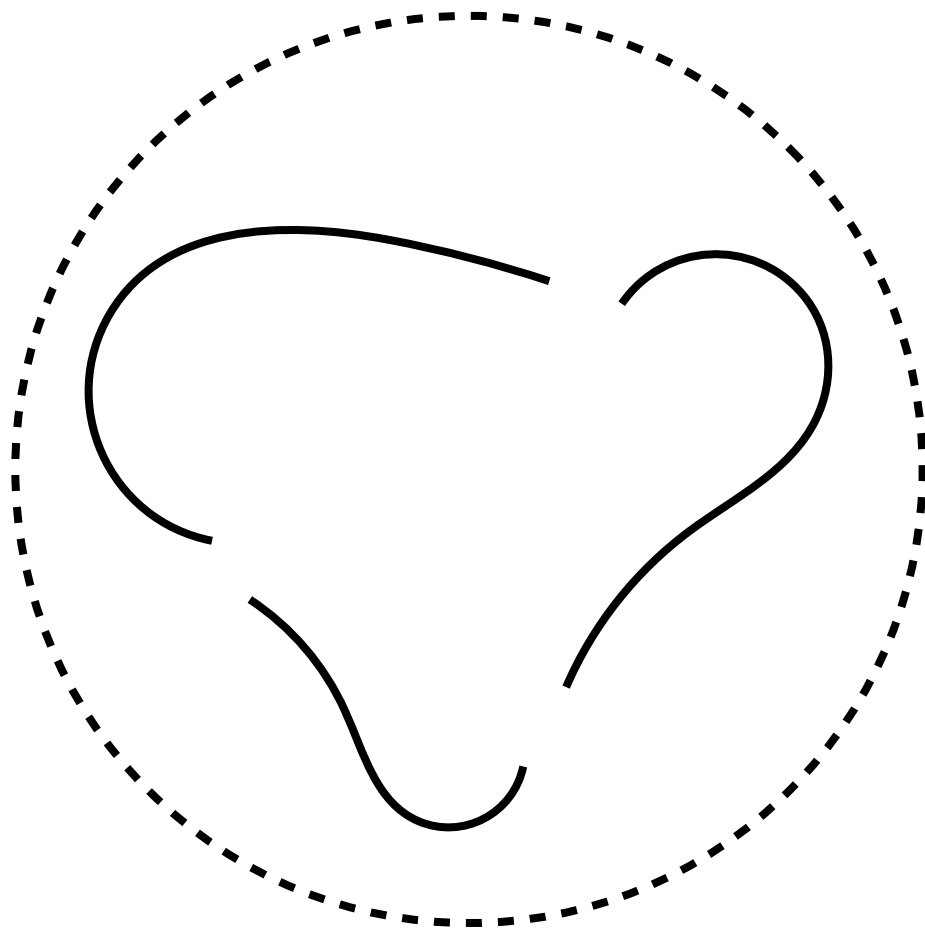
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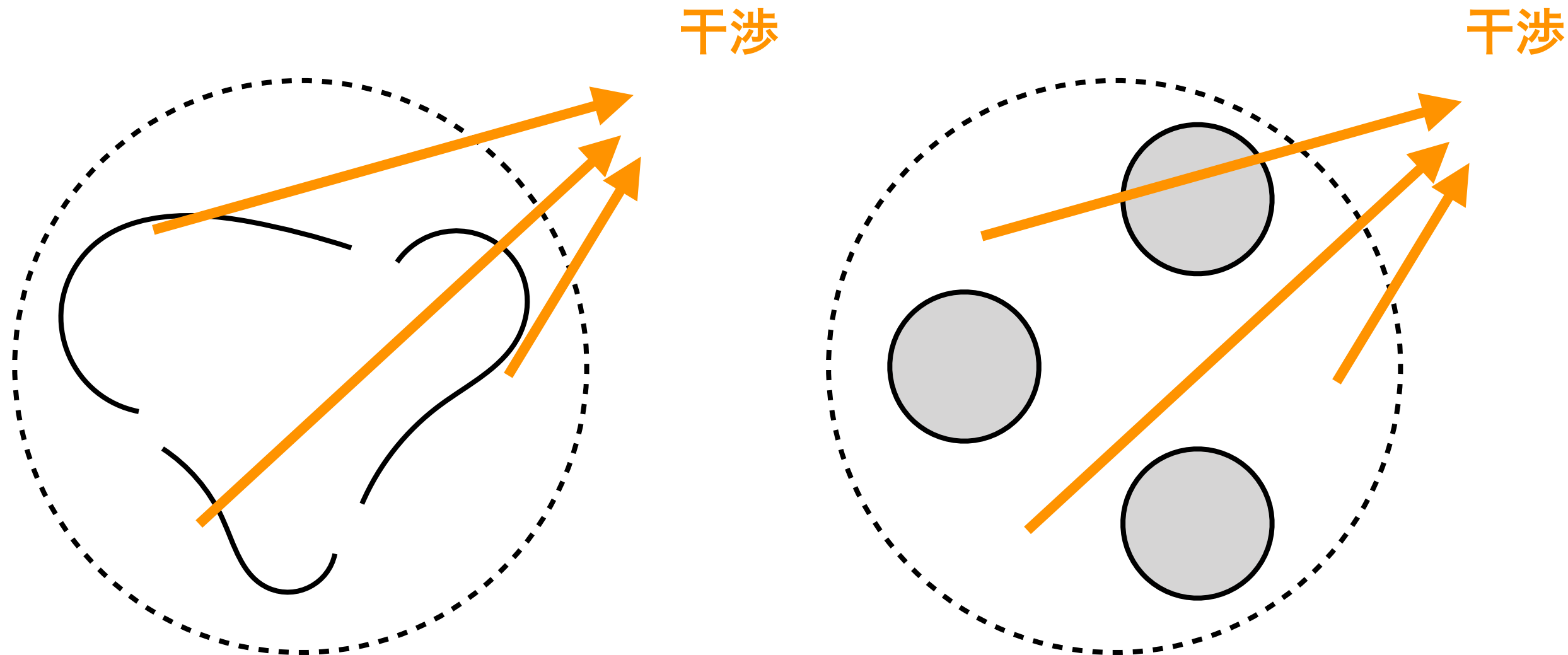
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散乱領域の幾何学的構造

- 弦がどこで切れるかの重ね合わせと解釈できる
- 閉弦や多点散乱 → カオスが出る可能性



3/3章 散乱領域と幾何学的構造 まとめ

- 散乱領域の写真を撮る
- タキオン+タキオン $\rightarrow$ タキオン+タキオン
高励起弦 $\rightarrow$ タキオン+タキオン
 - 散乱領域は二重/多重スリットのような一直線の構造
 - 閉弦の散乱や多点散乱でカオスが出る可能性

目次

序 章	概観	[5枚]
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第2章	弦の散乱データとフラクタル構造	[12枚]
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終 章	まとめ	[3枚]
		[42枚]

弦の散乱は
カオスか？

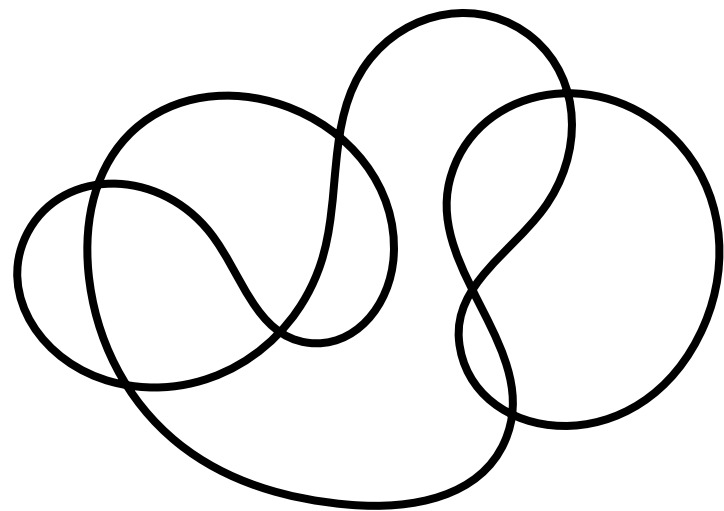
ボソン開弦のツリーレベルにカオスは無し

高次補正や閉弦の多点散乱にカオスを期待

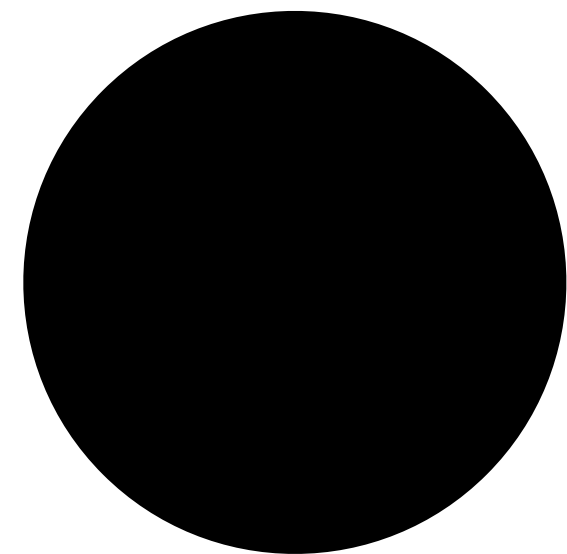
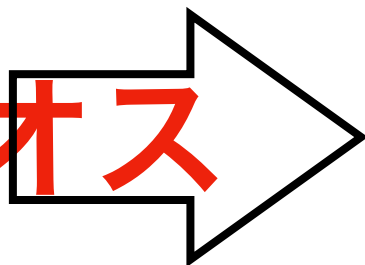
- 弦の散乱データのフラクタル性
 - 散乱データが微細に分裂するがフラクタルでは無い
 - 高次補正でカオスが出る可能性
- 弦の散乱領域の幾何学的構造
 - 散乱領域は二重/多重スリットのような一直線の構造
 - 閉弦の散乱や多点散乱でカオスが出る可能性
- 予想の検証
- 遅延時間の解析
- 改良された鞍点近似による高次補正の計算 等々

夢

カオスで**ミクロ**と**マクロ**の隔たりを埋める



弦のカオス



ブラックホ

予備スライド

[関連] ブラックホールカオス

粒子軌道は地平面の摂動に鋭敏

[Polchinski 15. See also Polchinski 16. Dray, 't Hooft 84. 't Hooft 96.]

ブラックホール時空のホログラフィ

[Shenker, Stanford 14, 15. Kitaev (KITP seminar) 14, 15]

ブラックホールはFast Scrambler

[Sekino, Susskind 08]

ブラックホールはカオス上限を飽和する

[Maldacena, Shenker, Stanford 15]

他多数…

ヒント 弦の散乱振幅の鞍点近似

[Gross, Mende, 87, Gross, Manes, 89, Mende, Ooguri, 90]

$$\mathcal{A} = \sum_{G=0}^{\infty} g^{2G-2+M} \mathcal{A}_G \quad \Rightarrow \quad \mathcal{A}_{\text{resum}}$$

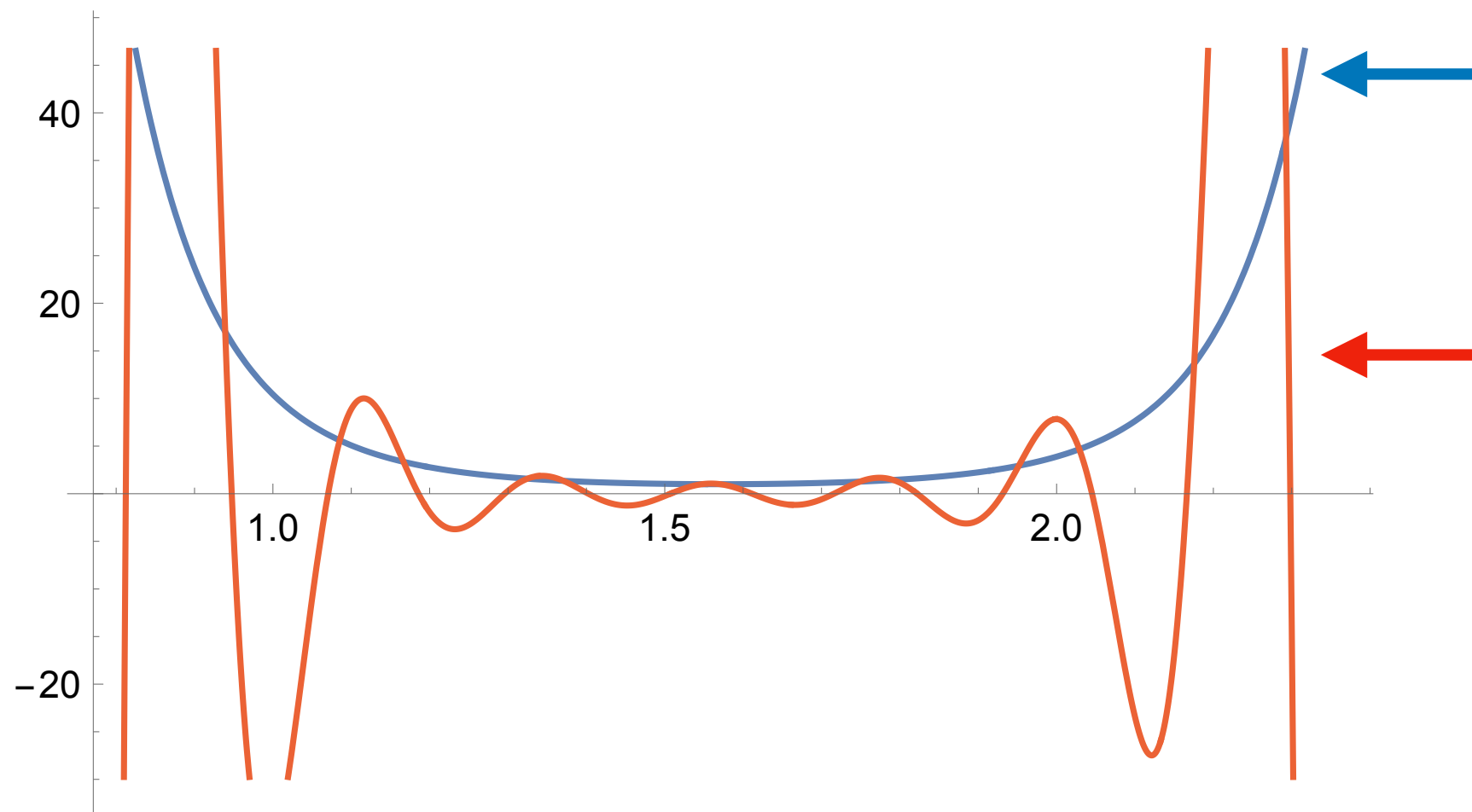
$$\mathcal{A}_G = \int \mathcal{D}g \mathcal{D}X \exp \left[-I_{\text{string}} + i \int P \cdot X \right]$$

先行研究は、主要な鞍点(の候補)を見つけた

カオスを知るには、より進んで、
非自明な鞍点を見つける必要あり

プロトタイプ Veneziano振幅

$$\begin{aligned} \mathcal{A}_{st} &= \int_0^1 dw \, w^{-\alpha(s)-1} (1-w)^{-\alpha(t)-1} \\ &= \int_0^1 dw \, e^{-(\alpha(s)+1) \ln w - (\alpha(t)+1) \ln(1-w)} \end{aligned}$$



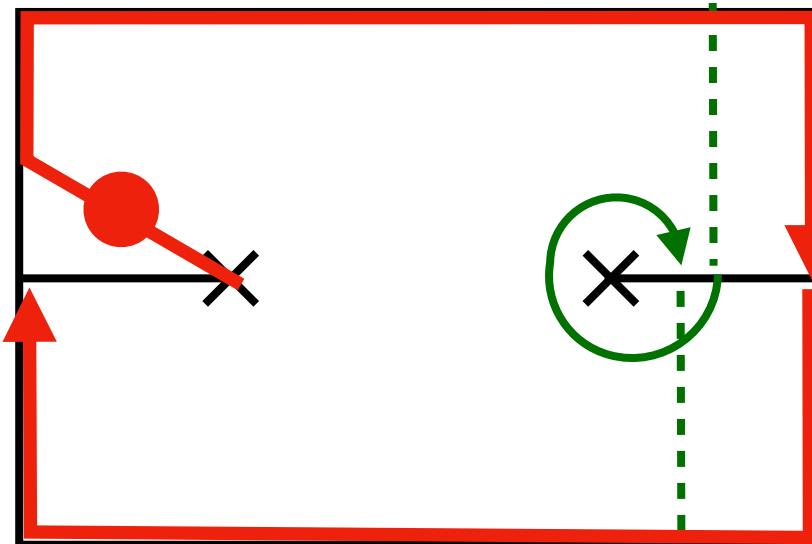
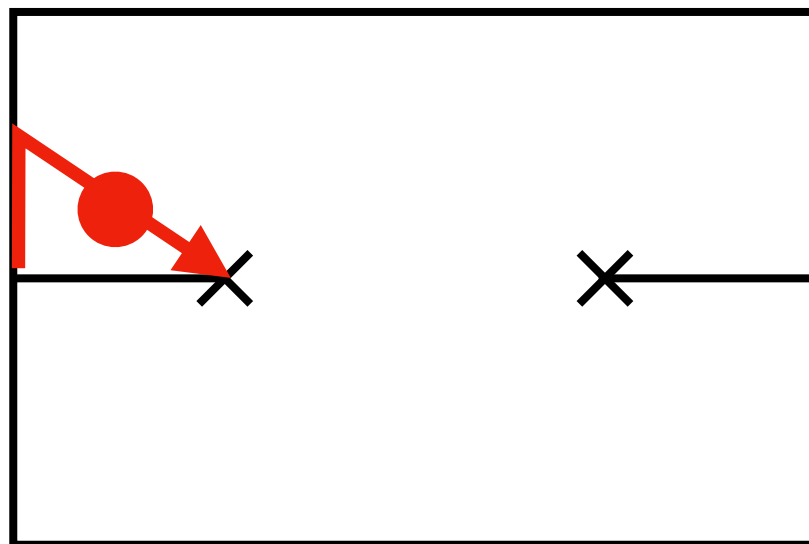
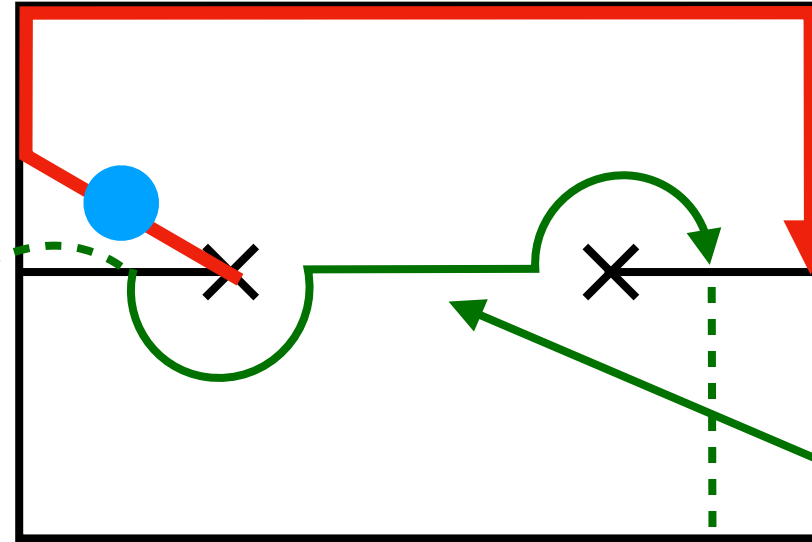
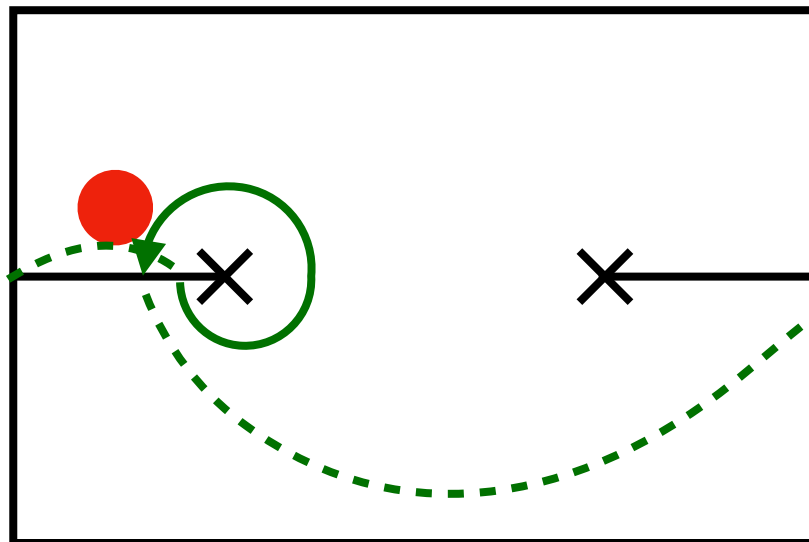
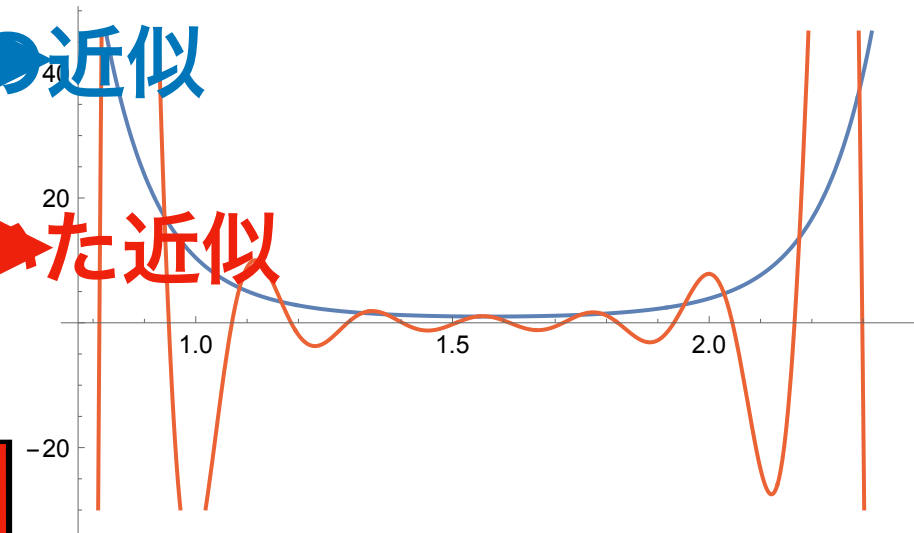
形式的な鞍点

正しい解析接

非自明な鞍点から零や極が分かる

自明な鞍点のみを用いた近似

非自明な鞍点も用いた近似



$i\varepsilon$ 処方

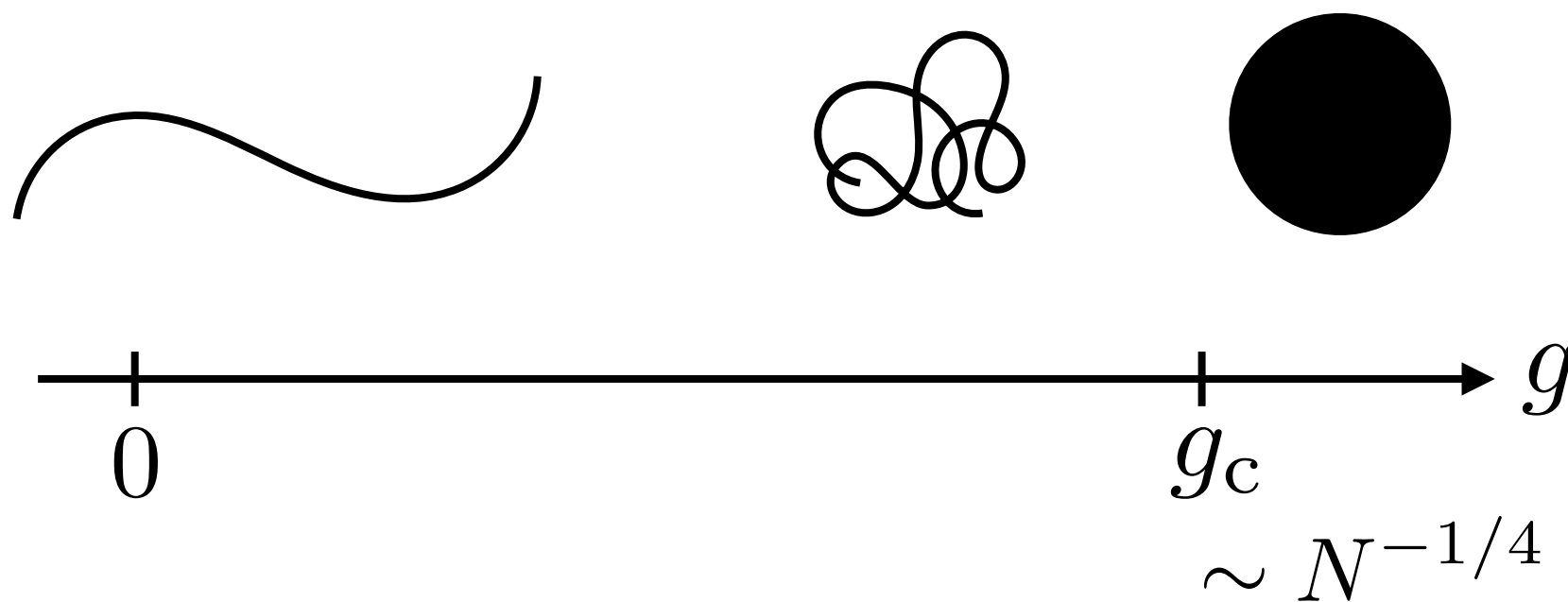
カオスの源を弦に期待

高励起弦とブラックホールとで、状態数が対応する

[Horowitz, Polchinski, 97, 98. Amati, Russo, 99. Halyo, Kol, R
[See also Susskind (review) 21. Gross, Rosenhaus 21]

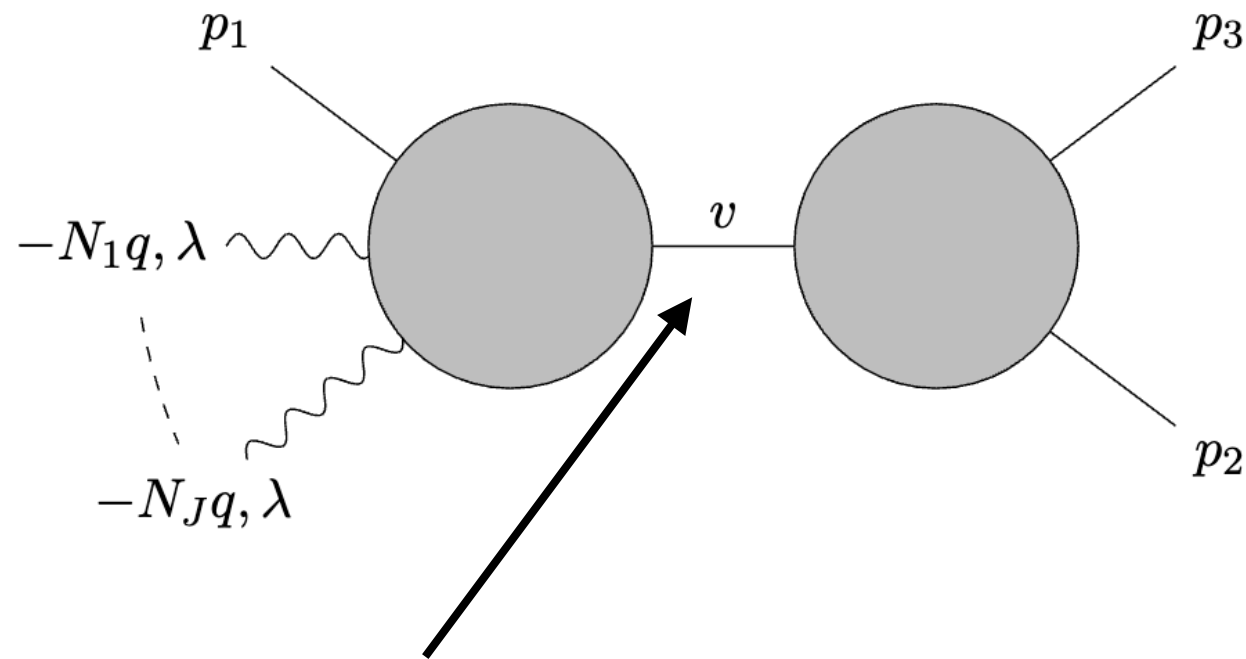
$$\begin{aligned} S_{\text{string}} &\sim \ln e^M \\ &\sim \ln e^{\sqrt{N}} \\ &\sim \sqrt{N} \end{aligned}$$

$$\begin{aligned} S_{\text{BH}} &\sim GM^2 \\ &\sim g^2 N \\ &\sim \sqrt{N} \end{aligned}$$



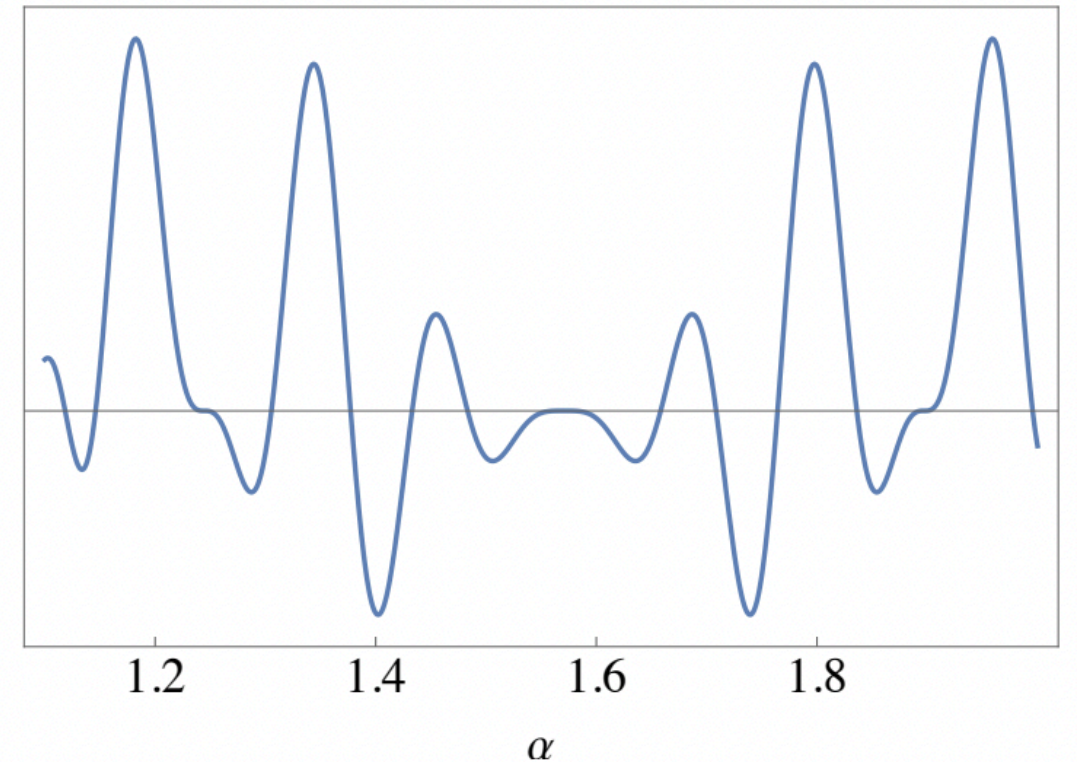
弦だからこそ、ブラックホールの熱的

実際、高励起な弦の崩壊振幅は複雑



高励起弦の極の寄与を見る

質量 : $M^2 = 2(N - 1), N = \sum_a N_a$
角運動量 : J



[Gross, Rosenhaus, 21. Rosenhaus, 21, 22]

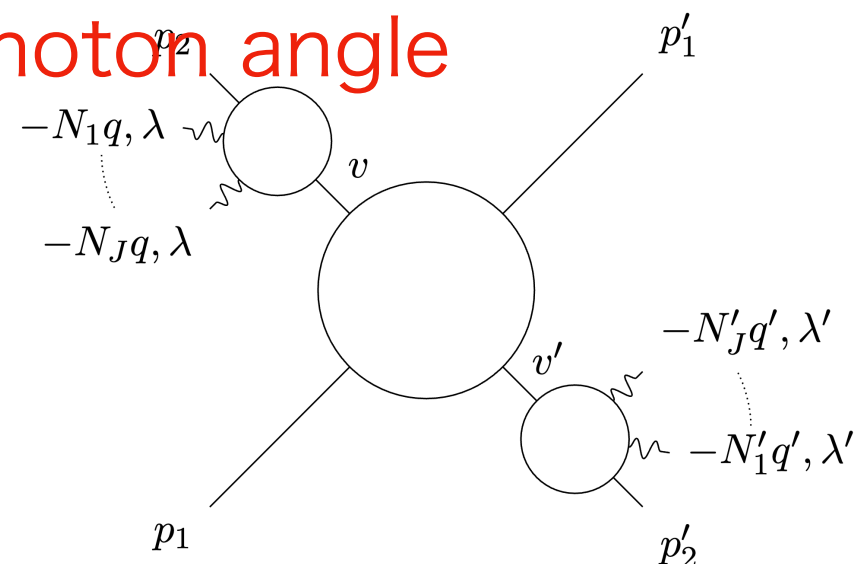
→ より定量的なカオスの評価が望まれる

Computation of HES scattering amplitude

Spherical coordinate

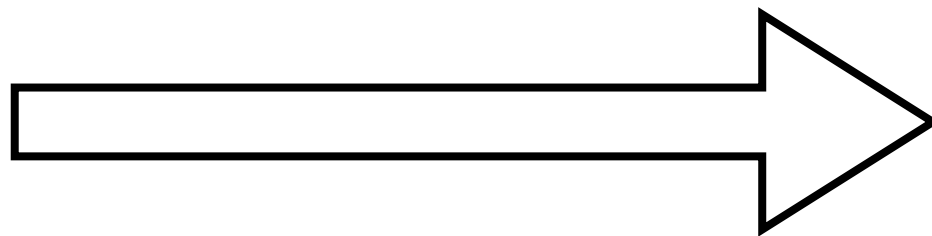
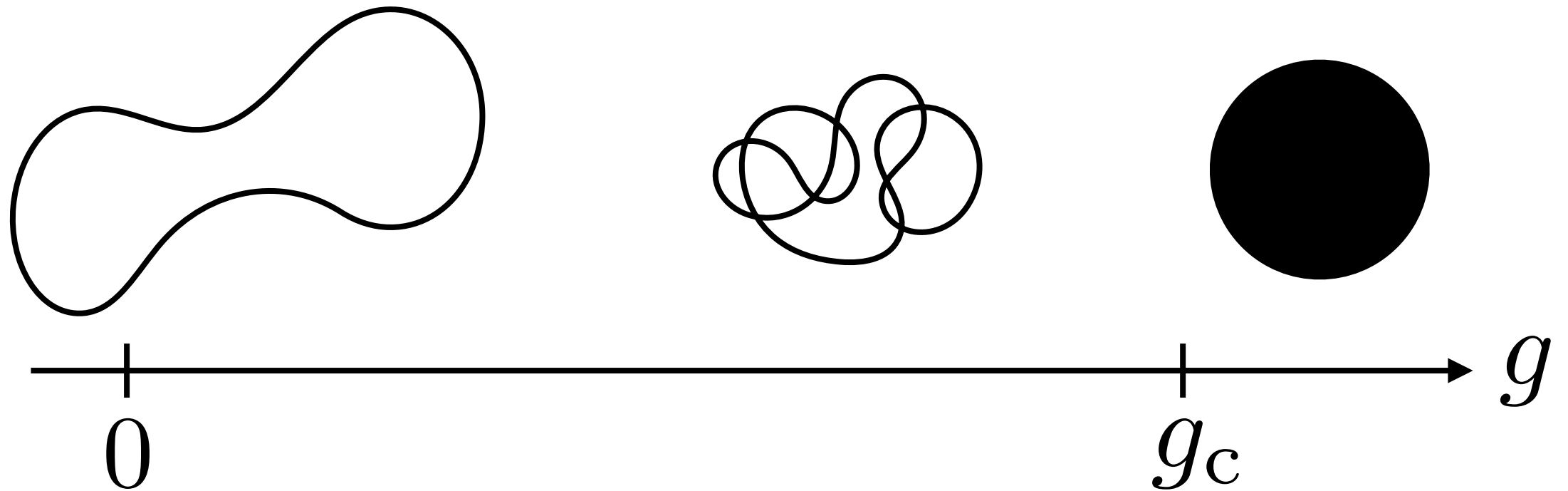
$$\begin{aligned}
 -q &= \frac{1}{\sqrt{2(N-1)+p^2-p\cos\theta}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, & -(-q') &= \frac{1}{\sqrt{2(N-1)+p'^2-p'\cos\theta'}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \\
 \lambda &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, & -\lambda' &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \\
 p_1 &= \begin{pmatrix} \sqrt{p^2-2} \\ p\sin\theta \\ 0 \\ p\cos\theta \end{pmatrix}, & -p'_1 &= \begin{pmatrix} \sqrt{p'^2-2} \\ p'\sin\theta'\cos\varphi' \\ p'\sin\theta'\sin\varphi' \\ p'\cos\theta' \end{pmatrix}, \\
 p_1 + p_2 - Nq &= \begin{pmatrix} \sqrt{s} \\ 0 \\ 0 \\ 0 \end{pmatrix}, & -p'_1 - p'_2 + N'q' &= \begin{pmatrix} \sqrt{s} \\ 0 \\ 0 \\ 0 \end{pmatrix}.
 \end{aligned}$$

Measure in/out tachyon angle
with respect to photon angle



Our goal

Highly Excited String



Maximally

Motivated by

[Horowitz, Polchinski, 97, 98. Amati, Russo, 99. Halyo, Kol, Rajaraman, Susskind 97]

[Susskind (review) 21. Chen, Maldacena, Witten, 21. Gross, Rosenhaus 21, Rosenhaus 21, 22]

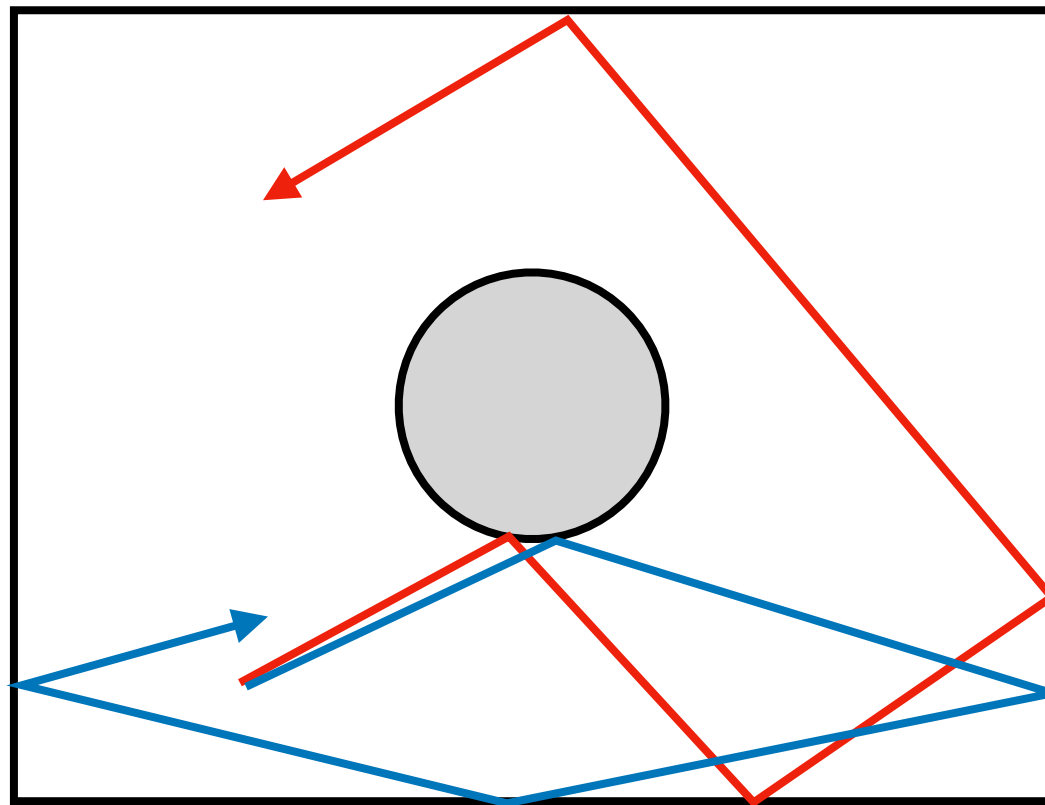
[Shenker, Stanford 14, 15. Kitaev (KITP seminar) 14, 15]

[Sekino, Susskind 08. Maldacena, Shenker, Stanford 15] many related

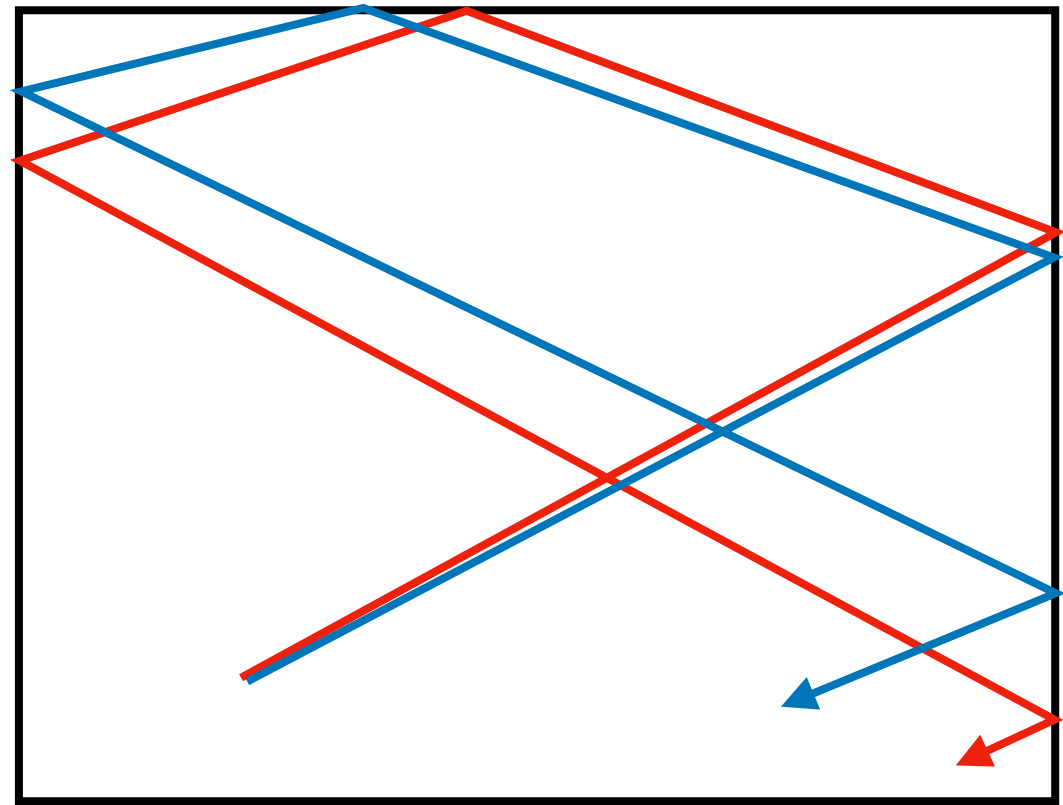
Chaos in dynamical system

- Stretching
 - Folding
- ⇒ Exponential growth of deviation
Finite phase space

Chaotic



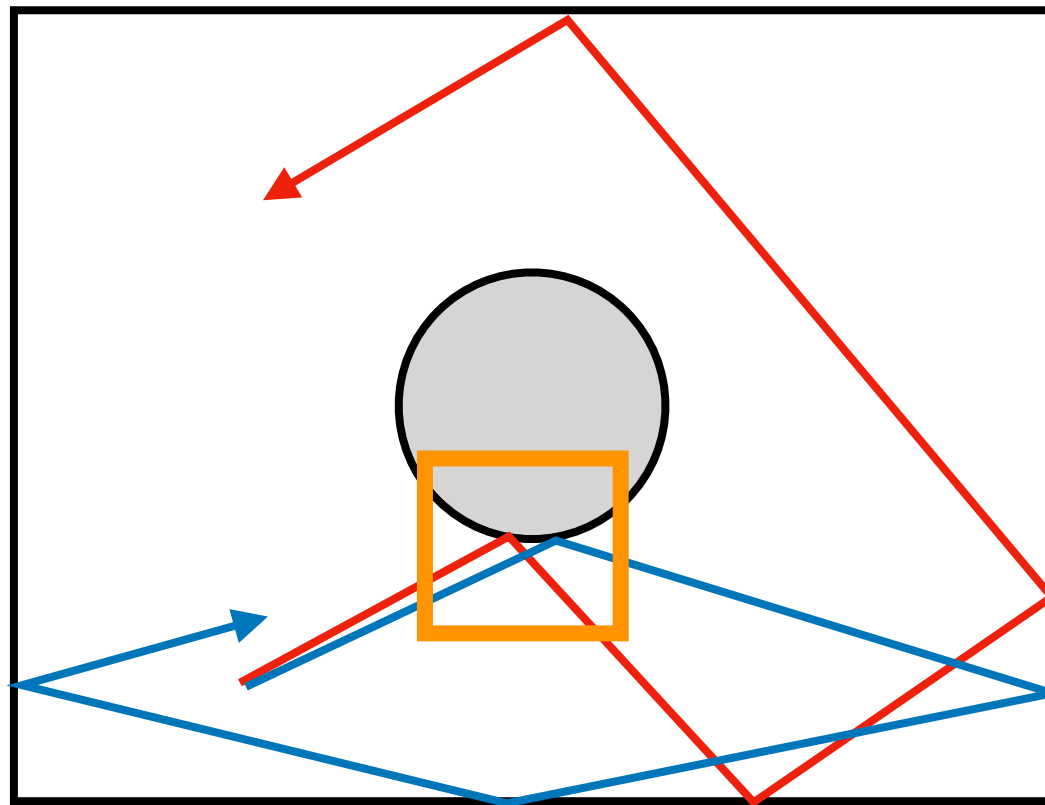
Not chaotic



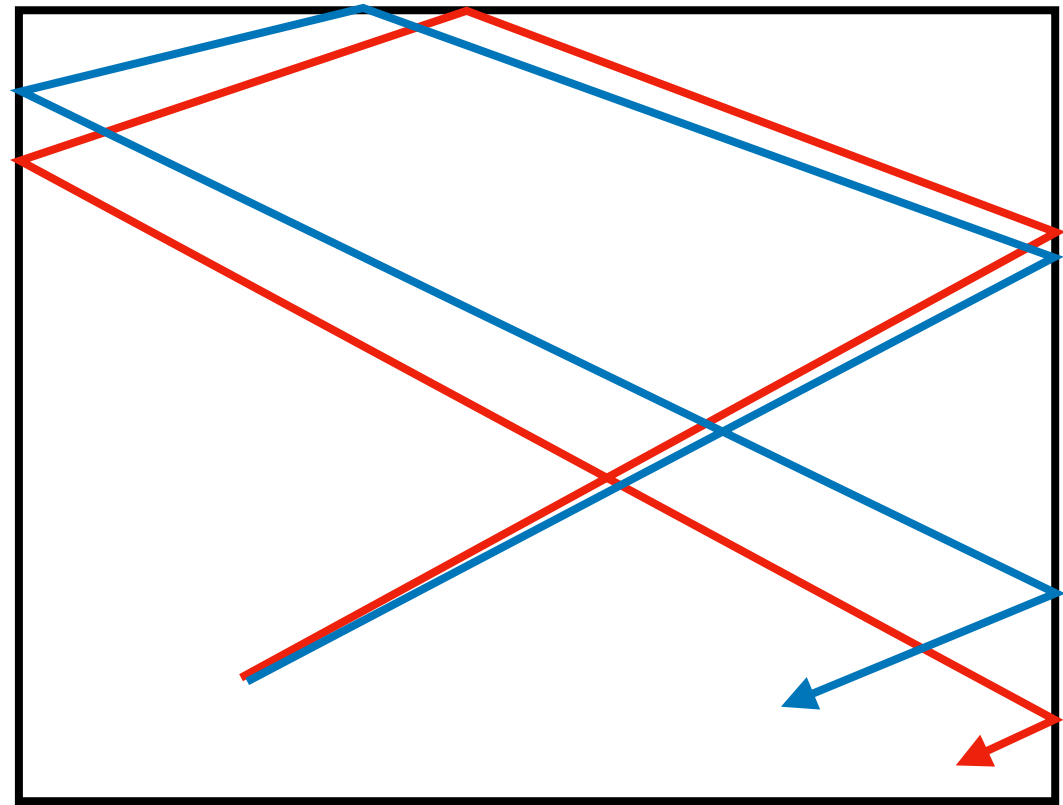
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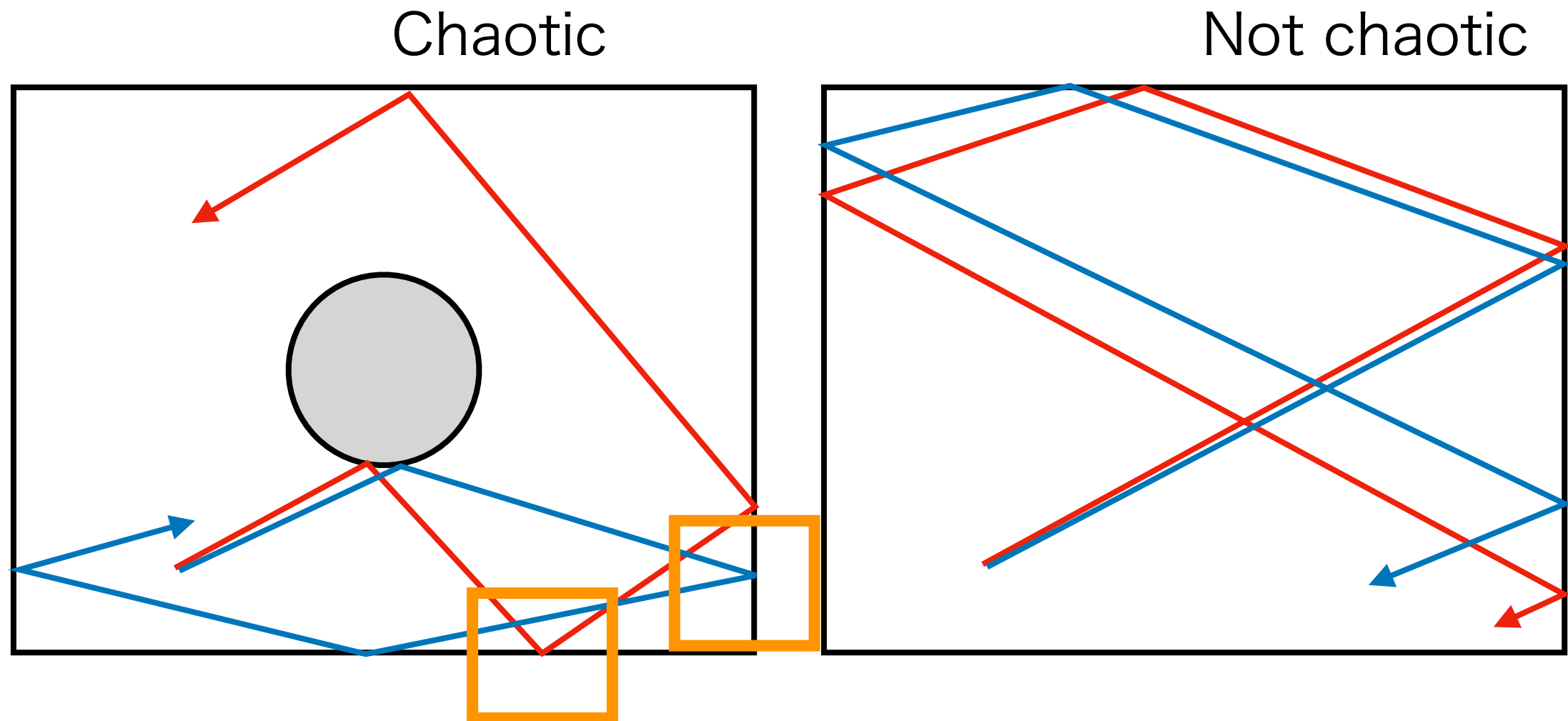


Not chaotic



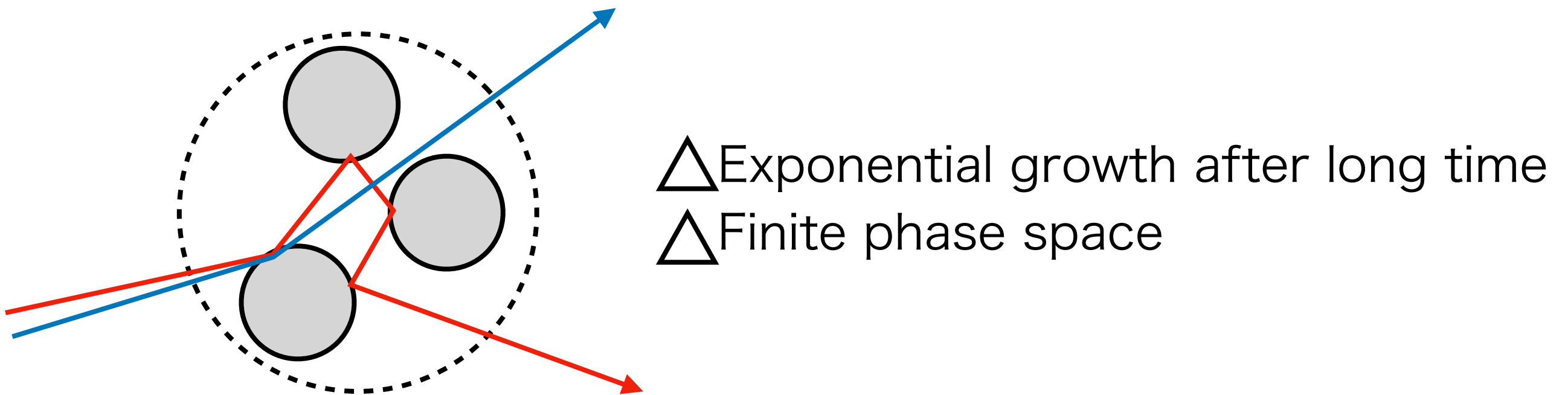
Chaos in dynamical system

- Stretching
 - Folding
- ⇒ Exponential growth of deviation
Finite phase space



Transient chaos

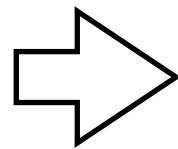
Chaotic behavior during **finite time**



Alternative characterizations are needed

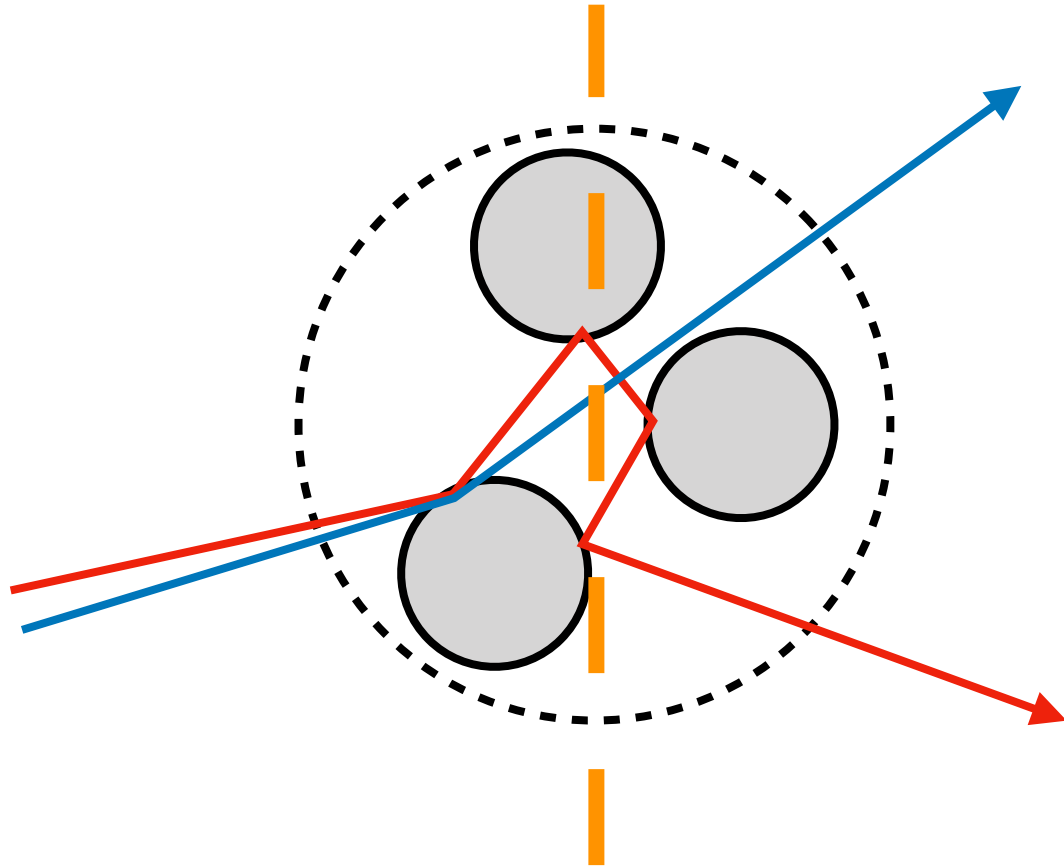
Chaotic scattering

- Folding

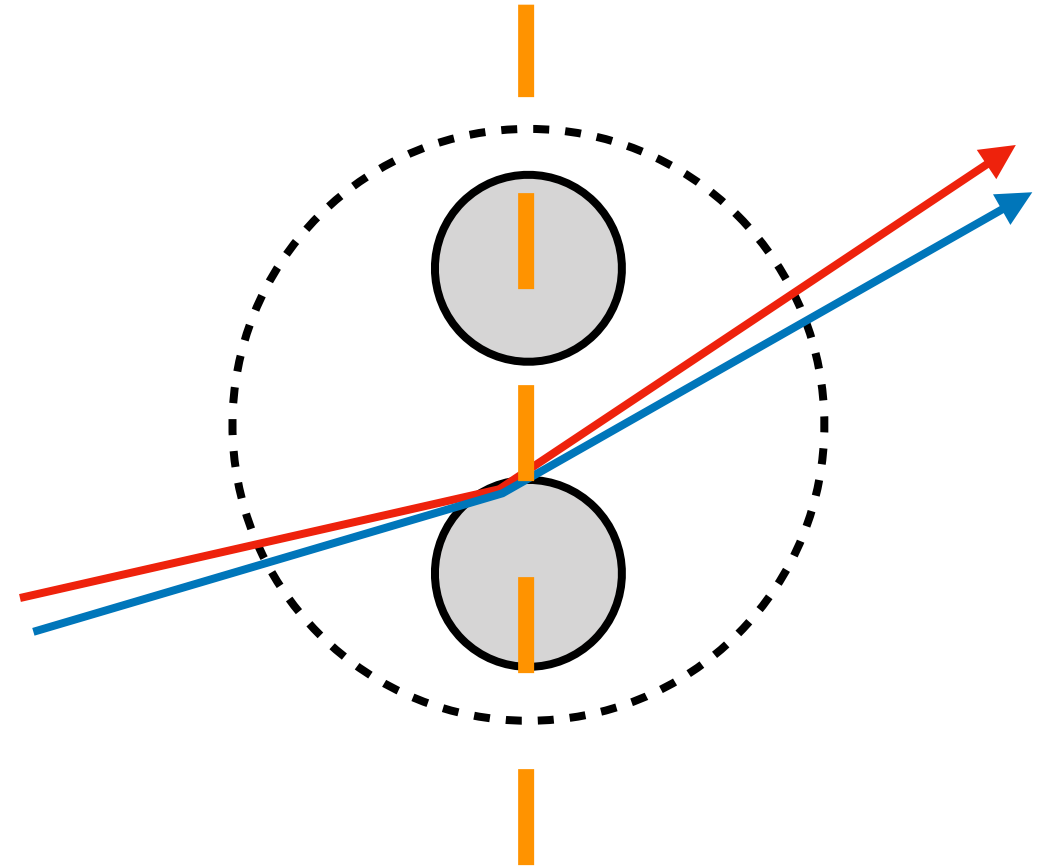


**Non-trivial geometrical structure
which is NOT on the straight line**

Chaotic



Not chaotic



Chaotic scattering

- Stretching $\Rightarrow$ Sensitive to initial angle $\Rightarrow$ **Fractal**

