Web of dualities on non-orientable surfaces

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About me

Introduction and results

Duality web:

Suppose you are given a $(1+1)$D bosonic theory T_B with $\mathbb{Z}_2$ symmetry.

$\to$ Topological manipulations:

- Gauging,
- Fermionizing,
- Stacking.

$\to$ Duality web [1][2]

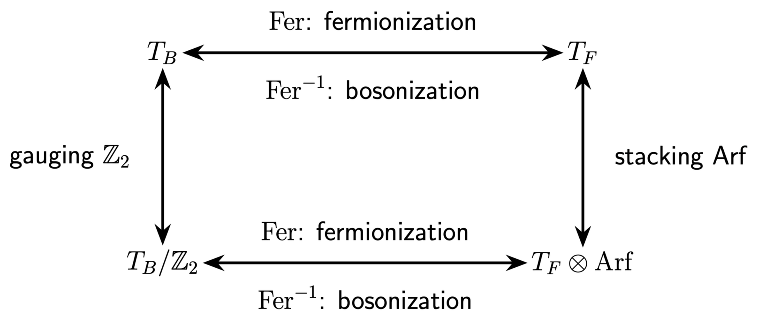

Figure: Orientable duality web.

Slogan: QFT with reflection/time-reversal symmetry
= QFT on non-orientable manifolds

$\to$ We study the non-orientable duality web from several aspects:

- Symmetry TFT $\to$ See my talk at RFCMP and our paper
- Group structure of topological manipulations
- Sectors of Hilbert space

RFCMP (Japanese)

Duality web on orientable surfaces

Gauging: Introduce a background gauge field A $\to$ make it dynamical.

$\to$ When $G=\mathbb{Z}_2$, introduce $a\in H^1(\Sigma;\mathbb{Z}_2)$ and sum over it!

e.g. For $\Sigma=T^2$, with $H^1(T^2;\mathbb{Z}_2)\cong\{0,x,y,x+y\}$, we obtain

$$Z_{T_B/\mathbb{Z}_2}=\frac{1}{2}(Z_{T_B}[0]+Z_{T_B}[x]+Z_{T_B}[y]+Z_{T_B}[x+y])$$
$$=\frac{1}{2}(\quad+\quad+\quad+\quad).$$

$\to$ We may obtain a dual $\mathbb{Z}_2$ symmetry by gauging.

$$Z_{O^B(T_B)}[a]:=\frac{1}{|H^1(\Sigma;\mathbb{Z}_2)|^{1/2}}\sum_{b\in H^1(\Sigma;\mathbb{Z}_2)}(-1)^{\int a\cup b}Z_{T_B}[b]$$

e.g. For $\Sigma=T^2$, we obtain

$$(\quad)_{O^B(T_B)}=\frac{1}{2}(\quad-\quad+\quad-\quad)_{T_B}$$

Fermionization: Map $\mathrm{Fer}: Z_{T_B}[a]\mapsto Z_{T_F}[\eta]$ where $\eta\in\mathrm{Spin}(\Sigma)$, i.e.

$$Z_{\mathrm{Fer}(T_B)}[\eta]\left(:=\frac{1}{|H^1(\Sigma;\mathbb{Z}_2)|^{1/2}}\sum_{a\in H^1(\Sigma;\mathbb{Z}_2)}\exp(\pi i q_\eta(a))Z_{T_B}[a]\right)$$

Here q_η is called a quadratic refinement (see our paper for details).

$\to$ Then one can compute

$$Z_{\mathrm{Fer}\circ O^B\circ\mathrm{Fer}^{-1}(T_F)}[\eta]=Z_{\mathrm{Arf}}[\eta]Z_{T_F}[\eta].$$

Here $Z_{\mathrm{Arf}}[\eta]\in\{\pm1\}$ is an invertible fermionic TQFT.

Physically, the Arf invariant is the partition function of the Kitaev chain.

Mathematically, the Arf invariant gives a homomorphism $\Omega_2^{\mathrm{Spin}}(\mathrm{pt})\xrightarrow{\sim}\mathbb{Z}_2$.

What changes on non-orientable surfaces

How can we characterize non-orientability?

$\to$ $\exists$ topological defect $P\in H_d(M^{d+1};\mathbb{Z}_2)$ s.t.

$w_1:=\mathrm{PD}(P)\in H^1(M^{d+1};\mathbb{Z}_2)$. [3]

Remark: w_1 is the obstruction to orientability.
i.e. $w_1\neq0\Leftrightarrow$ the spacetime is non-orientable.

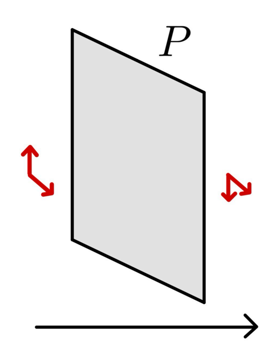

Figure: Domain wall P.

	orientable	non-orientable
w_1	trivial	nontrivial
str	Spin	Pin^-
Inv	$Z_{\mathrm{Arf}}[\eta]\in\mathbb{Z}_2$	$Z_{\mathrm{ABK}}[\eta]\in\mathbb{Z}_8$
SPT	trivial	$(-1)^{\int a\cup w_1}$, $(-1)^{\int w_1\cup w_1}$

Table: Summary: orientable vs non-orientable

$\to$ We can define operations S_1^B, S_2^B as follows:

$$Z_{S_1^B(T_B)}[a]:=(-1)^{\int a\cup w_1}Z_{T_B}[a],\quad Z_{S_2^B(T_B)}[a]:=(-1)^{\int w_1\cup w_1}Z_{T_B}[a]$$

Brief comment on SymTFT

The Fourier transform in QM and gauging in our setup:

- $\mathcal{F}(\psi)(p)=\tilde{\psi}(p)=\frac{1}{\sqrt{2\pi}}\int dxe^{-ipx}\psi(x)$.
- $Z_{O^B(T_B)}[a]=\frac{1}{|H^1(\Sigma;\mathbb{Z}_2)|^{1/2}}\sum_b(-1)^{\int a\cup b}Z_{T_B}[b]$.

$\to$ $\mathcal{F}(\psi)(p)=\langle p|\psi\rangle=\langle p|\int dx|x\rangle\langle x|\psi\rangle=\frac{1}{\sqrt{2\pi}}\int dxe^{-ipx}\psi(x)$

where $\langle p|x\rangle=\frac{1}{\sqrt{2\pi}}e^{-ipx}$.

$\to$ SymTFT gives analogous descriptions of topological manipulations.

$\to$ We construct SymTFT for O^B, S_1^B, S_2^B, Fer on non-orientable surfaces.

Group structure behind the duality

Group structure:

$$(O^B)^2=(S_1^B)^2=(O^BS_1^B)^8=1$$

$\to$ $D_8\cong\mathbb{Z}_8\rtimes\mathbb{Z}_2$, i.e.,
$\langle s,t\mid s^2=t^2=(st)^8=1\rangle$

Remark: $w_1=0\Rightarrow D_8\to\mathbb{Z}_2$

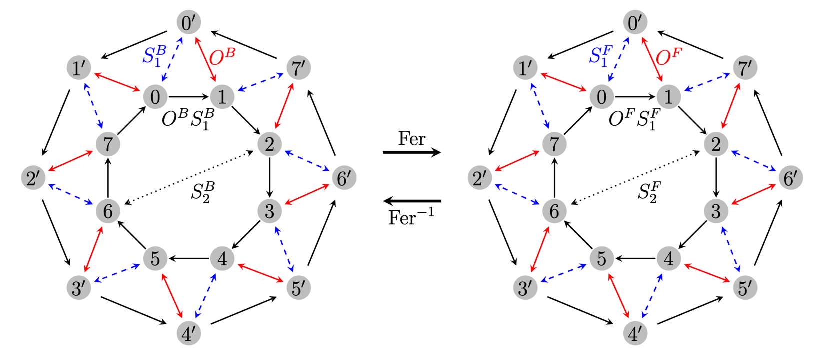

Figure: Non-orientable duality web.

Duality among sectors of the Hilbert space

In terms of $g\in\mathbb{Z}_2$, the Hilbert space on S^1 can be decomposed as

$$\mathcal{H}_{\text{all}}=\mathcal{H}\oplus\mathcal{H}^g\quad\text{with}\quad\begin{cases}\mathcal{H}=\mathcal{H}_{\text{even},+}\oplus\mathcal{H}_{\text{even},-}\oplus\mathcal{H}_{\text{odd},+}\oplus\mathcal{H}_{\text{odd},-}\\ \mathcal{H}^g=\mathcal{H}^g_{\text{even},+}\oplus\mathcal{H}^g_{\text{even},-}\oplus\mathcal{H}^g_{\text{odd},+}\oplus\mathcal{H}^g_{\text{odd},-}\end{cases}$$

Here even/odd and $+/-$ denote the eigenvalues of $g,P\in\mathrm{End}(\mathcal{H}_{\text{all}})$:

$$g\psi=\begin{cases}\psi & \psi\in\mathcal{H}_{\text{even}}\\ -\psi & \psi\in\mathcal{H}_{\text{odd}}\end{cases}\qquad P\psi=\begin{cases}\psi & \psi\in\mathcal{H}_{+}\\ -\psi & \psi\in\mathcal{H}_{-}\end{cases}$$

T_B		untwisted	twisted
even	$P=+1$	S_+	U_+
	$P=-1$	S_-	U_-
odd	$P=+1$	T_+	V_+
	$P=-1$	T_-	V_-

Table: Sectors of T_B on S^1.

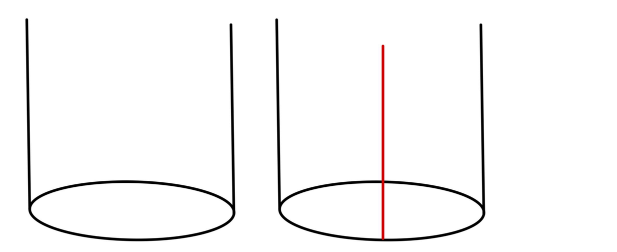

Figure: Untwisted/twisted Hilbert sp.

Suppose we have a torus partition function $Z_{T_B}[T^2;0]=\mathrm{Tr}^{T_B}_{\mathcal{H}}(e^{-lH})$

A partition function with an insertion of $P:\mathcal{H}_{\text{all}}\to\mathcal{H}_{\text{all}}$

$$Z_{T_B}[K;0]=\mathrm{Tr}^{T_B}_{\mathcal{H}}(Pe^{-lH})=$$

can be interpreted as a Klein bottle partition function.

In terms of $g:\mathcal{H}_{\text{all}}\to\mathcal{H}_{\text{all}}$ where $g\in\mathbb{Z}_2$, we obtain

$$Z_{T_B}[T^2;0]=\mathrm{Tr}^{T_B}_{\mathcal{H}}(e^{-lH})=\qquad Z_{T_B}[K;0]=\mathrm{Tr}^{T_B}_{\mathcal{H}}(Pe^{-lH})=$$
$$Z_{T_B}[T^2;x]=\mathrm{Tr}^{T_B}_{\mathcal{H}}(g\,e^{-lH})=\qquad Z_{T_B}[K;x]=\mathrm{Tr}^{T_B}_{\mathcal{H}}(gPe^{-lH})=$$
$$Z_{T_B}[T^2;y]=\mathrm{Tr}^{T_B}_{\mathcal{H}^g}(e^{-lH})=\qquad Z_{T_B}[K;y]=\mathrm{Tr}^{T_B}_{\mathcal{H}^g}(Pe^{-lH})=$$
$$Z_{T_B}[T^2;x+y]=\mathrm{Tr}^{T_B}_{\mathcal{H}^g}(g\,e^{-lH})=\qquad Z_{T_B}[K;x+y]=\mathrm{Tr}^{T_B}_{\mathcal{H}^g}(gP\,e^{-lH})=\quad.$$

For example, we can compute the odd, $+$, twisted sector of $O^B(T_B)$ as

$$\mathrm{Tr}^{O^B(T_B)}_{\mathcal{H}^g}\left(\frac{1-g}{2}\frac{1+P}{2}e^{-lH}\right)$$
$$=\frac{1}{4}(\quad-\quad+\quad-\quad)_{O^B(T_B)}$$
$$=\frac{1}{4}(\quad-\quad-\quad+\quad)_{T_B}$$
$$=\mathrm{Tr}^{T_B}_{\mathcal{H}^g}\left(\frac{1-g}{2}\frac{1-P}{2}e^{-lH}\right)$$

$O^B(T_B)$		$\mathcal{H}$	$\mathcal{H}^g$
even	$P=+1$	S_+	T_+
	$P=-1$	S_-	T_-
odd	$P=+1$	U_+	V_-
	$P=-1$	U_-	V_+

Table: Sectors of $O^B(T_B)$ on S^1.

which turns out to be the odd, $-$, twisted sector of T_B V_-.

$\to$ We performed the same analysis for other topological manipulations.

Future work

- Gauging symmetry on non-orientable manifolds in general.
- Non-invertible symmetry arising from ABK stacking invariance.

$\to$ Much remains to be understood!

This poster is mainly based on arXiv:2601.09067 with K. Tsuji

[1]:Karch et al. 1902.05550
[2]:Spieler 2504.20021
[3]:Thorngren 1404.4385