Conserved charge in general relativity

Sinya AOKI
Center for Gravitational Physics and Quantum Information,
Yukawa Institute for Theoretical Physics, Kyoto University

Center for Gravitational Physics and Quantum Information
Yukawa Institute for Theoretical Physics, Kyoto University

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

KYOTO UNIVERSITY KYOTO JAPAN FOUNDED 1897

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References

SA, T. Onogi and S. Yokoyama, "Charge conservation, Entropy Current, and Gravitation", Int. J. Mod. Phys. A36 (2021)2150201.

SA and K. Kawana, "Entropy and its conservation in expanding Universe", International Journal of Modern Physics A38 (2023) 2350072 [arXiv:2210.03323 [hep-th]].

SA, T. Onogi and T. Yamaoka, "Energies and a gravitational charge for massive particles in general relativity", [arXiv:2305.09849 [gr-qc]].

SA, Y. Hidaka, K. Kawana and K. Shimada, work in progress.

I. Introduction

Motivation

Questions

Is there a covariant conserved quantity in general relativity ?

If exists, what is its physical meaning ?

Some conclusions from our previous studies

1. There exists no covariant definition of conserved energy in general relativity, due to Noether's 2nd theorem.

2. A (matter) energy covariantly defined in general relativity is not conserved in general.

I will not discuss this anymore in this talk, due to a limitation of time.

For more details, please take a look at

SA and T. Onogi, "Conserved non-Noether charge in general relativity: Physical definition vs. Noether's 2nd theorem", Int. J. Mod. Phys. A36 (2022) 2250129,

This talk

We propose a new covariant conserved quantity in general relativity.

We call it a “gravitational charge”.

Contents

II. A conserved gravitational charge in general relativity

Construction of a conserved current

We start with the following decomposition of the energy-momentum tensor (EMT)

$$T_{\mu\nu} = \rho n_\mu n_\nu + P_{\mu\nu}, \quad P_{\mu\nu} n^\nu = n^\mu P_{\mu\nu} = 0$$

ρ: energy density

$P_{\mu\nu}$: pressure tensor

n^μ: a time-like unit vector

Hawking-Ellis type I

c.f. This decomposition is not generic but standard for massive classical matters.

We construct a conserved current from this EMT using $n^\nu(x)$ as

$$J^\mu(x) := T^\mu{}_\nu(x)\beta(x)n^\nu(x) = -\rho(x)\beta(x)n^\mu(x)$$

This definition is coordinate independent.

This current is required to satisfy

$$\nabla_\mu J^\mu = \nabla_\mu (T^\mu{}_\nu \beta n^\nu) = 0$$

Conservation condition

We can determine β by solving the following equation. $J^\mu = -\rho\beta n^\mu$

$$\nabla_\mu J^\mu = -\nabla_\mu(\rho\beta n^\mu) = -n^\mu\partial_\mu(\rho\beta) - \rho\beta K = 0 \qquad K := K^\mu{}_\mu,$$

$K^\nu{}_\mu := \nabla_\mu n^\nu$ (would be) extrinsic curvature of a hyper-surface normal to n^μ

We introduce a parameter τ (proper time) to solve the above equation:

$$n^\mu(x(\tau)) := \frac{dx^\mu(\tau)}{d\tau} \longrightarrow n^\mu\partial_\mu = \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu} = \frac{d}{d\tau}$$

PDE becomes ODE:

$$\frac{d}{d\tau}\rho\left(x(\tau)\right)\beta\left(x(\tau)\right) = -\rho\left(x(\tau)\right)\beta\left(x(\tau)\right)K\left(x(\tau)\right)$$

We can easily solve this ODE on each $x(\tau)$ as

$$\rho\left(x(\tau)\right)\beta\left(x(\tau)\right) = \rho\left(x(\tau_0)\right)\beta\left(x(\tau_0)\right)\exp\left[-\int_{\tau_0}^{\tau} d\eta\, K\left(x(\eta)\right)\right]$$

We need an initial condition at $\tau = \tau_0$. Among many choices, we propose to take

$$\rho\left(x(\tau_0)\right)\beta\left(x(\tau_0)\right) = \text{constant in space}$$

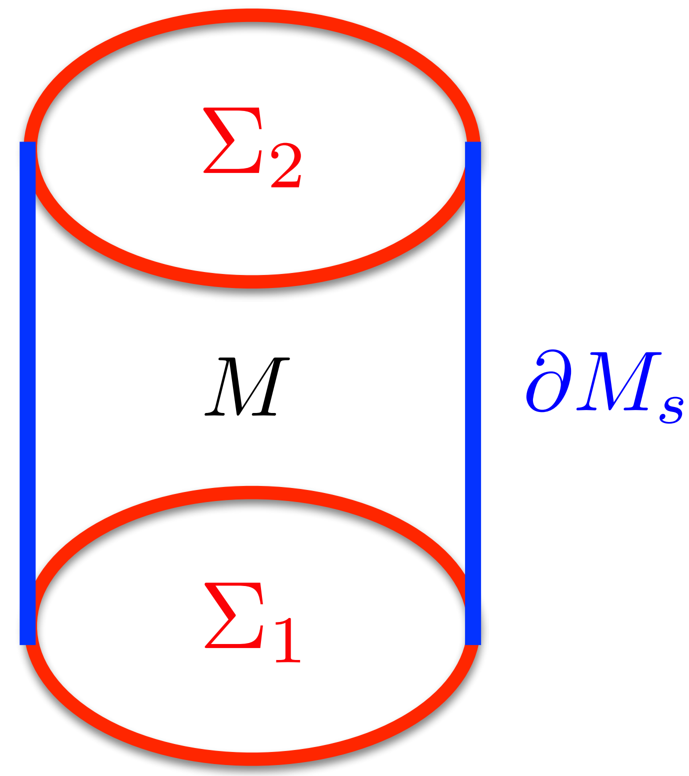
Conserved charge

We integrate $\sqrt{-g}\,\nabla_\mu J^\mu = \partial_\mu(\sqrt{-g}J^\mu) = 0$ over a spacetime region M as

$$0 = \int_M d^dx\,\sqrt{-g}\nabla_\mu J^\mu = \int_{\partial M} d\Sigma_\mu J^\mu = Q(\Sigma_2) - Q(\Sigma_1) + \int_{\partial M_s} d\Sigma_\mu J^\mu$$

$$Q(\Sigma) := \int_\Sigma d\Sigma_\mu J^\mu$$ covariant

$$\partial M = \partial M_s \oplus \Sigma_2 \oplus \Sigma_1$$



If $d\Sigma_\mu J^\mu = 0$ on ∂M_s (space boundary)

$\longrightarrow$ $Q(\Sigma_2) = Q(\Sigma_1)$ conserved

Covariant & conserved charge = Gravitational charge

$$Q(\Sigma) = -\int_\Sigma d\Sigma_\mu n^\mu(x)\rho(x)\beta(x)$$

Σ: spacelike-hyper surface

The gravitational charge is a Noether charge of the matter action invariant under a global transformation $\delta x^\mu = \epsilon\beta(x)n^\mu(x)$.

SA, "Noether's 1st theorem with local symmetries", PTEP 2023 (2023)1, 013B03.

What is a physical meaning of the gravitational charge ?

III. Gravitational charge for massive particles

SA, T. Onogi and T. Yamaoka, “Energies and a gravitational charge for massive particles in general relativity”, [arXiv:2305.09849 [gr-qc]].

Free massive particles with gravitational interactions

The EMT for free massive particles with gravitational interaction is explicitly given by

$$T^{\mu\nu}(x) = \frac{1}{\sqrt{-g(x)}} \sum_{n=1}^{N} m_n \int ds_n\, v_n^\mu(s_n) v_n^\nu(s_n) \delta^{(4)}(x - x_n(s_n))$$

$v_n^\mu(s_n) := \frac{dx_n^\mu(s_n)}{ds_n}$ 4 velocity, which satisfies $g_{\mu\nu} v_n^\mu v_n^\nu = -1$

s_n is a proper time for the n-th particle.

On the other hand, it is very difficult to exactly solve Einstein equation.

Determination of β

Since it is enough to determine $\beta(x)$ at $x \simeq x_n(s_n)$ where particles exit, we take

$$\beta(x) n^\mu(x) \simeq \beta_n(s_n) v_n^\mu(s_n) + O\left(|x - x_n|\right)$$

The conserved current becomes

$$\sqrt{-g} J^\mu = -\sum_{n=1}^{N} m_n \int ds_n\, \beta(s_n) v_n^\mu(s_n) \delta^{(4)}(x - x_n(s_n))$$

The current conservation implies

$$0 = \partial_\mu(\sqrt{-g} J^\mu) = -\sum_{n=1}^{N} m_n \int ds_n\, \beta(s_n) v_n^\mu \partial_\mu \delta^{(4)}(x - x_n(s_n)) = \sum_{n=1}^{N} m_n \int ds_n\, \beta(s_n) \frac{d}{ds_n} \delta^{(4)}(x - x_n(s_n))$$

$$= -\sum_{n=1}^{N} m_n \int ds_n\, \frac{d\beta(s_n)}{ds_n} \delta^{(4)}(x - x_n(s_n))$$

$\longrightarrow$ $$\frac{d\beta_n(s_n)}{ds_n} = 0$$ $\longrightarrow$ $$\beta_n(x^0) = -\beta_n^0$$ initial condition

Gravitational charge

$$Q := \int d^{d-1}x \sqrt{-g} J^0 = \sum_n m_n \beta_n^0 \int ds_n v_n^0(s_n) \delta(x^0 - x_n^0(s_n))$$

$$= \sum_n m_n \beta_n^0 \int dx_n^0(s_n) \delta(x^0 - x_n^0(s_n)) = \sum_n m_n \beta_n^0$$

Our proposal for the initial condition that $\rho(x(\tau_0))\beta(x(\tau_0))$ = constant leads to

$$m_n \beta_n^0 = \text{constant} = 1$$

(There are N independent conserved charges depending on initial conditions.)

$$\longrightarrow \qquad Q = N$$

Gravitational charge = particle number

This result looks trivial, but it is non-trivial to obtain this trivial but exact result from the gravitational charge constructed from the EMT.

IV. Gravitational charge in expanding Universe

SA and K. Kawana, "Entropy and its conservation in expanding Universe",
International Journal of Modern Physics A38 (2023) 2350072 [arXiv:2210.03323 [hep-th]].

SA, Y. Hidaka, K. Kawana and K. Shimada, work in progress.

Homogeneous and isotropic expanding Universe

The metric for homogeneous and isotropic expanding Universe is given by

$$ds^2 = -(dx^0)^2 + a^2(x^0)\tilde{g}_{ij}dx^i dx^j$$

The corresponding EMT is described by a perfect fluid as

$$T^0{}_0 = -\rho(x^0), \ T^i{}_j = P(x^0)\delta^i_j, \ T^0{}_j = T^i{}_0 = 0$$

The covariant conservation, $\nabla_\mu T^\mu{}_\nu = 0$, implies $\quad \dot{\rho} + (d-1)(\rho + P)\frac{\dot{a}}{a} = 0$

A matter energy is calculated from the EMT as

$$E(x^0) := -\int d^{d-1}x\,\sqrt{-g}\,T^0{}_0 = V_{d-1}a^{d-1}\rho, \quad V_{d-1} := \int d^{d-1}x\,\sqrt{\tilde{g}}.$$

The matter energy is not conserved as

$$\dot{E} = E\left[\frac{\dot{\rho}}{\rho} + (d-1)\frac{\dot{a}}{a}\right] = -(d-1)\frac{\dot{a}}{a}\frac{P}{\rho}E \neq 0$$

This is a good example for an absence of conserved energy in general relativity.

Gravitational charge

For simplicity, we consider a constant equation of state (EoS) as

$$P(x^0) = w\rho(x^0)$$

Einstein equation (or covariant conservation) determines the energy density as

$$\rho(x^0) = \rho_0 \left(\frac{a_0}{a(x^0)}\right)^{(d-1)(1+w)}$$

while equation for β can be solved as

$$\beta(x^0) = \beta_0 \left(\frac{a(x^0)}{a_0}\right)^{(d-1)w}$$

By combining these, the gravitational charge is given by

$$Q(x^0) = \int d^{d-1}x\,\sqrt{-g}\rho(x^0)\beta(x^0) = V_{d-1}\rho_0\beta_0 a_0^{d-1}$$

conserved

$$\sqrt{-g} \sim a(x^0)^{d-1}$$

What is a physical meaning ?

Thermodynamics for radiation-like matters

Let us consider an entropy for radiation-like matters assuming that the system is in equilibrium.

The fundamental relation of thermodynamics takes a form as

$$S(U,V) = UG\left(\frac{V}{U}\right)$$

An entropy $S(U, V)$ is concave (凸) function for U (energy) and V (volume).

By definition, we obtain

$$\frac{1}{T} := \frac{\partial S}{\partial U} = G(x) - xG'(x) \qquad \frac{P}{T} := \frac{\partial S}{\partial V} = G'(x)$$

The EoS that $P = w\rho = \frac{w}{x}$ implies $\quad (1+w)xG'(x) = wG(x)$

The solution is given by $\quad G(x) = c_0 x^{\frac{w}{1+w}} \quad$ with a positive constant c_0

We finally obtain the fundamental relation as

$$S(U,V) = c_0 U^{\frac{1}{1+w}} V^{\frac{w}{1+w}}$$

concavity (凸) requires $w \geq 0$

Confirmation of thermodynamic relations

An entropy density satisfies a thermodynamic relation as

$$s := \frac{S}{V} = c_0 \left(\frac{U}{V}\right)^{\frac{1}{1+w}} = \frac{\rho + P}{T}$$

$$\frac{1}{T} := \frac{\partial S}{\partial U} = \frac{c_0}{1+w}\left(\frac{V}{U}\right)^{\frac{w}{1+w}}$$

$$\frac{P}{T} := \frac{\partial S}{\partial U} = \frac{c_0 w}{1+w}\left(\frac{U}{V}\right)^{\frac{1}{1+w}}$$

An energy density satisfies a generalized Stefan-Boltzmann law as

$$\rho := \frac{U}{V} = \sigma_d T^{\frac{1+w}{w}}$$

$$\sigma_d := \left(\frac{c_0}{1+w}\right)^{\frac{1+w}{w}}$$

For radiation $w = \frac{1}{d-1}$ $\longrightarrow$ $\rho = \sigma_d T^d$

We confirm that S is indeed a thermodynamic entropy.

Assumption on equilibrium is justified.

Physical interpretation of the gravitational charge

If we take U and V for the expanding Universe as

$$U(x^0) := E(x^0) = V_{d-1}\rho_0 \frac{a_0^{(d-1)(1+w)}}{a(x^0)^{(d-1)w}} \qquad V(x^0) := a(x^0)^{d-1}\int d^{d-1}x\sqrt{\tilde{g}} = V_{d-1}a(x^0)^{d-1}$$

energy of Universe

volume of Universe

the entropy S becomes time-independent as

$$S(U,V) = c_0 U^{\frac{1}{1+w}} V^{\frac{w}{1+w}} = c_0 V_{d-1}^{\frac{1}{1+w}} \rho_0^{\frac{1}{1+w}} \frac{a_0^{d-1}}{a(x^0)^{(d-1)\frac{w}{1+w}}} V_{d-1}^{\frac{w}{1+w}} a(x^0)^{(d-1)\frac{w}{1+w}} = c_0 V_{d-1}\rho_0^{\frac{1}{1+w}} a_0^{d-1}$$

On the other hand, the gravitational charge is also time-independent as already seen:

$$Q(x^0) = V_{d-1}\rho_0\beta_0 a_0^{d-1}$$

Therefore we conclude

$$Q(x^0) = S(U,V) \qquad \text{with} \qquad c_0 = \beta_0 \rho_0^{\frac{w}{1+w}}$$

gravitational charge = entropy !

V. Conclusion

1. We propose a new conserved charge (gravitational charge) in general relativity.

2. Massive particles: gravitational charge = a number of particles

3. Expanding Universe: gravitational charge = entropy

    1. $\beta(x^0)$ is an (time-dependent) inverse temperature as

$$\beta(x^0) = \frac{1+w}{T}$$

4. (Future) More examples for gravitational charges will be considered to understand their physical meaning.

Thank you for your attention !