Everything Everywhere All at Once:

Holographic Entropy Inequalities, Entanglement Wedge Nesting, Topology of Error Correction, Black Holes, Cubohemioctahedron (and maybe the Toric Code)

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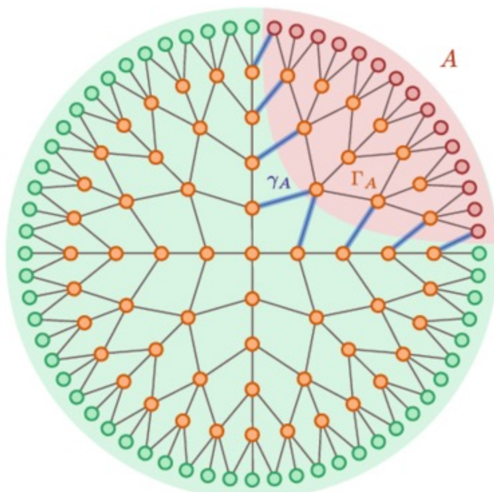
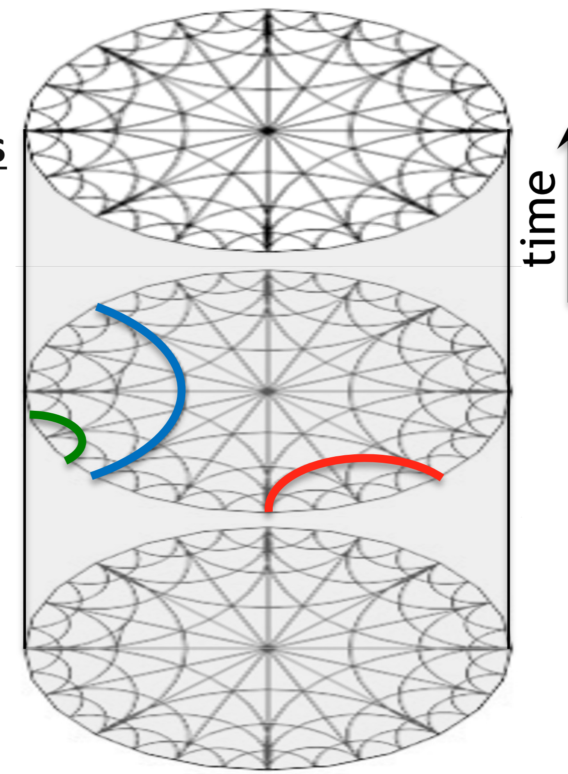
Minimal cut prescription…

- …for computing von Neumann entropies of subsystems:



(image from Cheng et al.,
"**Random Tensor Networks** with Non-trivial Links")

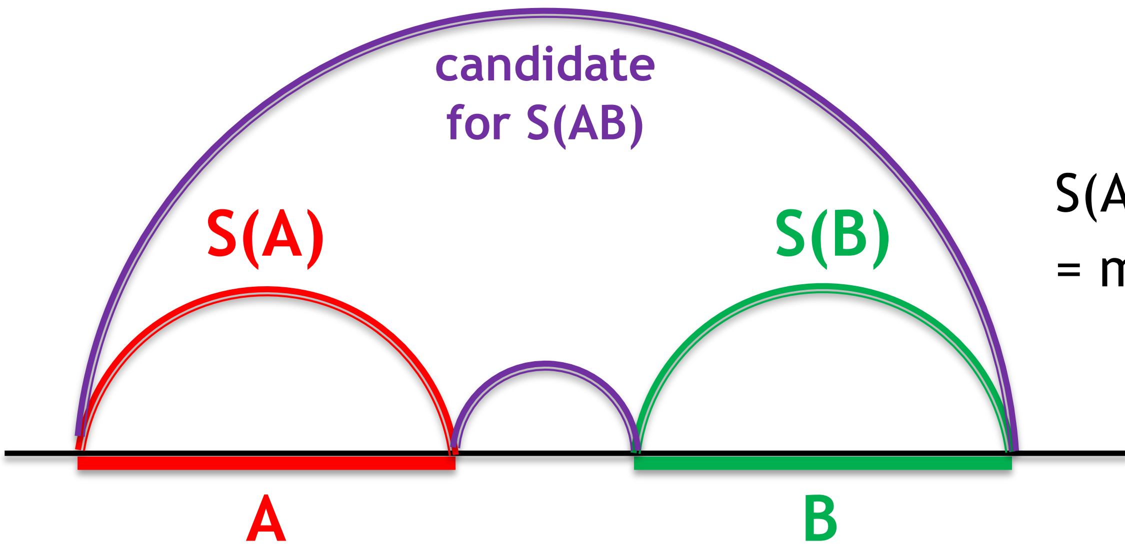
States dual to semiclassical spacetimes
in holographic duality
(Ryu-Takayanagi, 2006)



- Not all states obey it
- But the ones that do are an important class of states
- Today's talk is about this class of quantum states

Why min-cut imposes entropic constraints

- Use the minimal cut prescription to compute S(A), S(B), S(AB):



S(AB)
= min {S(A) + S(B),
the other candidate}

- Either way: S(AB) =< S(A) + S(B)
- Holographic "proof" of subadditivity

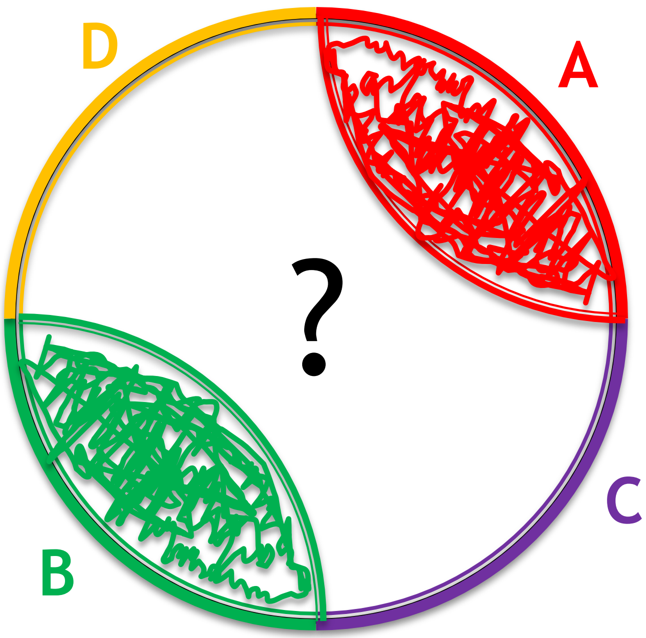
- Many other relations also follow **from the min-cut prescription**

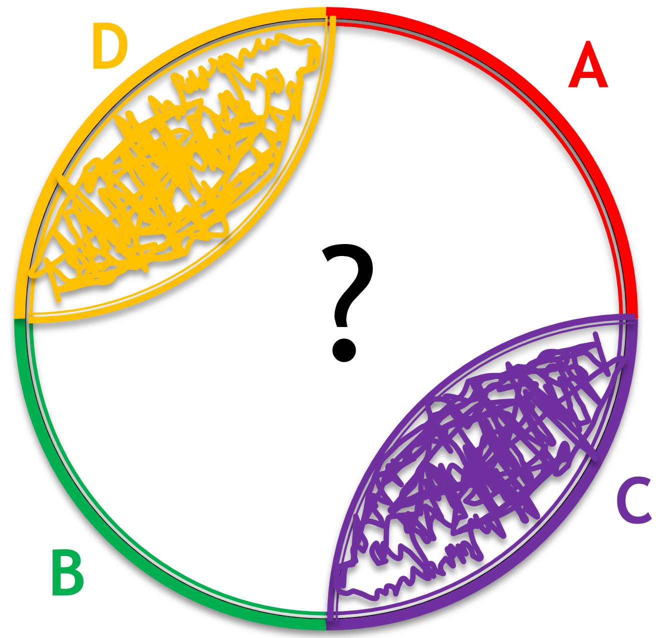
This talk is about these "holographic entropy inequalities."

They are **necessary conditions** for min-cut to work.

Holographic Erasure Correction

Almheiri, Dong, Harlow (2014)





- S(AB) = S(A) + S(B)
- central region protected against erasure of A, B

- S(CD) = S(C) + S(D)
- central region protected against erasure of C, D

<u>Interpreting our inequalities:</u> Saturation of a holographic inequality tells us which bulk regions are protected against which boundary erasure.

Known Entropy Inequalities

Cuenca (2019)

Table 1: Representatives for each of the 8 inequality orbits of the holographic entropy cone $\mathcal{C}_5$ for 5 regions. Respectively, their orbit lengths are 15, 20, 45, 72, 10, 60, 60 and 90, thus defining 372 facets for $\mathcal{C}_5$ in a 31-dimensional entropy space.

1. $S_A + S_B \geq S_{AB}$

2. $S_{AB} + S_{AC} + S_{BC} \geq S_A + S_B + S_C + S_{ABC}$

3. $S_{ABC} + S_{ADE} + S_{BCDE} \geq S_A + S_{BC} + S_{DE} + S_{ABCDE}$

4. $S_{ABC} + S_{ABD} + S_{ACE} + S_{BDE} + S_{CDE} \geq S_{AB} + S_{AC} + S_{BD} + S_{CE} + S_{DE} + S_{ABCDE}$

5. $S_{ABC} + S_{ABD} + S_{ABE} + S_{ACD} + S_{ACE} + S_{ADE} + S_{BCE} + S_{BDE} + S_{CDE} \geq S_{AB} + S_{AC} + S_{AD} + S_{BE} + S_{CE} + S_{DE} + S_{BCD} + S_{ABCE} + S_{ABDE} + S_{ACDE}$

6. $3S_{ABC} + 3S_{ABD} + S_{ABE} + S_{ACD} + 3S_{ACE} + S_{ADE} + S_{BCD} + S_{BCE} + S_{BDE} + S_{CDE} \geq 2S_{AB} + 2S_{AC} + S_{AD} + S_{AE} + S_{BC} + 2S_{BD} + 2S_{CE} + S_{DE} + 2S_{ABCD} + 2S_{ABCE} + S_{ABDE} + S_{ACDE}$

7. $2S_{ABC} + S_{ABD} + S_{ABE} + S_{ACD} + S_{ADE} + S_{BCE} + S_{BDE} \geq S_{AB} + S_{AC} + S_{AD} + S_{BC} + S_{BE} + S_{DE} + S_{ABCD} + S_{ABCE} + S_{ABDE}$

8. $S_{AD} + S_{BC} + S_{ABE} + S_{ACE} + S_{ADE} + S_{BDE} + S_{CDE} \geq S_A + S_B + S_C + S_D + S_{AE} + S_{DE} + S_{BCE} + S_{ABDE} + S_{ACDE}$

BC and Yunfei Wang (2022)

$$
\begin{aligned}
& S_{ABDE} + S_{ABDF} + S_{ABEG} + S_{ADEF} + S_{ADEG} \\
& + S_{ACDE} + S_{ACDF} + S_{ACEG} + S_{BDEF} + S_{BDEG} \\
& + S_{BCDE} + S_{BCDF} + S_{BCEG} + S_{CDEF} + S_{CDEG} \\
& \geq \\
& S_{ABC} + S_{ADE} + S_{ADF} + S_{AEG} + S_{BDE} + S_{BDF} + S_{BEG} + S_{CDE} + S_{CDF} + S_{CEG} \\
& + S_{ABDEF} + S_{ABDEG} + S_{ACDEF} + S_{ACDEG} + S_{BCDEF} + S_{BCDEG}
\end{aligned}
\tag{2.1}
$$

and 373 other ones --- Cuenca, Hubeny, Jia (last week)

New Inequalities

Toric inequalities are defined for m and n, which are both odd. They take the following form:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}} \tag{1.6}$$

We characterize and explore these inequalities in Section 2.2, then prove them in Section 5.1. We exemplified how terms of (1.6) can be arranged on a discretized torus in inequality (1.3). As we explain in Section 2.2.1, that spatial arrangement has further, even more compelling features.

Projective plane inequalities are defined for $m = n$. They read:

$$\frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{m}\left(S_{A_i^{(k)} B_{i+k}^{(m-k)}} + S_{A_i^{(k)} B_{i+k-1}^{(m-k)}}\right) \geq \sum_{k=1}^{m}\sum_{i=1}^{m} S_{A_i^{(k-1)} B_{i+k-1}^{(m-k)}} + S_{A_1^{(m)}} \tag{1.7}$$

Notation:

$$A_i^{(k)} = A_i A_{i+1} \dots A_{i+k-1} \quad \text{and} \quad B_j^{(l)} = B_j B_{j+1} \dots B_{j+l-1}$$

$$A_i^{\pm} \equiv A_i^{((m\pm1)/2)} \quad \text{and} \quad B_j^{\pm} \equiv B_j^{((n\pm1)/2)}$$

To give you a feeling:

$$
\begin{aligned}
& S_{A_1A_2A_3B_1} + S_{A_3A_4A_5B_1} + S_{A_5A_1A_2B_1} + S_{A_2A_3A_4B_1} + S_{A_4A_5A_1B_1} \\
+ & S_{A_1A_2A_3B_2} + S_{A_3A_4A_5B_2} + S_{A_5A_1A_2B_2} + S_{A_2A_3A_4B_2} + S_{A_4A_5A_1B_2} \\
+ & S_{A_1A_2A_3B_3} + S_{A_3A_4A_5B_3} + S_{A_5A_1A_2B_3} + S_{A_2A_3A_4B_3} + S_{A_4A_5A_1B_3} \\
& \geq \\
& S_{A_4A_5B_1} + S_{A_1A_2B_1} + S_{A_3A_4B_1} + S_{A_5A_1B_1} + S_{A_2A_3B_1} \\
+ & S_{A_4A_5B_2} + S_{A_1A_2B_2} + S_{A_3A_4B_2} + S_{A_5A_1B_2} + S_{A_2A_3B_2} \\
+ & S_{A_4A_5B_3} + S_{A_1A_2B_3} + S_{A_3A_4B_3} + S_{A_5A_1B_3} + S_{A_2A_3B_3} \\
& + S_{A_1A_2A_3A_4A_5}
\end{aligned}
\tag{2.9}
$$

The inequalities live on two-dimensional manifolds

LHS terms are faces

$$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}}$$

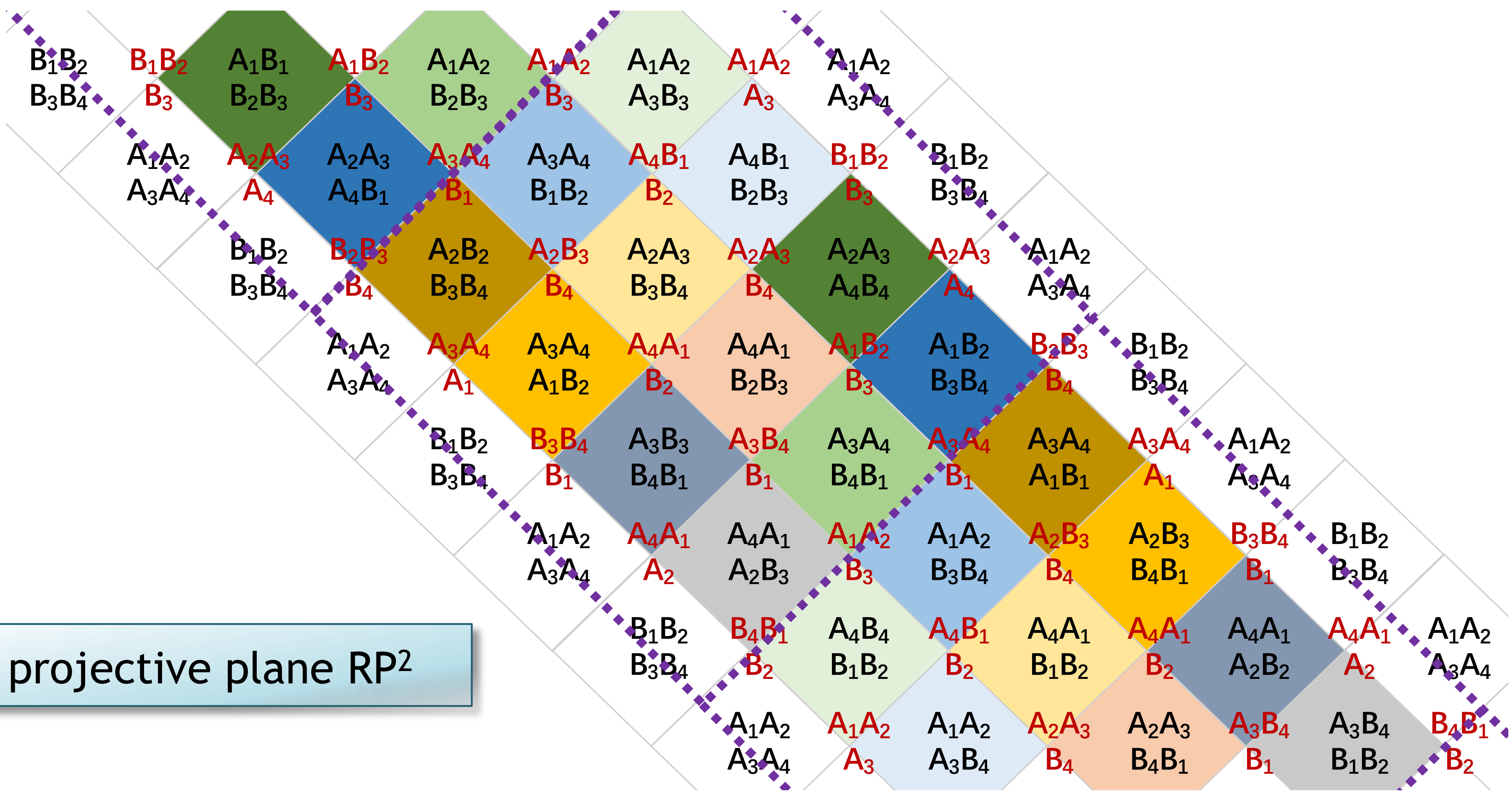
RHS terms are nodes

The inequalities live on two-dimensional manifolds

faces

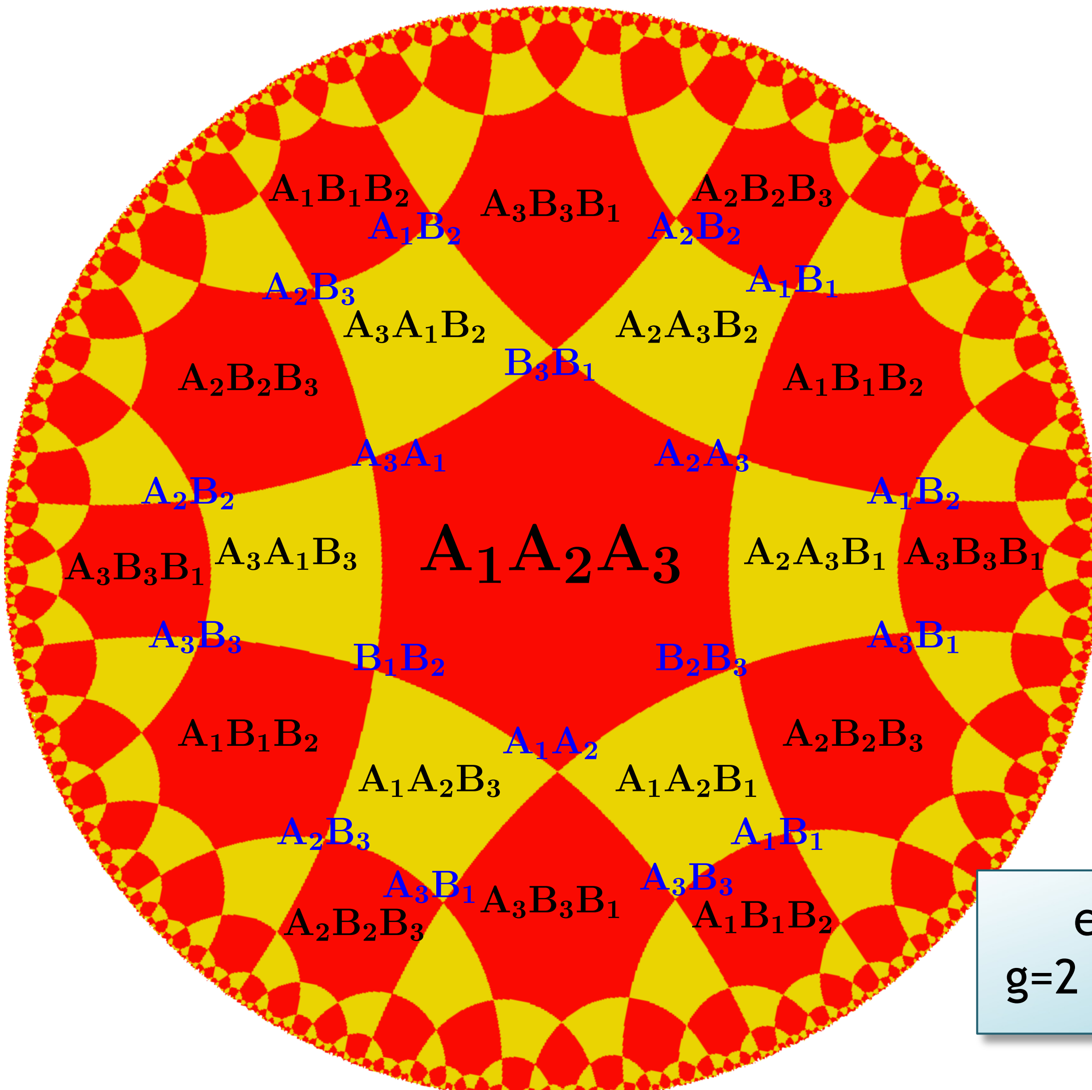
$$\frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{m}\left(S_{A_i^{(k)}B_{i+k}^{(m-k)}} + S_{A_i^{(k)}B_{i+k-1}^{(m-k)}}\right) \geq \sum_{k=1}^{m}\sum_{i=1}^{m} S_{A_i^{(k-1)}B_{i+k-1}^{(m-k)}} + S_{A_1^{(m)}}$$

nodes



projective plane RP^2

One other inequality defines a polytope called "cubohemioctahedron"



faces

$$
\begin{aligned}
&3S_{A_1B_1B_2} + 3S_{A_2B_2B_3} + 3S_{A_3B_3B_1} \\
&+ S_{A_1A_2B_1} + S_{A_2A_3B_2} + S_{A_3A_1B_3} \\
&+ S_{A_1A_2B_3} + S_{A_2A_3B_1} + S_{A_3A_1B_2} \\
&+ S_{A_1A_2A_3} \\
&\geq \\
&S_{A_1A_2} + S_{A_2A_3} + S_{A_3A_1} \\
&+ 2S_{A_1B_1} + 2S_{A_2B_2} + 2S_{A_3B_3} \\
&+ 2S_{A_1B_2} + 2S_{A_2B_3} + 2S_{A_3B_1} \\
&+ S_{B_1B_2} + S_{B_2B_3} + S_{B_3B_1}
\end{aligned}
$$

nodes

embedded on the g=2 Riemann surface

Spatial organization of terms fixed by Entanglement Wedge Nesting

LHS terms are faces

$$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}}$$

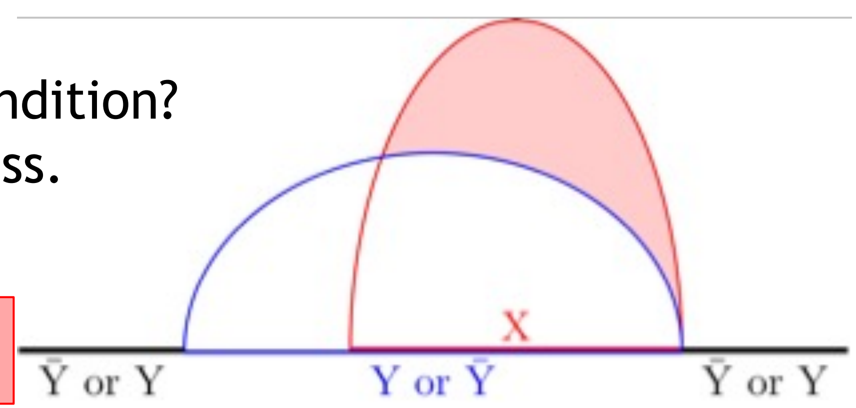
RHS terms are nodes

A_2A_3 A_4B_1	A_2A_3 B_1	A_1A_2 A_3B_1	A_1A_2 B_1	A_5A_1 A_2B_1
A_5A_1 B_3	A_4A_5 A_1B_3	A_4A_5 B_3	A_3A_4 A_5B_3	A_3A_4 B_3
A_2A_3 A_4B_2	A_2A_3 B_2	A_1A_2 A_3B_2	A_1A_2 B_2	A_5A_1 A_2B_2

- A node (RHS term) is incident to a face (LHS term) if and only if they **do not cross**:

They are disjoint or one is a subset of the other

- What is the significance of this condition? Their minimal cuts also do not cross.



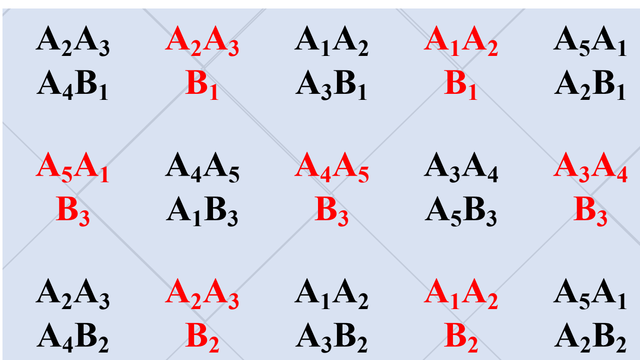
Entanglement Wedge Nesting: This cannot happen!

Connectivity in the graph

LHS terms are faces

$$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}}$$

RHS terms are nodes



$$\sum_{edges} CMI(edge) \geq S(A_1A_2 \ldots A_m)$$

Why do we have this structure?

- Look at one edge in this graph:

  $+\ S(A_1A_2A_3B_2) + S(A_3A_4A_5B_3)$
  $-\ S(A_1A_2B_2) - S(A_4A_5B_3)$

- When these regions form a pure state, this is a conditional mutual information:

  $+\ (A_1A_2B_2)(A_3) + (A_1A_2B_2)(B_1)$
  $-\ (A_1A_2B_2) - (A_1A_2B_2)(A_3)(B_1)$

- Every edge is non-negative by strong subadditivity!

Toward an interpretation

LHS terms are faces

$$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}}$$

RHS terms are nodes

$$\sum_{edges} CMI(edge) \geq S(A_1 A_2 \dots A_m)$$

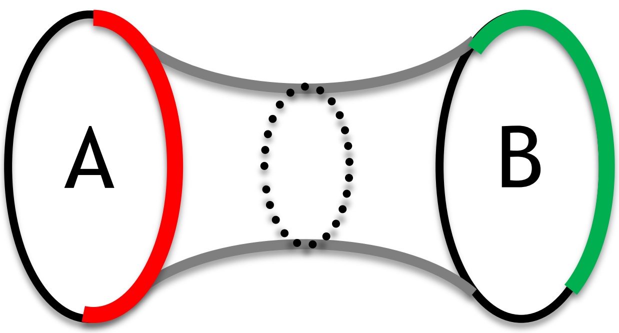
Why? I'll try to give an answer on two levels:

- What does it mean,
  or what can it mean,
  in holographic duality?
- What makes it true?
  Topology!

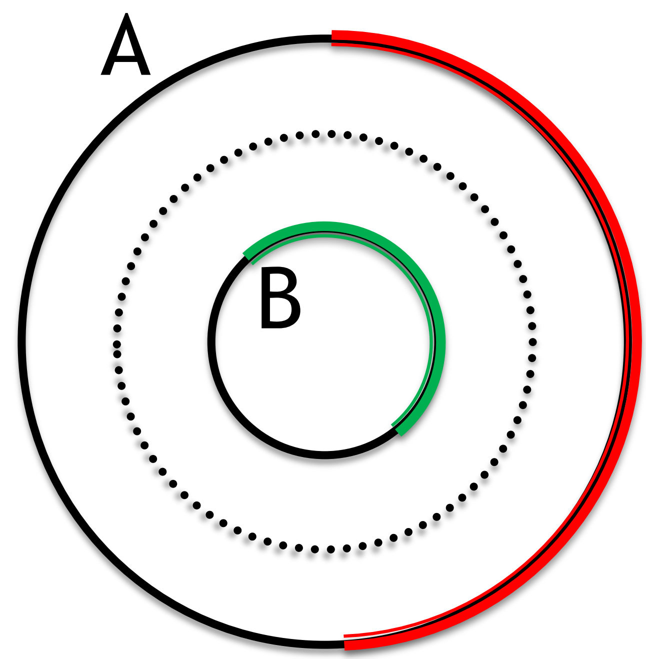
What does it mean in holography?

- Consider a pure state on many A_is and B_js
- Picture them as living on two circles:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}}$$



- Holographically, this is a two-sided black hole. The dotted thing is a black hole horizon. This is the term $S(A_1A_2\ldots A_m)$ on RHS.
- Let's redraw the same picture "from above"
- Let's look at any term like $S(A_i^+ B_j^-)$
- Generally, it can be in one of several phases.



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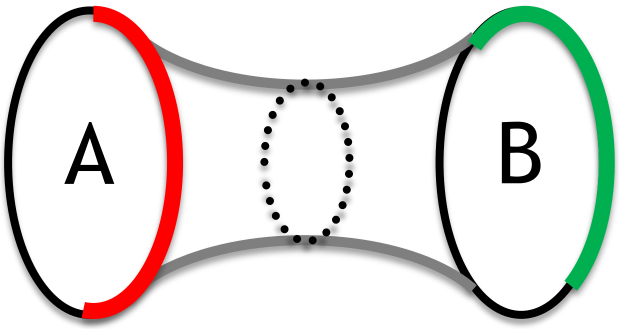

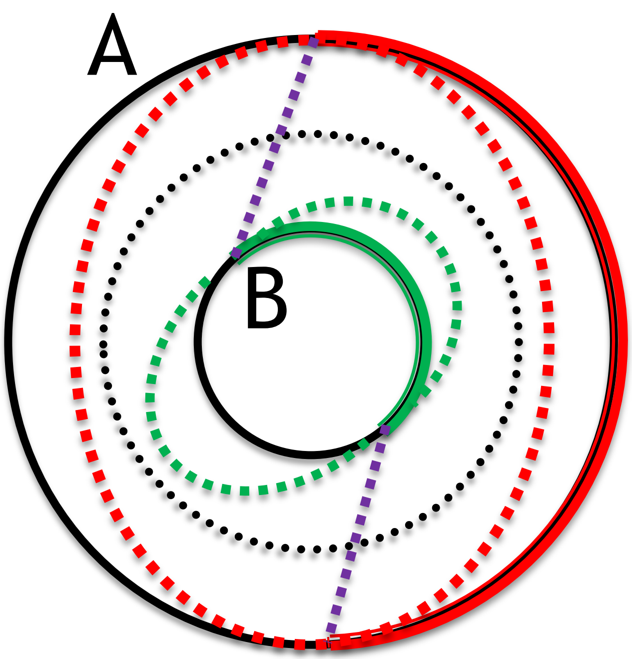

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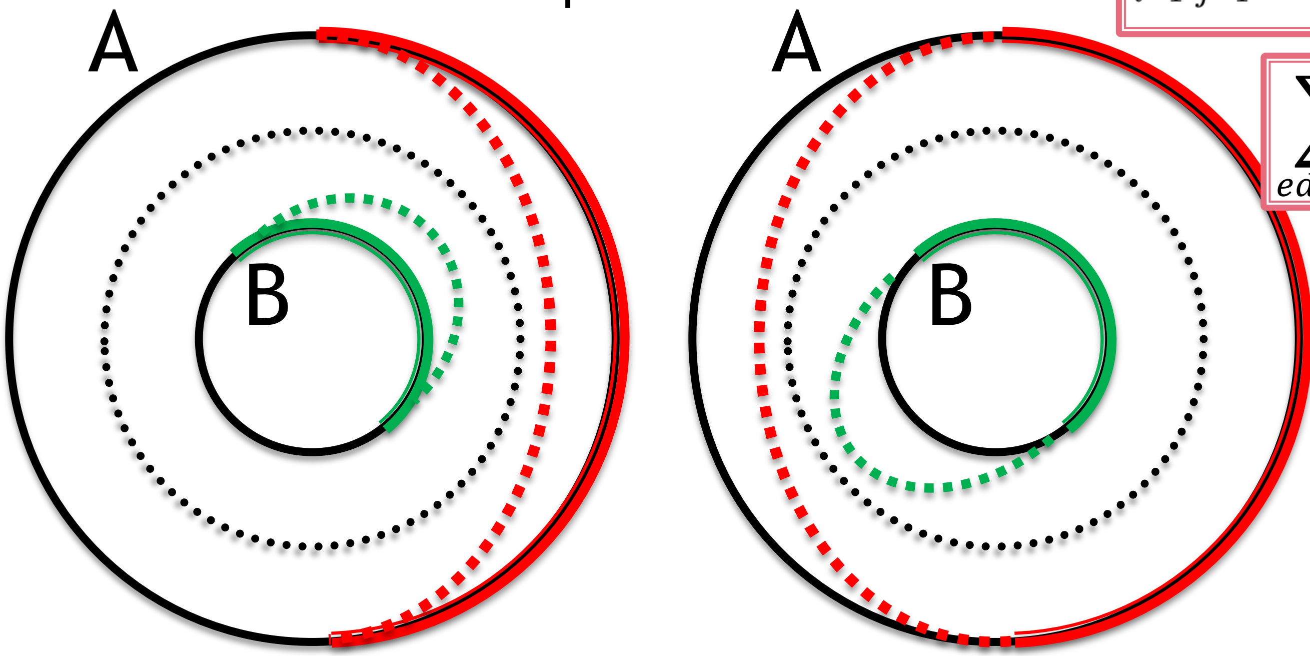
What does it mean in holography?

$$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}}$$

$$\sum_{edges} CMI \geq S(A_1 A_2 \ldots A_m)$$

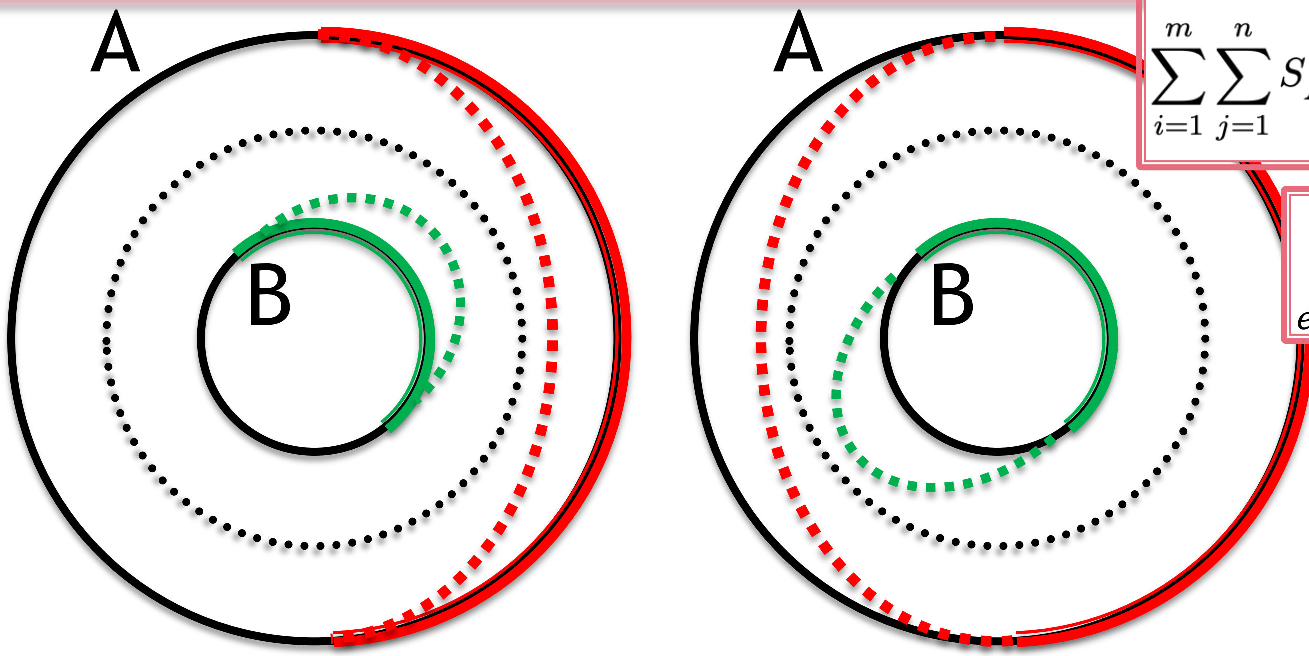
- Suppose $S(A_i^+ B_j^-)$ can only be in one of these two phases:



- Each CMI(edge) looks like:
  $CMI = S(A_1A_2B_2A_3) + S(A_1A_2B_2B_1) - S(A_1A_2B_2) - S(A_1A_2B_2A_3B_1)$
- We vary the right endpoint of A_i^+ and left endpoint of B_j^-. This vanishes in each phase!

The only non-zero CMI(edge) occur when terms live in different phases.

What does it mean in holography?



$$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}}$$

$$\sum_{edges} CMI \geq S(A_1 A_2 \dots A_m)$$

ongoing work with Bowen Chen, Ricardo Espindola, Dachen Zhang

- Take a continuum limit with MANY A_is and B_js.
- The inequality localizes on the phase transition. Many different behaviors are possible, depending on phase structure.
- For example, one possibility is:

$$\oint_{\text{phase transition}} d(\text{right endpt of } A^+)\,\frac{\partial S}{\partial(\text{right endpt of } A^+)} \geq S(A_1 A_2 \dots A_m)$$

What does it mean in holography?

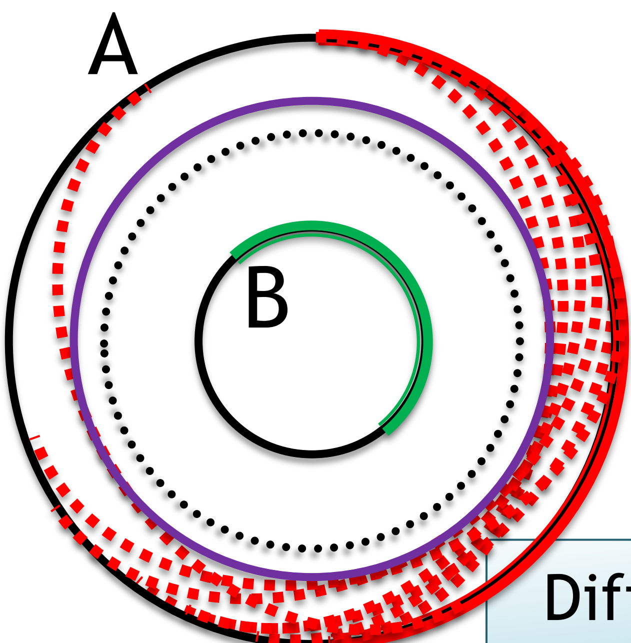
$$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}}$$

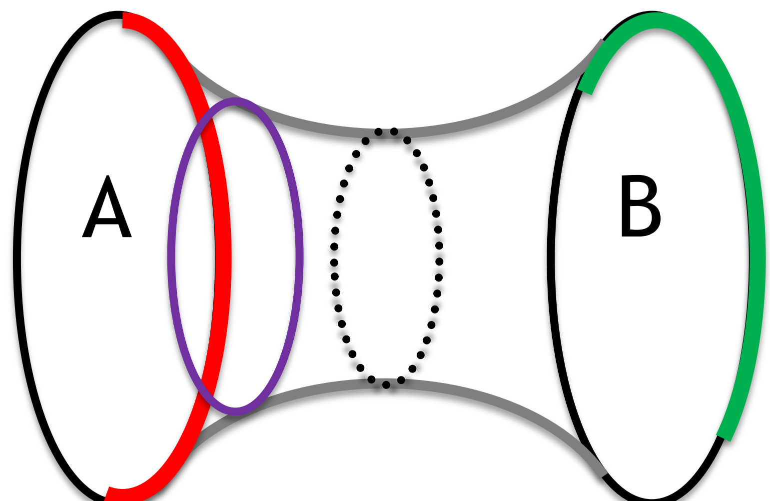
$$\sum_{edges} CMI \geq S(A_1 A_2 \ldots A_m)$$

$$\oint_{\text{phase transition}} d(\text{right endpt of } A^+) \frac{\partial S}{\partial(\text{right endpt of } A^+)} \geq S(A_1 A_2 \ldots A_m)$$

- This has a nice geometric interpretation: (differential entropy)

Purple curve >= Dotted curve





Differential Entropy >= Black Hole Entropy

What makes the inequalities true?

- I will try to convince you that the answer is:

  topology

  It is the topology of an error correcting scheme.

- I will sketch the proof of the inequalities.
  It is entirely topological in character.

Proof: General strategy

Bao, Nezami, Ooguri, Stoica, Sully (2015)

- Take an inequality, for example:
  S(AB) + S(BC) + S(AC) >= S(A) + S(B) + S(C) + S(ABC)

	x			$y = f(x)$			
	AB	BC	AC	A	B	C	ABC
O	0	0	0	0	0	0	0
	0	0	1	0	0	0	1
	0	1	0	0	0	0	1
C	0	1	1	0	0	1	1
	1	0	0	0	0	0	1
A	1	0	1	1	0	0	1
B	1	1	0	0	1	0	1
	1	1	1	0	0	0	1

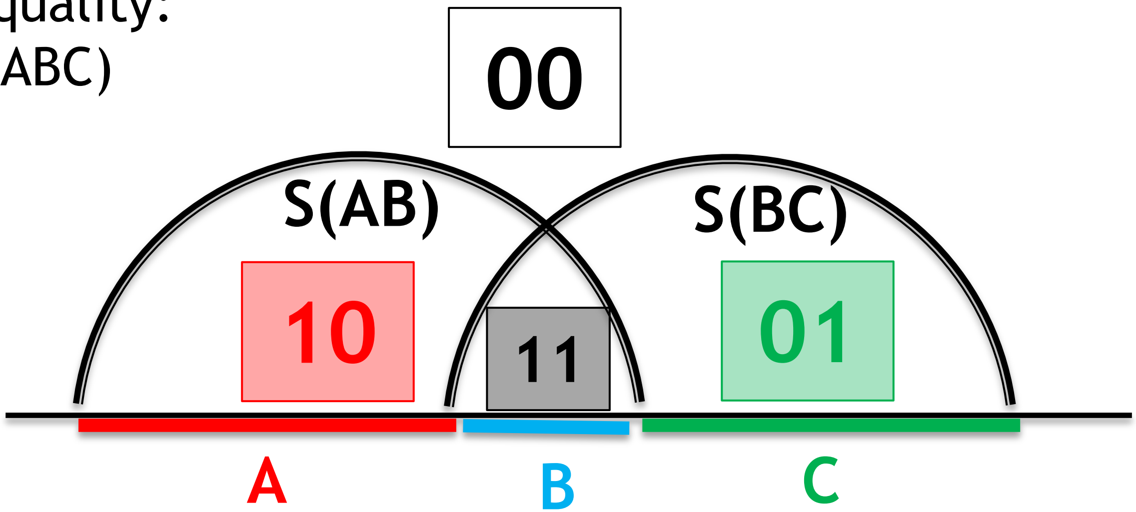
- Construct a table like so, such that
  $|x - x'| >= |f(x) - f(x')|$
- Every region indicator must map to a region indicator.

Why is this a proof?

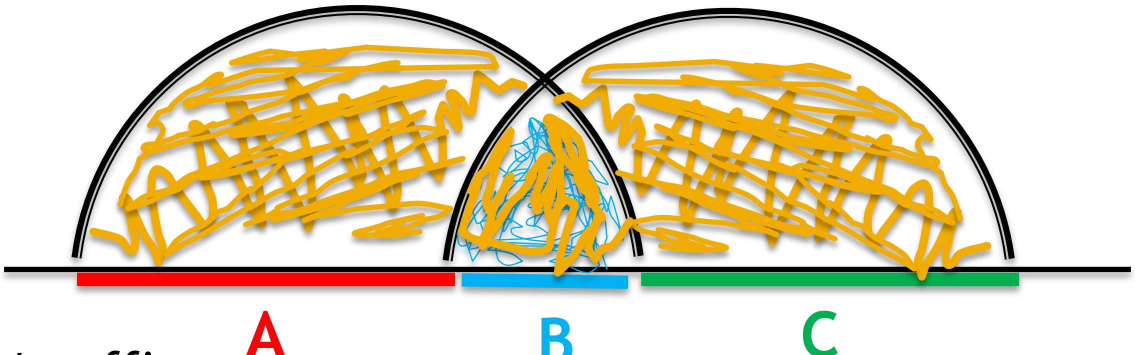
How the strategy works

- Take an even simpler inequality:
  S(AB) + S(BC) >= S(B) + S(ABC)

	x		$y = f(x)$	
	AB	BC	B	ABC
O	0	0	0	0
C	0	1	0	1
A	1	0	0	1
B	1	1	1	1



- Area x | x' contributes
  |x – x'| on LHS
  |f(x) – f(x')| on RHS



- So |x – x'| >= |f(x) – f(x')| suffices

How to find contractions for these inequalities?

Toric inequalities are defined for m and n, which are both odd. They take the following form:

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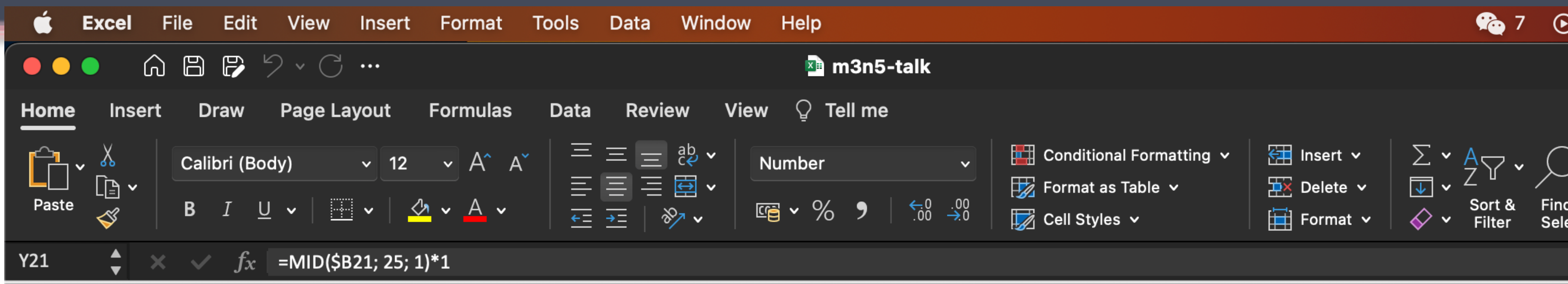
Notation:

$$A_i^{(k)} = A_i A_{i+1} \dots A_{i+k-1} \quad \text{and} \quad B_j^{(l)} = B_j B_{j+1} \dots B_{j+l-1}$$

$$A_i^{\pm} \equiv A_i^{((m\pm1)/2)} \quad \text{and} \quad B_j^{\pm} \equiv B_j^{((n\pm1)/2)}$$

By brute force?



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH	AI
1				ABCF	CDEF	EABF	BCDF	DEAF	ABCG	CDEG	EABG	BCDG	DEAG	ABCO	CDEO	EABO	BCDO	DEAO		DEF	ABF	CDF	EAF	BCF	DEG	ABG	CDG	EAG	BCG	DEO	ABO	CDO	EAO	BCO	ABCDE
2	9573	000	unique	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	32743	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1		0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
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7	31523	000	unique after	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1		0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0
8	29820	000	unique after	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0		0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
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10	32736	000	unique after	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
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19	31254	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1		0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0
20	28644	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0		0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
21	28486	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1		0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0
22	31110	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0		0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0
23	30952	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1		0	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0
24	27496	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0		0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0
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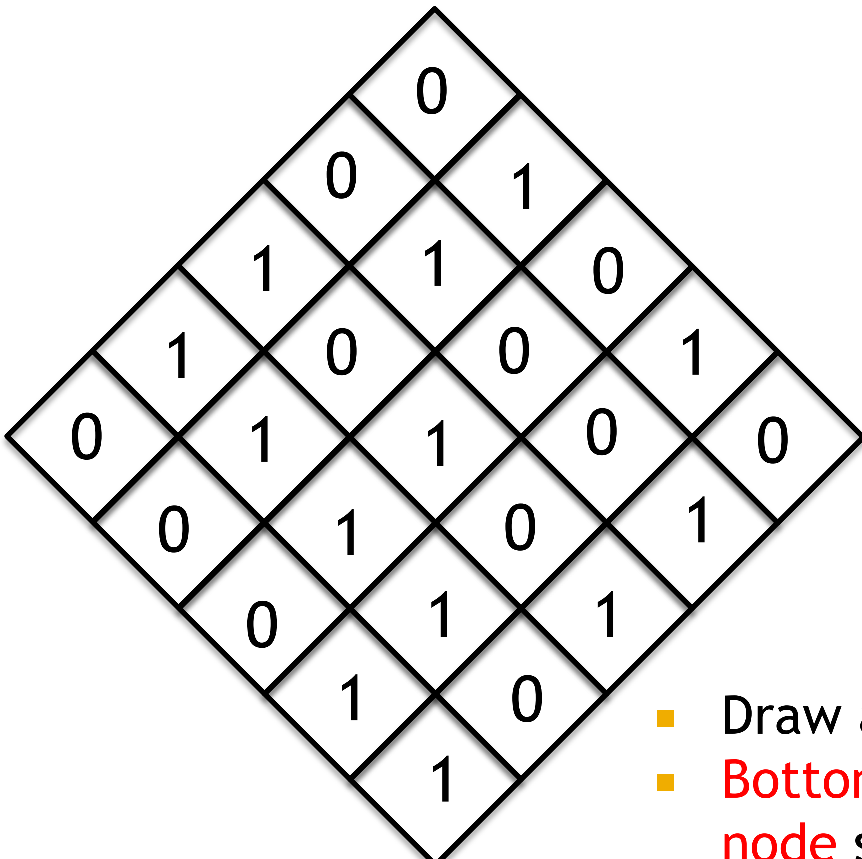
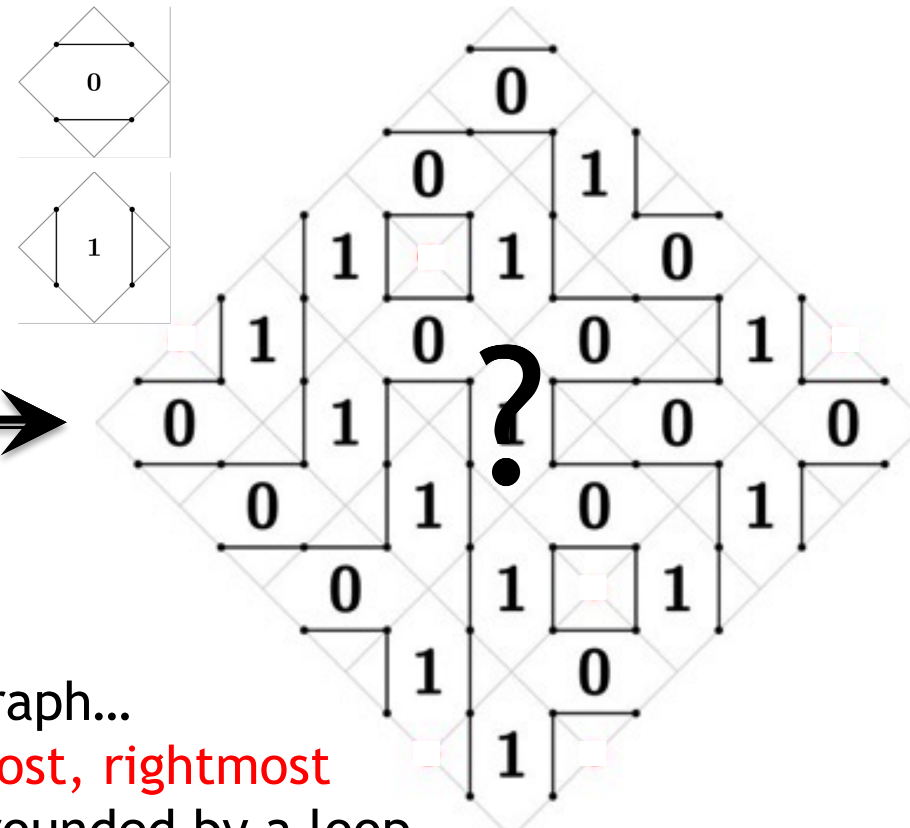
33	2	000	*	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1		0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0
34	32769	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0		0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
35	9352	000	unique	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	5442	000	unique	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	27141	000	unique after	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1		0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
38	8964	000	unique	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Explicitly by exploiting topology…

faces $$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}}$$ nodes



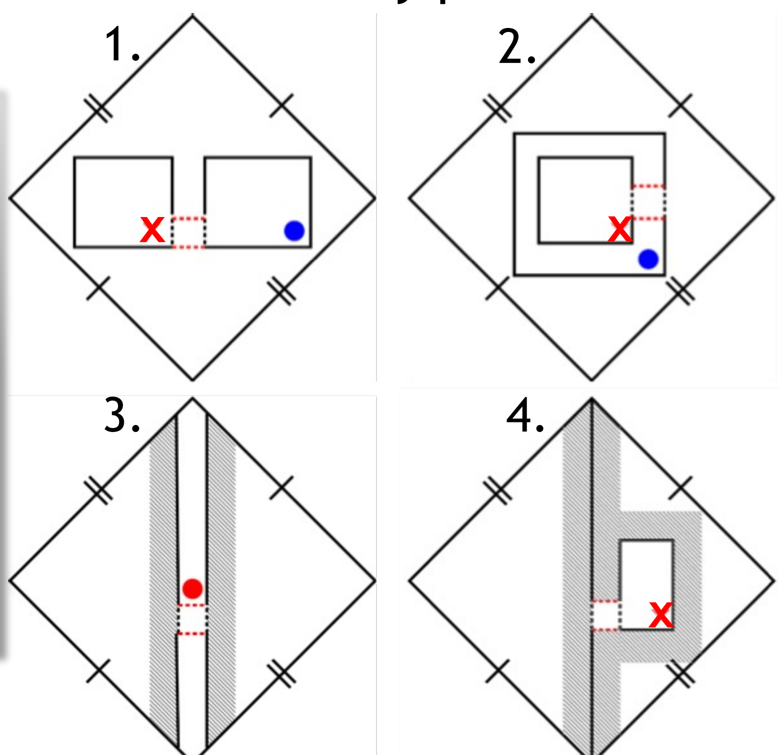


- Draw a graph…
- Bottommost, rightmost node surrounded by a loop gets a 1, all others get a 0

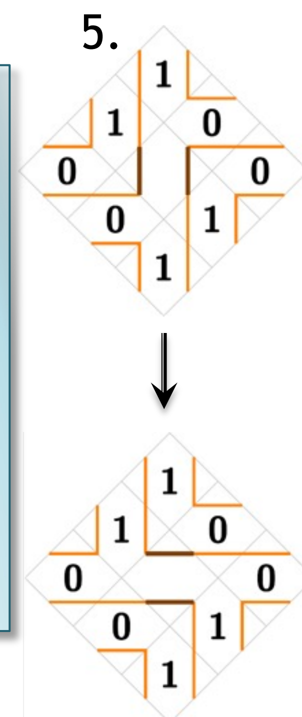
Why is this a contraction?

- A loop that wraps the torus is not a boundary of a region on the torus.
- To show |x – x'| >= |f(x) – f(x')|
  consider the effect of flipping any one bit in x.
- These are the only possibilities:

2 loops <-> 1 loop



1 loop <-> 1 loop



Nodes where f(x) changes are marked in **red**.

There is at most one in every picture.

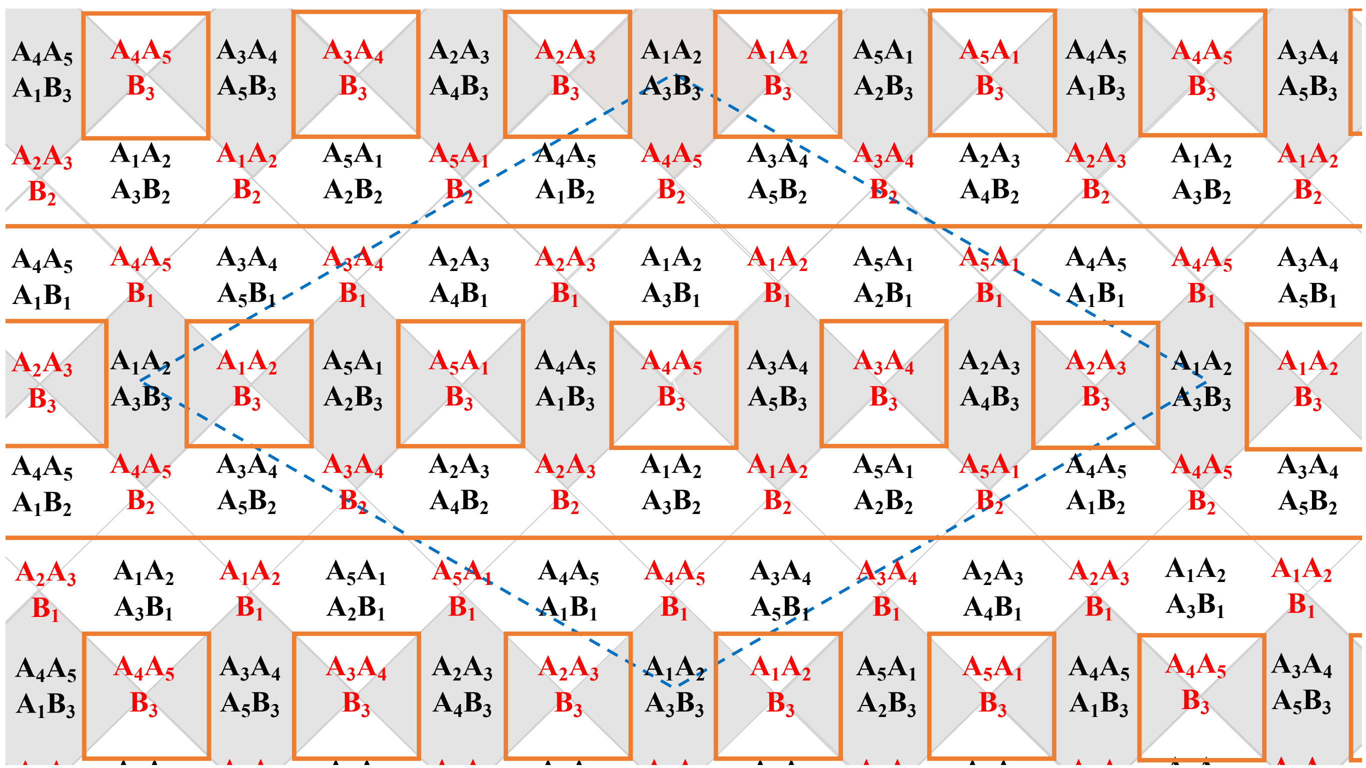
Therefore
x -> f(x)
is a contraction.

Therefore
the inequalities are true.

Loose ends

- This was for the toric inequalities.
  The proof for the RP^2 inequalities is very similar.
- I didn't say anything about the special (black hole) term $S(A_1A_2 \ldots A_m)$.
  It's fixed by a simple parity argument.
- We also have the condition:
  region indicator -> region indicator

  It works:



Everything Everywhere All at Once:

- Two new infinite families of holographic entropy inequalities
- They naturally live on (discretized) 2d manifolds.
  One of the resulting polytopes is the cubohemioctahedron.
- The spatial organization on the 2d manifolds
  is dictated by entanglement wedge nesting.
- The argument is entirely topological.
  The topology characterizes a nesting pattern.
- The effect is an enhancement in the erasure-correcting property
  of a holographic code.
- This enhancement is sometimes manifested
  as the entropy of a black hole.
- The loops on the torus are reminiscent of the toric code.

THANK YOU!

Everything Everywhere All at Once:

Holographic Entropy Inequalities, Entanglement Wedge Nesting, Topology of Error Correction, Black Holes, Cubohemioctahedron (and maybe the Toric Code)

Bartłomiej Czech

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Institute for Advanced Study Tsinghua University

with Sirui Shuai, Yixu Wang, Daiming Zhang

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