# Terrestrial Holography:

## Or, how one piece of space can encode a larger one

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### Bulk Entanglement Wedges

In AdS/CFT, given a boundary subregion $R$, we can reconstruct bulk operators that lie in the entanglement wedge $E(R)$ of $R$. Recent conjectures suggest that this should be extended to bulk subregions. We focus on time symmetric slices.

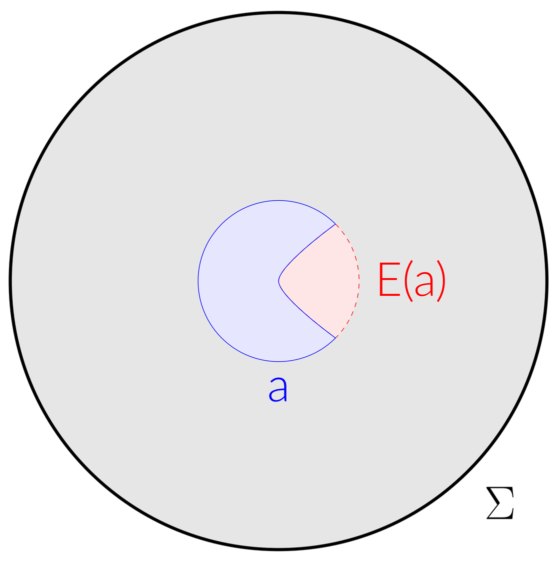



Figure 1. On a static slice $\Sigma$, a causally complete region a and its entanglement wedge E(a)

A wedge is a subset of spacetime such that $a = a''$. This wedge is uniquely determined by $\Sigma \cap \eth a \equiv \eth a$, the "corner" of $a$. A *bulk entanglement wedge* $E(a)$ of any wedge $a$ is:

1. $a \subset E(a)$
2. $E(a)$ has the smallest generalized entropy of all such wedges.

This reduces to the usual RT formula if $a$ approaches the boundary.

### Edge Modes of the Wedge: Defining a Subsystem Consistently

Even in $U(1)$ lattice gauge theory, the Hilbert space does not factorize by Gauss' Law.

This leads directly to the introduction of edge modes when restricting to a subsystem.

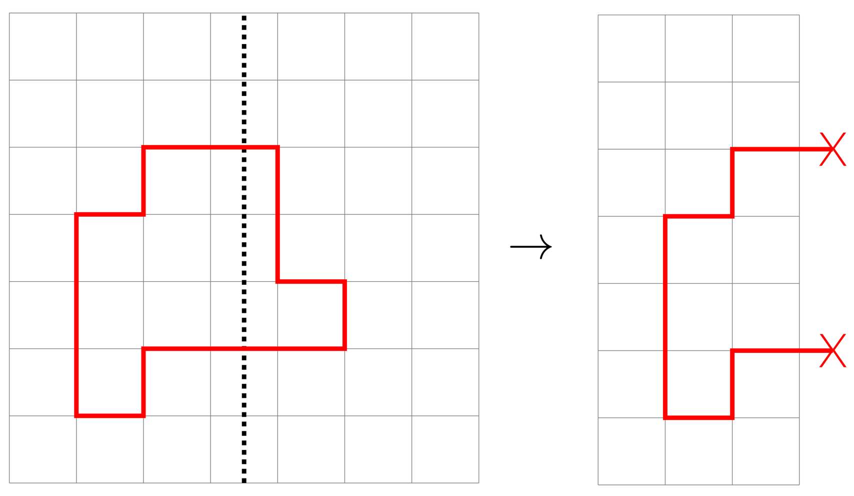


Figure 2. A Wilson line is cut open when taking a subsystem, leading to auxiliary edge mode degrees of freedom.

In gravity, edge modes correspond to the coordinate embedding $Y^a(\sigma)$ of $\eth a$.

### Extended Phase Space: No Boundary Conditions Needed

System of $a$, relative to a specific embedding $\phi : a \to \mathcal{M}$, is specified by an action:

$$\mathcal{S} = \int_a \phi^* L[g, Y] + \int_{\eth a} \phi^* \ell[g, Y].$$

A diffeomorphism which moves the embedding is described by *field dependent* $\chi$:

$$\sigma \to Y(\sigma) + \chi(\sigma).$$

Under such variations, $\delta$ *does not commute with* $\phi$

$$\delta \int_a \phi^* L = \int_a \phi^*(\delta L + \mathcal{L}_\chi L).$$

Let $\xi$ be a generator of a diffeomorphism which is potentially allowed to move $\eth a$.

$$H_\xi = H_\xi^{bulk} + H_\xi^{\eth a} = \int_\Sigma C_\xi + \int_{\eth a} Q_\xi$$

Because we are using *extended* phase space:

$$\{H_\xi, H_\eta\} = -H_{[\xi,\eta]}$$

*independent of boundary conditions*

### The Algebra of the Wedge Before Imposing The Constraints

Maximal subalgebra of $\mathfrak{diff}(M)$ which supports nonzero charges:

$$\mathfrak{ucs} = \mathfrak{diff}(\eth a) \ltimes \mathfrak{gl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$$

Given a generator $\xi \in \mathfrak{g}$, define $g = \exp\{\xi\} \in G$. This leads to a representation of $G$:

- $V_g = \exp\left(iH_\xi^{bulk}\right)$: diffeomorphisms on $a$
- $U_g = \exp(i\eth a_\xi)$: diffeomorphisms on edge modes
- $W_g$: corner symmetries (physical transformations, not gauged)
- Kinematical operators: $\mathcal{O}_{kin} = \{\mathcal{A}_a, VU(G), W(G)\}''$
- Kinematical Hilbert space: $\mathcal{H}_a \otimes L^2(G)$

### References

[1] Raphael Bousso and Geoff Penington. Entanglement wedges for gravitating regions. *Physical Review D*, 107(8), apr 2023.
[2] Luca Ciambelli, Robert G. Leigh, and Pin-Chun Pai. Embeddings and integrable charges for extended corner symmetry. *PRL*, 128(17), Apr 2022.
[3] William Donnelly and Laurent Freidel. Local subsystems in gauge theory and gravity. *Journal of High Energy Physics*, 2016(9), sep 2016.
[4] Kristan Jensen, Jonathan Sorce, and Antony Speranza. Generalized entropy for general subregions in quantum gravity, 2023.

### The Algebra of the Wedge After Imposing The Constraints

Gauge invariance under diffeomorphisms implies commuting with the constraint.

$$\mathcal{E}_a = \{\hat{a} \in \mathcal{O}_{kin} | \forall g \in G, (VU)_g \hat{a} (VU)_g^\dagger = \hat{a}\}''$$

More explicitly:

$$\hat{a} = \int d\mu_g V_g^\dagger a V_g \otimes |g\rangle\langle g|$$

$$(VU)_h \hat{a} (VU)_h^\dagger = \int d\mu_g V_{gh^{-1}}^\dagger a V_{gh^{-1}} |gH^{-1}\rangle\langle gH^{-1}| = \int d\mu_g V_g^\dagger a V_g \otimes |g\rangle\langle g| = \hat{a}$$

$$(VU)_h W_g (VU)_h^\dagger = W_g$$

Dynamical algebra is $\mathcal{E}_a = \{\hat{\mathcal{A}}_a, W(G)\}''$.

This holds to all orders in perturbation theory.

### Computing The Entropy of a State Using The Replica Trick

One can show $\mathcal{E}_a$ defined above is type II, so traces and density matrices are well defined. We compute the entropy of a state $\hat{\Phi} = |\Psi\rangle \otimes \psi(g)$ in this algebra. Define $|\Psi_g\rangle = V_g |\Psi\rangle$. Using the replica trick:

$$S(\rho_{\hat{\Psi}}) = \lim_{n \to 1} \frac{1}{1-n} \log \operatorname{tr}_{\mathcal{E}^{\otimes n}}(\rho_{\hat{\Psi}}^{\otimes n} \tau_{\mathcal{E}})$$

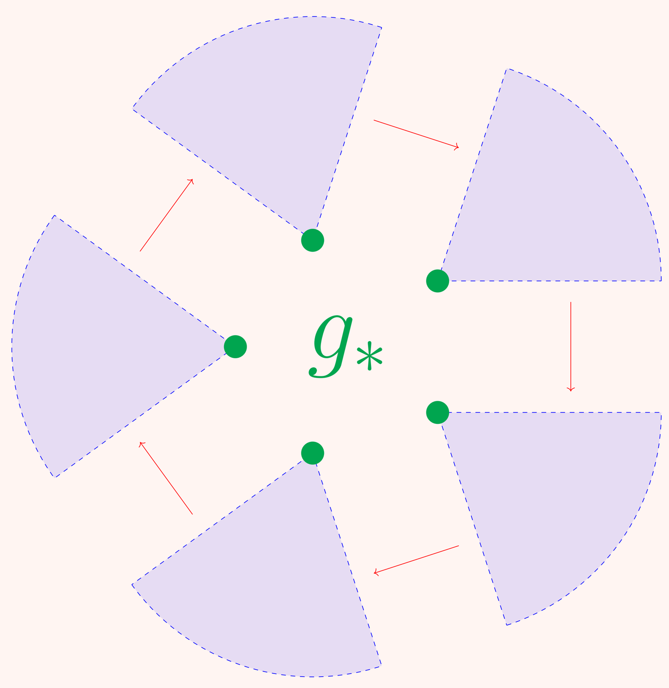
$$= \lim_{n \to 1} \frac{1}{1-n} \log \left[ \int \prod_{i=1}^n \left[d\mu_i |\psi(g_i)|^2\right] \langle \Psi^n | V_{\vec{g}}^\dagger \tau_{\mathcal{A}} V_{\vec{g}} | \Psi^n \rangle \right]$$

Make the following assumptions:

- Replica symmetry (take diagonal of $n$ $d\mu_g$ integrals)
- Saddle approximation (dominated by some $g_*$)
- Hamiltonians $H_g$ are bounded below ($a \subset \phi_g(a)$)

$$S(\rho_{\hat{\Psi}}) = -\partial_n \left[\langle \Psi_{g_*}^n | \tau_{\mathcal{A}} | \Psi_{g_*}^n \rangle\right]_{n=1} - \int d\mu_g |\psi(g)|^2 \log |\psi(g)|^2$$

### Path Integral Interpretation of This Calculation




Compute Replica Trick for Minimal Embedding (Saddle Approximation)

$$S(\rho_{\hat{\Psi}}) \approx -\partial_n \left[\frac{\langle \Psi_{g_*}{}^n | \tau_{\mathcal{A}} | \Psi_{g_*}{}^n \rangle}{\langle \Psi_{g_*}{}^n | \Psi_{g_*}{}^n \rangle}\right]_{n=1}$$

This Reduces to the Usual Formula:

$$S(\rho_{\hat{\Psi}}) = \left[\frac{\partial I_n}{\partial n}\bigg|_{n=1} - I_1\right](g_*) = S_{gen}(\phi_{g_*}(a))$$

$g_*$ represents embedding which minimizes $S_{gen}$

$$E(a) = \phi_{g_*}(a) \supset a$$

### Discussion

- Bulk subregions $a$ have an entanglement wedge $E(a)$ which is generally larger than $a$.
- Extended phase space allows for integrable charges independent of boundary conditions.
- This allows for quantization of the charges even if the corner $\eth a$ moves under a diffeomorphism.
- We use these charges to construct the gauge invariant operators $\mathcal{E}_a$.
- We compute the entropy of an arbitrary quantum-classical state $\hat{\Psi} = \Psi\psi(g)$.
- These form a basis: there is no loss of generality in a saddle point analysis.
- We find a renormalized entropy that agrees with the generalized entropy.
- This saddle represents the entanglement wedge $E(a) = \phi_{g_*}(a)$ of $a$.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