Proof of 4D & 5D A_n AGT conjecture at β=1

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Based on Qian Shen, Zi-Hao Huang, Shao-Ping Hu, Qing-Jie Yuan and KZ, arXiv:2405.13676.
Qing-Jie Yuan, Shao-Ping Hu, Zi-Hao Huang and KZ, arXiv:2305.11839.
Hong Zhang and Yutaka Matsuo, arXiv:1110.5255.

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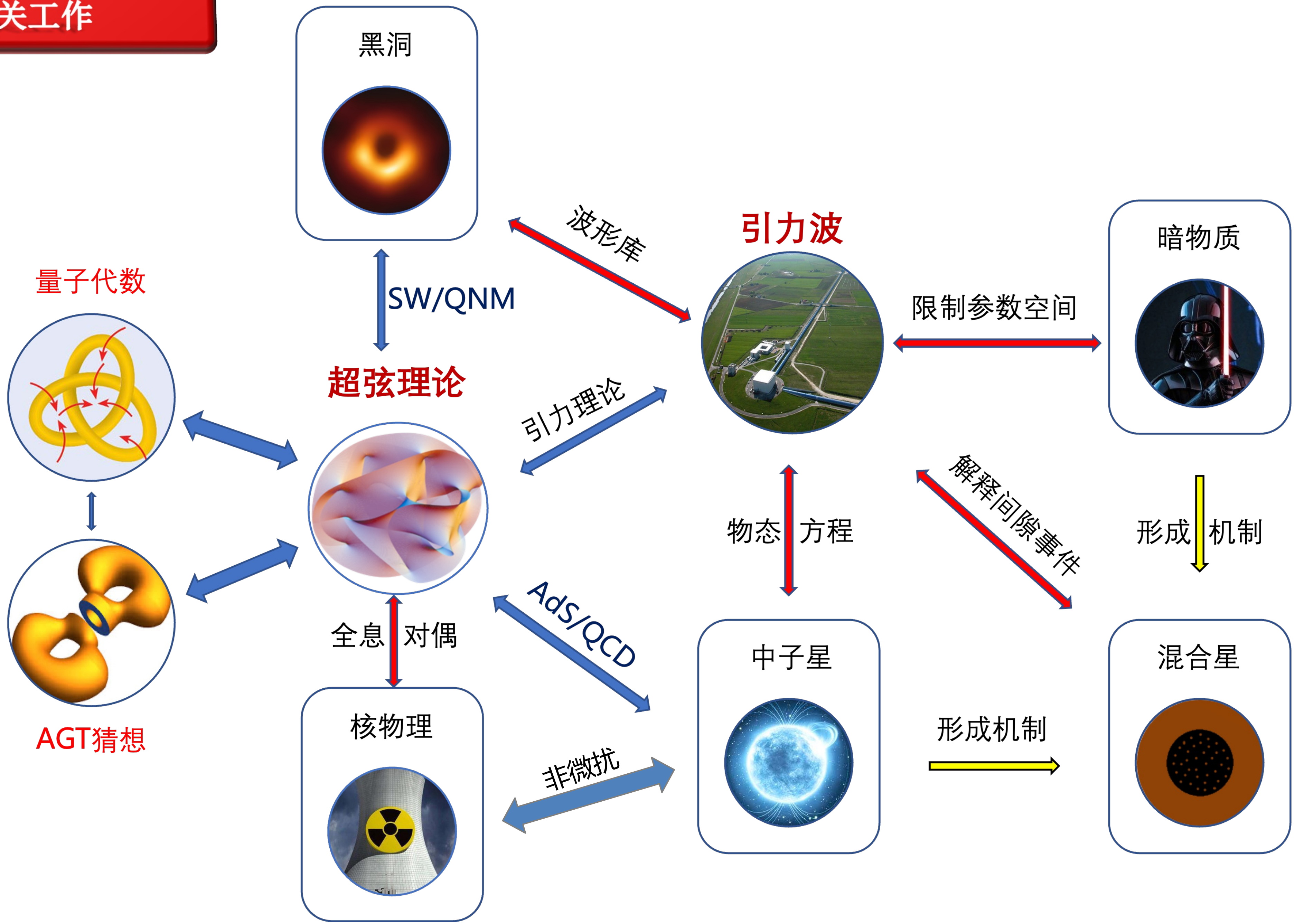

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Proof of 5D A_n AGT conjecture at $\beta = 1$

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JHEP — Published for SISSA by Springer

Received: *March 13, 2024*
Accepted: *May 6, 2024*
Published: *May 31, 2024*

Duality between Seiberg-Witten theory and black hole superradiance

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Published for SISSA by Springer

Received: October 11, 2023
Accepted: February 3, 2024
Published: February 20, 2024

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Yang Lei,[a,1] Hongfei Shu,[b,c,d,1] Kilar Zhang[e,f,1] and Rui-Dong Zhu[a,1]

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Kilar Zhang[a,b], Ling-Wei Luo[c], Jie-Shiun Tsao[c], Chian-Shu Chen[d,e], Feng-Li Lin[c,*]

Published for SISSA by Springer

Received: February 7, 2023
Accepted: August 21, 2023
Published: August 29, 2023

ABCD of qq-characters

Satoshi Nawata,[a] Kilar Zhang[b,c] and Rui-Dong Zhu[d]

A proof of A_n AGT conjecture at $\beta = 1$

Qing-Jie Yuan,[a] Shao-Ping Hu,[a] Zi-Hao Huang[a] and Kilar Zhang[a,b,c,1]

Dark I-Love-Q

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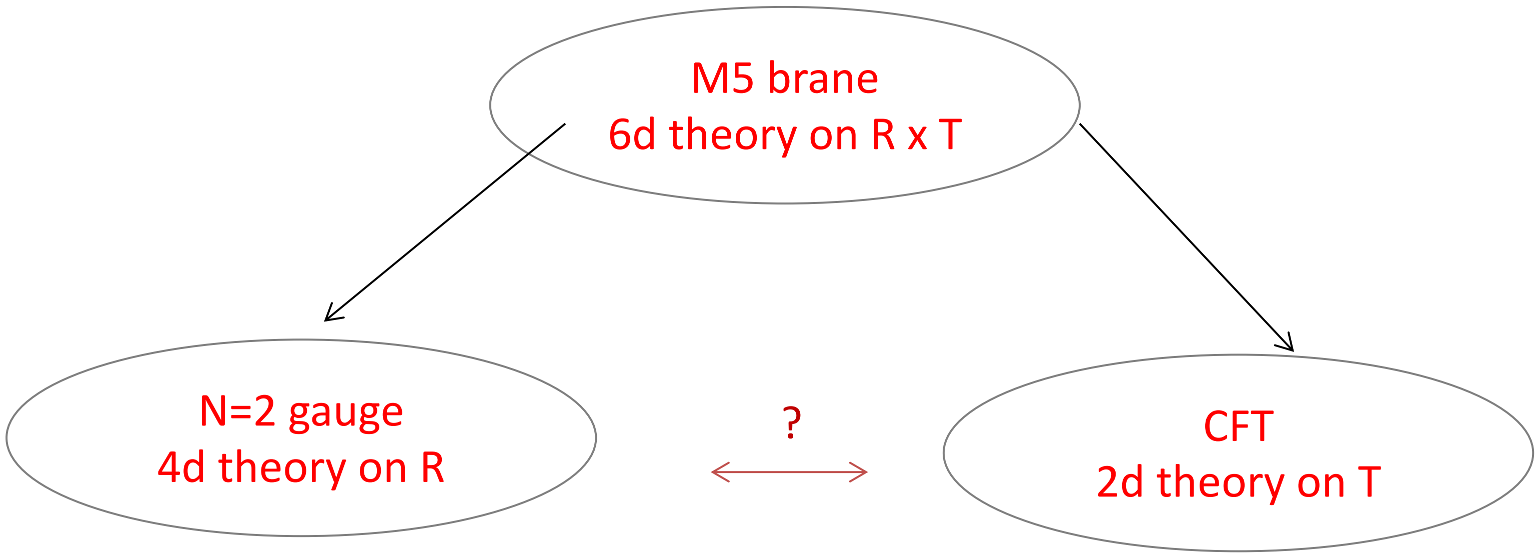
Introduction

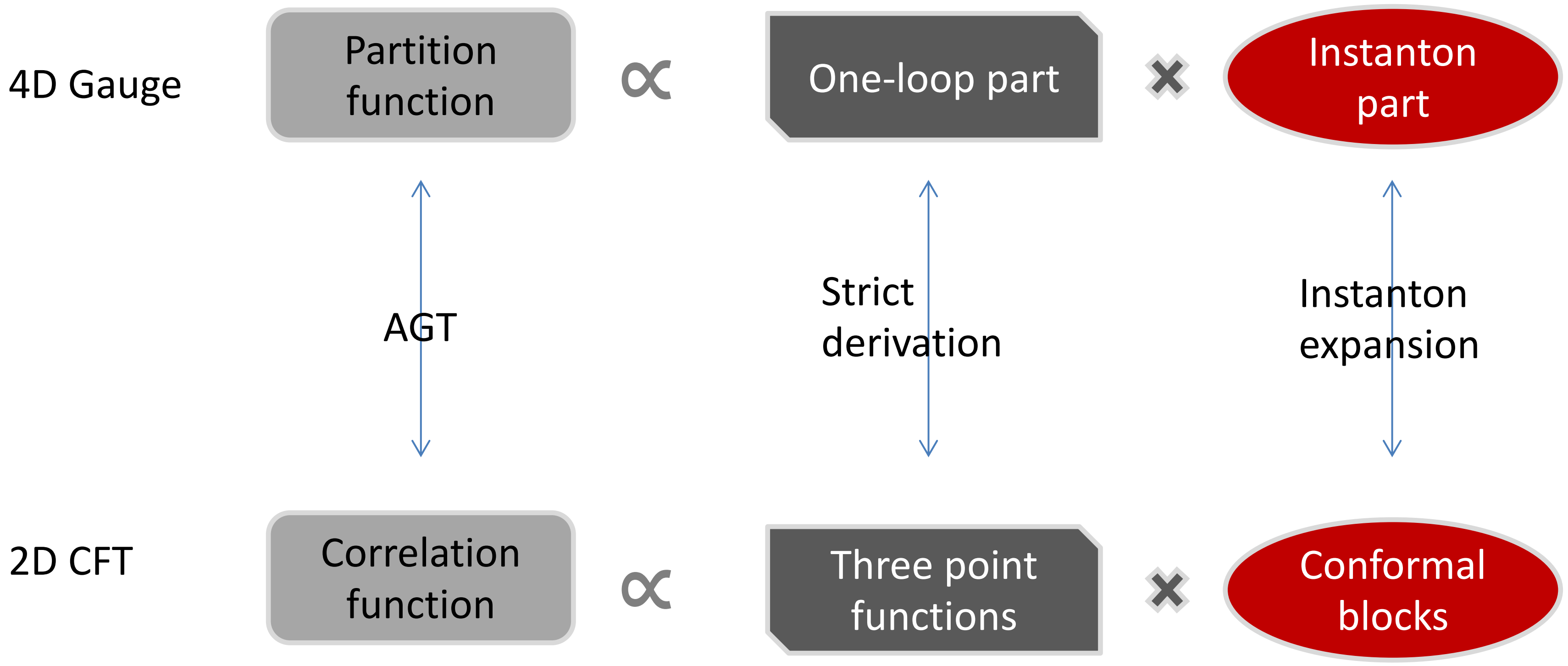
In 2002, Nekrsov performed a technique called Ω deformation in the reduction from 6D $\mathcal{N}=1$ gauge theory to 4D $\mathcal{N}=2$ gauge theory, and implied its connection with 2D conformal theory.



He found exact formulae of the partition function (Nekrasov partition function) of the $\mathcal{N}=2$ gauge theory, and showed that it reproduces the prepotential as determined by the Seiberg–Witten curve.

Introduction

AGT conjecture



SU(2) [Alday–Gaiotto–Tachikawa '09]

SU(N) [Wyllard '09]

Introduction

We have been working on the proof of AGT conjecture through two different ways.

DIRECT APPROACH

Conformal blocks

↓

Dotsenko-Fateev integral

↓

Selberg integral of Jack polynomials

↓

Instanton part of partition function

We calculated the conformal block in the form of Dotsenko–Fateev integral and reduce it in the form of Selberg integral of N Jack polynomials.

We found a formula for such Selberg average which satisfies some nontrivial consistency conditions and showed that it reproduces the SU(N) version of AGT conjecture.

$\beta = 1$ SU(2) [A. Mironov et. al. '10]

$\beta = 1$ SU(N) [Zhang Matsuo '11]

Introduction

Recursive approach

2D CFT

4D Gauge

Conformal blocks

AGT conjecture

$$\langle\langle \vec{Y}|V|\vec{W}\rangle\rangle = Z$$

Partition function

Satisfy Ward identity

trivial

nontrivial

Constrained by a recursion Relation

$$\sum \hat{\mathcal{O}} \langle\langle \vec{Y}|V|\vec{W}\rangle\rangle = 0$$

$$\sum \hat{\mathcal{O}} Z = 0$$

$\beta = 1$ SU(N) [Kanno−Matsuo−Zhang '12]

arbitrary β SU(N) [Kanno−Matsuo−Zhang '13]

A brief review of AGT conjecture and Nekrasov formula

Nekrasov partition function

Consider lifting the $\mathcal{N}=2$ four dimensional theory to $\mathcal{N}=(1,0)$ six dimensional theory, and then compactify the six dimensional $\mathcal{N}=1$ SUSY gauge theory on the manifold with the topology **T2** $\times$ ***R4*** with the metric :

$$ds^2 = r^2 dz d\bar{z} + g_{\mu\nu}\left(dx^\mu + V^\mu dz + \bar{V}^\mu d\bar{z}\right)\left(dx^\nu + V^\nu dz + \bar{V}^\nu d\bar{z}\right)$$

where $V^\mu = \Omega^\mu_\nu x^\nu,\ \bar{V}^\mu = \bar{\Omega}^\mu_\nu x^\nu,$ and

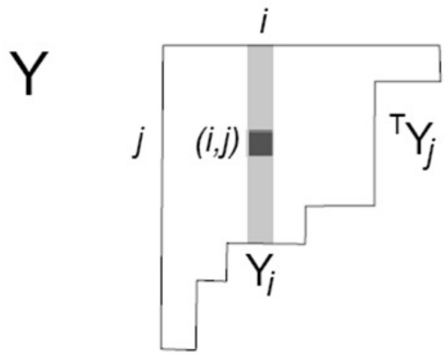
$$\Omega^{\mu\nu} = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}, \quad \bar{\Omega}^{\mu\nu} = \begin{pmatrix} 0 & \bar{\epsilon}_1 & 0 & 0 \\ -\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\epsilon}_2 \\ 0 & 0 & -\bar{\epsilon}_2 & 0 \end{pmatrix}$$

The action of the four dimensional theory in the limit *$r \to 0$ is not that of the pure supersymmetric Yang–Mills* theory on *R4*. Rather, it is a deformation of the latter by the Ω−dependent terms. It is called an *$\mathcal{N}=2$ theory in the Ω-background*.

$$\beta = -\epsilon_1/\epsilon_2$$

Nekrasov partition function

With his idea of the Ω–background, Nekrasov calculated the following partition function



$$Z(\tau, a, m, \epsilon) = \int_{\phi(\infty)=a} D\Phi DAD\lambda \ldots e^{-S(\Omega)}$$

It has the important property that it gives the prepotential of the Seiberg-Witten theory in the limit $\epsilon_1 = -\epsilon_2 = \hbar \to 0$

$$F(\tau, a, m) = \lim_{\hbar \to 0} \hbar^2 \log Z_{\text{full}}(\tau, a, m; \hbar, -\hbar)$$

$$Z_{\text{full}}(q; a, m; \epsilon) = Z_{\text{tree}} Z_{\text{1loop}} Z_{\text{inst}}, \quad Z_{\text{inst}}(q; a, m; \epsilon) = \sum_{\mathbf{Y}} \mathbf{q}^{\mathbf{Y}} z(\mathbf{Y}, a, m)$$

where the instanton is labeled by a N-tuple of Young diagrams: $\mathbf{Y} := (\vec{Y}^{(1)}, \cdots, \vec{Y}^{(n)})$, (Fig. 1). The parameter a (resp. m) represents the diagonalized VEV of vector multiplets (resp. mass of hypermultiplets) whereas $q_i = e^{\pi i \tau_i}$ is the instanton expansion parameter for ith gauge group $SU(N_i)$, $\mathbf{q}^{\mathbf{Y}} := \prod_{i=1}^{n} q_i^{|\vec{Y}^{(i)}|}$. The total partition function is decomposed into a product of the contributions of the perturbative parts Z_{tree}, $Z_{1-\text{loop}}$ and non-perturbative instanton correction Z_{inst}. The latter is further decomposed into a sum of sets of Young diagrams. $\vec{Y}^{(i)} = (Y_1^{(i)}, \cdots, Y_{N_i}^{(i)})$ is a collection of N_i Young diagram which parameterizes the fixed points of instanton moduli space for i th gauge group $U(N_i)$.

Nekrasov partition function

Single gauge group case

$$Z_{\text{full}}(q;a,\mu;\epsilon) = Z_{\text{tree}} Z_{\text{1loop}} Z_{\text{inst}}, \quad Z_{\text{inst}}(q;a,m;\epsilon) = \sum_{\vec{Y}} q^{|\vec{Y}|} N^{\text{inst}}_{\vec{Y}}(a,\mu),$$

$$N^{\text{inst}}_{\vec{Y}}(a,\mu) = z_{\text{vect}}(\vec{Y},a) \prod_{i=1}^{2N} z_{\text{fund}}(\vec{Y},\mu_i) = \frac{\prod_{s=1}^{N} \prod_{k=1}^{2N} f_{Y_s}(\mu_k + a_s)}{\prod_{t,s=1}^{N} g_{Y_t,Y_s}(a_t - a_s)},$$

linear quiver gauge case

with gauge group $SU(N_1) \times \cdots \times SU(N_n)$.

$$Z^{\text{Nek}} = \sum_{\vec{Y}^{(1)},\cdots,\vec{Y}^{(n)}} q_i^{|\vec{Y}^{(i)}|} \bar{V}_{\vec{Y}^{(1)}} \cdot Z_{\vec{Y}^{(1)}\vec{Y}^{(2)}} \cdots Z_{\vec{Y}^{(n-1)}\vec{Y}^{(n)}} \cdot V_{\vec{Y}^{(n)}}$$

$$\begin{aligned}
Z_{\vec{Y}^{(i)}\vec{Y}^{(i+1)}} &= Z(\vec{a}^{(i)},\vec{Y}^{(i)};\vec{a}^{(i+1)},\vec{Y}^{(i+1)};\mu^{(i)}),\\
\bar{V}_{\vec{Y}^{(1)}} &= Z(\vec{\lambda},\vec{\emptyset};\vec{a}^{(1)},\vec{Y}^{(1)};\mu^{(0)}),\\
V_{\vec{Y}^{(n)}} &= Z(\vec{a}^{(n)},\vec{Y}^{(n)};\vec{\lambda}',\vec{\emptyset};\mu^{(n)}),
\end{aligned}$$

AGT conjecture

For a Liouville theory on a sphere, the four-point correlation function of *V at* positions *∞, 1, q, 0 is*

$$\langle V_{\beta_0}(\infty)V_{m_0}(1)V_{m_1}(q)V_{\beta_1}(0)\rangle = \int \frac{d\beta}{2\pi} C(\beta_0^*, m_0, \beta)C(\beta^*, m_1, \beta_1) \left| q^{\Delta_\beta - \Delta_{m_1} - \Delta_{\beta_1}} \mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q) \right|^2$$

$$\propto \int a^2 da \left| Z_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q) \right|^2$$

Where C(β 1, β 2, β 3) is the three point function given by the DOZZ formula.

The function *F* carries the coordinate (*q*) dependence and reflects the contributions of the conformal descendants. It is called conformal block.

$$Z_{\text{inst}}^{U(2),N_f=4}(a, m_0, \tilde{m}_0, m_1, \tilde{m}_1) = (1-q)^{2m_0(Q-m_1)} \mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)$$

it is Checked $\mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)$ is the conformal block of a virasoro algebra with central charge c = 1 + 6Q2 at position ∞, 1, q, 0,

Up to order q11

AGT conjecture

SU(N) generalization

$$\langle V_{\alpha_4}(\infty)V_{\alpha_3}(1)V_{\alpha_2}(q)V_{\alpha_1}(0)\rangle$$

The conformal block of this
correlation function is written in the form,

$$\mathcal{F}_{\alpha_4,\alpha_3,\alpha_2,\alpha_1}(q) = \sum_{\vec{Y}} q^{|\vec{Y}|} N_{\vec{Y}}^{\mathrm{Toda}}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$$

Correlation functions of Toda theory and Selberg Formula

SU(N) Toda

Bosons $\phi(z) = (\phi_1(z), \cdots, \phi_N(z))$

$$\phi_i(z)\phi_j(0) \sim \delta_{ij} \ln(z)$$

Symmetry: W_N algebra

$$R_N = :\prod_{m=1}^{N} \left(Q\frac{d}{dz} - i(h_m, \partial_z \varphi) \right): = \sum_k W^{(k)}(z) \left(Q\frac{d}{dz} \right)^{N-k}$$

$$c = (N-1)(1 + N(N+1)Q^2)$$

$W^{(k)}$ satisfies W_N algebra (a nonlinear algebra) with central charge

$$V_{\vec{\alpha}}(z) =: e^{(\alpha, \phi(z))} :$$

$$Q_j^{(\pm)} = \int \frac{dz}{2\pi i} V_j^{(\pm)}(z) = \int \frac{dz}{2\pi i} : e^{\alpha_{\pm}(e_j, \phi(z))} :$$

Dotsenko-Fateev integral

$$Z_{\mathrm{DF}}(q) =$$
$$\left\langle\!\!\left\langle : e^{(\tilde{\alpha_1},\phi(0))} :: e^{(\tilde{\alpha_2},\phi(q))} :: e^{(\tilde{\alpha_3},\phi(1))} :: e^{(\tilde{\alpha_4},\phi(\infty))} : \prod_{a=1}^{N-1} \left(\int_0^q : e^{b(e_a,\phi(z))} : dz \right)^{N_a} \left(\int_1^\infty : e^{b(e_a,\phi(z))} : dz \right)^{\tilde{N}_a} \right\rangle\!\!\right\rangle$$

We apply *Wick's theorem* to evaluate the correlator

$$\left\langle\!\left\langle : e^{(\tilde{\alpha}_1,\phi(z_1))} : \ \ldots \ : e^{(\tilde{\alpha}_n,\phi(z_n))} : \right\rangle\!\right\rangle = \prod_{1\le i<j\le n} (z_j - z_i)^{(\tilde{\alpha}_i,\tilde{\alpha}_j)}$$

$$Z_{\mathrm{DF}}(q) = q^{(\alpha_1,\alpha_2)/\beta}(1-q)^{(\alpha_2,\alpha_3)/\beta} \prod_{a=1}^{N-1} \prod_{I=1}^{N_a} \int_0^q dz_I^{(a)} \prod_{J=N_a+1}^{N_a+\tilde{N}_a} \int_1^\infty dz_J^{(a)} \prod_{i<j}^{N_a+\tilde{N}_a} (z_j^{(a)} - z_i^{(a)})^{2\beta} \quad \times$$
$$\times \prod_i^{N_a+\tilde{N}_a} (z_i^{(a)})^{(\alpha_1,e_a)} (z_i^{(a)} - q)^{(\alpha_2,e_a)} (z_i^{(a)} - 1)^{(\alpha_3,e_a)} \prod_{a=1}^{N-2} \prod_i^{N_a+\tilde{N}_a} \prod_j^{N_{a+1}+\tilde{N}_{a+1}} (z_j^{(a+1)} - z_i^{(a)})^{-\beta} \quad .$$

Selberg integral

$$\int_{[0,1]^k} |\Delta(x)|^{2\gamma} \prod_{i=1}^{k} x_i^{\alpha-1}(1-x_i)^{\beta-1}\,\mathrm{d}x = \prod_{i=1}^{k} \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma+1)}{\Gamma(\alpha+\beta+(i+k-2)\gamma)\Gamma(\gamma+1)}$$

When $k = 1$ the Selberg integral simplifies to the Euler beta integral

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \Re(\alpha) > 0,\ \Re(\beta) > 0,$$

Here we consider its $AN-1$ extension ($AN-1$ Selberg integral):

$$S_{\vec{u},\vec{v},\beta} = \int dx \prod_{a=1}^{N-1} \left[\left|\Delta\left(x^{(a)}\right)\right|^{2\beta} \prod_{i=1}^{N_a} \left(x_i^{(a)}\right)^{u_a} \left(1-x_i^{(a)}\right)^{v_a} \right] \prod_{a=1}^{N-2} \left|\Delta\left(x^{(a)}, x^{(a+1)}\right)\right|^{-\beta}$$

Selberg integral

$$I^{A_n}_{N_1,\ldots,N_n}(\mathcal{O};u_1,\ldots,u_n,v;\beta)$$
$$\equiv \int\limits_{C_\beta^{N_1,\ldots,N_n}[0,1]} \mathcal{O}(x^{(1)},\ldots,x^{(n)}) \prod_{r=1}^{n}\prod_{i=1}^{N_r}(x_i^{(r)})^{u_r}(1-x_i^{(r)})^{v_r}$$
$$\times \prod_{r=1}^{n}|\Delta(x^{(r)})|^{2\beta}\prod_{r=1}^{n-1}|\Delta(x^{(r)},x^{(r+1)})|^{-\beta}\,\mathrm{d}x^{(1)}\cdots\mathrm{d}x^{(n)}$$

Seamus P. Albion, Eric M. Rains, and S. Ole Warnaar, 2021

For $\beta = 1$

$$C^{N_1,\ldots,N_n} = C_1^{N_1}\times\cdots\times C_n^{N_n}, \quad \text{where} \quad C_r^{N_r} = \underbrace{C_r\times\cdots\times C_r}_{N_r\ \text{times}}.$$

$$I^{A_n}_{N_1,\ldots,N_n}(\mathscr{O};u_1,\ldots,u_n,v)$$
$$\equiv \frac{1}{(2\pi\mathrm{i})^{N_1+\cdots+N_n}}\int\limits_{C^{N_1,\ldots,N_n}} \mathscr{O}(x^{(1)},\ldots,x^{(n)}) \prod_{r=1}^{n}\prod_{i=1}^{N_r}(x_i^{(r)})^{u_r}(x_i^{(r)}-1)^{v_r}$$
$$\times \prod_{r=1}^{n}\Delta^2(x^{(r)})\prod_{r=1}^{n-1}\Delta^{-1}(x^{(r)},x^{(r+1)})\,\mathrm{d}x^{(1)}\cdots\mathrm{d}x^{(n)}$$

Reduction to Selberg integral

$$Z_{DF}(q) = \sum_{\vec{Y}} q^{|\vec{Y}|} \left\langle \prod_{a=1}^{N} j_{Y_a}^{(\beta)}(-r_k^{(a)} - \frac{v'_{a+}}{\beta}) \right\rangle_+ \left\langle \prod_{a=1}^{N} j_{Y_a}^{(\beta)}(\tilde{r}_k^{(a)} + \frac{v'_{a-}}{\beta}) \right\rangle_-$$

$j_{Y_a}^{(\beta)}$: Jack polynomials

$\langle \cdots \rangle_\pm$: Selberg average

$$p_k^{(a)} := \sum_i (x_i^{(a)})^k$$

Cauchy–Stanley identity

$$\exp(\beta \sum_{k=1}^{\infty} \frac{1}{k} p_k p'_k) = \sum_R j_R^{(\beta)}(p) j_R^{(\beta)}(p')$$

Jack polynomials

Jack polynomials are characterized by the fact that they are the eigenfunctions of Calogero-Sutherland Hamiltonian written in the form,

$$\mathcal{H} = \sum_{i=1}^{M} D_i^2 + \beta \sum_{i<h} \frac{z_i + z_j}{z_i - z_j}(D_i - D_j), \quad D_i := z_i \frac{\partial}{\partial z_i} .$$

The explicit form of low level ones are listed below;

$$J_{[1]}^{(\beta)}(p_k) = p_1 \,,$$
$$J_{[2]}^{(\beta)}(p_k) = \frac{p_2 + \beta p_1^2}{\beta + 1}, \quad J_{[11]}^{(\beta)}(p_k) = \frac{1}{2}(p_1^2 - p_2) \,,$$
$$J_{[3]}^{(\beta)}(p_k) = \frac{2p_3 + 3\beta p_1 p_2 + \beta^2 p_1^3}{(\beta+1)(\beta+2)}, \quad J_{[21]}^{(\beta)}(p_k) = \frac{(1-\beta)p_1 p_2 - p_3 + \beta p_1^3}{(\beta+1)(\beta+2)}, \quad J_{[111]}^{(\beta)}(p_k) = \frac{1}{6}p_1^3 - \frac{1}{2}p_1 p_2 + \frac{1}{3}p_3 .$$

Selberg integral

Schur functions,

$$\chi_\lambda(x) = \frac{\det_{1\leqslant i,j\leqslant n}\left(x_i^{\lambda_j+n-j}\right)}{\Delta(x)}$$

Seamus P. Albion, Eric M. Rains, and S. Ole Warnaar, 2021

$$\left\langle \prod_{r=1}^{n+1} \chi_{Y^{(r)}}\left[x^{(r)} - x^{(r-1)}\right] \right\rangle_{u_1,\dots,u_n,v}^{N_1,\dots,N_n}$$
$$= \prod_{r=1}^{n+1} \prod_{1\leqslant i<j\leqslant \ell_r} \frac{Y_i^{(r)} - Y_j^{(r)} + j - i}{j-i} \prod_{r,s=1}^{n+1} \prod_{i=1}^{\ell_r} \frac{(A_{r,s} - N_{s-1} + N_s - i + 1)_{Y_i^{(r)}}}{(A_{r,s} + \ell_s - i + 1)_{Y_i^{(r)}}}$$
$$\times \prod_{1\leqslant r<s\leqslant n+1} \prod_{i=1}^{\ell_r} \prod_{j=1}^{\ell_s} \frac{Y_i^{(r)} - Y_j^{(s)} + A_{r,s} + j - i}{A_{r,s} + j - i},$$

AGT conjecture from Selberg integral

Reduction to Selberg integral

Before we go to the details conclusion first

$$Z_{\text{inst}}(q) = Z_{\text{DF}}(q)$$

$$N^{\text{inst}}_{\vec{Y}} = N^{\text{Toda}}_{\vec{Y}}$$

$$N^{\text{inst}}_{\vec{Y}} \equiv N^{\text{inst}}_{\vec{Y}+} N^{\text{inst}}_{\vec{Y}-}, \quad N^{\text{Toda}}_{\vec{Y}} \equiv N^{\text{Toda}}_{\vec{Y}+} N^{\text{Toda}}_{\vec{Y}-},$$

$$N^{\text{inst}}_{\vec{Y}+} \equiv \frac{\prod_{s=1}^{n+1}\prod_{k=1}^{n+1} f_{Y_s}(\mu_k + a_s)}{\prod_{t,s=1}^{n+1} G_{Y_t,Y_s}(a_t - a_s)} \prod_{s=1}^{n+1}\left\{(-1)^{|Y_s|}\sqrt{\frac{G_{Y_s,Y_s}(0)}{G_{Y_s,Y_s}(1-\beta)}}\right\},$$

$$N^{\text{Toda}}_{\vec{Y}\pm} \equiv \left\langle \prod_{a=1}^{n+1} j^{(\beta)}_{Y_a}\left(-r^{(a)}_k - \frac{v'_{a\pm}}{\beta}\right)\right\rangle_{\pm} = \prod_{a=1}^{n+1}\sqrt{\frac{G_{Y_a,Y_a}(0)}{G_{Y_a,Y_a}(1-\beta)}}\left\langle \prod_{a=1}^{n+1} J^{(\beta)}_{Y_a}\left(-r^{(a)}_k - \frac{v'_{a\pm}}{\beta}\right)\right\rangle_{\pm}$$

Known results on Selberg average

SU(2) **case:** The relevant Selberg averages for one and two Jack polynomials were obtained by Kadell.

$$\left\langle J_Y^{(\beta)}(p)\right\rangle_{u,v,\beta}^{SU(2)} \qquad\qquad \left\langle J_A^{(\beta)}(p+w)J_B^{(\beta)}(p)\right\rangle^{SU(2)}$$

SU(*n*+1) **case:**

The one-Jack Selberg integral for *SU*(*n*+1) could be calculated by the formula offered by Warnaar.

$$\left\langle J_B^{(\beta)}(p_k^{(n)})\right\rangle_{\vec{u},\vec{v},\beta}^{SU(n+1)}$$

He also gives A_2 two Jack integral $\left\langle J_R^{(\beta)}(p_k^{(1)})J_B^{(\beta)}(p_k^{(2)})\right\rangle_{u,v,\beta}^{SU(3)}$

A conjecture on Selberg average

To evaluate we need Selberg average of $(n+1)$ Jack polynomials. While we do not perform the integration so far, we find a formula for $\beta = 1$ which reproduces known results and satisfies consistency conditions.

The Jack polynomial for $\beta = 1$ is called Schur polynomial. $J_Y^{(\beta)}|_{\beta=1} = \chi_Y.$

Conjecture : *We propose the following formula of Selberg average for $n+1$ Schur polynomials,*

$$
\begin{aligned}
&\left\langle \chi_{Y_1}(-p_k^{(1)} - v_1') \cdots \chi_{Y_r}(p_k^{(r-1)} - p_k^{(r)} - v_r') \cdots \chi_{Y_{n+1}}(p_k^{(n)}) \right\rangle^{SU(n+1)}_{\vec{u},\vec{v},\beta=1} \\
&= \prod_{s=1}^{n} \left\{ (-1)^{|Y_s|} \times \frac{[v_s + N_s - N_{s-1}]_{Y_s'}}{[N_s + N_{s-1}]_{Y_s'}} \times \prod_{1\le i<j\le N_{s-1}+N_s} \frac{(j-i+1)_{Y_{si}'-Y_{sj}'}}{(j-i)_{Y_{si}'-Y_{sj}'}} \right\} \times \prod_{1\le i<j\le N_n} \frac{(j-i+1)_{Y_{(n+1)i}-Y_{(n+1)j}}}{(j-i)_{Y_{(n+1)i}-Y_{(n+1)j}}} \\
&\times \prod_{1\le t<s\le n+1} \left\{ \frac{[v_t + u_t + \cdots + u_{s-1} + N_t - N_{t-1}]_{Y_t'}}{[v_t - v_s + u_t + \cdots + u_{s-1} + N_t - N_{t-1} - N_s]_{Y_t'}} \times \frac{[-v_s + u_t + \cdots + u_{s-1} - N_s + N_{s-1}]_{Y_s}}{[v_t - v_s + u_t + \cdots + u_{s-1} - N_{t-1} - N_s + N_{s-1}]_{Y_s}} \right. \\
&\left. \times \prod_{i=1}^{N_t} \prod_{j=1}^{N_{s-1}} \frac{v_t - v_s + u_t + \cdots + u_{s-1} + N_t - N_{t-1} - N_s + N_{s-1} + 1 - (i+j)}{v_t - v_s + u_t + \cdots + u_{s-1} + N_t - N_{t-1} - N_s + N_{s-1} + 1 + Y_{ti}' + Y_{sj} - (i+j)} \right\},
\end{aligned}
$$

Proof

$$\left\langle \prod_{r=1}^{n+1} \chi_{Y^{(r)}} \left[x^{(r-1)} - x^{(r)}\right] \right\rangle_+ = \left. \frac{\prod_{s=1}^{n+1} (-1)^{|Y^{(s)}|} \prod_{r=1}^{n+1} \prod_{s=1}^{n+1} f_{Y^{(s)}}(\mu_r + a_s)}{\prod_{r,s=1}^{n+1} G_{Y^{(r)},Y^{(s)}}(a_r - a_s)} \right|_{\beta=1},$$

and the second part,

$$\left\langle \prod_{r=1}^{n+1} \chi_{Y^{(r)}} \left[y^{(r)} - y^{(r-1)}\right] \right\rangle_- = \left. \frac{\prod_{s=1}^{n+1} (-1)^{|Y^{(s)}|} \prod_{r=n+2}^{2n+2} \prod_{s=1}^{n+1} f_{Y^{(s)}}(\mu_r + a_s)}{\prod_{r,s=1}^{n+1} G_{Y^{(r)},Y^{(s)}}(a_r - a_s)} \right|_{\beta=1}.$$

Proof

Lemma 1

$$\prod_{1\leqslant i<j\leqslant l_Y}\frac{(Y_i-Y_j+\beta(j-i))_\beta}{(\beta(j-i))_\beta}\prod_{i=1}^{l_Y}\frac{1}{(\beta(l_Y-i+1))_{Y_i}}=\frac{1}{G_{Y,Y}(0)}$$

Lemma 2

$$\prod_{i=1}^{l_Y}(-z-\beta i+\beta)_{Y_i}=(-1)^{|Y|}f_Y(z)$$

Lemma 3

$$\prod_{i=1}^{l_Y}\prod_{j=1}^{l_W}\frac{Y_i-W_j-x+j-i}{-x+j-i}\prod_{i=1}^{l_Y}\frac{1}{(-x+l_W-i+1)_{Y_i}}\prod_{i=1}^{l_W}\frac{1}{(x+l_Y-i+1)_{W_i}}$$
$$=\left.\frac{(-1)^{|Y|+|W|}}{G_{Y,W}(x)G_{W,Y}(-x)}\right|_{\beta=1}.$$

Proof

Lemma 1

$$\prod_{1\leqslant i<j\leqslant l_Y} \frac{(Y_i - Y_j + \beta(j-i))_\beta}{(\beta(j-i))_\beta} \prod_{i=1}^{l_Y} \frac{1}{(\beta(l_Y - i + 1))_{Y_i}} = \frac{1}{G_{Y,Y}(0)}$$

Lemma 2

$$\prod_{i=1}^{l_Y} (-z - \beta i + \beta)_{Y_i} = (-1)^{|Y|} f_Y(z)$$

Lemma 3

$$\prod_{i=1}^{l_Y} \prod_{j=1}^{l_W} \frac{Y_i - W_j - x + j - i}{-x + j - i} \prod_{i=1}^{l_Y} \frac{1}{(-x + l_W - i + 1)_{Y_i}} \prod_{i=1}^{l_W} \frac{1}{(x + l_Y - i + 1)_{W_i}}$$
$$= \left. \frac{(-1)^{|Y|+|W|}}{G_{Y,W}(x) G_{W,Y}(-x)} \right|_{\beta=1} .$$

prove the conjecture
have already been made. Some have been successful but have mostly proven the conjecture under some special assumptions/choice of parameters. The present proof is relatively general, but still considers the algebra A_n only and the case beta=1.

As far as I can follow the derivation, the proof presented in the paper seems correct and provides a good contribution to understanding of this rather specialized subject. However, in my view this is not a result that deserves publication in PRL. My reason for this is that the paper is too technical and would be better suited for a more technical/specialized journal. It also presents a really specialized result, the proof a very particular conjecture in a very particular case. Even if the conjecture itself is significant in establishing a relationship between Gauge Theories and Conformal Field theory, its proof for such an specific case seems only of interest to specialists, which supports my suggestion of submitting to a more specialized journal where some aspects of the proof could be expanded upon, once the page limit is removed.

β parameter. Such a proof is not only interested for mathematicians, but may also further elucidate the underlying physical mechanism. In this aspect, there seems not much meaningful efforts for a long time. Therefore it is delightful to see that the authors are making some good progress in this paper.

The paper considers the case of $\beta=1$, and builds on earlier works on the A_1 case [18] and a conjectural formula in [19] for general A_n cases by one of the authors. The key is the evaluations of some Selberg integrals, studied recently by some mathematicians in [20].
With the new technical tool, the authors are able to prove the AGT conjecture for the general A_n cases. I don't find obvious gaps in the proof. Since one of the authors has done some good research on the topic in the past, the current results are reasonable and trustworthy, and may be useful to prove the case of general β parameter in the future. I recommend publication of the paper.

5D

$$Z_{\text{inst}}(\mathfrak{q}) = \sum_{\vec{Y}} \mathfrak{q}^{|\vec{Y}|} N_{5d}^{\text{inst}}(\vec{a}, \mu),$$

$$N_{5d}^{\text{inst}} = \prod_{r,s=1}^{n+1} \frac{\mathbb{F}_{Y^{(s)}}(\mu_r + a_s, q^{-1})\mathbb{F}_{Y^{(s)}}(\mu_{n+1+r} + a_s, q)}{\mathbb{G}_{Y^{(r)},Y^{(s)}}(a_r - a_s, q)\mathbb{G}_{Y^{(s)},Y^{(r)}}(a_s - a_r + 1 - \beta, q^{-1})},$$

$$\mathbb{F}_Y(z, q) = \prod_{(i,j)\in Y} (1 - q^{z+\beta(i-1)-(j-1)}),$$

$$\mathbb{G}_{Y,W}(x, q) = \prod_{(i,j)\in Y} (1 - q^{x+\beta(Y'_j - i)+(W_i - j)+\beta}).$$

5D

$$[n]_q \equiv \frac{1-q^n}{1-q}.$$

$$[n_1]_q[n_2]_q \neq [m_1]_q[m_2]_q$$

q-deformed Selberg integral

$$
\begin{aligned}
&I^{A_n}_{N_1,\dots,N_n}(\mathcal{O};u_1,\dots,u_n,v;\beta)\\
&=\frac{1}{N_1!\cdots N_n!(2\pi\mathrm{i})^{N_1+\cdots+N_n}}\int_{\mathbb{T}^{N_1+\cdots+N_n}}\mathcal{O}(x^{(1)},\dots,x^{(n)})\prod_{i=1}^{N_n}\frac{(q^{a_n}/x_i^{(n)},q^{1-a_n}x_i^{(n)};q)_\infty}{(q^{v+1}/x_i^{(n)},x_i^{(n)};q)_\infty}\prod_{r=1}^{n-1}\prod_{i=1}^{N_r}(x_i^{(r)})^{a_r}\\
&\times\prod_{r=1}^{n}\prod_{1\leqslant i<j\leqslant N_r}\frac{(x_i^{(r)}/x_j^{(r)},x_j^{(r)}/x_i^{(r)};q)_\infty}{(tx_i^{(r)}/x_j^{(r)},tx_j^{(r)}/x_i^{(r)};q)_\infty}\prod_{r=1}^{n-1}\prod_{i=1}^{N_r}\prod_{j=1}^{N_{r+1}}\frac{((qt)^{1/2}x_j^{(r+1)}/x_i^{(r)};q)_\infty}{((q/t)^{1/2}x_j^{(r+1)}/x_i^{(r)};q)_\infty}\frac{\mathrm{d}x^{(1)}}{x^{(1)}}\cdots\frac{\mathrm{d}x^{(n)}}{x^{(n)}},
\end{aligned}
$$

In [11] the Selberg integral formula for $n+1$ Schur polynomials is proposed. We now find a new q-deformed Selberg integral average formula, containing a product of $n+1$ Schur functions:

$$
\begin{aligned}
&\left\langle\prod_{r=1}^{n+1}\chi_{Y^{(r)}}\left[x^{(r)}-x^{(r-1)}\right]\right\rangle^{N_1,\dots,N_n}_{u_1,\dots,u_n,v}\\
&=\prod_{r=1}^{n+1}\prod_{(i,j)\in Y^{(r)}}\frac{1-q^{-N_r+N_{r-1}+i-j}}{1-q^{-Y_i^{(r)}-Y_j'^{(r)}+i+j-1}}\prod_{1\leq r<s\leq n+1}\prod_{i=1}^{L_{Y^{(r)}}}\prod_{j=1}^{L_{Y^{(s)}}}\frac{1-q^{-Y_i^{(r)}+Y_j^{(s)}-A_{r,s}-j+i}}{1-q^{-A_{r,s}-j+i}}\\
&\times\prod_{1\leq r<s\leq n+1}\prod_{(i,j)\in Y^{(r)}}\frac{1-q^{-A_{r,s}+N_{s-1}-N_s+i-j}}{1-q^{-A_{r,s}-L_{Y^{(s)}}+i-j}}\prod_{(i,j)\in Y^{(s)}}\frac{1-q^{-A_{r,s}+N_r-N_{r-1}-i+j}}{1-q^{-A_{r,s}+L_{Y^{(r)}}-i+j}}.
\end{aligned}
$$

q-deformed conformal blocks

$$\left\langle\!\!\left\langle \prod_{r=1}^{n} \left\{ \prod_{i=1}^{N_r} (\mathfrak{q} x_i^{(r)}; q)_{v_{r-}} \prod_{j=1}^{\tilde{N}_r} (\mathfrak{q} y_i^{(r)}; q)_{v_{r+}} \right\} \prod_{r,s=1}^{n} \prod_{i=1}^{N_r} \prod_{j=1}^{\tilde{N}_s} \left(\mathfrak{q} x_i^{(r)} y_j^{(s)}; q \right)_{\beta}^{C_{rs}} \right\rangle_{+} \right\rangle_{-}$$

$$\sum_{\vec{Y}} \mathfrak{q}^{|\vec{Y}|} \left\langle \prod_{r=1}^{n+1} \chi_{Y^{(r)}} \left[x^{(r-1)} - x^{(r)} \right] \right\rangle_{+} \left\langle \prod_{r=1}^{n+1} \chi_{Y^{(r)}} \left[y^{(r)} - y^{(r-1)} \right] \right\rangle_{-}$$

$$\left\langle \prod_{r=1}^{n+1} \chi_{Y^{(r)}}\left[x^{(r-1)} - x^{(r)}\right]\right\rangle_{+} \left\langle \prod_{r=1}^{n+1} \chi_{Y^{(r)}}\left[y^{(r)} - y^{(r-1)}\right]\right\rangle_{-}$$

$$= \prod_{r,s=1}^{n+1} \frac{\mathbb{F}_{Y^{(s)}}(\mu_r + a_s, q)\mathbb{F}_{Y^{(s)}}(\mu_{n+1+r} + a_s, q^{-1})}{\mathbb{G}_{Y^{(r)},Y^{(s)}}(a_r - a_s, q)\mathbb{G}_{Y^{(s)},Y^{(r)}}(a_s - a_r + 1 - \beta, q^{-1})}.$$

Lemma

$$\prod_{i=1}^{L_Y}\prod_{j=1}^{L_W} \frac{1 - q^{x+Y_i-W_j+j-i}}{1 - q^{x+j-i}} \prod_{(i,j)\in Y} \frac{1}{1 - q^{x+L_W-i+j}} \prod_{(i,j)\in W} \frac{1}{1 - q^{x-L_Y+i-j}}$$

$$= \left.\frac{1}{\mathbb{G}_{Y,W}(-x, q^{-1})\mathbb{G}_{W,Y}(x,q)}\right|_{\beta=1}.$$

Conclusion