Massachusetts Institute of Technology

Center for Theoretical Physics

Near-Extremal Black Hole Entropies from Replica Matrices

Sergio Hernández-Cuenca

based on forthcoming work

Related:
Chandrasekaran-Engelhardt-Fischetti-SHC [2207.09472]
and Engelhardt-SHC-Verheijden [WIP]

Quantum Information, Quantum Matter and Quantum Gravity

YITP, Kyoto University

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Introduction

A Word on Holographic Expectations

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- Some standard bulk operations: KK-truncations, EFT in semiclassical gravity, near-horizon limits, spacetime complexifications, sums over topologies...
- How do these operations affect the boundary dual?
  Perhaps not inconceivable that neglecting bulk details leads to disorder-averaging on the boundary

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- A tool that the CMT has long utilized to study complicated systems is disorder averages
- If boundary ensemble averages make bulk gravitational physics more tractable, then let's just use this to our advantage
- We just must make sure we are computing the quantities that are sensible for an ensemble of theories!

Spectral Ensembles

A Simple Thermodynamic Fact

Common definitions:

- Spectral density: $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ (may be distributional)
- Ground state energy: $E_0 \equiv \inf \operatorname{supp} \rho$
- Partition function: $Z(\beta) \equiv \int_0^\infty dE\, \rho(E)\, e^{-\beta E}$ (Laplace transform)
- Free energy: $F(\beta) \equiv -\frac{1}{\beta} \log Z(\beta)$
- Thermal entropy: $S(\beta) \equiv \beta^2 F'(\beta) = (1 - \beta\partial_\beta) \log Z(\beta)$

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Proof sketch: Write $S(\beta) = \log Z(\beta)\,(1 - f(\beta))$ with $f(e^x) = \partial_x \log(-\log Z(e^x))$

- Use dominated convergence to show $\lim_{\beta\to\infty} Z(\beta) = 0$
- Use Bernstein–Widder to show $\lambda \equiv \lim_{\beta\to\infty} f(\beta) \leq 1$
- Show $\lambda = 1$ implies $\rho \supset n\delta$ with $n > 0$ and $\lim_{\beta\to\infty} S(\beta) = \log n$

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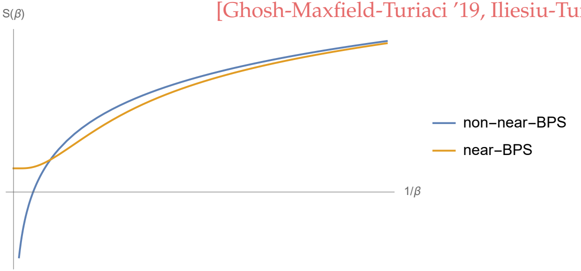
- Negativity of $S(\beta)$ at large β $\Longleftrightarrow$ Neglect of the ground state
- Generic (non-near-BPS) near-extremal black holes suffer from this due to quantum corrections that are relevant at $\beta \gtrsim \mathcal{O}(S_0)$:

[Ghosh-Maxfield-Turiaci '19, Iliesiu-Turiaci '20]



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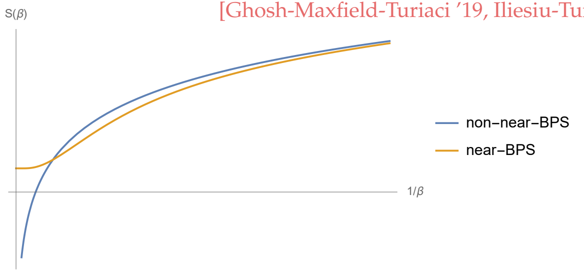

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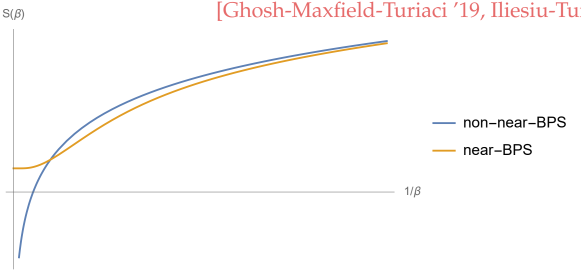

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- Near-BPS black holes ruled by $\mathcal{N} = 4$ super-JT: $\rho(E) \supset e^{S_0}\delta(E)$ (e.g. $4D\,\mathcal{N} = 2$ SUGRA w/ $\Lambda = 0$ or $(4,4)$ SUGRA in AdS_3)

[Heydeman-Iliesiu-Turiaci-Zhao '20]

Possible Resolutions

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4. We are not even computing a sensible quantity to begin with:
   Regardless of the above, this is a correct statement from the perspective of the matrix integrals dual to JT theories
   [Saad-Shenker-Stanford '19, Stanford-Witten '19]

Dissecting Spectral Ensembles

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  - Sensibly for near-BPS because $p_0 \propto \delta$ (SUSY-protected E_0)
- Emphasis: no matter to what order, $\mathcal{O}(S_0)$, $\mathcal{O}(e^{-S_0})$, $\mathcal{O}(e^{-e^{S_0}})$, or even exactly, one computes $\langle Z(\beta) \rangle$: $S(\beta)$ will generically be pathological because it is simply the wrong quantity!

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- f_a is a proxy to f_q when f is self-averaging, $f_q/f_a = 1 + \mathcal{O}(1/N)$
- Example quantity: entropy, linear in the free energy, for which:

$$F_a(\beta) \equiv -\frac{1}{\beta} \log\langle Z(\beta) \rangle \qquad \text{and} \qquad F_q(\beta) \equiv -\frac{1}{\beta} \langle \log Z(\beta) \rangle$$

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- Explorations of $\langle \log Z(\beta) \rangle$ using the gravitational PI:
    - Full quantum JT [Engelhardt-Fischetti-Maloney '20]
    - Semiclassical JT+matter [Chandrasekaran-Engelhardt-Fischetti-SHC '22] [Engelhardt-SHC-Verheijden WIP]

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- But... is (semiclassical) gravity even able to capture $\langle \log Z(\beta) \rangle$? What if it only differs from $\log \langle Z(\beta) \rangle$ at $\mathcal{O}(e^{-e^{S_0}})$? **Ask RMT!**

Replica Matrices

Matrix Integrals

- An $N \times N$ matrix ensemble is specified by a partition function:

$$\mathcal{Z} \equiv \int dM \, e^{-N \, \mathrm{Tr} \, V(M)}$$

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- Symmetries on M determine the form of dM upon diagonalization and turn $\mathcal{Z}$ into a canonical eigenvalue integral:

$$\mathcal{Z} = \int d\Lambda \; \mu(\Lambda) \, e^{-N \operatorname{Tr} V(\Lambda)}, \qquad \Lambda \equiv \{\lambda_k\}_{k=1}^{N}$$

  e.g. for the Wigner-Dyson ensembles, $V(x) = \frac{1}{2}x^2$ (Gaussian) and

$$\mu(\Lambda) \propto \prod_{1 \le i < j \le N} \left|\lambda_i - \lambda_j\right|^b \prod_{k=1}^{N} d\lambda_k, \qquad b = 1, 2, 4 \;\; (O, U, S)$$

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- Our observables of interest are moments of $Z(\beta) \equiv \operatorname{Tr} e^{-\beta M}$:

$$\langle Z(\beta)^m \rangle \equiv \int d\Lambda \, \mu(\Lambda) \, (\operatorname{Tr} e^{-\beta \Lambda})^m \, e^{-N \operatorname{Tr} V(\Lambda)}$$

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- Standard RMT machinery concerns obtaining the ensemble-averaged spectrum $\langle \rho(\lambda) \rangle \equiv \left\langle \sum_{k=1}^{N} \delta(\lambda - \lambda_k) \right\rangle$
  - Topological recursion of loop equations gives $\mathcal{O}(1/N)$ expansion
  - Orthogonal polynomials and the string equation give both $\mathcal{O}(1/N)$ and $\mathcal{O}(e^{-N})$ contributions

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- Common problem: no matter how exactly $\langle \rho(\lambda) \rangle$ is obtained, this just gives the annealed $\log\langle Z(\beta) \rangle$. Must compute $\langle Z(\beta)^m \rangle$ directly and perform the $m \to 0$ replica trick to obtain $\langle \log Z(\beta) \rangle$[1]

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- Ideally leading order is enough, so look for saddles at large N

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LARGE-N SADDLES

- Writing $\langle Z(\beta)^m \rangle = \int d\Lambda\, e^{-I_m(\Lambda)}$, the matrix action to extremize is:

$$I_m(\Lambda) \equiv N \sum_{k=1}^{N} V(\lambda_k) - \sum_{j<k} \log \left|\lambda_k - \lambda_j\right|^b - m \log \sum_{k=1}^{N} e^{-\beta \lambda_k}$$

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- But recall pathologies occurred at large $\beta = \mathcal{O}(N)$, for which

$$\log \sum_{k=1}^{N} e^{-\beta \lambda_k} = -\beta \min\{\Lambda\} + \cdots = \mathcal{O}(N)$$

  so expect a new saddle in this regime!

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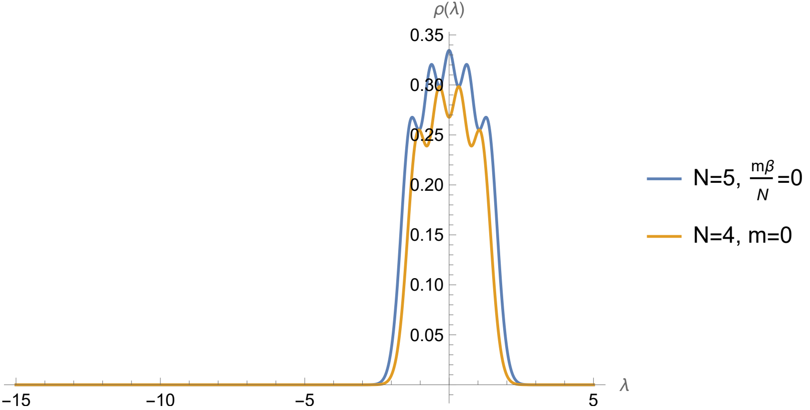
- Need a more refined ρ ansatz to find such double-limit saddles
- Get intuition from the GUE by computing $\rho(\lambda)$ exactly at small N (obtained by integrating over all eigenvalues but one in $\langle Z(\beta)^m \rangle$)

Intuition from the GUE

Plot exact $\rho(\lambda)$ from the $\langle Z(\beta)^m \rangle$ integral for $N = 4, 5$ and varying β:



For the picky: eigenvalues are rescaled by $\lambda \to \sqrt{N}\lambda$ using the corresponding N, and ρ is normalized to 1 for $N = 5$ and to $4/5$ for $N = 4$

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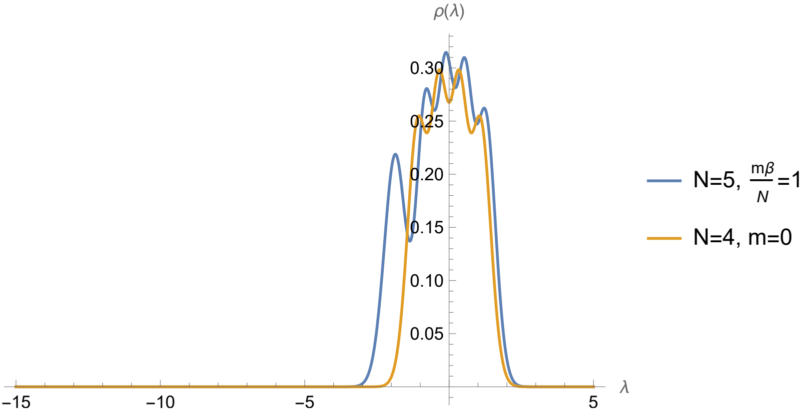

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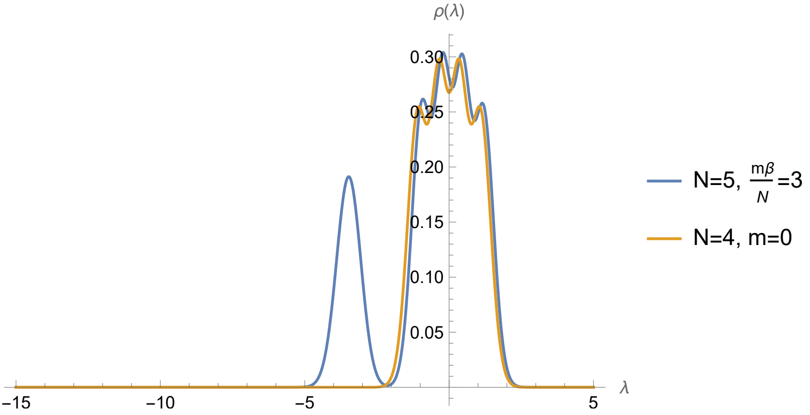

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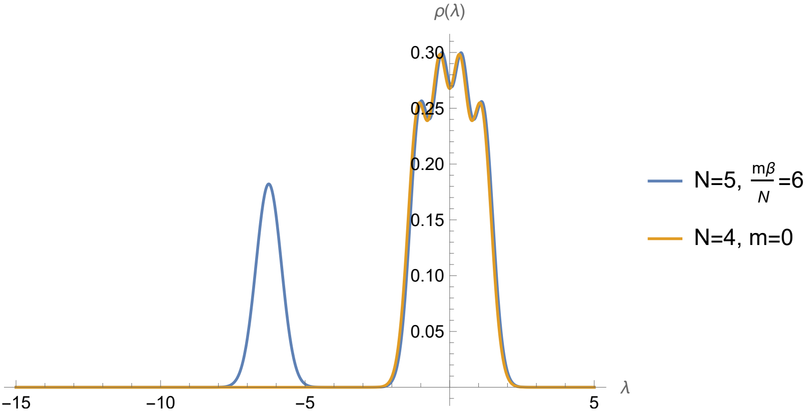

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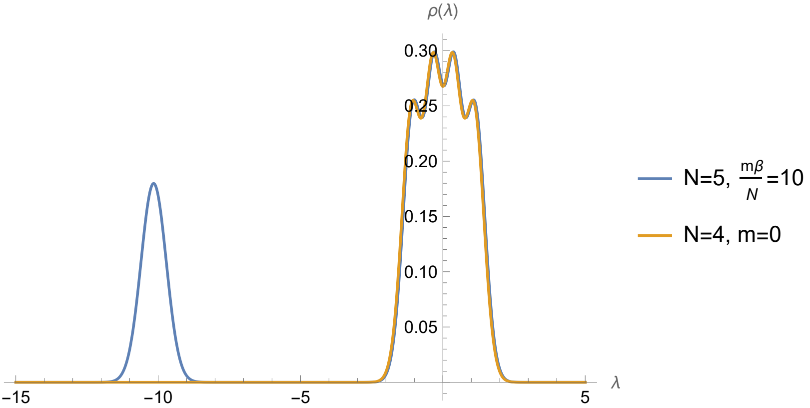

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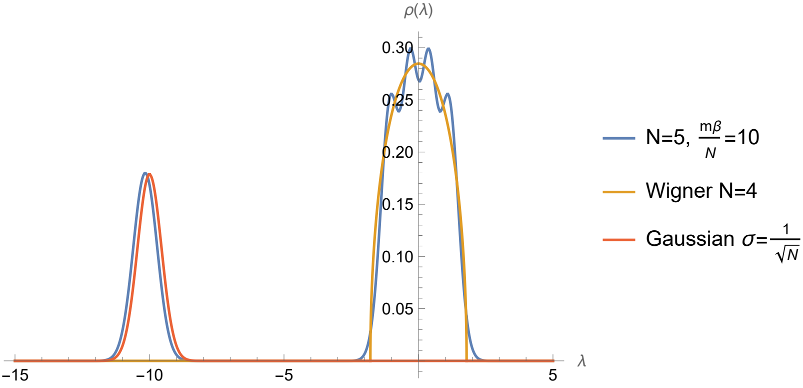

Large-(N, β) saddle $\rho_0(\lambda)$ for $\langle Z(\beta)^m \rangle$ approaches the Wignerian $m = 0$ saddle for $N - 1$ plus a moving δ-function for $\min\{\Lambda\}$ sensitive to $\frac{m\beta}{N}$

GENERAL LARGE-(N, β) ANSATZ

- Without loss of generality, perform extremization on:

$$\rho(\lambda) = \delta(\lambda - \lambda_0) + \sigma(\lambda)$$

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- General strategy: Solve eq. 1 for σ, then use it to solve eq. 2 for λ_0

Example: Wigner-Dyson (Gaussian)

- Large-(N, β) saddle ρ_0 for $\langle Z(\beta)^m \rangle$:

$$\sigma(\lambda) = \begin{cases} \frac{N}{b\pi}\sqrt{2b\left(1-\frac{1}{N}\right)-\lambda^2} & \text{if } |\lambda| \leq \sqrt{2b\left(1-\frac{1}{N}\right)} \\ 0 & \text{otherwise} \end{cases}$$

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- The saddle result $\langle Z(\beta)^m \rangle = \exp\Big(-\Big(I_m[\rho_0^{(m)}] - I_0[\rho_0^{(0)}]\Big)\Big)$ is:

$$e^{N\left(\sqrt{2b}\kappa\sqrt{1+\frac{2\kappa^2}{b}}+b\sinh^{-1}\sqrt{\frac{2\kappa^2}{b}}\right)}\left(1+\frac{m\,e^{-\frac{N\sqrt{8b}\kappa}{m}}}{\sqrt{2b}\kappa}I_1\left(\frac{N\sqrt{8b}\kappa}{m}\right)\right)^m$$

Consistency Checks

- For small $\kappa = \frac{m\beta}{2N}$, we recover a factorized Wigner I_1 result

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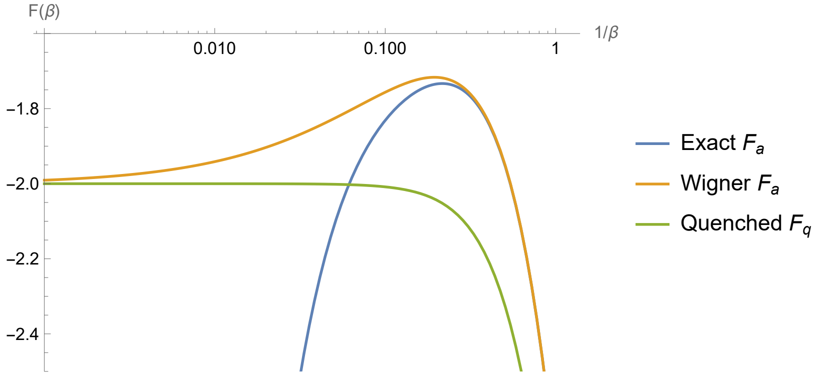
- Key question: does the large-(N, β) saddle succeed in computing the quenched free energy through the $m \to 0$ replica trick?

GUE Results

Free energies for the GUE with $N = 10$:



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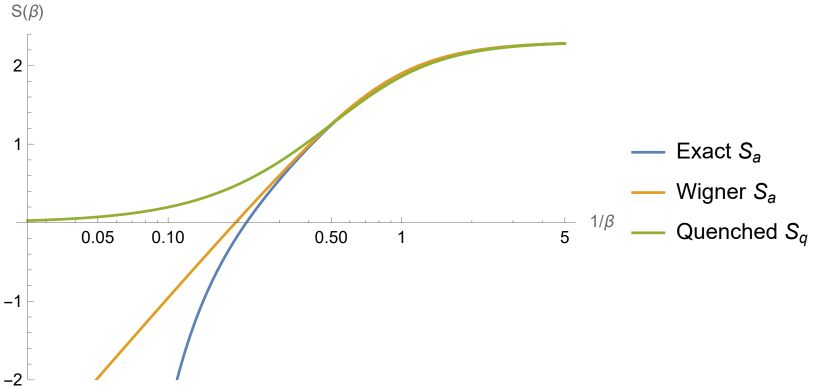

Sanity check: $\lim_{\beta\to\infty} S_q = 0$ (non-degenerate ground) and $\lim_{\beta\to 0} S_q = \log N$

The large-(N, β) saddle gives a well behaved quenched entropy!

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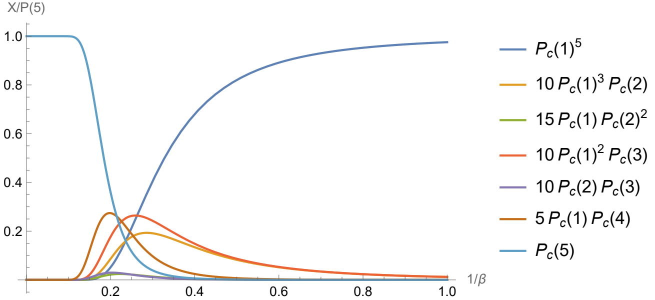

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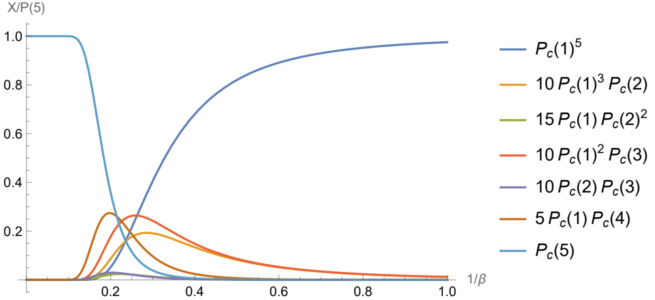

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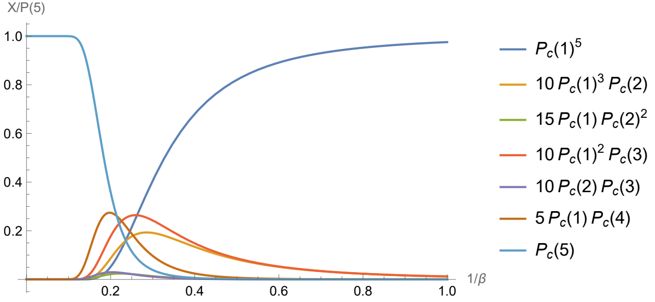

- This suggests only the largest cumulant contributes at large β (no RSB). However, this $m > 1$ intuition is misleading: recall $m\beta$ often appear together, so the $m \to 0$ limit competes with large β

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- Expect leading order nonsense for more general quantities when annealing instead of quenching: late-time correlation function decay, infinite extremal wormhole throat sizes, …
- Leading order in matrix integrals suffices to compute quenched quantities. How does semiclassical gravity capture these?

MAIN OPEN QUESTION

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Thank you!

GENERAL REMARKS

- The large-(N, β) saddle that computes $\langle Z(\beta)^m \rangle$ exists for any matrix potential and ensemble measure, and the strategy for solving the saddle equations works as well
- These low-temperature matrix saddles exist for all m and allow to compute arbitrary moments which generically do not factorize (cf. high temperatures, for which $\langle Z(\beta)^m \rangle \approx \langle Z(\beta) \rangle^m$. These will give nontrivial cumulants $\langle Z(\beta)^m \rangle_c$ (see later)
- The quenched logarithm that results turns out to take a remarkably simple form at leading order for any ensemble:

$$\langle \log Z(\beta) \rangle = \log\big(n\, e^{-\beta \lambda_0} + Z_W(\beta)\big)$$

  w/ $Z_W(\beta)$ the LT of the Wignerian ρ_0 and $\lambda_0 + \text{gap} = \inf \operatorname{supp} \rho_0$
- For double-scaled matrix integrals the DSL can be taken at the end and the same results apply (one can also use the leading resolvent and the effective potential for the saddle EoM)