Tensor renormalization group approach to the four-dimensional lattice gauge theories

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Contents

Research motivation

QFT on a lattice

✔ Quantum field theory (QFT) is a fundamental tool to describe various physical phenomena

- We would like to solve the QFT to understand the physics and make a theoretical prediction. **One of the ways is to solve the path integral of the QFT**

✔ Once we consider the QFT on a lattice, we can regard the path integral just as a multiple integral

$$\int \prod_{x \in \mathbb{R}^d} \mathrm{d}\phi(x)\ \mathrm{e}^{-S[\phi]} \quad \rightarrow \quad \int \prod_{n \in \Lambda_d} \mathrm{d}\phi(n)\ \mathrm{e}^{-S_{\mathrm{lat}}[\phi]}$$

✔ The QFT on a lattice provides us with a mathematically rigorous starting point and **we can investigate it based on the procedure of statistical physics**

Standard numerical approach for QFT on a lattice

✔ Monte Carlo (MC) simulation

- The MC is based on the probabilistic interpretation of the given Boltzmann weight $\mathrm{e}^{-S[\phi]}$
- Lattice QCD is one of the most successful applications of the MC

✔ There are several difficulties in the MC simulation, though

- We encounter the **sign problem** when $\mathrm{e}^{-S[\phi]}$ takes negative or complex value
- Bosons are easily dealt with the MC, but **fermions** are not
  In the path integral formalism, fermions are described by the Grassmann numbers, which obey the anti-commutation $\psi\phi + \phi\psi = 0$

Different approach?

✔ There are many systems suffering from the sign problem

- QCD at finite density

✔ Many unrevealed physics in such systems

- Thermodynamic limit (or zero-temperature limit) is almost inaccessible w/ the standard MC approach

✔ We need different numerical methods which can give us insights for these systems

- **How about tensor network?**

A quick overview of higher-dimensional TRG

Tensor network & Lattice field theory

✔ **A method to investigate quantum many-body system expressing an objective function as a tensor contraction (= tensor network)**

Orús, APS Physics 1(2019)538-550
Bañuls-Cichy, Rep. Prog. Phys. 83(2020)024401
Meurice-Sakai-Unmuth–Yockey, Rev. Mod. Phys. 94(2022)025005
Okunishi-Nishino-Ueda, J. Phys. Soc. Jap. 91(2022)062001

✔ TN method provides us with various ways to investigate lattice QFT

- **Hamiltonian formalism**

  Describe a state vector as a TN, which is **variationally optimized**

  Cf. **DMRG**, **TEBD**

  White, PRL69(1992)2863-2866, White, PRB48(1993)10345-10356
  Vidal, PRL91(2003)147902, Vidal, PRL98(2007)070201
  Cf. Various talks in this workshop

- **Lagrangian formalism**

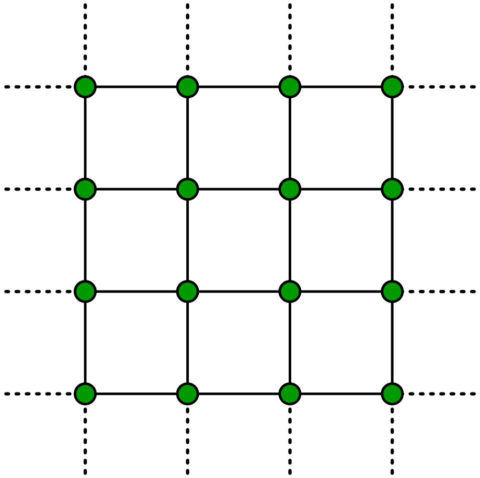
  Describe a path integral as a TN, which is **approximately contracted**

  Cf. **TRG**, **TNR, loop-TNR, GILT**

  Levin-Nave, PRL99(2007)120601
  Evenbly-Vidal, PRL115(2015)180405, Evenbly, PRB95(2017)045117
  Yang-Gu-Wen, PRL118(2017)110504
  Hauru-Delcamp-Mizera, PRB97(2018)045111

Pros and Cons

- ✔ Tensor renormalization group (TRG) approximately contracts a given TN based on the idea of real-space renormalization group
  - No sign problem
  - Thermodynamic limit
  - Grassmann variables
  - Path integral
  - **Higher dimension than $d = 2$**



- ✔ **Higher-dimensional TN computation is challenging**
  - Further algorithmic development is necessary
    - Improvement of the TRG based on the removement of short-range correlations
    - Lessons form other TN methods such as TTN, PEPS, isoTNS, etc

Cf. L. Vanderstraeten, 9/26
Cf. F. Pollmann, 9/27

Status of (3+1)D TN calculations

Hamiltonian formalism	Lagrangian formalism
• QED at finite density Magnifico+	• Ising model SA+ • Staggered fermion w/ strongly coupled U(N) Milde+ • Complex ϕ^4 theory at finite density SA+ • Nambu—Jona-Lasinio model at finite density SA+ • Real ϕ^4 theory SA+ • $\mathbb{Z}_2$ & $\mathbb{Z}_3$ gauge-Higgs at finite density SA-Kuramashi

✔ So far, the (3+1)D TN calculations have been driven by the Lagrangian formalism w/ the TRG approach

✔ Development of **parallel computing method** specialized for individual algorithms to reduce their execution time per process

SA+, PoS(LATTICE2019)138
Yamashita-Sakurai, CPC278(2022)108423

✔ **Application of ML techniques and/or GPU** is a recent hot topic

Liao+, PRX9(2019)031041
R. G. Jha-Samlodia, CPC(2023)108941

Procedure of TRG approach

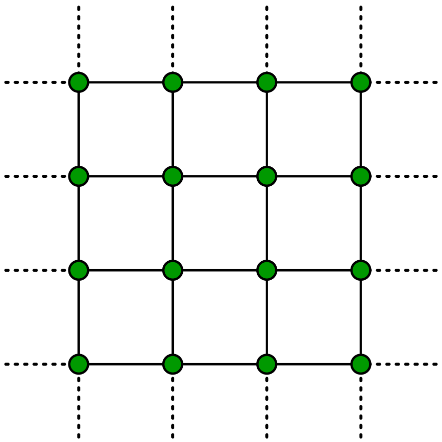
1) Represent the path integral as a tensor network

- Lattice QFTs can be easily represented by TN

Cf. Y. Meurice, 9/15

Meurice-Sakai-Unmuth–Yockey, Rev. Mod. Phys. 94(2022)025005
Meurice, "Quantum Field Theory, A quantum computation approach"

$Z \quad \rightarrow$



2) Take contractions approximately

- Various algorithms are proposed to achieve this mission
- In 2D, we can also use other schemes to take contractions approximately

Cf. iTEBD for 2D classical Ising model: Orús-Vidal, PRB78(2008)155117

Higher-order TRG (HOTRG)

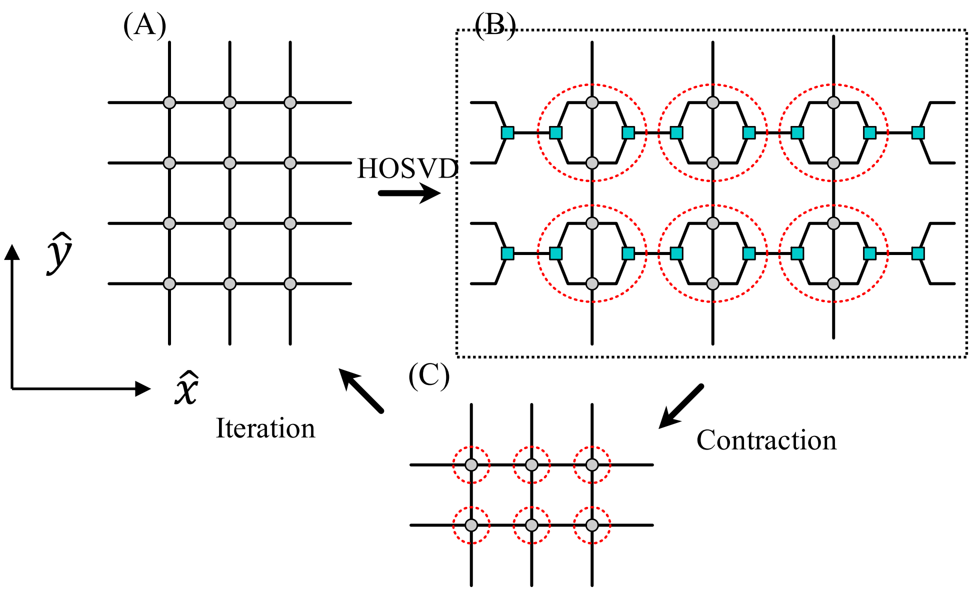
Xie-Chen-Qin-Zhu-Yang-Xiang, PRB86(2012)045139

Memory	Complexity
$O(D^{2d})$	$O(D^{4d-1})$

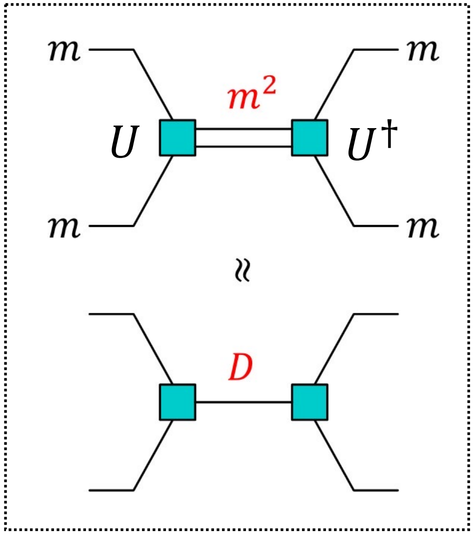
✔ Applicable to any d-dimensional lattice

✔ # of tensors are reduced to half

Iterating this CG n times, we can approximately contract 2^n tensors



Sequential coarse-graining along with each direction

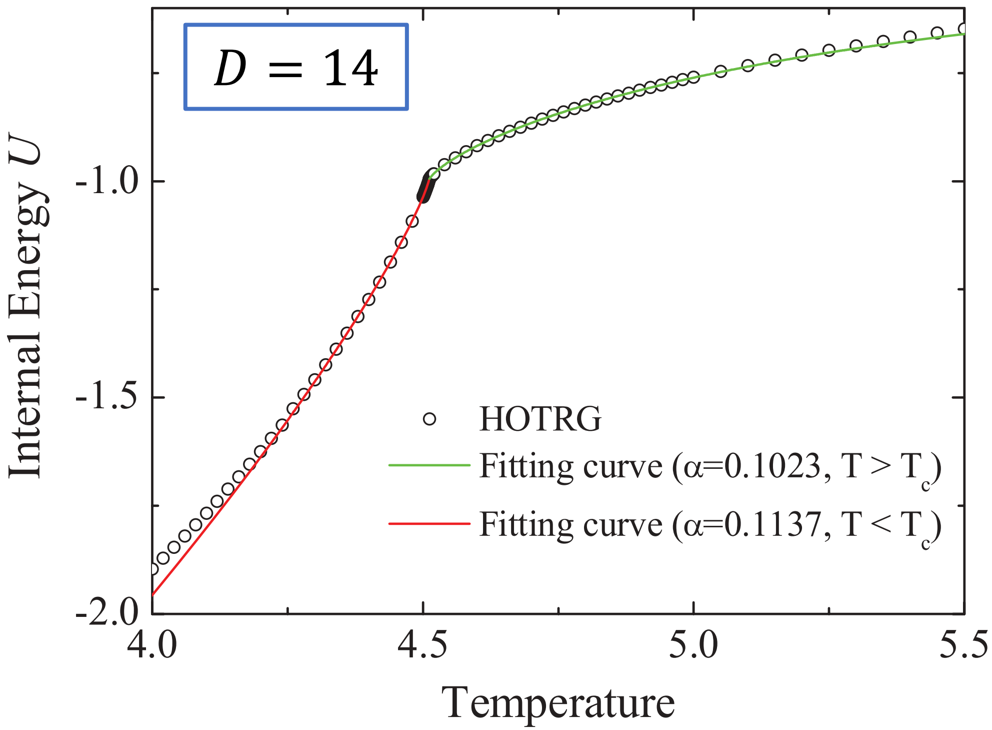


***D*: bond dimension**

Example: 3D Ising model w/ HOTRG

Xie-Chen-Qin-Zhu-Yang-Xiang, PRB86(2012)045139

✔ **Critical point is precisely located** with relatively small bond dimension



Critical point

Method	T_c
HOTRG (D = 16, from U)	4.511544
HOTRG (D = 16, from M)	4.511546
Monte Carlo[37]	4.511523
Monte Carlo[38]	4.511525
Monte Carlo[39]	4.511516
Monte Carlo[35]	4.511528
Series expansion[40]	4.511536
CTMRG[12]	4.5788
TPVA[13]	4.5704
CTMRG[14]	4.5393
TPVA[16]	4.554
Algebraic variation[41]	4.547

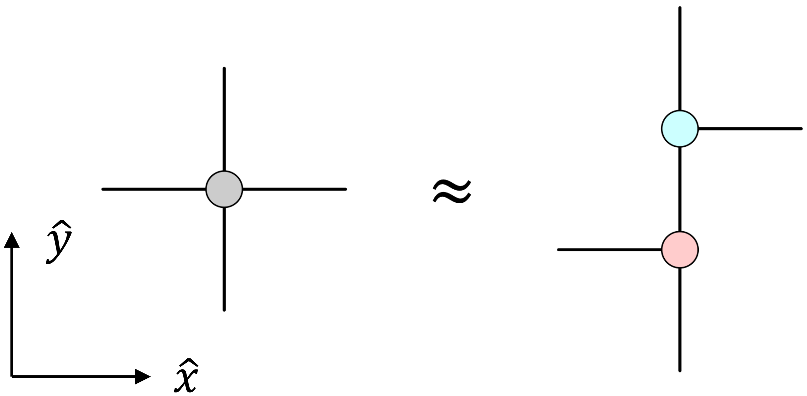
Anisotropic TRG (ATRG)

Adachi-Okubo-Todo, PRB102(2020)054432

✔ Applicable to any d-dimensional lattice

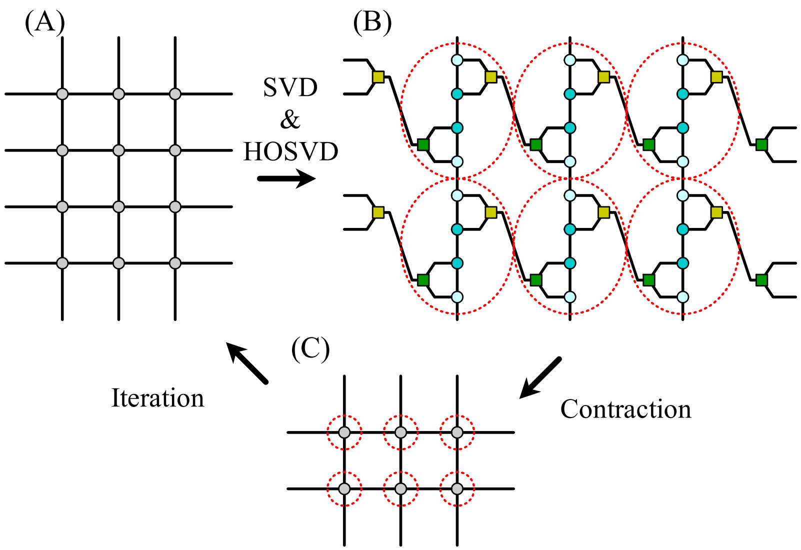
✔ **More economic than the HOTRG**

Memory	Complexity
$O(D^{d+1})$	$O(D^{2d+1})$



$$T_{xyx'y'} \approx \sum_{i=1}^{D} A_{xyi} B_{x'y'i}$$

ATRG considers the block-spin transformation within lower-rank tensors. (Memory: $O(D^{2d}) \to O(D^{d+1})$)

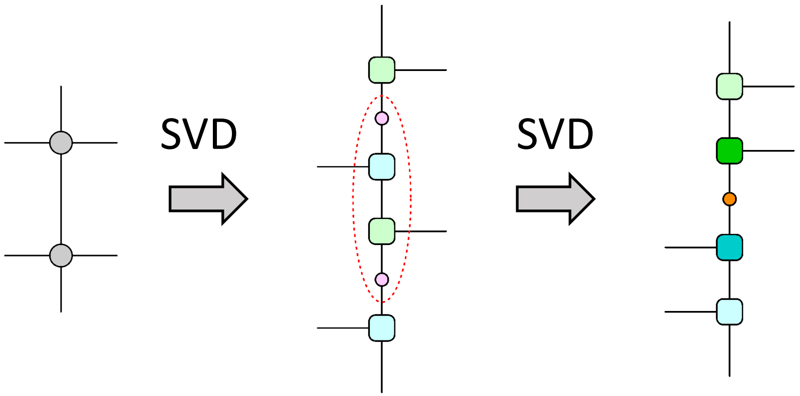


# of tensors are reduced to half

Canonical form in ATRG

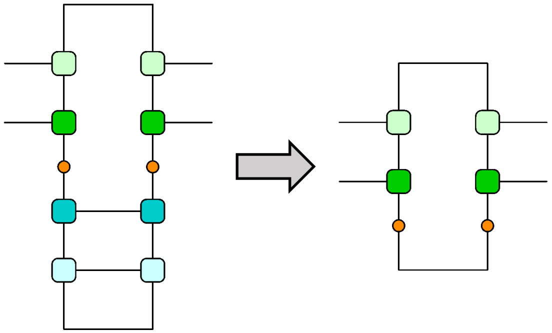
✔ ATRG converts two adjacent tensors into a **canonical form**

- Canonical form is an important idea in MPS Schollwöck, Annals of Physics 326(2011)96-192
  Cf. F. Pollmann, 9/27



✔ **"Reduced density matrix" is simplified thanks to the canonical form**

- Highly helpful in practical computations

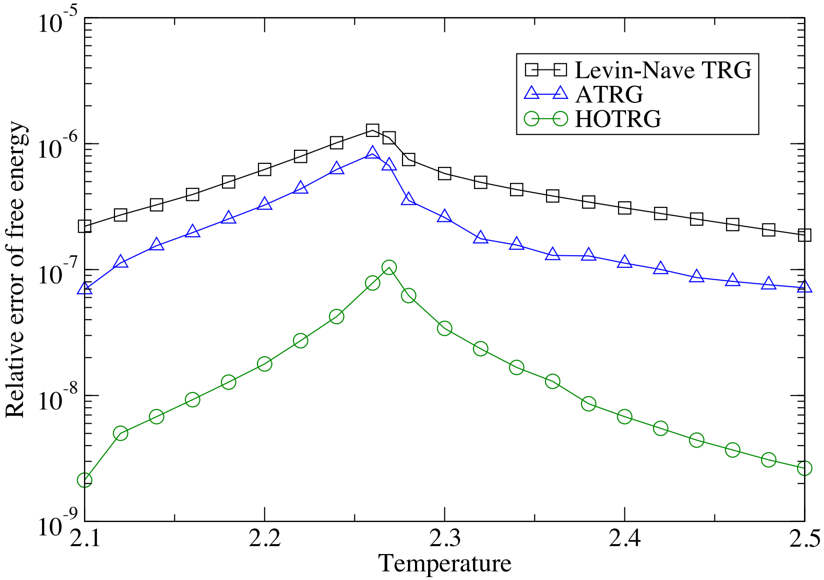


Benchmarking w/ 2D Ising model

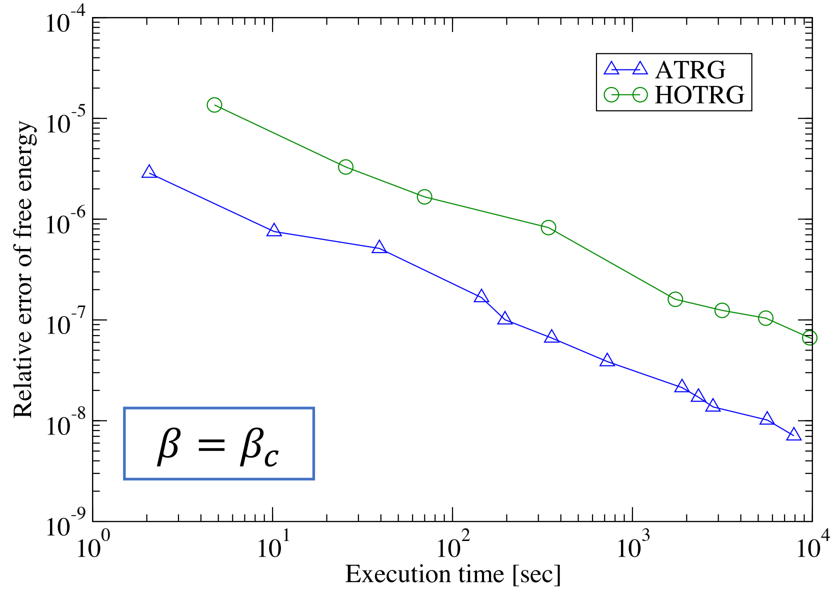
✔ HOTRG & ATRG improve the accuracy of the original (LN-)TRG at the same D
The exact solution is well reproduced

✔ **ATRG shows better performance than the HOTRG at the same execution time**

Comparison of three types of TRG
w/ $D = 24$



Relative error vs execution time



Grassmann TRG approach

Gu-Verstraete-Wen, arXiv:1004.2563

✔ Any TRG algorithm can be applied for fermions
Fermionic path integral can be expressed as a tensor network generated by **Grassmann tensors**

Gu, PRB88(2013)115139
Shimizu-Kuramashi, PRD90(2014)014508
Takeda-Yoshimura, PTEP2015(2015)043B01
Meurice, PoS LATTICE2018(2018)231
Bao's thesis, PhD, Uwaterloo
SA-Kadoh, JHEP10(2021)188

$$\mathcal{T}_{\eta_1\eta_2\eta_3\cdots} = \sum_{i_1,i_2,i_3,\cdots} T^{i_1 i_2 i_3\cdots}\eta_1^{i_1}\eta_2^{i_2}\eta_3^{i_3}\cdots$$

✔ A clear correspondence btw tensors and Grassmann tensors

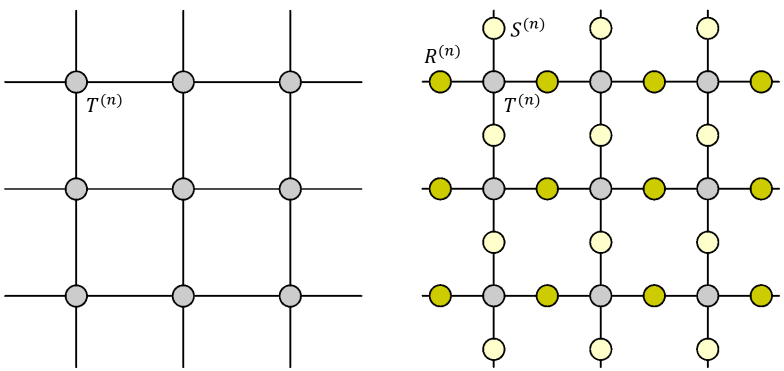
	Tensor	Grassmann tensor
index	integer	Grassmann number
contraction	$\Sigma_i \cdots$	$\int\int \mathrm{d}\bar{\eta}\mathrm{d}\eta \mathrm{e}^{-\bar{\eta}\eta}\cdots$

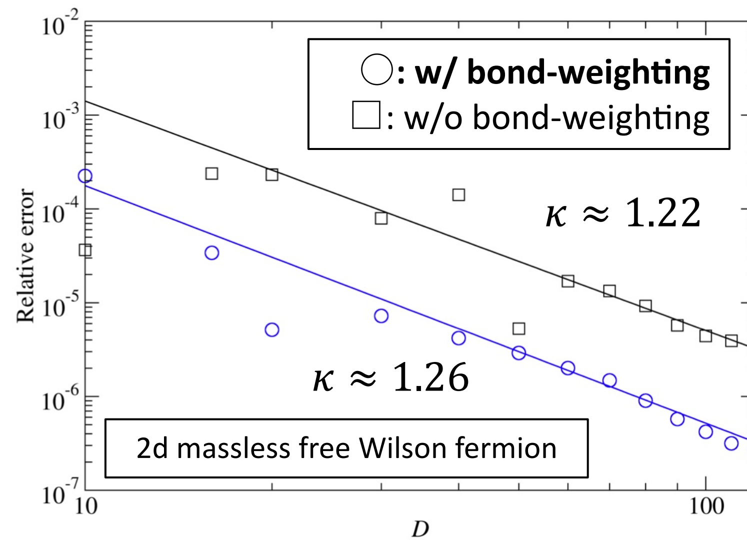
$$\mathrm{e}^{A\bar{\boldsymbol{\psi}}_{\boldsymbol{n}}\boldsymbol{\psi}_{\boldsymbol{n}+\boldsymbol{\mu}}} = \left(\int\int \mathrm{d}\bar{\eta}_n \mathrm{d}\eta_n \mathrm{e}^{-\bar{\eta}_n\eta_n}\right)\exp\left[-\sqrt{A}\bar{\boldsymbol{\psi}}_{\boldsymbol{n}}\eta_n + \sqrt{A}\bar{\eta}_n\boldsymbol{\psi}_{\boldsymbol{n}+\boldsymbol{\mu}}\right]$$

Several public codes of Grassmann TRG

✔ **Grassmann bond-weighted TRG** https://github.com/akiyama-es/Grassmann-BTRG by SA

- Originally proposed for spin models — Adachi-Okubo-Todo, PRB105(2022)L060402
- Works also well for fermions — SA, JHEP11(2022)030





$$\delta f \sim D^{2\kappa} \text{ w/ } \kappa = 1.344 \text{ when } c = 1$$

Tagliacozzo+, PRB78(2008)024410
Pollmann+, PRL102(2009)255701

✔ **GrassmannTN** https://github.com/ayosprakob/grassmanntn by A. Yosprakob

- Python package for Grassmann TRG computations — Yosprakob, arXiv:2309.07557

TRG study of (3+1)D $\mathbb{Z}_2$ & $\mathbb{Z}_3$ gauge-Higgs models

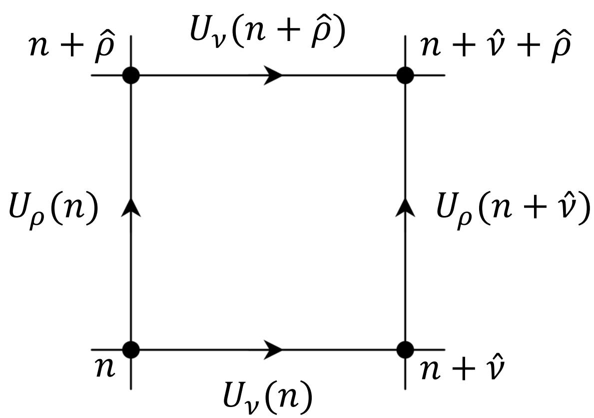
First application of TRG to (3+1)D LGT

SA-Kuramashi, JHEP05(2022)102

SA-Kuramashi, arXiv:2304.07934 (To appear in JHEP)

$\mathbb{Z}_n$ gauge-Higgs model in the unitary gauge

$$S = -\beta \sum_n \sum_{\nu>\rho} \mathrm{Re}[U_\nu(n)U_\rho(n+\hat{\nu})U_\nu^*(n+\hat{\rho})U_\rho^*(n)]$$
$$-\eta \sum_n \sum_\nu \left[\mathrm{e}^{\mu\delta_{\nu,d}}\sigma^*(n)U_\nu(n)\sigma(n+\hat{\nu}) + \mathrm{e}^{-\mu\delta_{d,4}}\sigma^*(n)U_\nu^*(n-\hat{\nu})\sigma(n-\hat{\nu})\right]$$



$U_\nu(n)(\in \mathbb{Z}_n)$: link variable living on edges
$\sigma(n)(\in \mathbb{Z}_n)$: matter field living on sites
β: inverse gauge coupling
η: spin-spin coupling
μ: chemical potential

✔ Unitary gauge fixing: $\sigma^*(n)U_\nu(n)\sigma(n+\hat{\nu}) \mapsto U_\nu(n)$

$$S = -\beta \sum_n \sum_{\nu>\rho} \mathrm{Re}[U_\nu(n)U_\rho(n+\hat{\nu})U_\nu^*(n+\hat{\rho})U_\rho^*(n)]$$
$$-2\eta \sum_n \sum_\nu \left[\cosh(\mu\delta_{\nu,4})\mathrm{Re}U_\nu(n) + \mathrm{i}\sinh(\mu\delta_{\nu,4})\mathrm{Im}U_\nu(n)\right]$$

Motivation of studying $\mathbb{Z}_n$ gauge-Higgs model

✔ **The simplest lattice gauge theory coupling to a matter field**

- A good target to see whether the TRG is efficient for the (3+1)D lattice gauge theory or not

✔ **The model possesses the critical endpoint (CEP)**

- QCD at finite temperature and density also has the CEP
  Can we use the TRG to specify the precise location of CEP?

✔ **We can consider the model at finite density**

- We can investigate how the CEP moves by finite chemical potential
  The $\mathbb{Z}_2$ model is free from the sign problem but the $\mathbb{Z}_3$ model is not

Cf. TRG studies of gauge-Higgs models in 2D
Unmuth–Yockey+, PRD98(2018)094511, Bazavov+, PRD99(2019)114507, Butt+, PRD101(2020)094509

✔ We use the ATRG w/ parallel computation (slicing fundamental tensors)

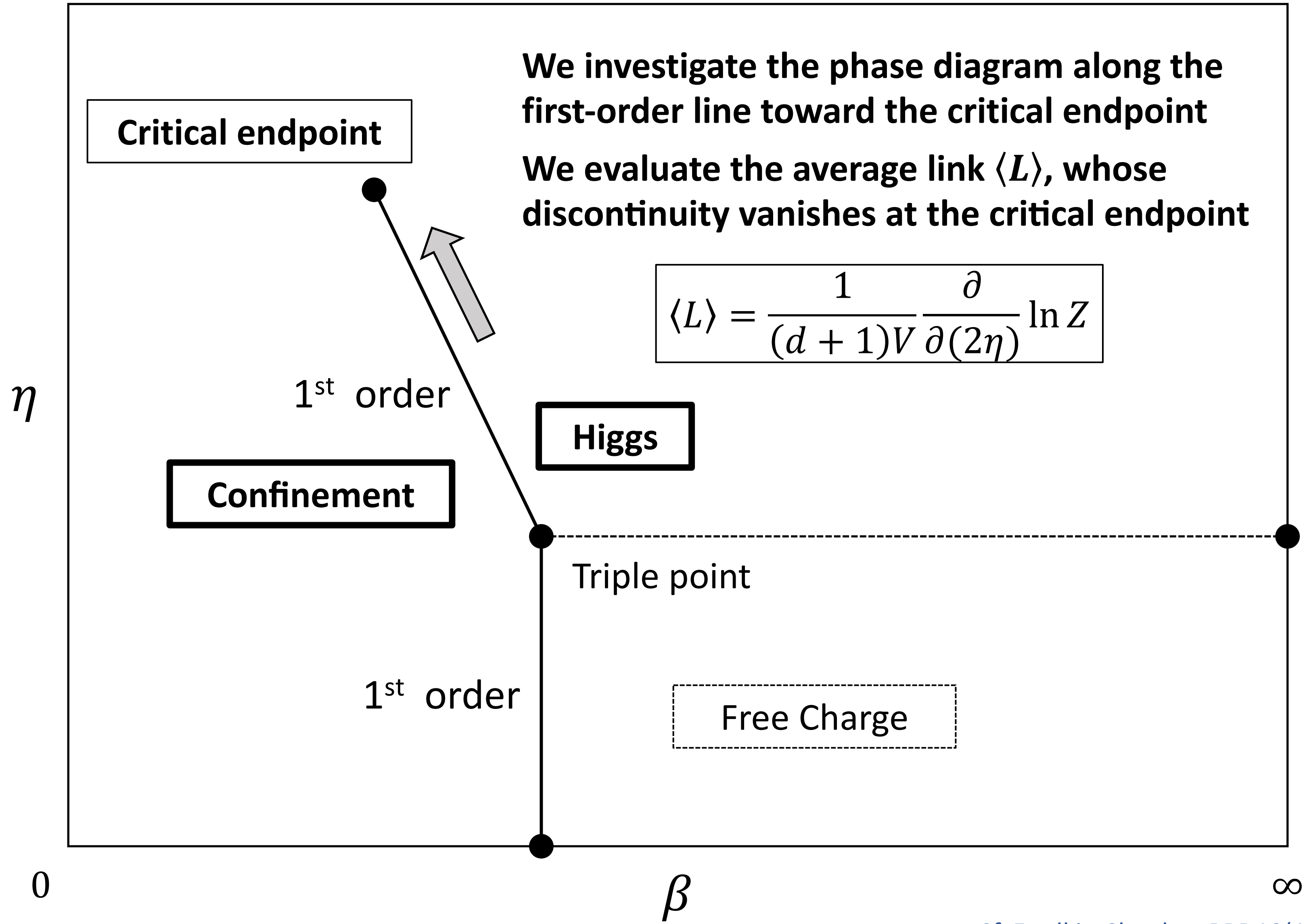
SA+, PoS(LATTICE2019)138

Phase diagram of the (3+1)D model at $\mu = 0$

We investigate the phase diagram along the first-order line toward the critical endpoint

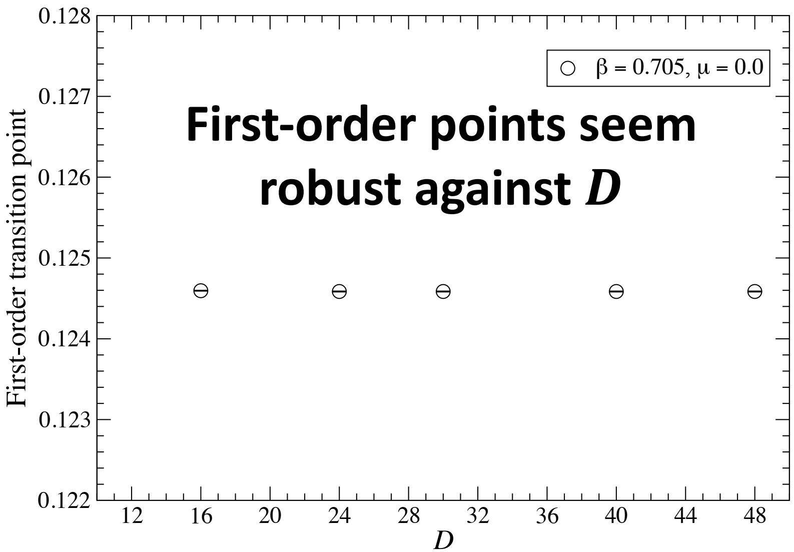
We evaluate the average link $\langle L \rangle$, whose discontinuity vanishes at the critical endpoint

$$\langle L \rangle = \frac{1}{(d+1)V} \frac{\partial}{\partial(2\eta)} \ln Z$$

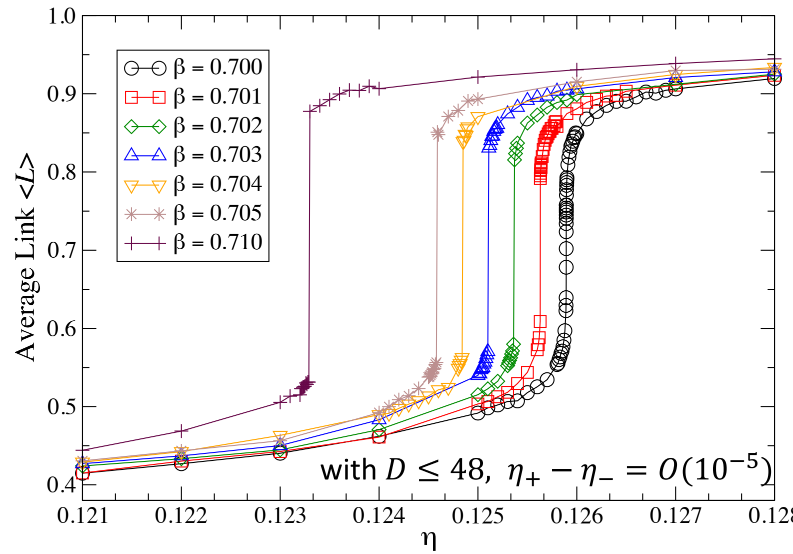


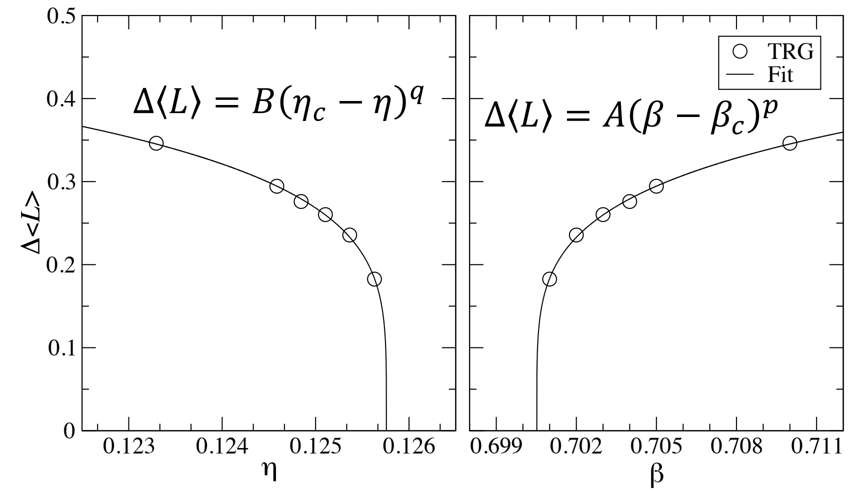
Cf. Fradkin-Shenker, PRD19(1979)3682

Study of the (2+1)D model at $\mu = 0$



Bond dimension

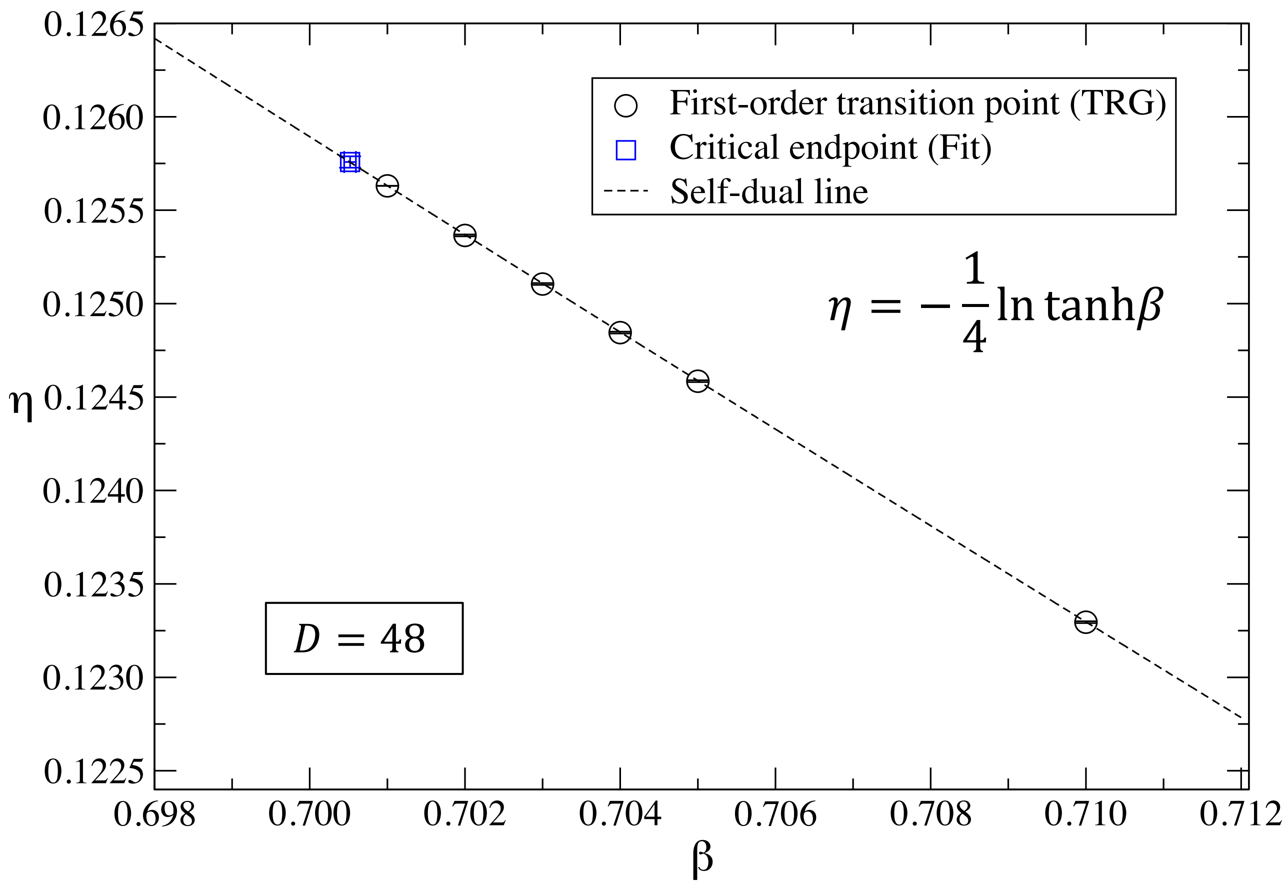




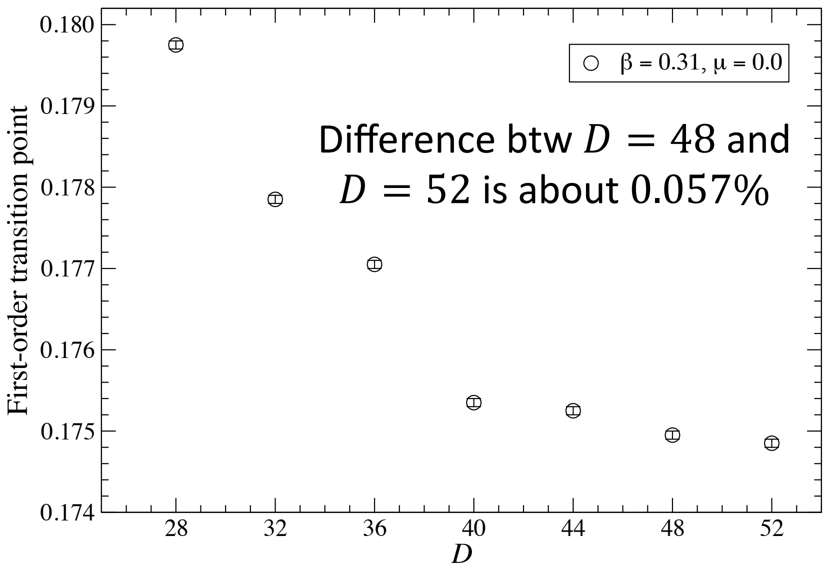
MC Somoza+, PRX11(2021)041008	$\beta_c \approx 0.701$
TRG this work	$\beta_c = 0.70051(7)$

Comparison with the self-dual line

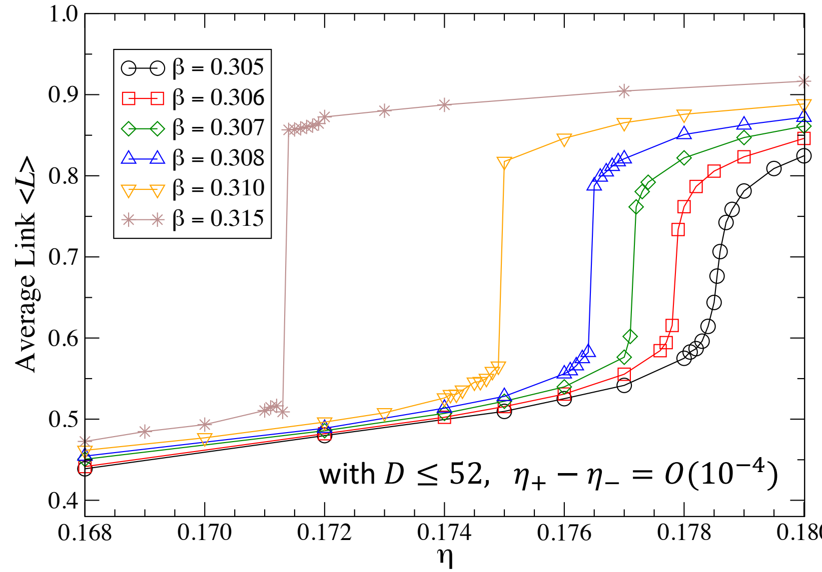
✔ **All transition points are well located on the self-dual line**



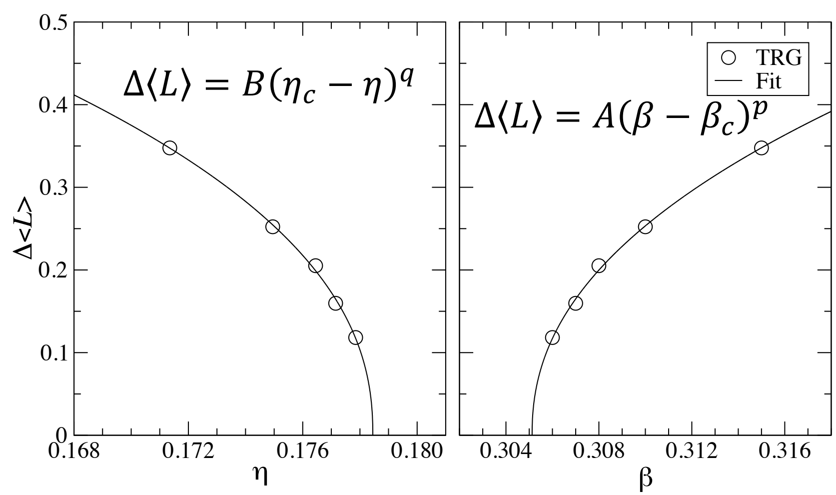
(3+1)D model at vanishing density



Bond dimension



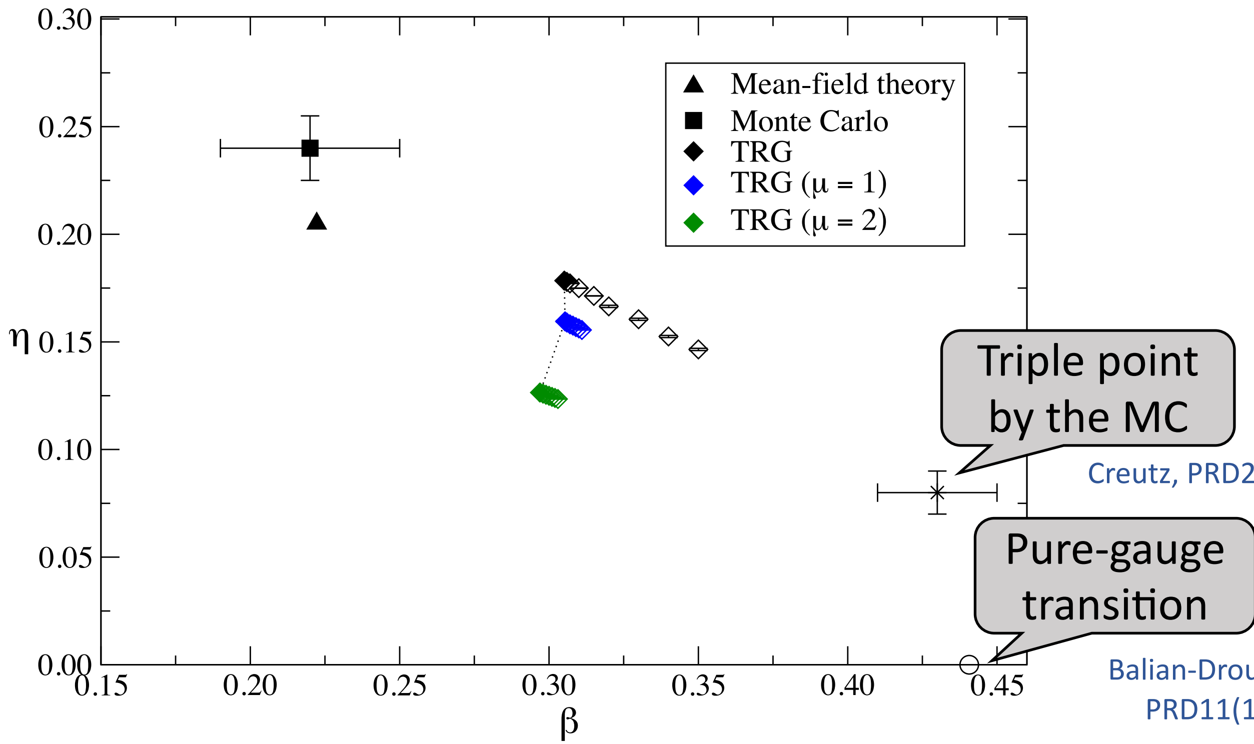
Mean-field Brezin-Drouffe, NPB200(1982)93	$(\beta_c, \eta_c) = (0.22, 0.205)$
MC on $V = 8^4$ Creutz, PRD21(1980)1006	$(\beta_c, \eta_c) = (0.22(3), 0.24(2))$
TRG w/ $D = 52$ this work	$(\beta_c, \eta_c) = (0.3051(2), 0.1784(2))$



Status of the phase diagram near the CEP

✔ **It seems that TRG and MC share a similar first-order line at $\mu = 0$**

✔ Revisiting the CEP by the modern MC simulation should be meaningful



Creutz, PRD21(1980)1006

Balian-Drouffe-Itzykson, PRD11(1975)2098

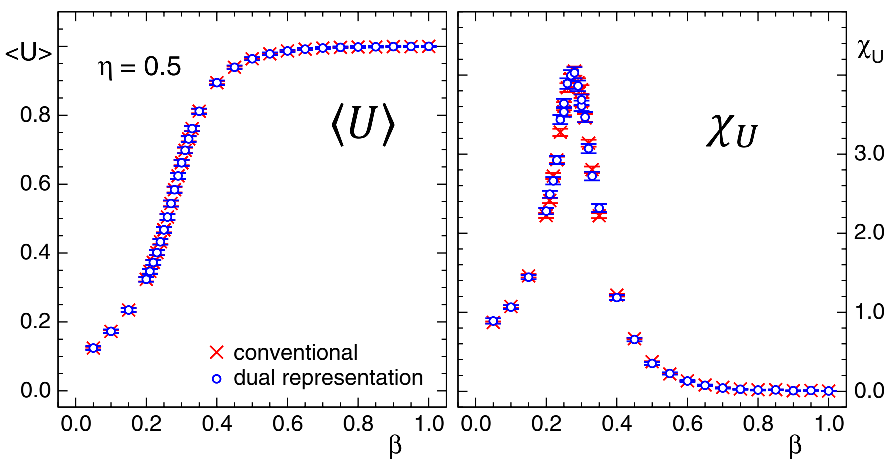
(3+1)D $\mathbb{Z}_3$ model at vanishing density 1/2

✔ Comparison btw MC and TRG via the average plaquette $\langle U \rangle$ and its susceptibility

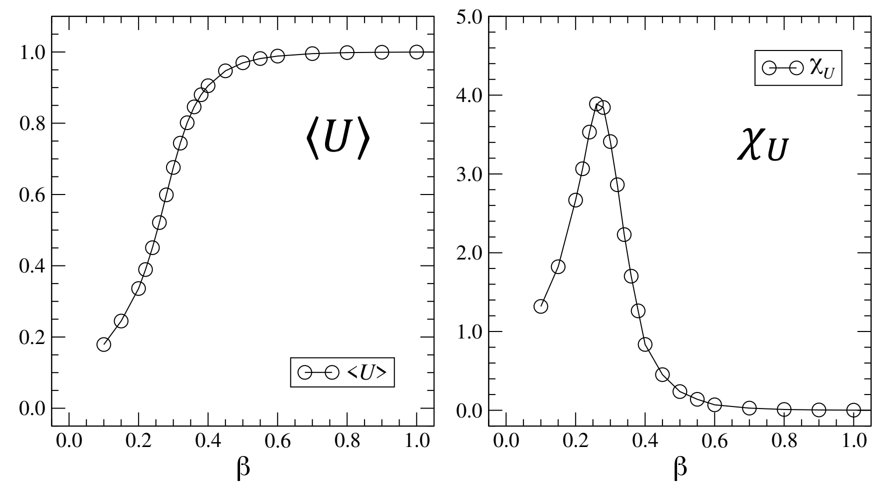
- Good agreement just w/ $D = 45$ at finite-η regime $\quad \langle U \rangle = -\frac{1}{6V}\frac{\partial}{\partial \beta}\ln Z$
- The susceptibility of $\langle U \rangle$ is obtained by numerical difference in case of TRG

Monte Carlo

Gattringer-Schmidt, PRD86(2012)094506



TRG w/ $D = 45$

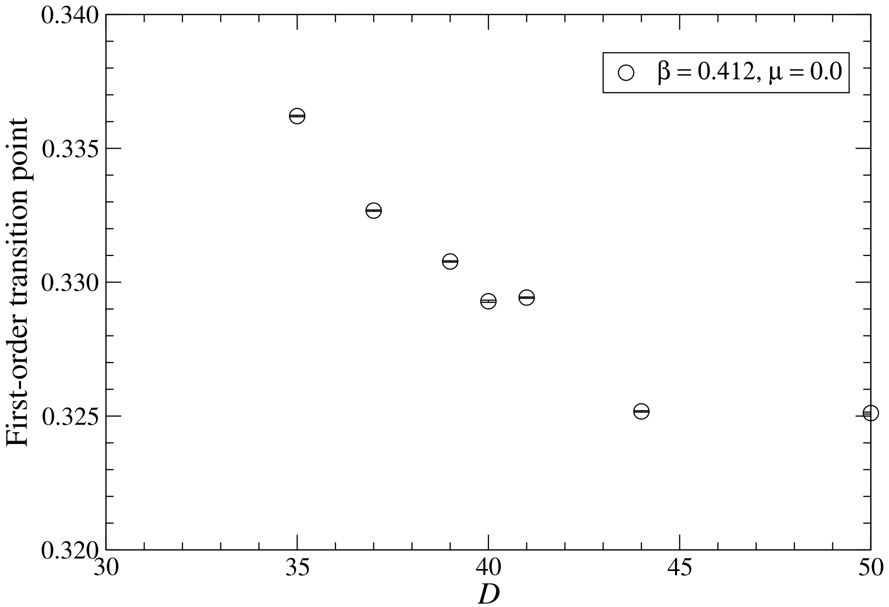


$\eta = 0.5, \mu = 0$

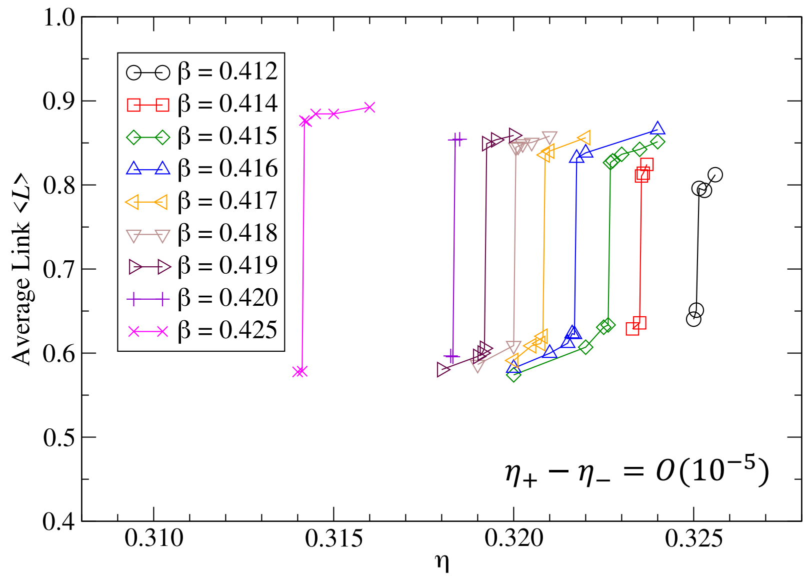
(3+1)D $\mathbb{Z}_3$ model at vanishing density 2/2

- ✔ Location of the transition point seems converging w.r.t D
  - Relative error btw $D = 44$ and $D = 50$ is **0.019%**
- ✔ $\Delta\langle L\rangle$ becomes smaller when β becomes smaller as expected

D-dependence in the first-order transition point



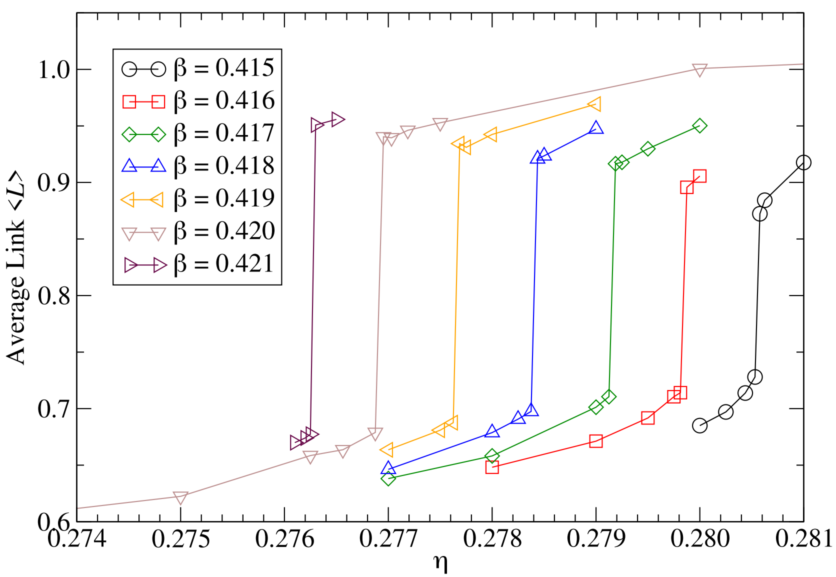
Average link@$D = 50$



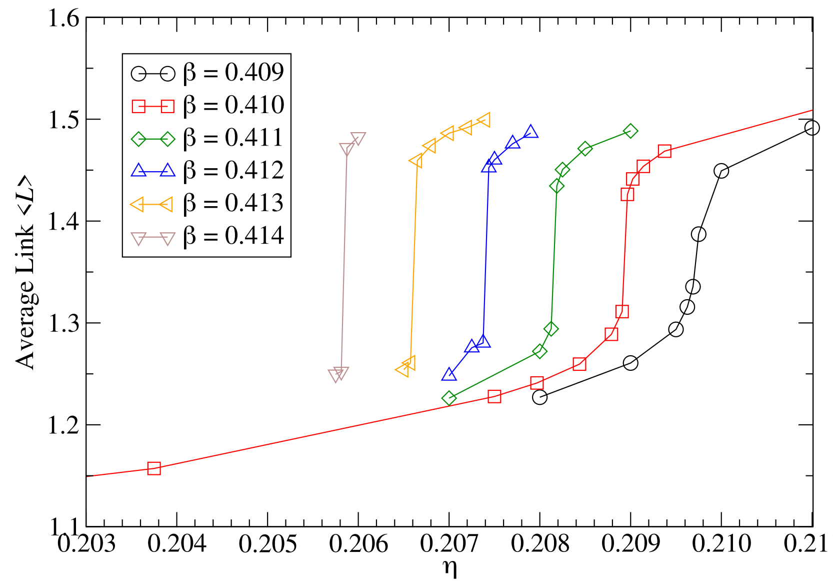
(3+1)D $\mathbb{Z}_3$ model at finite density

✔ Again, $\Delta\langle L\rangle$ becomes smaller when β becomes smaller as expected

Average link@$\mu = 1$



Average link@$\mu = 2$

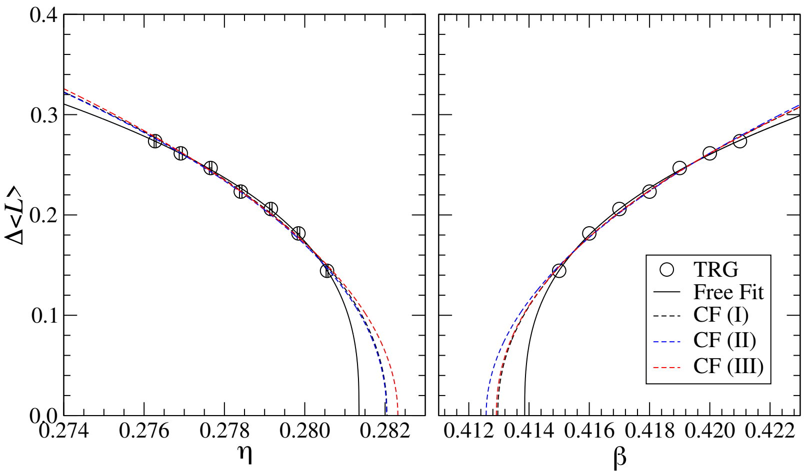


CEP in (3+1)D $\mathbb{Z}_3$ model at finite μ

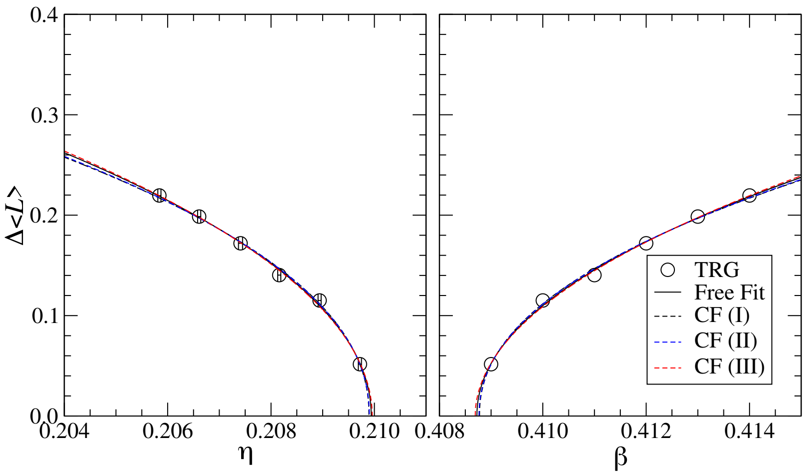
✔ CEP is determined via the similar fit to the $\mathbb{Z}_2$ model

- Fit by $\Delta\langle L\rangle = A(\beta-\beta_c)^p$ and $\Delta\langle L\rangle = B(\eta_c-\eta)^q$
- According to the mean field theory, $p = q = 0.5$
- The simultaneous fit among different μ suggests $\boldsymbol{p = 0.46(2)}, \boldsymbol{q = 0.46(3)}$

$\mu = 1$



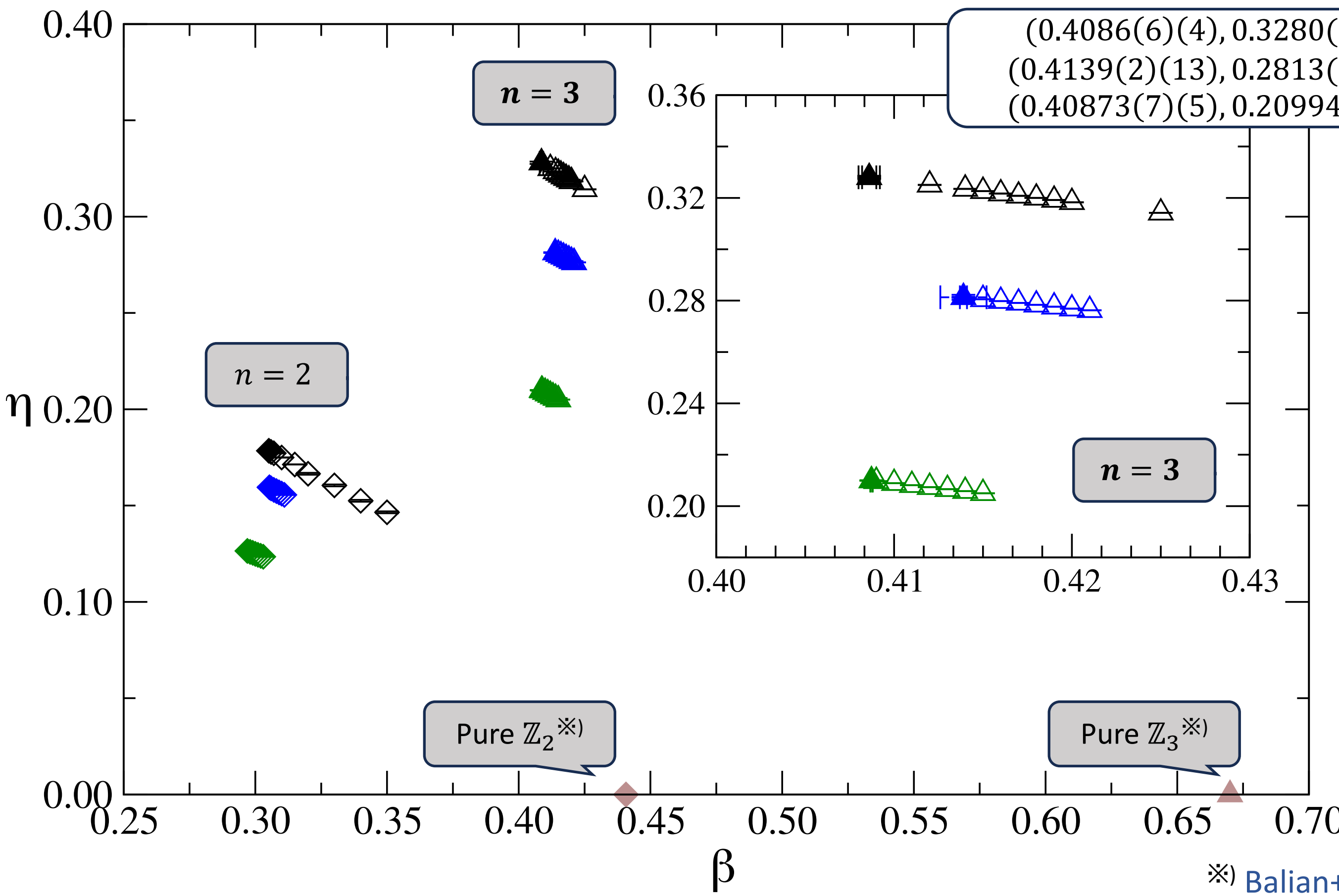
$\mu = 2$



Phase diagram of $\mathbb{Z}_n$ gauge-Higgs model ($n = 2,3$)

✔ n-dependence in the resulting CEP is consistent w/ previous studies

Cf. U(1) gauge-Higgs studies, Baig-Clua, PRD57(1998)3902, Franzki+, PRD57(1998)6625



Black : $\mu = 0$
Blue : $\mu = 1$
Green: $\mu = 2$

※) Balian+, PRD10(1974)3376

Summary

- ✔ TRG is a typical TN algorithm, which enables us to perform TN contraction approximately using the idea of RSRG
- ✔ TRG w/ parallel computation has been a good way to investigate higher-dimensional QFT on a thermodynamic lattice
- ✔ Several public codes for Grassmann TRG
- ✔ **The first application of TRG for (3+1)D LGT** has been made
  We have obtained the TRG estimates of **CEP in $\mathbb{Z}_2$ & $\mathbb{Z}_3$ gauge-Higgs model at finite density**

Future Perspective

✔ **A next interesting (challenging) target can be the (3+1)D QED**

- Variational approach based on the tree TN for the (3+1)D lattice QED ($L \leq 8$)

Magnifico+, Nature Commun. 12(2021)1

✔ **How can we approach $D \to \infty$?**

- This problem should be addressed

Cf. Finite-entanglement scaling:
Tagliacozzo+, PRB78(2008)024410
Pollmann+, PRL102(2009)255701
Cf. L. Vanderstraeten, 9/26

✔ **Although TRG is based on Lagrangian formalism, some problems are shared with quantum computations based on Hamiltonian formalism**

- TRG may give us insights from the viewpoint of classical computation and vice versa

- How can we deal with higher-dimensional non-abelian gauge theories with TN?
  Cf. TRG approach for SU(N) gauge theory

Fukuma-Kadoh-Matsumoto, PTEP2021(2021)123B03
Hirasawa+, JHEP12(2021)011
Kuwahara-Tsuchiya, PTEP2022(2022)093B02

- Multi-flavor fermions?

SA, PRD108(2023)034514
Yosprakob-Nishimura-Okunishi, arXiv:2309.01422