ENTANGLEMENT OF INTERNAL DEGREES OF FREEDOM AND HOLOGRAPHY

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- S.R.Das, A. Kaushal, G. Mandal and S.P. Trivedi : J.Phys. A53 (2020) 44, 444002.
- S.R.Das , A. Kaushal, S. Liu , G. Mandal and S.P. Trivedi : JHEP 04 (2021) 225
- S.R. Das, S. Hampton and S.Liu : JHEP 06 (2022) 046
- S.R. Das, A. Jevicki & J. Zheng – JHEP 12 (2022) 052
- S.R. Das, A. Kaushal, G. Mandal, K. Nanda, M. Radwan & S.P. Trivedi- JHEP 04 (2023) 141
- H. Bohra, S.R. Das, G. Mandal, M. Radwan & S.P. Trivedi – *in progress*

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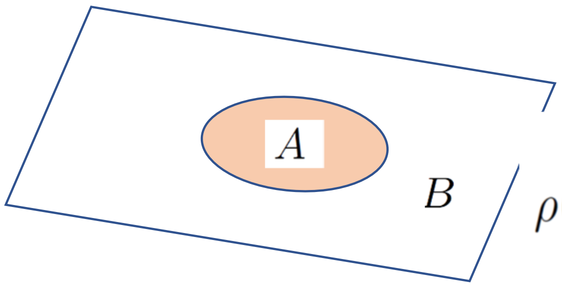
Entanglement of Internal Degrees of Freedom

- A familiar kind of entanglement in quantum field theories is entanglement in base space – this is the notion of entanglement of the degrees of freedom localized in some subregion of the space on which the theory is defined, with the rest.



$$\rho(\phi_A(x), \phi'_A(x)) = \int \mathcal{D}\phi_B(x)\Psi[\phi_A(x), \phi_B(x)]\Psi^\star[\phi'_A(x), \phi_B(x)]$$

- In holographic theories this is evaluated by the Ryu-Takayanagi formula and its generalizations.

- Base space entanglement plays a key role in the emergence of a smooth gravitational dual – this is understood when the bulk is AdS.
- However, there are known (non-AdS/ non-CFT) dualities where the dual theory is 0+1 dimensional – gauged quantum mechanics of matrices.
- Examples

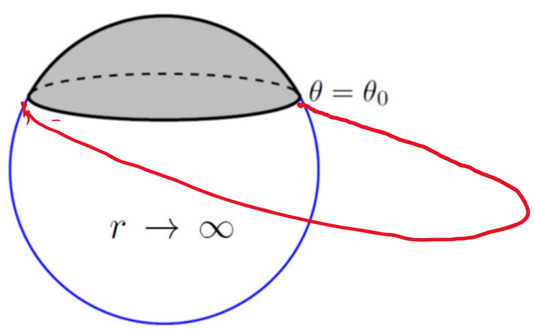
  (1) Single Matrix Quantum Mechanics ⟺ two-dimensional string theory,

  (2) BFSS Matrix Model ⟺ IMSY limit of D0 brane background

  (3) BMN Matrix Model ⟺ IIA strings in pp-wave backgrounds
- We would like to understand whether entanglement plays a key role in these examples as well.
- Now there is no base space. Therefore, this kind of entanglement must involve the matrix degrees of freedom.

- The first part of this talk will deal with the question: is there a precise and gauge invariant notion of entanglement in such cases ?
- We will discuss such a precise notion – TARGET SPACE ENTANGLEMENT –

- The second part of my talk deals with a somewhat different, though related question.
- In most known examples of holography, the bulk space-time has an internal compact space, e.g. $AdS_m \times Y^n$. The internal space Y^n geometrizes a R symmetry.
- We would like to learn if a smooth Y^n is related to some kind of entanglement. Is there an entanglement wedge construction ?
- As a first step, we will explore possible meanings of RT surfaces which are anchored on a subregion of the internal space Y^n at the AdS boundary. This question was first investigated by *Mollabashi, Shiba, Takayanagi* – subsequently by *Karch & Uhleman.*
- We will approach this issue from a somewhat different point of view.



I : Target Space Entanglement

Single Matrix Quantum Mechanics

- Begin with single matrix gauged quantum mechanics

$$S = \beta \int dt \ \mathrm{Tr} \left[(\partial_t M - i[A_t, M])^2 - V(M) \right] \qquad \Psi(M_{ij})$$

$$\beta \sim N$$

- In the $A_t = 0$ gauge, the Gauss Law constraint requires the states to be singlets under the remaining global $U(N)$ symmetry.
- With a suitable potential this is the earliest model of gauge-gravity duality.
- We seek a useful notion of entanglement of the matrix degrees of freedom

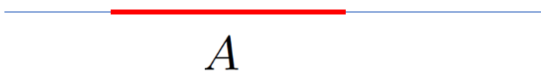
- An obvious guess is to consider the entanglement of a block in the matrix with the rest, e.g.

$$M = \begin{pmatrix} \boxed{\begin{matrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{matrix}} & \begin{matrix} M_{13} & M_{14} & M_{15} \\ M_{23} & M_{24} & M_{25} \end{matrix} \\ \begin{matrix} M_{31} & M_{32} \\ M_{41} & M_{42} \\ M_{51} & M_{52} \end{matrix} & \begin{matrix} M_{33} & M_{34} & M_{35} \\ M_{43} & M_{44} & M_{45} \\ M_{53} & M_{54} & M_{55} \end{matrix} \end{pmatrix}$$

- A priori this is problematic, since a gauge transformation mixes up the matrix elements.
- We seek a gauge invariant notion.

- The gauge invariant operators are of the form

$$\hat{C} = \mathrm{Tr}\left(\hat{M}^m \hat{\Pi}_M^n\right)_{\mathrm{order}}$$



- Now consider a projector associated to an interval A on the real line

$$(\hat{P}_A) = \int_A dx \ \delta(x\mathbf{I} - \hat{M})$$

This is a matrix valued operator

- The projected gauge invariant operators

$$\hat{C}_A = \mathrm{Tr}\left(\hat{P}_A M \hat{P}_A M \cdots \hat{P}_A M \hat{P}_A \hat{\Pi}_M \hat{P}_A \cdots \hat{\Pi}_M \hat{P}_A\right)$$

form a subalgebra.

- Expectation values of these operators are evaluated by a reduced density matrix with an associated von Neumann entropy.

- To understand the meaning of this, it is useful to fix the gauge further.
- Use the (time independent) $U(N)$ to diagonalize the matrix.

$$M_{ij} \to \mathrm{diag}[\lambda_1, \lambda_2, \cdots \lambda_N]$$

- The remaining gauge symmetry is permutations of the eigenvalues. The wave function needs to be symmetric under this Weyl group.
- There is a Fadeev-Popov jacobian which is square of the van der Monde determinant

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

- This can be absorbed in the wavefunction, which is now the wave function of N fermions.

$$\Psi(M_{ij}) \to \Psi(\lambda_i)$$

- Consider a simple operator

$$\mathrm{Tr}(M^n) = \sum_{i=1}^{N} (\lambda_i)^n$$

One body operator in Fermion theory

- The projected version is then

$$\mathcal{O}_n^P = \sum_{i=1}^{N} \int_A dx \delta(x - \lambda_i) \ \lambda_i^n$$

- Cosnider a simple operator

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One body operator in Fermion theory

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$$\mathcal{O}_n^P = \sum_{i=1}^{N} \int_A dx \delta(x - \lambda_i) \; \lambda_i^n$$

- Consider now the expectation value of such an operator for N=2, n=3

$$\begin{aligned}\langle \Psi | \mathcal{O}_3^P | \Psi \rangle = & \int_A d\lambda_1 \int_A d\lambda_2 \Psi^\star(\lambda_1, \lambda_2)(\lambda_1^3 + \lambda_2^3)\Psi(\lambda_1, \lambda_2) \\ & + 2 \int_A d\lambda_1 \int_{\bar{A}} d\lambda_2 \Psi^\star(\lambda_1, \lambda_2)(\lambda_1^3)\Psi(\lambda_1, \lambda_2)\end{aligned}$$

- This therefore measures the sum of powers of the eigenvalue only if it lies in the region of interest A.

- This means that the Hilbert space becomes a sum over sectors.
- A sector is labelled by the number of particles k which lie in the region of interest. (*S.R.D, G. Mandal & S. Trivedi*; *Mazenc & Ranard, 2019*)

$$\mathcal{H}_N = \oplus_k \mathcal{H}_{k,N-k}$$

- Acting on a state in this sector the operator has a non-trivial action only on those particles – so we can think in terms of an operator $\mathcal{O}_k$ which lives in this smaller Hilbert space.
- Similar to gauge theories (*Casini & Huerta; Soni & Trivedi; Aoki, Iritani, Nozaki, Numasawa & Shiba)* and symmetry resolved entanglement (cf. talks by *Rene Myer* and *Di Giulio*)

- The entire answer may be written in terms of reduced density matrices

$$\langle\Psi|\mathcal{O}^P|\Psi\rangle = \sum_{k=1}^{N-1} \mathrm{Tr}_A\left[\tilde{\rho}_k \mathcal{O}_k\right]$$

- Where the trace is over the smaller Hilbert space

$$\langle\lambda_a|\hat{\rho}_k|\lambda'_a\rangle = \binom{N}{k} \int \prod_{\alpha=k+1}^{N} [d\lambda_\alpha]\ \Psi^\star\left(\lambda'_a, \lambda_\alpha\right)\ \Psi\left(\lambda_a, \lambda_\alpha\right)$$

- Note that each of the $\tilde{\rho}_k$ is not normalized. Its trace is the probability for k particles to lie in A

$$p_k = \mathrm{tr}_k \tilde{\rho}_k$$

- The full reduced density matrix is block diagonal – each block corresponds to a sector

$$\rho = \begin{pmatrix} \tilde{\rho}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \tilde{\rho}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\rho}_N \end{pmatrix} \qquad \mathrm{tr}\rho = 1$$

- The associated von Neumann entropy is then given by

$$S = -\mathrm{tr}(\rho \log \rho) = -\sum_{k=0}^{N} \mathrm{tr}_k(\tilde{\rho}_k \log \tilde{\rho}_k)$$

- This generalizes to situations where the whole system is in a mixed state.

- The entropy can be re-expressed in terms of normalized density matrices

$$S = -\sum_k p_k \log p_k - \sum_k p_k \mathrm{tr}_k \hat{\rho}_k \log \hat{\rho}_k \qquad \hat{\rho}_k = \frac{1}{p_k}\tilde{\rho}_k$$

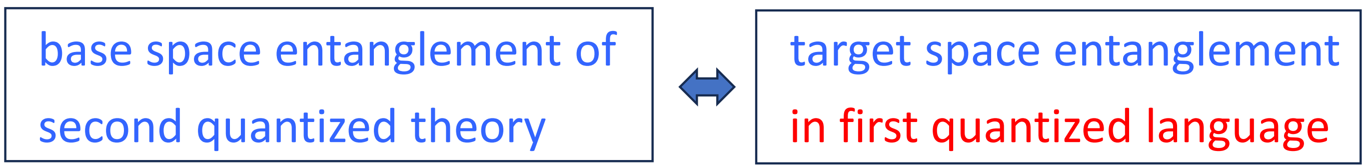
- The first term is a "classical" piece. The second term is the distillable part.

- The theory can be also expressed in a second quantized form

Fermi Level

$$H = \int d\lambda \left[\frac{1}{2\beta} \frac{d\Psi^\dagger}{d\lambda} \frac{d\Psi}{d\lambda} + \beta V(\lambda) \Psi^\dagger \Psi + \beta \mu_F \Psi^\dagger \Psi \right]$$

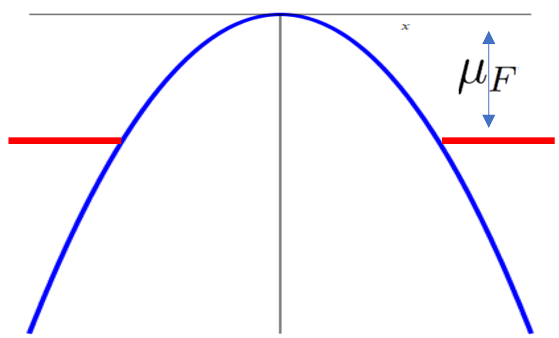
- In this fermion *field theory*, we could consider the entanglement of an interval in the space defined by λ in the filled fermi level ground state – as in any other field theory.



- A calculation of this entanglement in the second quantized language was done quite some time ago to address the question of a meaning of entanglement in String Theory. *(S.R. Das, 1995).* The result was somewhat surprising.
- Our recent works aims to extend this discussion to the problem of multiple matrices, for which there is no second quantized description.

Connection to 2d String Theory

- Matrix quantum mechanics in an inverted oscillator potential is in fact the non-perturbative description of this 1+1 dimensional string theory.



- The ground state is the filled Fermi sea. Low energy excited states are fermion-hole pairs near the Fermi level.

- As is usually the case when there are a large number of particles, it is useful to understand the physics in terms of the density of fermions – collective field

$$\rho(x,t) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \lambda_i(t)\right) = \partial_x \varphi(x,t)$$

- In the 1990's it was realized that this bosonic field is in fact the single dynamical mode of a 1+1 dimensional string theory. The low energy effective action for this theory is

$$S = \int dtdy\ \sqrt{g}e^{-2\Phi}\left[R - 4(\nabla\Phi)^2 + (\nabla T)^2 - V(T)\right]$$

- Upto a fuzziness of the order of the string scale, the field T is in fact the fluctuation of the collective field. (*S.R.D. & A. Jevicki, 1990*)

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- Upto a fuzziness of the order of the string scale, the field T is in fact the fluctuation of the collective field. (*S.R.D. & A. Jevicki, 1990*)
- At the perturbative level this was established long ago. (*Gross & Klebanov*; *Mandal, Sengupta & Wadia*; *Di Francesco & Kutasov*; *Polchinski*; *Naatsume & Polchinski…..*)
- In recent years this duality has been extended to include non-perturbative effects (*Balthazer, Rodriguez & Yin*; *Sen -2019-2023*).

- The Hamiltonian for the fluctuations of the collective field is

$$H = \int dq \left[\Pi_q^2 + \frac{\pi^2}{2}(\partial_q \eta)^2 + \frac{1}{\beta \rho_0^2}\left((\partial_q \eta)^3 + \Pi_\eta (\partial_q \eta) \Pi_\eta\right)\right] \qquad \beta \sim N$$

$$\rho_0 = \sqrt{2(\mu_F - V(\lambda)} \qquad dq = \int \frac{d\lambda}{\rho_0(\lambda)}$$

- This is a relativistic massless scalar with a position dependent coupling

$$g_s(q) = \frac{1}{\beta \rho_0(q)}$$

- This is the string coupling: it is space dependent since there is a non-trivial dilaton.
- The fermions themselves are D-particles (*McGreevy & Verlinde*; *Klebanov, Maldacena & Seiberg*, 2003)

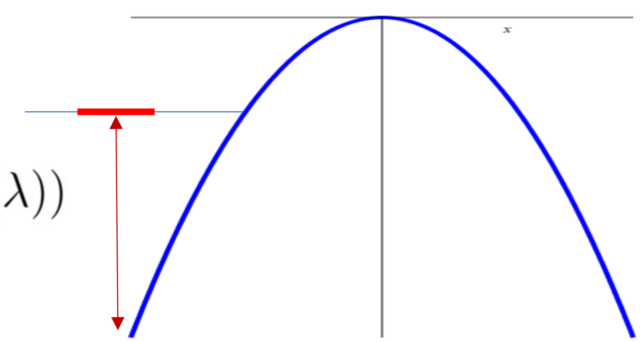
- In this fermionic field theory one can now calculate the standard base space entanglement entropy of a region – as we saw this is the Target Space entanglement in the first quantized language.
- For the inverted oscillator potential – for a region far from the hump (*S.R.D., 1995*; *E. Mazenc & S. Hartnoll, 2015*)

$$S_{EE} = \frac{1}{3}\log\frac{q_2 - q_1}{\sqrt{g_{eff}(q_1)g_{eff}(q_2)}}$$

$$g_{eff}(q) = \frac{1}{\beta\rho_0(\lambda)^2} = \frac{g_s}{\sinh^2 q}$$

- Looks like the result for a 2d c=1 CFT with the cutoff provided by the string coupling.

$$\rho^2(\lambda) \sim (\mu_F - V(\lambda))$$



- The appearance of the local fermi momentum as a cutoff is perfectly natural from the point of view of fermions.
- However, from the point of view of the String Theory this is interesting since the local fermi momentum is a position dependent coupling.
- From this "String Field Theory" point of view this finiteness is non-perturbative *(S.R.D, A. Jevicki, J. Zheng, 2023*).

- One might have thought that the cutoff should be the string length.
- There could be other notions of entanglement in String Theory where the cutoff would be indeed the string length (*Dabholkar, 1994*; *He, Numasawa, Takayanagi & Watanabe, 2014*; *Moitra and Dabholkar, 2023*).

Multiple Matrices and D0 Branes

- Consider quantum mechanics of many matrices, BFSS D0 brane theory

$$S = \frac{N}{2(g_s N) l_s} \mathrm{Tr} \int dt \left[\sum_{I=1}^{9} (D_t X^I)^2 - \frac{1}{l_s^4} \sum_{I \neq J = 1}^{9} [X^I, X^J]^2 \right] + \text{fermions}$$

- We cannot of course choose a gauge where all the matrices are diagonal - there is no second quantized description.
- However, the *same* construction using a projection operator provides a gauge invariant notion of entanglement in target space.

- Consider some Hermitian matrix operator made out of the matrices

$$f(\hat{X}^I)$$

- Now construct a matrix valued projection operator

$$\boxed{P_1 = \int_A dx\ \delta(x \cdot I - f(\hat{X}^I))}$$

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- Starting with the gauge invariant operators

$$\hat{\mathcal{O}} = Tr(\mathcal{O}) = Tr(\cdots \hat{X}^I \cdots \hat{\Pi}_J \cdots)$$

- We can now construct a subalgebra of projected operators

$$\hat{\mathcal{O}}^{P_1} = Tr(\cdots (\hat{X}^I)^{P_1} \cdots (\hat{\Pi}_J)^{P_1} \cdots)$$

- where

$$\hat{X}^I \to (\hat{X}^I)^{P_1} = \hat{P}_1 X^I \hat{P}_1 \qquad \hat{\Pi}_J \to (\hat{\Pi}_J)^{P_1} = \hat{P}_1 \hat{\Pi}_J \hat{P}_1$$

- We can then associate a reduced density matrix and a corresponding von Neumann entropy for this subalgebra.

- Since $f(\hat{X}^I)$ is a Hermitian operator itself we can now fix a gauge in which this matrix is diagonal.
- The meaning of this projection becomes clear in this gauge. The projection retains the eigenvalues of $f(\hat{X}^I)$ which lie in the region of interest.

- For example, consider the case

$$f(X^I) = X^1$$

- Then choose a gauge

$$\hat{X}^1 \to \text{diag}\left(\hat{\lambda}_1, \cdots \hat{\lambda}_N\right) \qquad \hat{\Pi}_1 \to \text{diag}\left(\hat{\pi}_1 \cdots \hat{\pi}_N\right)$$

- The remaining symmetries are Weyl transformations which permute the eigenvalues and mixes up the off-diagonal elements of the other matrices
- And a $U(1)^N$ which acts non-trivially on the off diagonal elements.
- These need to be imposed on the states.

- Once again there are sectors labelled by the number of eigenvalues of X^1, k which lie in the region of interest.
- What does this projection do to the other matrices which are not diagonal ?
- In the sector labelled by k it is easy to see that the projector in the matrix space is given by

$$P_1 = \begin{pmatrix} \mathbf{I}_{k\times k} & \mathbf{0}_{k\times(N-k)} \\ \mathbf{0}_{(N-k)\times k} & \mathbf{0}_{(N-k)\times(N-k)} \end{pmatrix}$$

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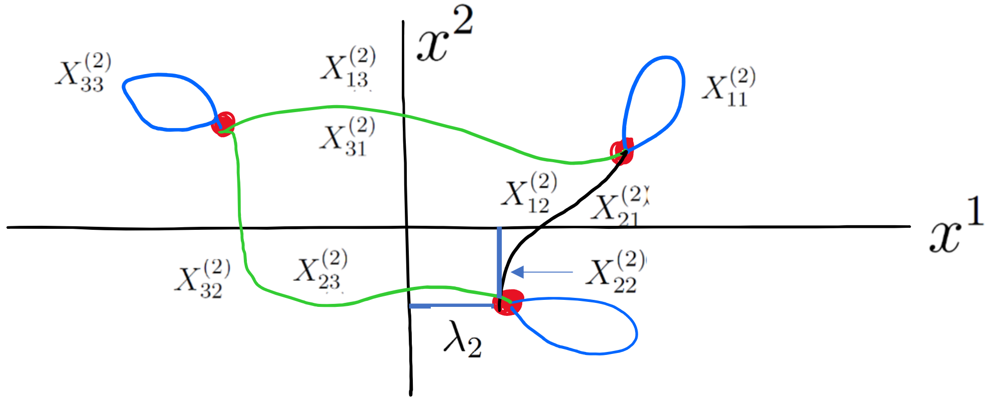
- Thus, we retain all operators in the $k \times k$ block.
- There is another projection which retains the off-diagonal blocks.

- For a projection which uses the Hermitian matrix $f(X^I)$ we can (roughly) associate a region in the target space x^I, bounded by the function $f(x^I)$

- Consider a typical snapshot of a configuration of the eigenvalues λ_α and the matrix elements $X^I_{\alpha\beta}$. Consider $N=3$, and the matrices

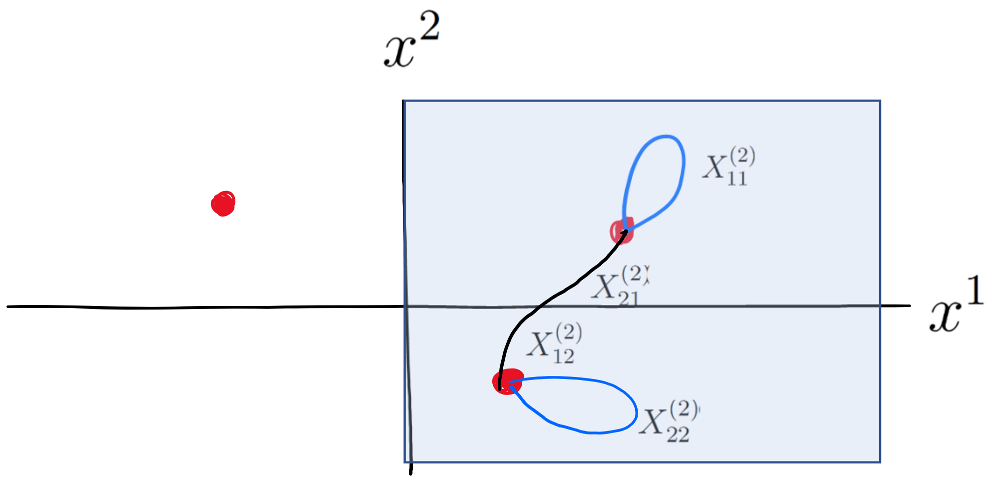
$$X^1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad X^2 = \begin{pmatrix} X^{(2)}_{11} & X^{(2)}_{12} & X^{(2)}_{13} \\ X^{(2)}_{21} & X^{(2)}_{22} & X^{(2)}_{23} \\ X^{(2)}_{31} & X^{(2)}_{32} & X^{(2)}_{33} \end{pmatrix}$$

- A configuration can be pictorially represented as



- For the constraint $x^1 > 0$ the projector $\hat{P}_1$ then keeps

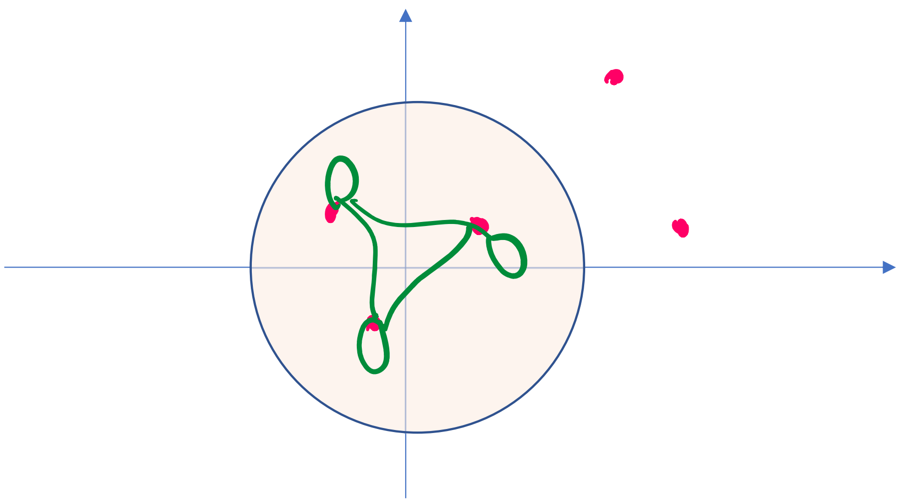
$$X^1 = \begin{pmatrix} \lambda_1 & 0 & \\ 0 & \lambda_2 & \\ & & \end{pmatrix} \qquad X^2 = \begin{pmatrix} X_{11}^{(2)} & X_{12}^{(2)} & \\ X_{21}^{(2)} & X_{22}^{(2)} & \\ & & \end{pmatrix}$$



- In a similar spirit, a constraint

$$f(\hat{X}^I) = \hat{R}^2 - r_0^2 \qquad\qquad \hat{R}^2 = \sum_{I=1}^{9} (\hat{X}^I)^2$$

- It is now useful to fix a gauge where R is diagonal.
- In this gauge the projection keeps all the matrix elements inside a "ball"



We will come to some caveats later

- In the ‘t Hooft limit the interpretation of the matrices is in terms of D0 branes.
- In low energy states where the D0 branes are well separated, the diagonal elements are roughly locations of the D0 branes. The off-diagonal elements are open strings joining the D0 branes (*Witten*).
- For a large number of coincident D0 branes forming a bound state, there is a dual geometry : the IMSY geometry (*Itzhaki, Maldacena, Sonnenschien & Yankelowicz*).

- We thus have a concrete gauge invariant notion of entanglement in target space.
- In a suitable limit this relates to regions of the bulk space as perceived by D branes.
- In a theory of gravity, it is not sensible to talk about a “region of space”, except in a semiclassical regime.
- Our notion is always well defined in the dual quantum theory – however it provides a notion of bulk entanglement in specific limits and in specific states, e.g. states where the D branes are well separated.

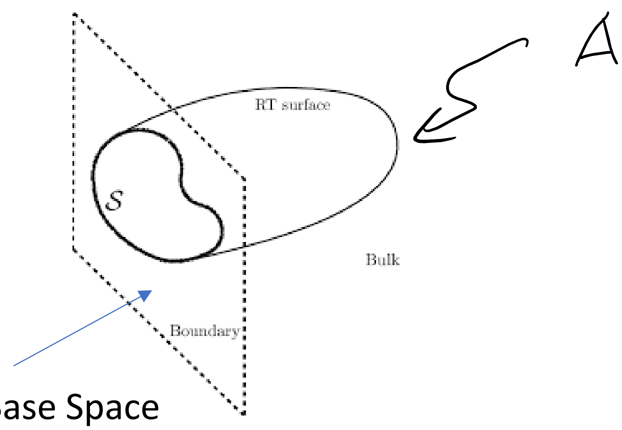
- It is tempting to speculate that the answer is proportional to the area of the boundary of the region in the space on which the D branes move.
- Since this quantifies the entanglement of $O(N^2)$ degrees of freedom we expect the result to be $\sim 1/G_N$ - in the gravity dual
- There is some evidence for this in the works of *Hampapura & Lawrence*, and by *Frenkel and Hartnoll* in a specific context.

- We illustrated the main ideas in theories where there is no base space.
- However, even when the dual theory has a base space, the theory is typically a large-N theory of matrices. The above constructions remain unchanged if the constraint is imposed at all points on the base space.
- However, there could be interesting combinations of base space and target space entanglement.

II Product spaces and Entanglement

Unusual Ryu-Takayanagi

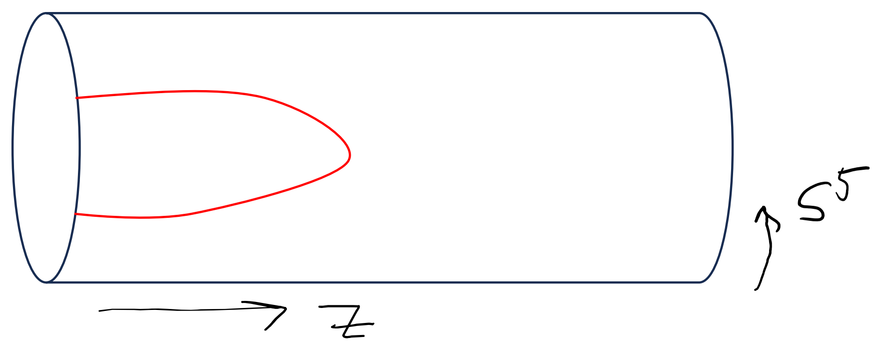
- For holographic field theories dual to asymptotically AdS space-times, the leading large-N entanglement entropy for a region of the base space is evaluated by the Ryu-Takayanagi formula



Field Theory Base Space

$$S = \frac{A}{4G_N}$$

- One might wonder: is there a meaning of the area of extremal surfaces on a spatial slice which are anchored on a region of the internal space, and smeared in the spatial directions in the boundary field theory

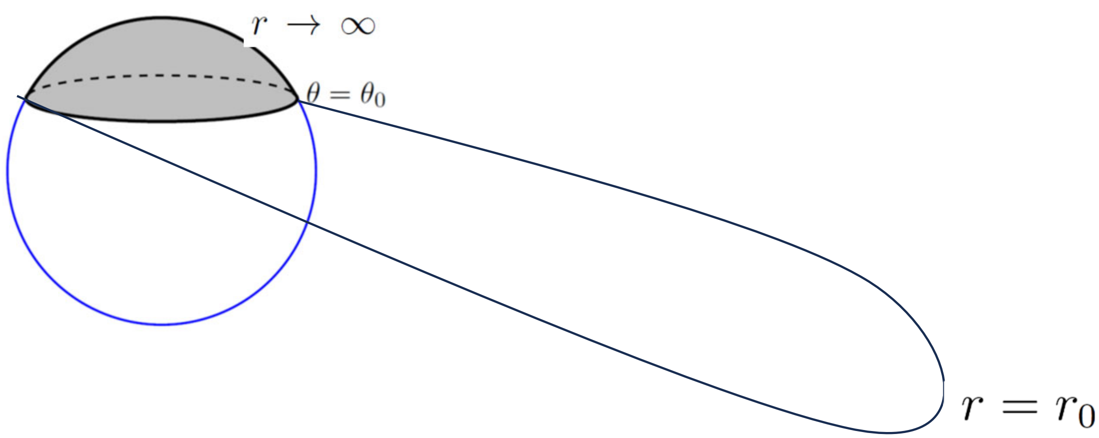


- Does this measure some kind of entanglement ? In what sense ?
- (*Mollabashi, Shiba, Takayanagi*; *Karch & Uhlemann)*

- It turns out, however, there is a result due to *Graham and Karch* which states that if there is an extremal surface anchored on the boundary ∂A of a sub-region of an internal manifold Y at the asymptotic boundary of AdS, ∂A itself must be an extremal surface inside Y
- Consider, e.g. $AdS_{n+2} \times S^m$.

$$ds^2 = \left[r^2[-dt^2 + d\vec{x}^2] + \frac{dr^2}{r^2} \right] + \alpha^2 \left[d\theta^2 + \sin^2\theta d\Omega^2_{m-1} \right]$$

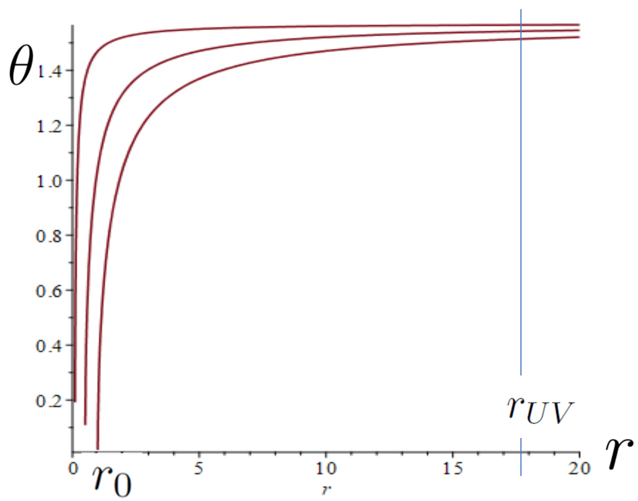
- A RT surface ending at the boundary of AdS_{n+2} on a latitude $\theta = \theta_0$ is described by a function $\theta(r)$.



Result : $\theta_0 = \frac{\pi}{2}.$

- This result follows entirely from an analysis of the equations satisfied by the surface near the asymptotic boundary of AdS.
- If the internal space is not compact, the extremal surface can end on any sub-region, but the volume of the subregion must diverge.

- This theorem can be evaded if we do not go all the way to the boundary of AdS, but stop at some r_{UV}. The dual theory has a finite UV cutoff.
- In fact, the following is an exact solution for the extremal surface for $AdS_{n+2} \times S^n$



$$\frac{r}{r_0} = \sec(\theta)$$

$$r_0 = r_{UV}\cos(\theta_0)$$

$$A = R^{m+n} V_B V_{S^{m-1}} \frac{r_{UV}^n}{n} \sin(\theta_0)^n.$$

- For small θ_0 this is a Volume Law.
- For generic m and n similar curves obtained by numerical solutions. For example for m=n=5 they have been numerically obtained in (*Mollabashi, Shiba, Takayanagi)*

- If the space is not a direct product, but a warped product, e.g.

$$ds^2 = \left[r^2[-dt^2 + d\vec{x}^2] + \frac{dr^2}{r^2} \right] + \alpha^2 \Phi(r)^2 \left[d\theta^2 + \sin^2\theta d\Omega^2_{m-1} \right]$$

$$\Phi(r) \sim (a + r)^\delta$$

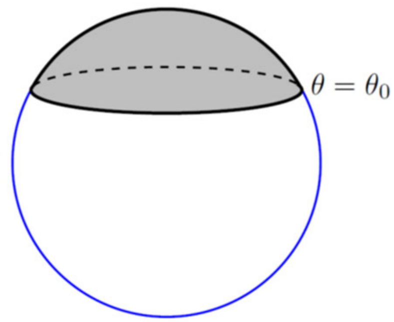
one can again evade the Graham-Karch result.

- These facts will play a role in what follows.

What would such a quantity mean

- *Mollabashi, Shiba, Takayanagi :* For a cap whose boundary is at $\theta = \theta_0$ the area quantifies the entanglement of a $SU(M)$ with $SU(N-M)$

$$\frac{M}{N} = \frac{\text{Volume of subregion}}{\text{Volume of } S^5}$$



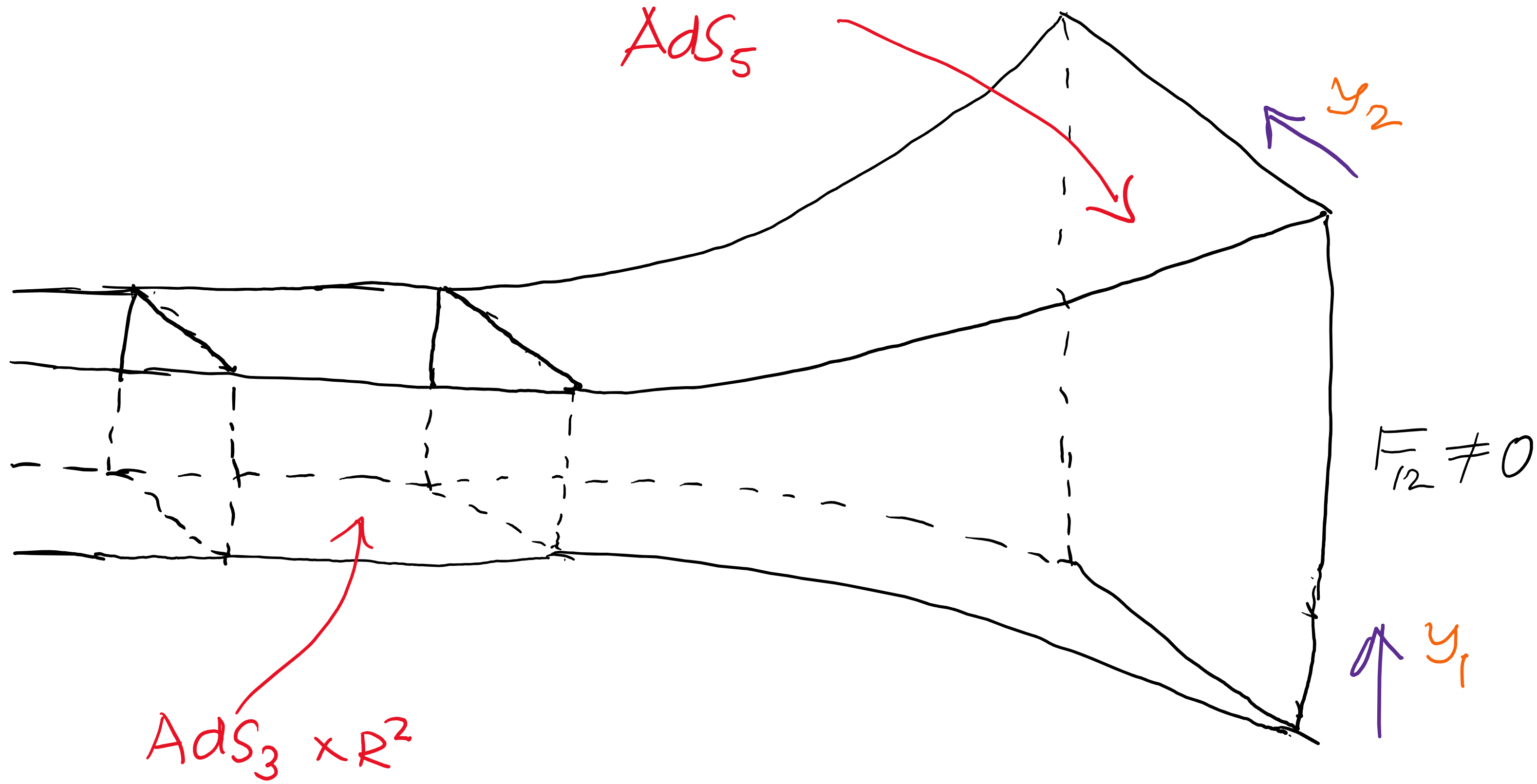
An answer for a Flow Geometry

- Our strategy to understand the meaning of these extremal surfaces is to embed such product geometries in higher dimensional asymptotically AdS space-times.
- Examples:

(1) Extremal AdS-RN, $AdS_{n+2} \rightarrow AdS_2 \times S^n$ or $AdS_{n+2} \rightarrow AdS_2 \times R^n$

(2) Dual of N=4 with a magnetic field *('d Hoker and Kraus*)

$$AdS_5 \rightarrow AdS_3 \times R^2$$


AdS5
y2
F12 ≠ 0
y1
AdS3 × R2

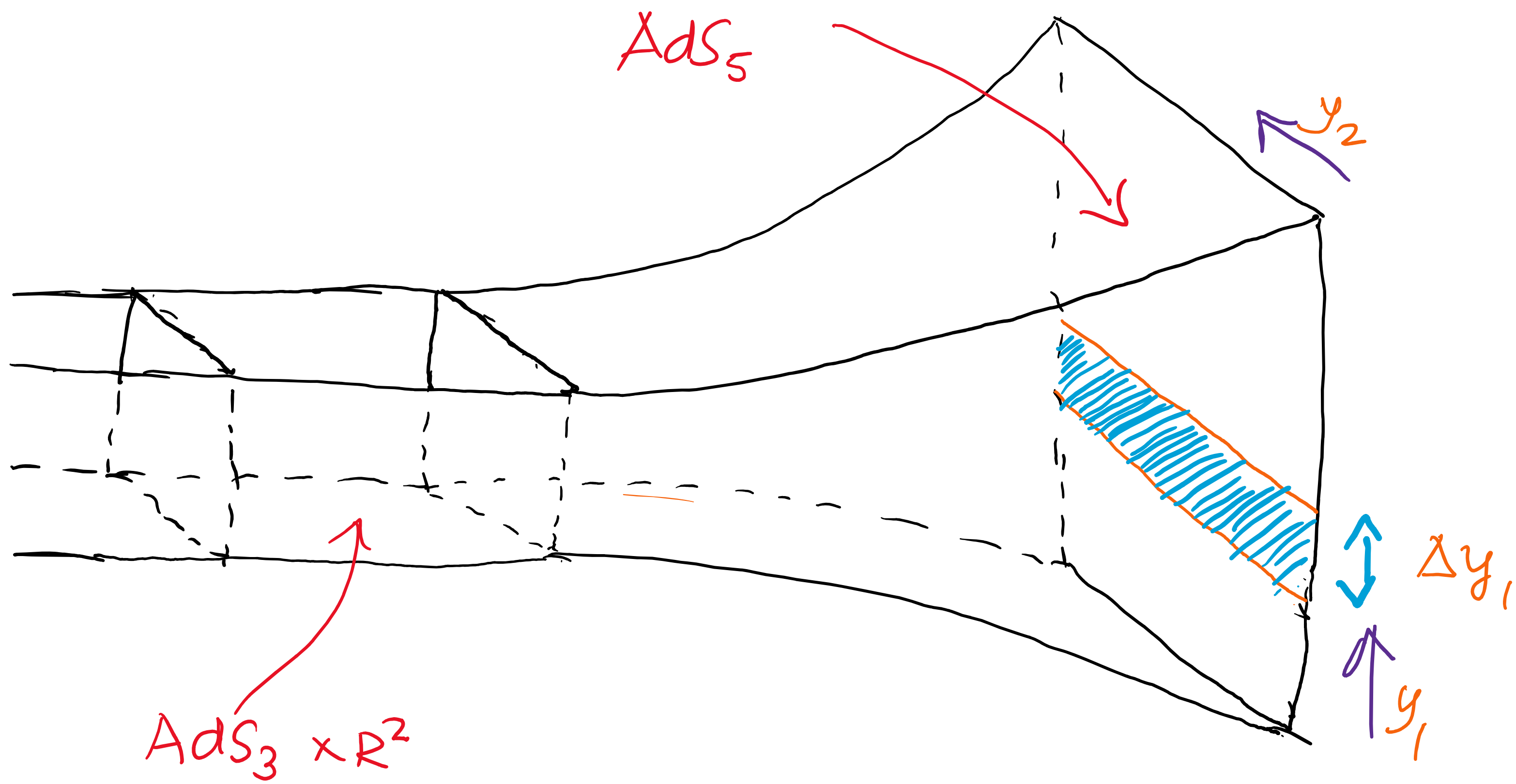
- The background is

$$ds^2 = \frac{dr^2}{L_0(r)^2} + 2L_0(r)dx^+dx^- + e^{2V_0(r)}(dx_1^2 + dx_2^2) \qquad F_{12} = \sqrt{3}$$

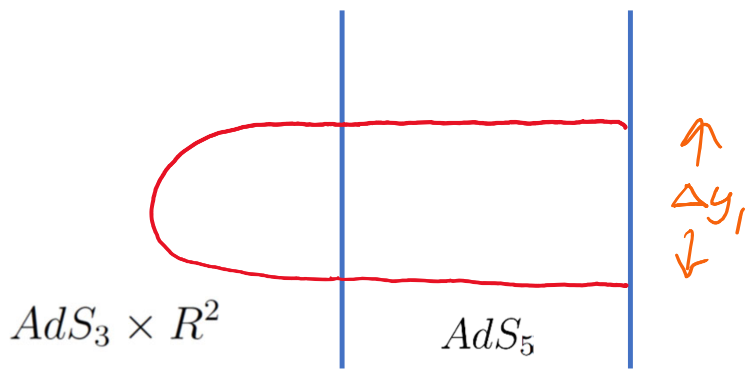
$r \to \infty$ $\qquad L_0(r) \to 2(r - r_0) \qquad e^{2V_0(r)} \to c_v(r - r_0) \qquad AdS_5$

$r \to 0$ $\qquad L_0(r) \to 2\sqrt{3}r \qquad e^{2V_0(r)} \to 1 \qquad AdS_3 \times R^2$

- Consider a strip on the boundary – and a RT surface anchored on it.



- One finds that when the strip width is large enough, the RT surface traverses the strip direction mostly after it enters the $AdS_3 \times R^2$ IR region.
- This means that the variation of the area with respect to the strip width is calculable in the IR geometry.



- This can be verified by an explicit calculation.

- In the full theory this RT surface is a standard RT surface – which is anchored on a region of the base space of the dual 3+1 dimensional field theory.
- However, from the point of view of the IR geometry this width dependent piece is anchored on a region of the internal part R^2 of the product space $AdS_3 \times R^2$

- For a general $AdS_{n+2} \times R^m$ with $n > 0$

$$ds^2 = \left[r^2[-dt^2 + d\vec{x}^2] + \frac{dr^2}{r^2}\right] + \alpha^2 \left[dy_1^2 + \cdots + dy_m^2\right]$$

- There is a critical value of the strip width $(\Delta y)_c = \pi(2n\alpha)^{-1}$
- For $\Delta y_1 < (\Delta y)_c$ the RT surface turns around at some location in the bulk. The area is

$$\boxed{2R^{(m+n)}\frac{V_{n+m-1}}{n}\alpha^{m-1}r_{UV}^n \sin(n\alpha\Delta y_1)}$$

Volume Law

- At $\Delta y_1 = (\Delta y)_c$ the turning point reaches the Poincare horizon
- For $\Delta y_1 > (\Delta y)_c$ the minimal surface has two disconnected pieces

$$\boxed{A_3 = 2R^{(m+n)}\frac{V_{n+m-1}}{n}\alpha^{m-1}r_{UV}^n}$$

Area Law

Volume of all the spatial directions except y_1

- There is now a clear meaning of this. In terms of the UV field theory, we are measuring the entanglement entropy of a region of the $x_1 - x_2$ plane, i.e. the entropy associated to the subalgebra of gauge invariant operators

$$\mathcal{O}_\Delta(x_1, x_2, x_3, t) \qquad \Delta = 2 + \sqrt{m^2 + 4}$$

with e.g. the restriction $0 < x_1 < \Delta x_1$

- In a theory dual to the IR geometry, the primary operators are the fourier transforms of these – but the conformal dimension now depends on the momentum.

$$\mathcal{O}_{k_1,k_2}(x_3,t) = \int dx_1 dx_2 \; e^{-i\vec{k}\cdot\vec{x}} \mathcal{O}_\Delta(x_1,x_2,x_3,t)$$

- In terms of the IR theory, we are calculating the EE associated with the subalgebra of operators defined by

$$\mathcal{O}_\Delta(x_1,x_2,x_3,t) = \int dk_1 dk_2 \; e^{i\vec{k}\cdot\vec{x}} \mathcal{O}_{k_1,k_2}(x_3,t) \qquad \Delta_k = 1 + \sqrt{m^2+1+\vec{k}^2}$$

- Where $0 < x_1 < \Delta x_1$

Base Space Entanglement in the UV ⟺ Entanglement in Internal Space in the IR

- The presence of the magnetic field leads to Landau levels, gapping out excitations which have momenta in the $x_1 - x_2$ directions.
- The low energy theory becomes a 1+1 dimensional theory.
- Since correlations along the $x_1 - x_2$ direction are short ranged, once the strip width reaches the scale of the inverse gap, the extensive part of the EE saturates.

- A more familiar example concerns the geometry of a RN extremal black hole, or a black brane

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2)$$

$$r^2 f(r) = (r - r_h)^2 \frac{(r^2 + 2rr_h + 3r_h^2)}{R_4^2}$$

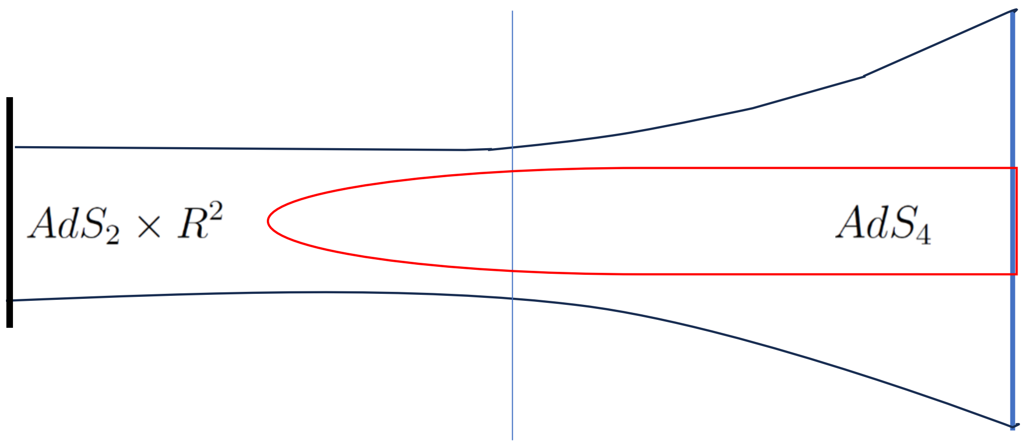
- This describes a flow from AdS_4 to $AdS_2 \times R^2$

$$r \gg r_h \qquad f(r) \to \frac{r^2}{R_4^2}$$

$$r \to r_h \qquad f(r) \to \frac{(r - r_h)^2}{R_2^2}$$

- In what follows we need to keep the warping factor, which becomes a linear dilaton in the $AdS_2 \times R^2$ geometry

- Now consider a usual RT surface anchored on a strip on the AdS_4 boundary.
- One finds that when the strip width is large enough, the RT surface traverses the strip direction mostly after it enters the $AdS_2 \times R^2$ IR region.
- This means that the variation of the area with respect to the strip width is calculable in the IR geometry.



- The result is, however, significantly different.
- Now, as we increase the strip width the turning point comes closer and closer to the horizon, and touches the horizon only in the limit of infinite width.
- In this limit of large enough strip width, the area of the RT surface is given by

$$A = \left[(\Delta y) L_x r_h^2 \right] + (\text{UV piece independentof } \Delta y)$$

- Unlike the higher dimensional examples, we discussed earlier the IR piece does not depend sensitively on where we put the dividing screen between the AdS_4 and the IR $AdS_2 \times R^2$

- The meaning of this area is the same as in the higher dimensional example : this relates to the subalgebra of fourier transforms of the primary fields in the IR geometry, restricted to a region.
- The EE is extensive since the theory has a chemical potential.
- The story is similar for $AdS_4 \rightarrow AdS_2 \times S^2$. Fourier replaced by spherical harmonics.

- The primary operators are fourier components of the operators of the UV theory

$$\mathcal{O}_{k_x,k_y}(t) \qquad\qquad \Delta_k = \frac{1}{2} + \sqrt{m^2 + \frac{1}{4} + \vec{k}^2}$$

- In this language, the extremal areas quantify entanglement entropies associated with linear combinations of operators

$$\mathcal{O}(x,y,t) = \int dk_x dk_y \; e^{i\vec{k}\cdot\vec{x}} \mathcal{O}_{k_x,k_y}(t)$$

- For $AdS_4 \to AdS_2 \times S^2$ replace Fourier by Spherical Harmonics.

What about product geometries ?

- What about $AdS_5 \times S^5$ which do not result from a flow, where do not have the crutch of a UV geometry where the internal space becomes base space ?
- It is tempting to suggest a similar explanation.
- For example, in the N=4 SYM the primary fields are

$$\mathcal{O}_{l,m}(\vec{x},t) = \mathrm{Tr}\left[X^{(I}X^{J}X^{K)} - (\mathrm{traces})\right]$$

- We now collect these to form operators which are functions of 5 angles

$$\mathcal{O}(\vec{x},\theta_i,t) = \sum_{l,m} Y_{l,m}(\theta_i)\mathcal{O}_{l,m}(\vec{x},t)$$

- Restricting these angles to a suitable subregion and taking products and sums leads to a subalgebra whose associated von Neumann entropy is what the RT surface evaluates.
- This suggestion has been made by *Karch & Uhleman*, and *Anous, Karczmarek, van Raamsdonk and Way* for a similar situation in D0 brane geometry.

- The operators $\mathcal{O}(\vec{x}, \theta_i, t)$ or $\mathcal{O}(\vec{x}, \vec{y}, t)$ which we defined above performs the projection after the color degrees of freedom are traced over.
- This is a very different quantity than TARGET SPACE ENTANGLEMENT – where the projection involved the color degrees of freedom.

Are these operators meaningful ?

- Are these operators in any sense approximately "local" on the internal space ?
- One might think that the locality of supergravity on the internal space should tell us something about this question.
- However, the CFT correlator is related to the behavior of the supergravity correlator at large invariant distances, NOT short invariant distances – 2 points on the boundary are always infinitely far away.

Correlators

- We can begin exploring this question by looking at CFT two-point functions coming from a scalar of mass m in Euclidean $AdS_{d+1} \times Y^n$

$$ds^2 = \frac{1}{z^2}[dz^2 + d\vec{x}^2] + g_{mn}(y)dy^m dy^n$$

- Need to compute the correlators of operators

$$\mathcal{O}(\vec{x}, \vec{y}) = \sum_{\ell, \vec{m}} \mathcal{O}_{\ell,\vec{m}}(\vec{x}) Y_{\ell,\vec{m}}(\vec{y})$$

using the CFT results

$$\langle \mathcal{O}(\vec{x}, \vec{y}) \mathcal{O}(\vec{x}', \vec{y}') \rangle = \sum_{\ell, \vec{m}} \langle \mathcal{O}_{\ell,\vec{m}}(\vec{x}) \mathcal{O}_{\ell,-\vec{m}}(\vec{x}') \rangle Y_{\ell,\vec{m}}(\vec{y}) Y_{\ell,-\vec{m}}(\vec{y}')$$

$$\begin{aligned} \mathcal{O}_{\ell,\vec{m}}(\vec{x}) &= \text{primary operator} \\ Y_{\ell,\vec{m}}(\vec{y}) &= \text{eigenvalue of Laplacian on } Y^n \\ &\quad\ \text{with eigenvalue } \ell^2 \end{aligned}$$

- Need to use correlators with normalizations given by AdS/CFT

$$\langle \mathcal{O}_{\ell,\vec{m}}(\vec{x})\mathcal{O}_{\ell',\vec{m}'}(\vec{x}')\rangle = C(\nu_{\ell,\vec{m}})\left(\frac{\epsilon}{|\vec{x}-\vec{x}'|}\right)^{2\Delta_{\ell,\vec{m}}} \qquad C(\nu_{\ell,\vec{m}}) = \frac{(2\nu_\ell)}{\pi^{\frac{d}{2}}}\frac{\Gamma(d/2+\nu_\ell)}{\Gamma(\nu_\ell)}$$

$$\Delta_\ell = d/2 + \nu_\ell \qquad \nu_\ell = \sqrt{d^2/4 + m^2 + \ell^2}$$

- Define

$$u \equiv 2\log\left(\frac{|\vec{x}-\vec{x}'|}{\epsilon}\right)$$

- It turns out that the correlator can be expressed as derivatives of the Euclidean propagator of a massive scalar on Y^n where u plays the role of the Euclidean time

$$D_{\ell,m}(u,\vec{y},\vec{y}') = \sum_{\ell,\vec{m}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega u}}{\omega^2+\ell^2+M^2} Y_{\ell,\vec{m}}(\vec{y}) Y_{\ell,\vec{m}}(\vec{y}')$$

- Note: this is not the same scalar we started out with. Its mass is

$$M = \sqrt{d^2/4+m^2}$$

- When $|\vec{y}-\vec{y}\,'| \gg 1/M$ dies off exponentially.
- This indicates that it is sensible to consider such operators in the discussion of entanglement in internal spaces.

Epilogue: Applications to Many Body Physics

- We have provided a definition of entanglement in a theory of N particles associated with a region of space on which the particles move.
- The concept of course generalizes to arbitrary number of dimensions. One application of our ideas is the problem of Integer Quantum Hall Effect (*S.R.D., S. Hampton & S. Liu, 2022)*
- When a second quantized formulation exists, this is usual base space entanglement.
- However, there are many-body systems where e.g. ground state wave functions (e.g. ground state) are known in a first quantized description, but not in a second quantized description. The concepts introduced above will be useful in these systems.

どうもありがとうございます。