On the Role of Designs in the Data-Driven Approach to Quantum Statistical Inference

Michele Dall'Arno (michele.dallarno.mv@tut.jp)

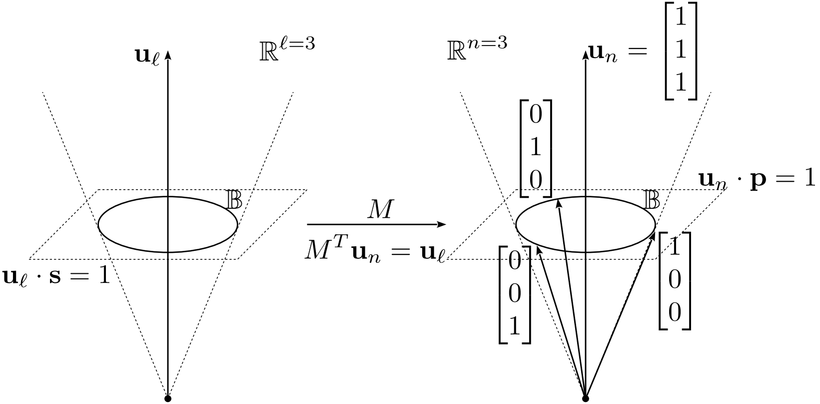
Department of Computer Science and Engineering, Toyohashi University of Technology, Toyohashi, Japan

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto, Japan

Hilbert's formalism vs Geometric formalism

	Hilbert formalism	**Geometrical formalism**
Linear space	Hermitian $d \times d$	$\mathbb{R}^{\ell=d^2}$
Unit effect	$\mathbb{1}_d$	$\mathbf{u}_\ell$
Normalization	$\mathrm{Tr}[\rho] = 1$	$\mathbf{u}_\ell \cdot \mathbf{s} = 1$
Measurements	$\sum_{j=1}^{n} \pi_j = \mathbb{1}_d$	$M^T \mathbf{u}_n = \mathbf{u}_\ell$
Inner product	Hilbert-Schmidt	Dot prod.
Born rule	$p_j = \mathrm{Tr}[\rho \pi_j]$	$\mathbf{p} = M\mathbf{s}$
Purity	$\mathrm{Tr}[\rho^2]$	$\lvert\mathbf{s}\rvert^2$

Measurements in the geometrical formalism



The fact that for any state **s** one has $\mathbf{u}_n^T M \mathbf{s} = 1$ is equivalent to the condition

$$M^T \mathbf{u}_n = \mathbf{u}_\ell.$$

Operational setup

A set of correlations (i.e. conditional probability distributions) between an input and an output is given.

We regard these correlations as generated by some (unspecified) quantum measurement upon the input of some (also unspecified) states.

A measurement is *consistent* with a correlation if there exist a state upon the input of which the measurement produces the correlation.

Our aim is to produce an inference for a consistent measurement.

However, the measurement consistent with any given correlations is in general not unique. How should we proceed?

Quantum measurement tomography

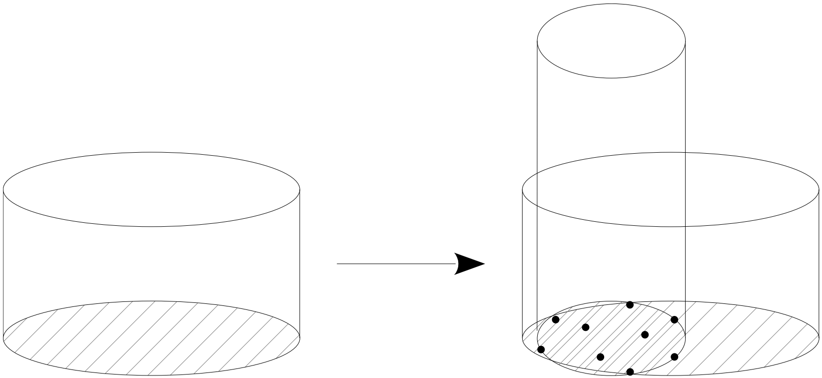
Quantum measurement tomography addresses this issue by additionally imposing that the given correlation has been generated by a given set of states.

The linearity of the theory allows to recover the measurement by linear inversion.

The set of states cannot itself be obtained via quantum state tomography, because the latter, in a symmetric fashion, would require an assumption on the measurement which, by definition of our problem, is instead unspecified and the target of the inference.

Tomography cannot “bootstrap” itself!

Bayesian inference of quantum measurements



Likelihood of set $\mathcal{P}$ of probability distributions given measurement M:

$$P\left(\mathcal{P}\middle|M\right) = \delta\left(\mathcal{P} \subseteq M\mathbb{S}\right) P\left(M^{+}\mathcal{P}\right) \frac{1}{\operatorname{vol} M\mathbb{S}}.$$

Bayesian inference of quantum measurements

- Assume states are i.i.d. and uniformly sampled, that is $P(M^+\mathcal{P}) = \prod_k P(M^+\mathbf{p}_k)$ is constant.
- Also, vol $M\mathbb{S} \propto \sqrt{\det M^T M}$.

$$P\left(\mathcal{P}\middle|M\right) \propto \delta\left(\mathcal{P} \subseteq M\mathbb{S}\right) \frac{1}{\sqrt{\det M^T M}}.$$

- Assume (improper) uniform prior $P(M)$.
- $\delta(\mathcal{P} \subseteq M\mathbb{S})$ is the consistency requirement.
- $\det M^T M$ is the committal degree.

$$\underset{\substack{M\in\mathbb{R}^{n\times\ell}\\ M^T\mathbf{u}_n=\mathbf{u}_\ell}}{\operatorname{argmax}} P\left(M\middle|\mathcal{P}\right) = \underset{\substack{M\in\mathbb{R}^{n\times\ell}\\ M^T\mathbf{u}_n=\mathbf{u}_\ell\\ \mathcal{P}\subseteq M\mathbb{S}}}{\operatorname{argmin}} \det M^T M.$$

Data-driven inference

Our approach is to infer the *minimally committal* measurements among all measurements consistent with the given correlation.

The committal degree of a measurement is the volume of its probability range.

Indeed, for any two measurements, range inclusion is a necessary and sufficient condition for one measurement to be able to simulate the other through a suitable statistical transformation.

Since this inferential protocol does not require any additional input other than the given correlation, it is referred to as *data-driven*.

Data-driven inference can be used to bootstrap tomography!

Data-driven inference

For any spanning set $\mathbb{S} \subseteq \mathbb{R}^\ell$ of admissible states and any set $\mathcal{P} \subseteq \mathbb{R}^n$ of probability distributions spanning an ℓ-dimensional subspace, we denote with $\mathcal{M}_\mathbb{S}(\mathcal{P})$ the set of quasi-measurements from $\mathbb{S}$ consistent with $\mathcal{P}$, that is

$$\mathcal{M}_\mathbb{S}(\mathcal{P}) := \left\{ M \in \mathbb{R}^{n\times\ell} \middle| M^T \mathbf{u}_n = \mathbf{u}_\ell \ \wedge \ \mathcal{P} \subseteq M\mathbb{S} \right\}.$$

Definition (Data-driven inference)

Upon the input of any set $\mathcal{P}$ of quasi-probability distributions spanning $\mathbb{R}^\ell$, the output of the data-driven inference map $\mathtt{ddi}_\mathbb{S}(\mathcal{P})$ is the set of quasi-measurements consistent with $\mathcal{P}$ with minimum-volume probability range, that is

$$\mathtt{ddi}_\mathbb{S}(\mathcal{P}) := \operatorname*{argmin}_{M \in \mathcal{M}_\mathbb{S}(\mathcal{P})} \det M^T M.$$

Spherical 2-designs

A probability distribution p over a set $\mathcal{S}$ of states is a tight frame if and only if

$$\sum_{\mathbf{s}\in\mathcal{S}} p(\mathbf{s})\,\mathbf{s}\otimes\mathbf{s} \propto \mathbb{1}_\ell \,.$$

A probability distribution p over a set $\mathcal{S}$ of states is a 2-design if and only if it is indistinguishable from the uniform distribution on the surface of $\mathbb{S}$ when given two copies.

Lemma

Any 2-design is a tight frame.

Characterization theorem for the data-driven inference

Theorem (Data-driven inference)

Upon the input of any set $\mathcal{P}$ of quasi-probability distributions spanning $\mathbb{R}^\ell$, quasi-measurement M belongs to the output of the data driven inference map $\texttt{ddi}_{\mathbb{S}}(\mathcal{P})$ if the counter-image $\mathcal{S}$ of $\mathcal{P}$, that is

$$\mathcal{P} = M\mathcal{S},$$

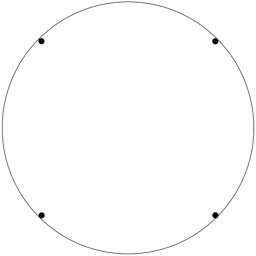
supports a spherical 2-design.

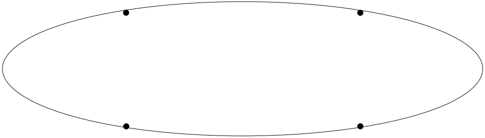
This represents a closed-form characterization of M whenever $\mathcal{P}$ contains ℓ linearly independent probability distributions. Indeed, in this case the only $\mathcal{S}$ that supports a spherical 2-design is the regular simplex, and linear inversion directly provides M.

The emergence of the role of designs

Designs have the property that their minimum volume enclosing ellipsoid is the unit ball.

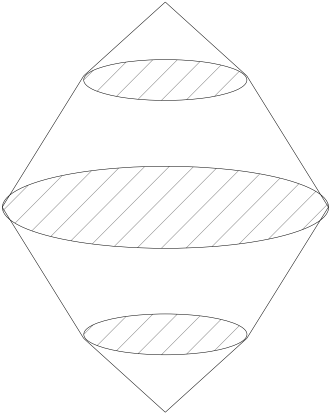
IC structures do NOT have this property in general.





How about the inference of states?

The Bayesian inference of states works similarly, just with a more cumbersome Math due to the more complex structure of the set of admissible effects.



A d-conical outer approximation of the set of admissible effects replaces the sperical outer approximation of the set of adimissible states.

Conclusion

Our results shines new light on the role played by symmetric, informationally complete (SIC) structures, and more generally designs, in the quantum Bayesian inferential process.

So far, such a role has been justified based on the symmetry of the tomographic reconstruction formula (inherited by the symmetry of the structures themselves) when such structures are adopted.

However, any informationally complete (not necessarily symmetric) structure is universal for tomographic reconstruction, albeit with a less-symmetric formula.

Instead, if the tomographic task is replaced with the data-driven inferential task we consider, as a consequence of our result not any informationally complete structure will do, and thus the role of designs emerges naturally.

References

- F. John, *Extremum problems with inequalities as subsidiary conditions*, 1948.
- D. Blackwell, *Equivalent Comparisons of Experiments*, 1953.
- P. M. Alberti and A. Uhlmann, *A problem relating to positive linear maps on matrix algebras*, 1980.
- M. Dall'Arno, F. Buscemi, A. Bisio, and A. Tosini, *Data-driven inference, reconstruction, and observational completeness of quantum devices*, 2020.
- Wojciech Słomczyński and Anna Szymusiak, *Morphophoric POVMs, generalised qplexes, and 2-designs*, 2020.
- Anna Szymusiak and Wojciech Słomczyński, *Can QBism exist without Q? Morphophoric measurements in generalised probabilistic theories*, 2023.
- M. Dall'Arno, *On the role of designs in the data-driven approach to quantum statistical inference*, 2023.