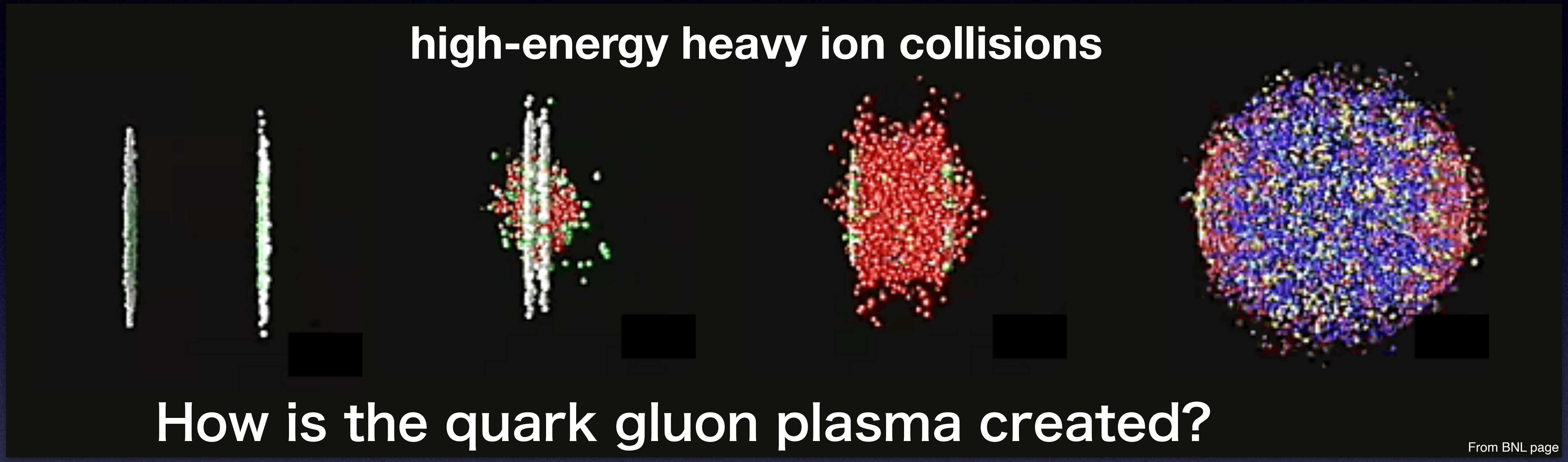
Hamiltonian lattice gauge theory based on spin networks

Yoshimasa Hidaka(KEK)

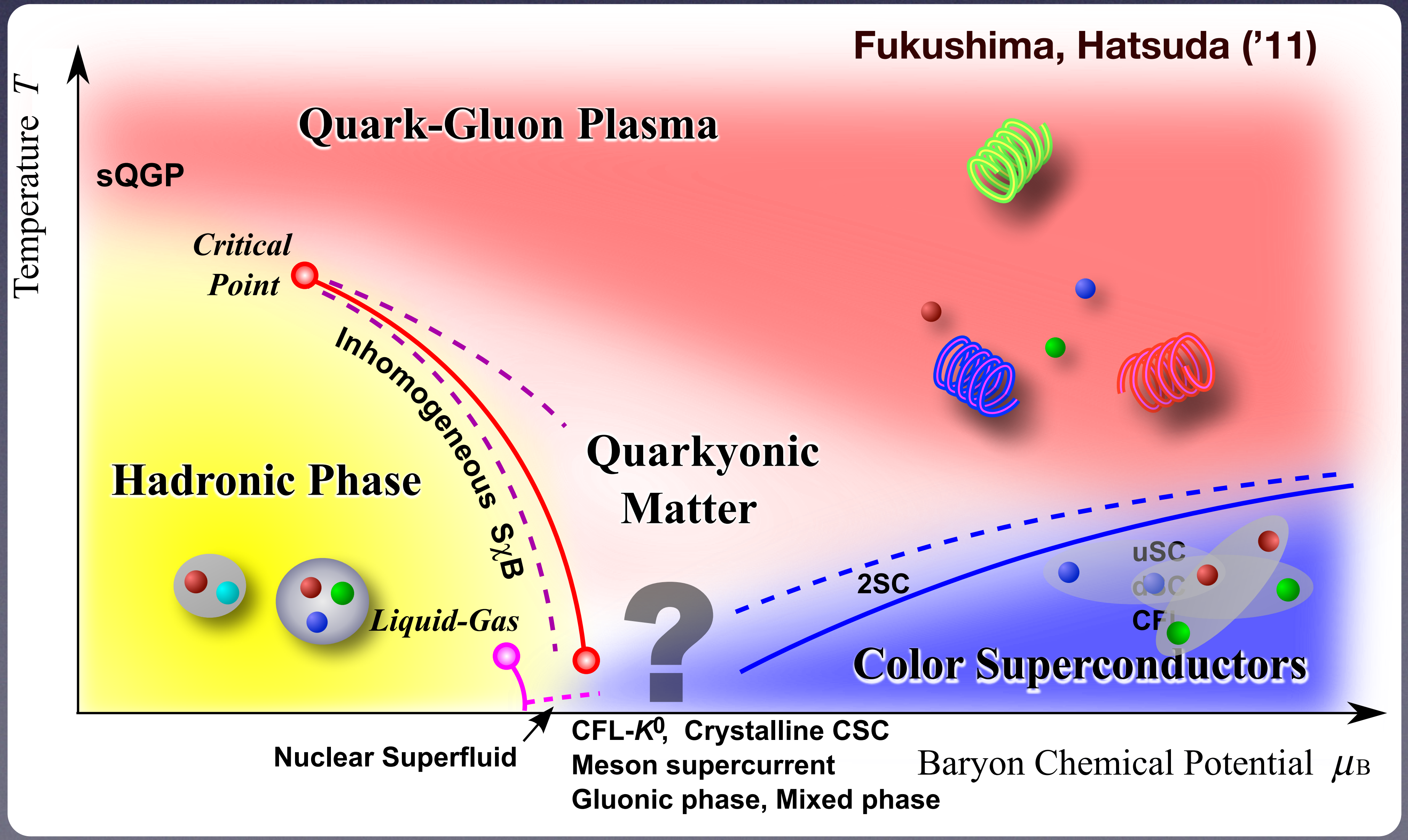
Based on
Hayata, YH, Phys. Rev. D 103 (2021) 9, 094502, 2305.05950, 2306.12324
Hayata, YH, Kikuchi Phys. Rev. D 104 (2021) 7, 074518
Hayata, YH, Nishimura, 2309.xxxxx

Motivation
Big problem in QCD

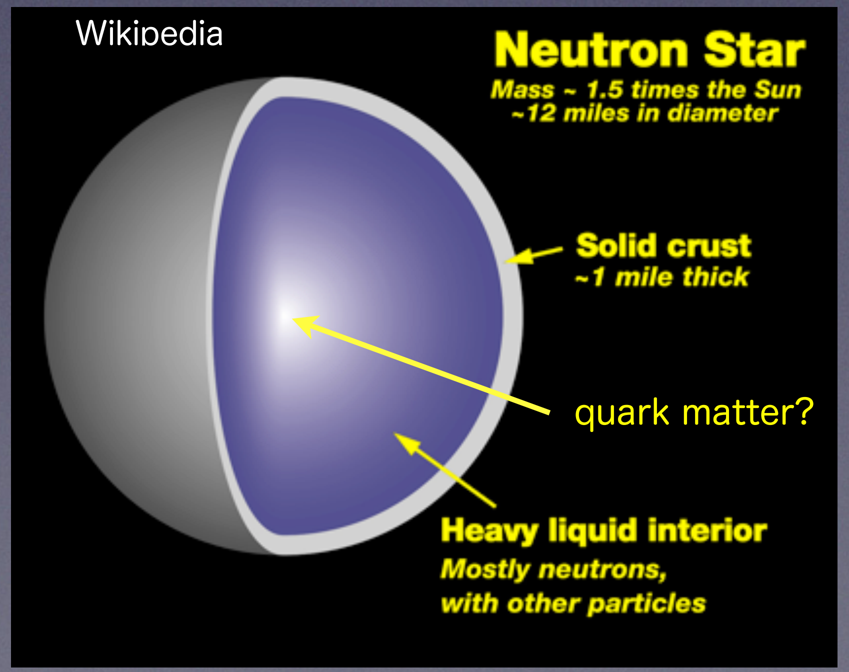
manybody dynamics of QCD



Dense QCD



What phases are realized in the interior of a neutron star?



Difficulty

Sign problem: Difficulties in first-principles calculations based on importance sampling

$$\langle O \rangle = \int \mathcal{D}A \, \det(D+m) e^{iS} O$$

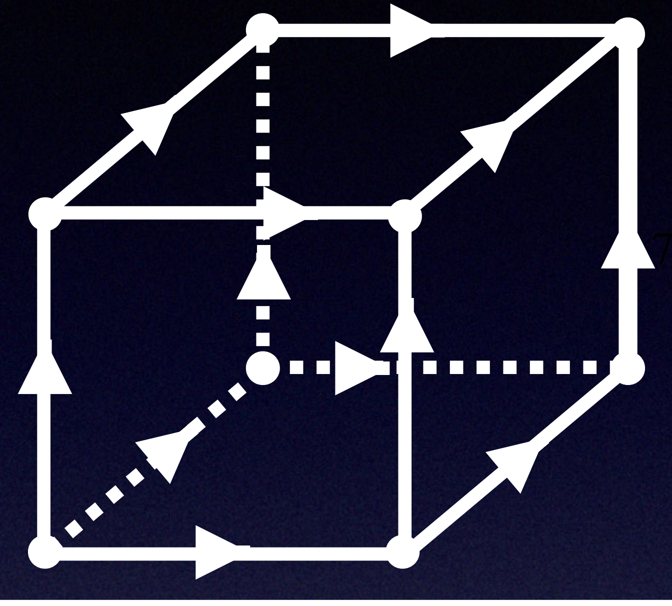
In real-time, finite-density problems, the weight is complex

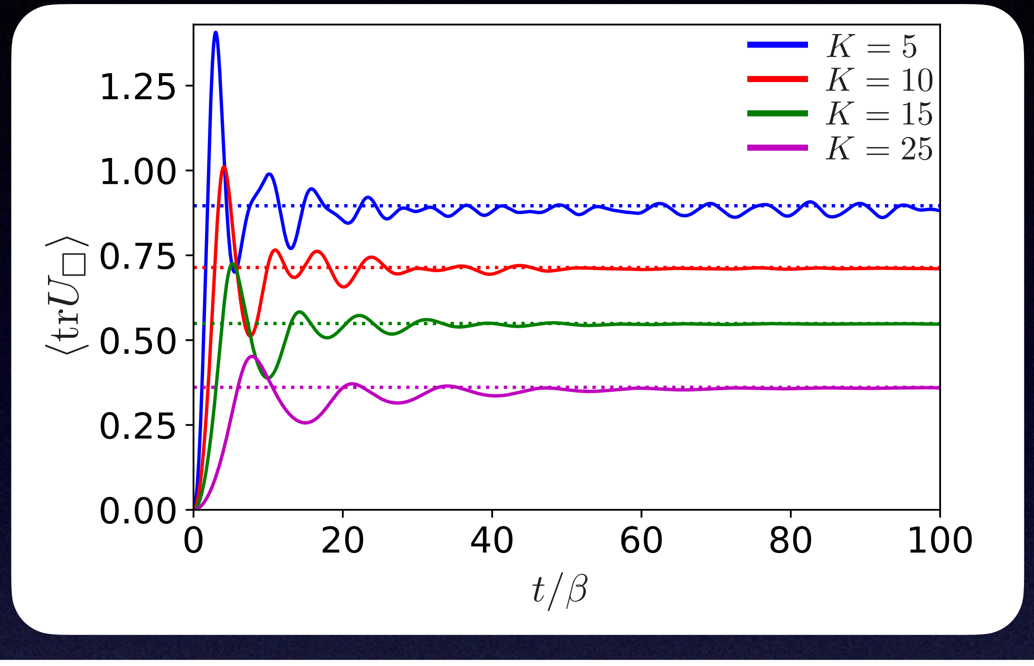
$$\not\approx \frac{1}{N} \sum_j O_j$$

Hamiltonian approach

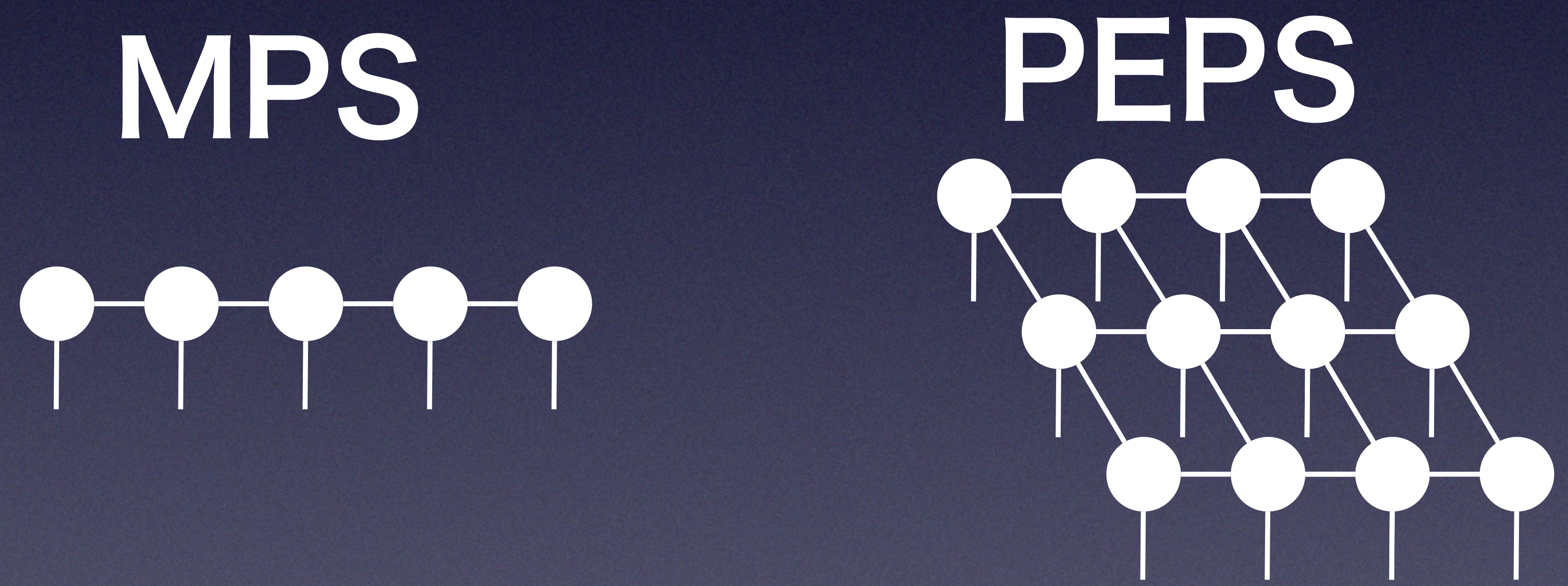
Directly solve Schrodinger equation to avoid sign problem

Smaller systems can be simulated directly

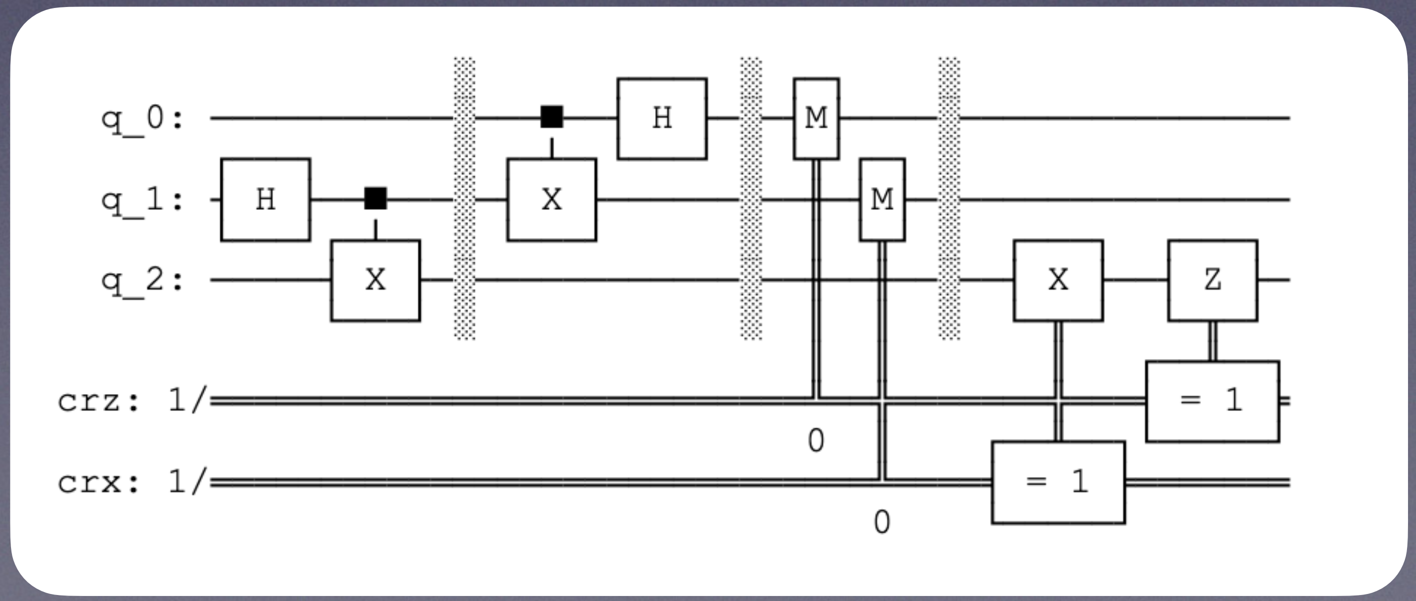




Tensor Networks



Quantum simulation



Difficulty of Hamiltonian gauge theory

Infinite degrees of freedom

Link variable is continuous (regularization required)

$$U \in \mathrm{SU}(N)$$

continuous

What approximation is compatible with gauge symmetry?

Large gauge redundancy

$$\dim \mathscr{H}_{\mathrm{phys}} \ll \dim \mathscr{H}_{\mathrm{total}}$$

need to solve Gauss law constraint

Outline

- **Formalism**
  - Kogut-Susskind Hamiltonian formalism
  - spin network = Gauge invariant Hilbert space
  - Quantum deformation as regularization
- **Application**
  - Confinement-deconfinement phase transition in mean field approximation (2306.12324)
  - Thermalization on a small lattice (Phys. Rev. D 103, 094502(2021))
  - Quantum scar (2305.05950)
  - Scrambling (Phys. Rev. D 104 (2021) 7, 074518)
  - Finite density in (1+1) dimensions (work in progress)
- **Summary**

Kogut-Susskind Hamiltonian formalism

$SU(N)$ Gauge theory ($A_0 = 0$ gauge)

Commutation relation $[A^i_n(x), E_{mj}(x')] = i\delta_{nm}\delta^i_j\delta(x - x')$

Gauge field **Electric field**

Hamiltonian $H = \int d^3x \left(\frac{g^2}{2} E^2(x) + \frac{1}{2g^2} B^2(x) \right)$

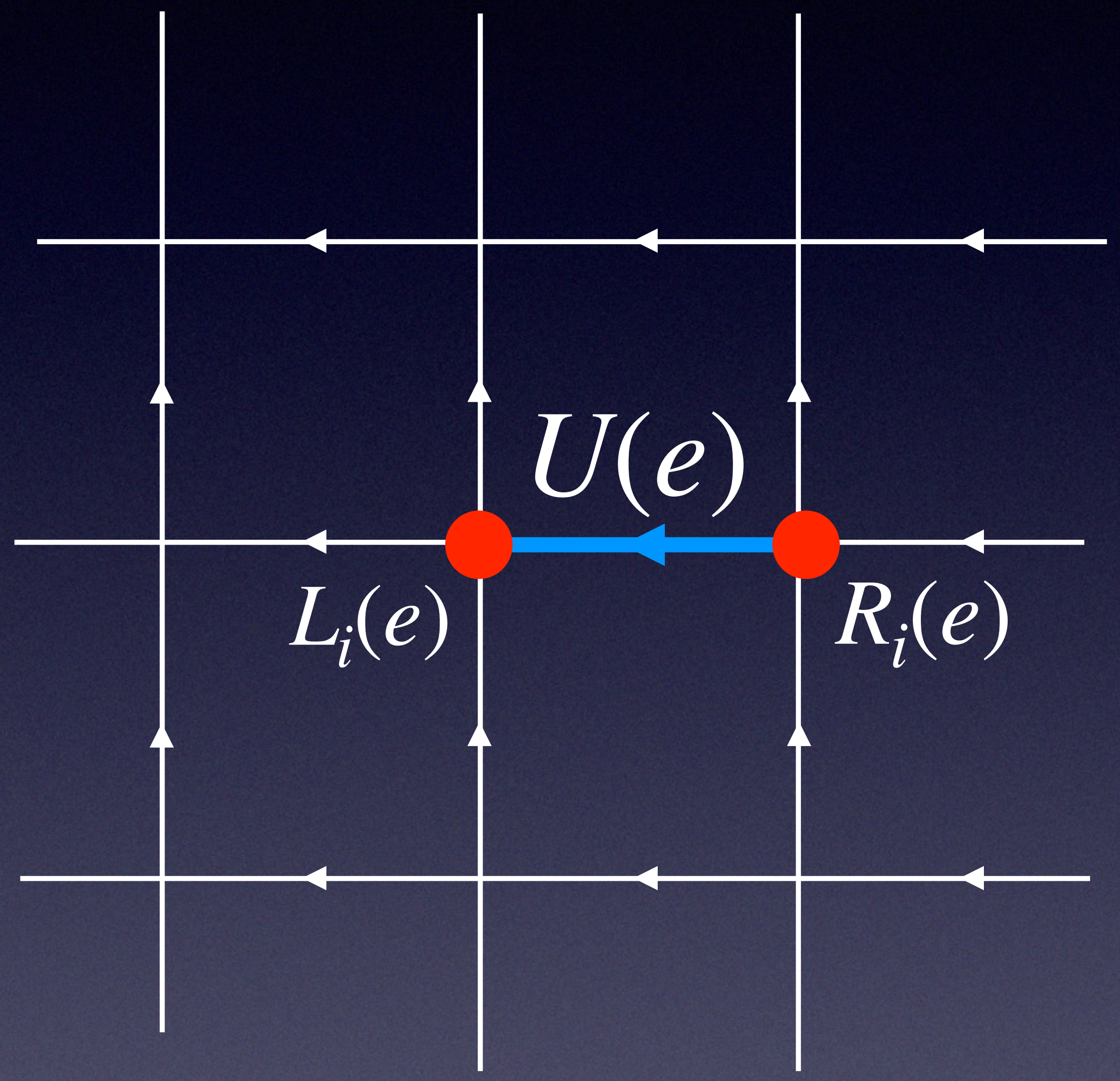
Magnetic field $B^i_l = \frac{1}{2}\epsilon_{lnm}(\partial_m A^i_n - \partial_m A^i_n + f^i_{jk} A^j_m A^k_n)$

Gauss law constraint $(\boldsymbol{D} \cdot \boldsymbol{E})^i \,|\, \Psi_{\text{phys}}\rangle = 0$

Kogut-Susskind Hamiltonian formalism

Kogut, Susskind, Phys. Rev. D 11, 395 (1975)

Time is continuous, space is discretized



$e^{i\int A} \to U(e)$:link variable $\in \mathrm{SU}(N)$ on edge e

$L_i(e), R_i(e)$:Left and right electric fields $\in \mathrm{su}(N)$

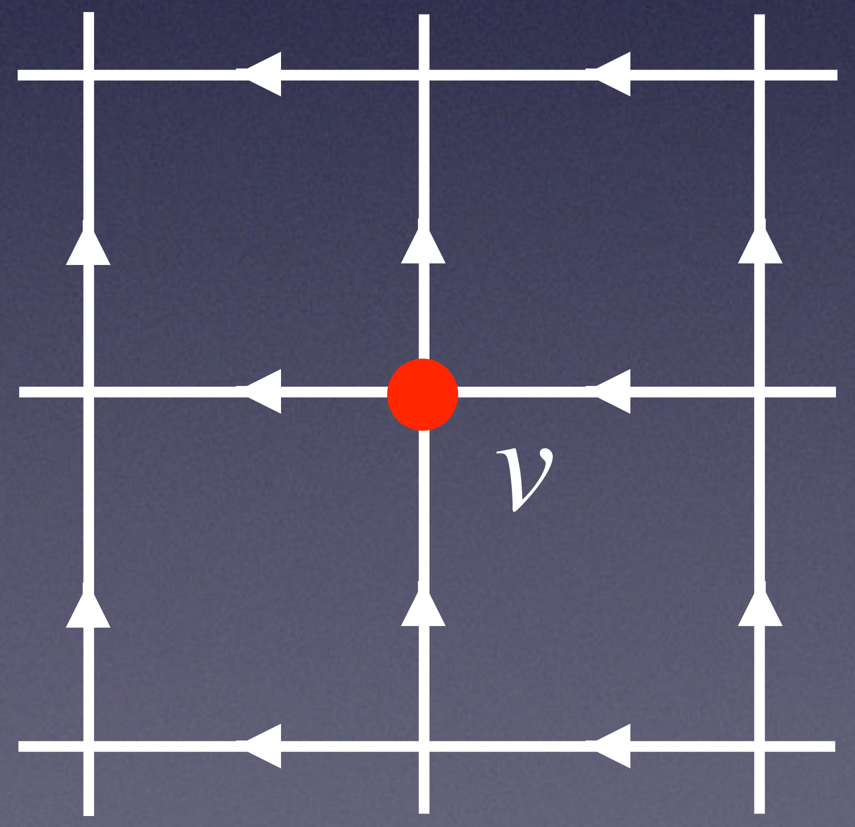
$L_i(e)$ and $R_i(e)$ are not independent

$[U_{\mathrm{adj}}(e)]_i^j L_j(e) = R_i(e)$ ⇒ $R_i^2(e) = L_i^2(e) =: E_i^2(e)$

Commutation relation

$$[A^i_n(x), E_{mj}(x')] = i\delta_{nm}\delta^i_j\delta(x - x')$$

$$[R_i(e), U(e')] = U(e)T_i\delta_{e,e'}$$
$$[L_i(e), U(e')] = T_iU(e)\delta_{e,e'}$$
$$[L_i(e), L_j(e')] = -\,\mathrm{i}f^k_{\ ij}L_k(e)\delta_{e,e'}$$
$$[R_i(e), R_j(e')] = \mathrm{i}f^k_{\ ij}R_k(e)\delta_{e,e'}$$



Gauss law constraint $(D \cdot E)^i\,|\Psi_{\mathrm{phys}}\rangle = 0$

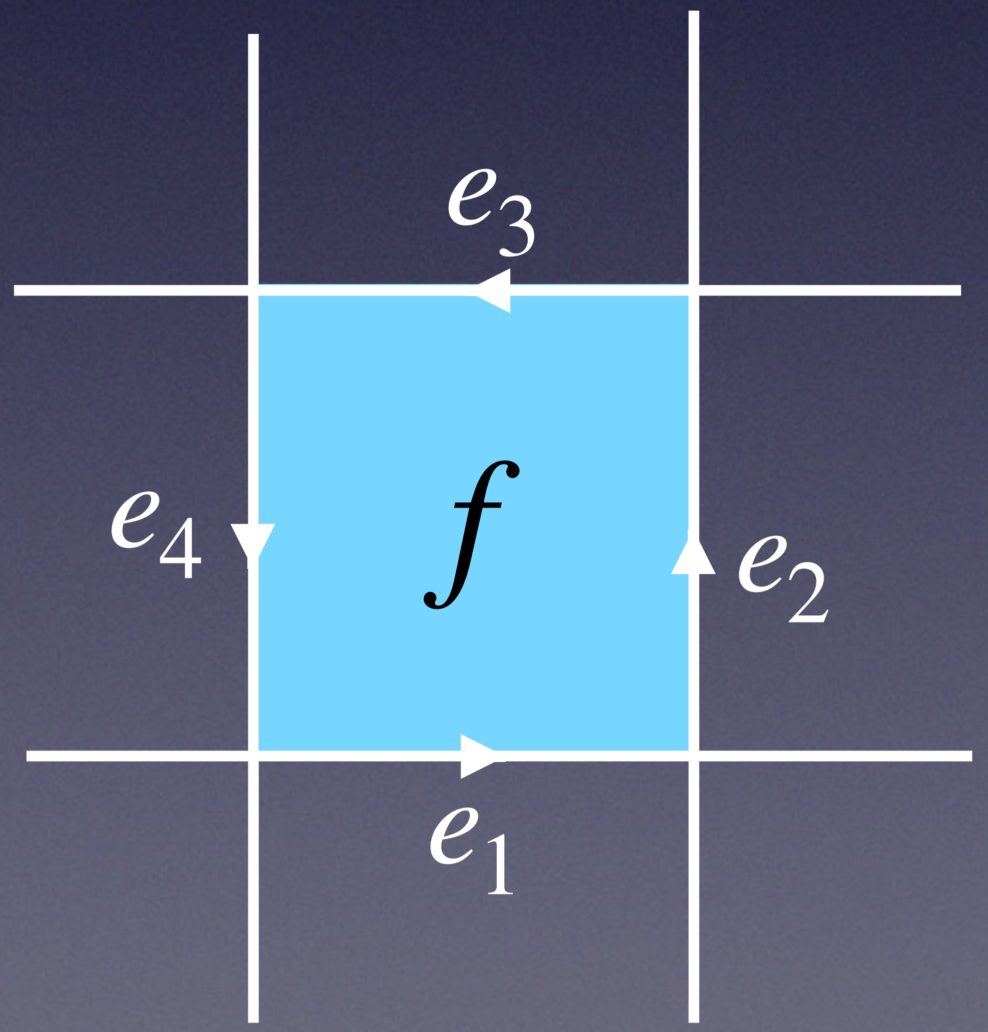
$$\left(\sum_{e\in C_1|\mathrm{s}(e)=v} R_i(e) - \sum_{e\in C_1|\mathrm{t}(e)=v} L_i(e)\right)|\Psi_{\mathrm{phys}}\rangle = 0$$

C_1**:set of edges,** s,t**: source and target functions**

Hamiltonian

$$H = \frac{1}{2}\sum_{e\in C_1}(E(e))^2 - \frac{K}{2}\sum_{f\in C_2}(\mathrm{tr}U(f) + \mathrm{tr}U^\dagger(f))$$

C_2:set of faces



$$U(f) := U(e_4)U(e_3)U(e_2)U(e_1)$$

In order to solve, we need

- Construction of basis
- Solve Gauss law constraint

  ⇒Spin netwoark in the case of SU(2)
- Calculate the action of an operator on a spin network
- Regularization

Two types of bases:

Eigenstates of Wilson lines

$$[U_a]^m_n \,|\, g\rangle = [\rho_a]^m_n(g)\,|\, g\rangle \qquad g \in \mathrm{SU}(2)$$

Wilson line with Rep a

representation matrix

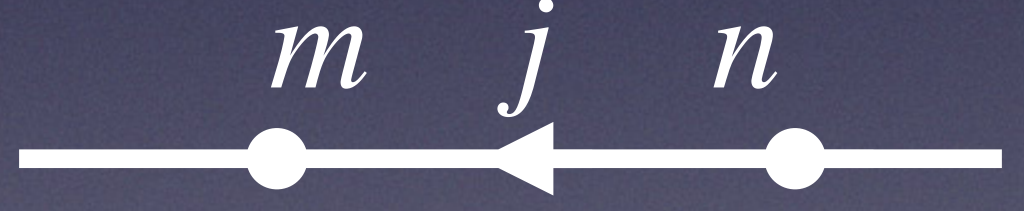
Eigenstates of Electric fields (for $SU(2)$)

$$R_i^2(e)\,|\, j, m, n\rangle = C_2(j)\,|\, j, m, n\rangle$$

Casimir $C_2(j) = j(j+1)$

$$R_3(e)\,|\, j, m, n\rangle = n\,|\, j, m, n\rangle$$

$$L_3(e)\,|\, j, m, n\rangle = m\,|\, j, m, n\rangle$$



$|\, j, m, n\rangle$

State can be generated by Wilson line

$$\sqrt{d_a}[U_a]^{m_a}_{n_a}|0,0,0\rangle = |j_a, m_a, n_a\rangle$$

$d_a = 2j_a + 1$ **Quantum dimension**

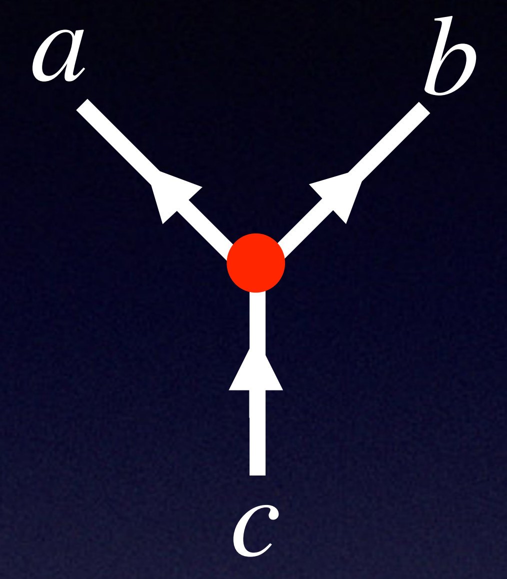
Graphical representation:

$\Leftrightarrow$ $\sqrt{d_a}[U_a]^{m_a}_{n_a}$ $\Leftrightarrow$ $|j_a, m_a, n_a\rangle$

operator **state**

Physical state: Spin network

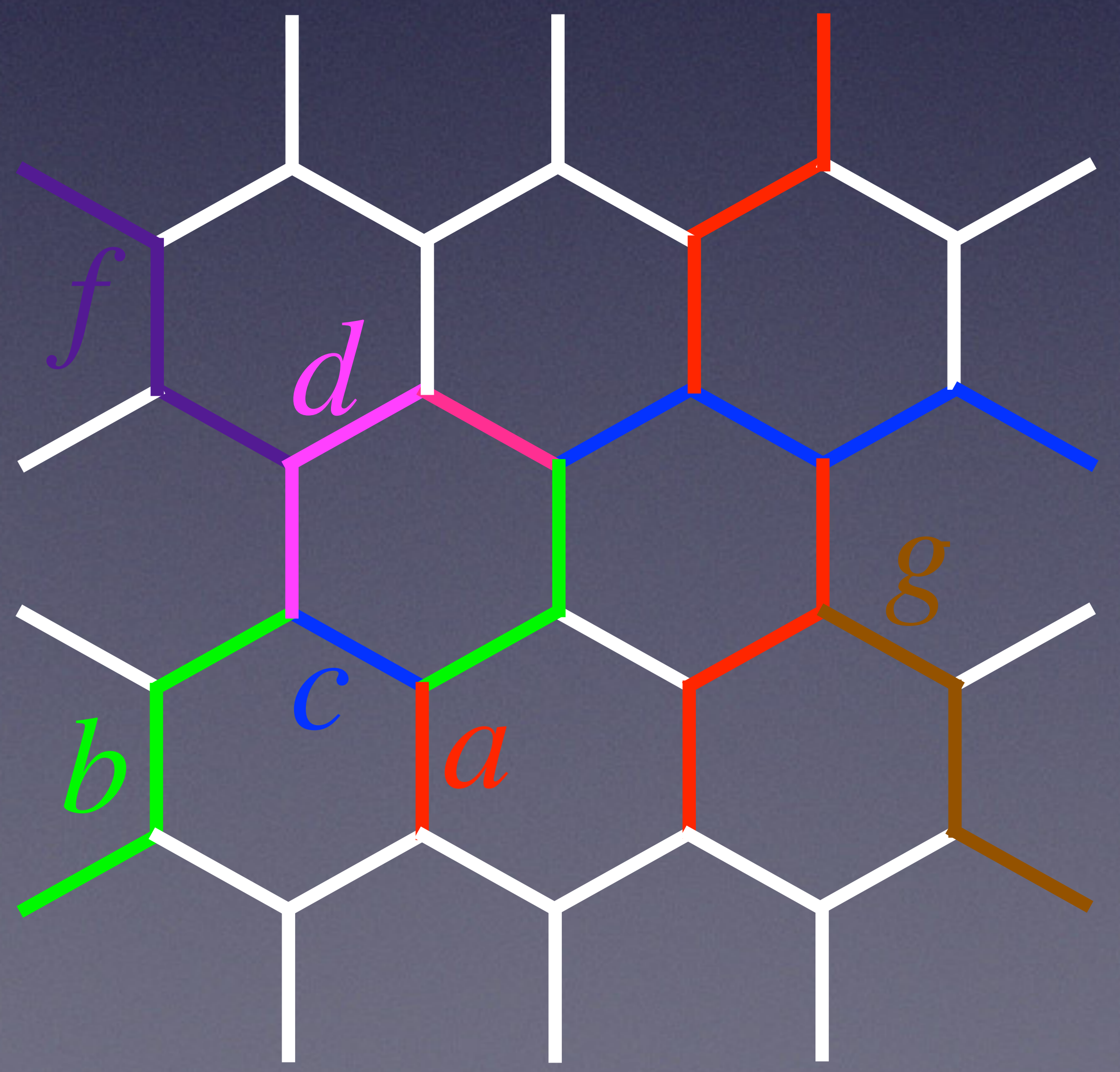
Gauss law constraint: $SU(2)$ invariant of each vertex



$$(R_i(c) - L_i(a) - L_i(b))\,|\Psi_{\text{phys}}\rangle = 0$$

$$\Rightarrow \sum_{n_a, n_b, m_c} \frac{1}{\sqrt{d_c}} \langle j_a n_a\, j_b n_b \,|\, j_c m_c\rangle \,|j_a, m_a, n_a\rangle\,|j_b, m_b, n_b\rangle\,|j_c, m_c, n_c\rangle$$

Clebsch–Gordan coefficients



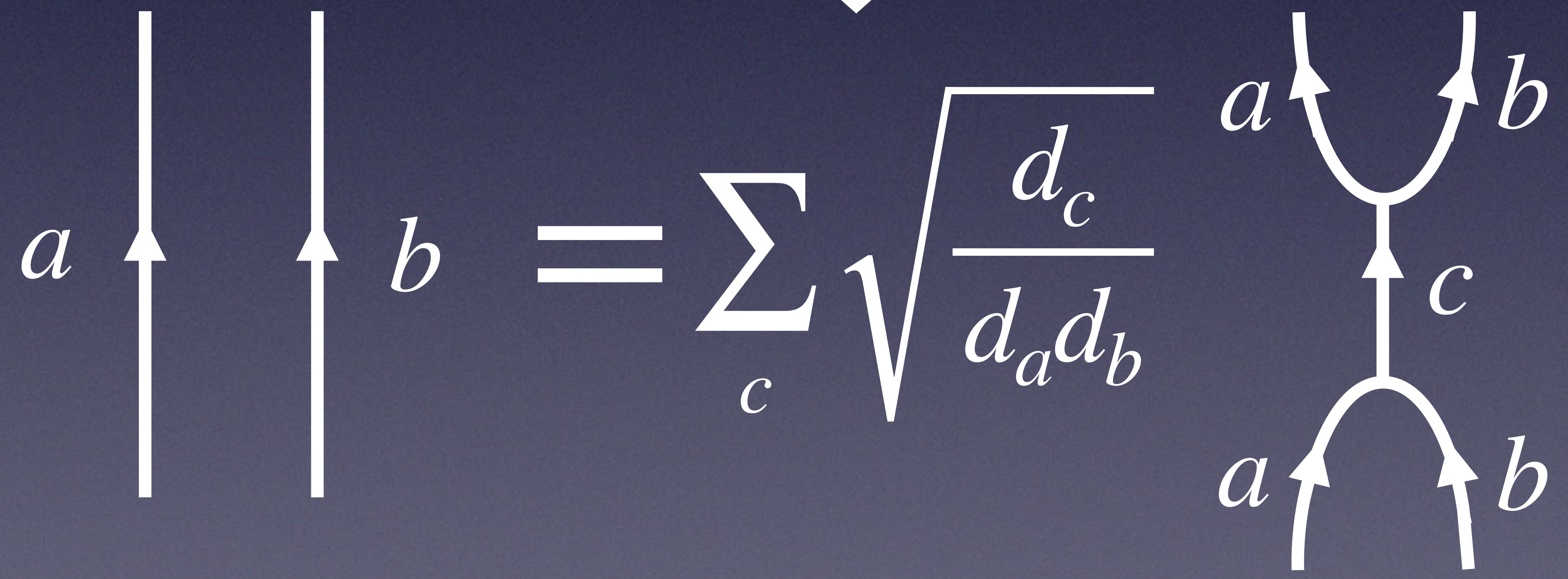
Physical state: spin network with three vertices

Angular momentum labels on the edges

At each vertex the labels satisfy the triangle inequality

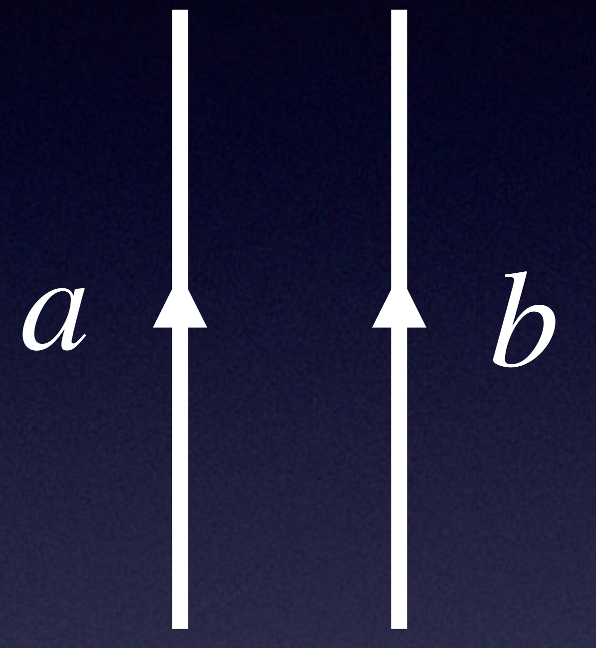
If the composition rules of the Wilson line are known, the action of an operator on a state is determined:

$$[U_a]^{m_a}_{n_a}[U_b]^{m_b}_{n_b} = \sum_{j_c, m_c, n_c} \langle j_a m_a\, j_b m_b \,|\, j_c, m_c \rangle \langle j_c, n_c \,|\, j_a n_a\, j_b n_b \rangle [U_c]^{m_c}_{n_c}$$

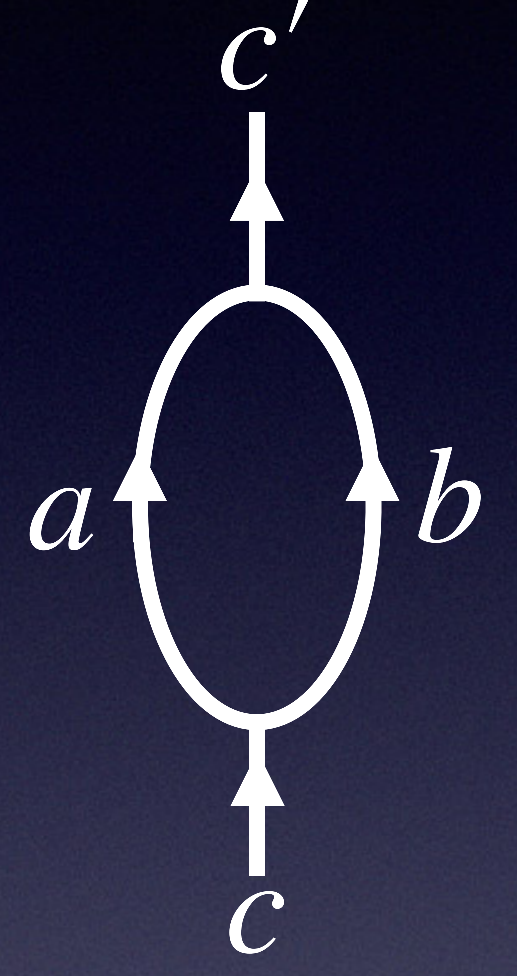


Algebra of Networks

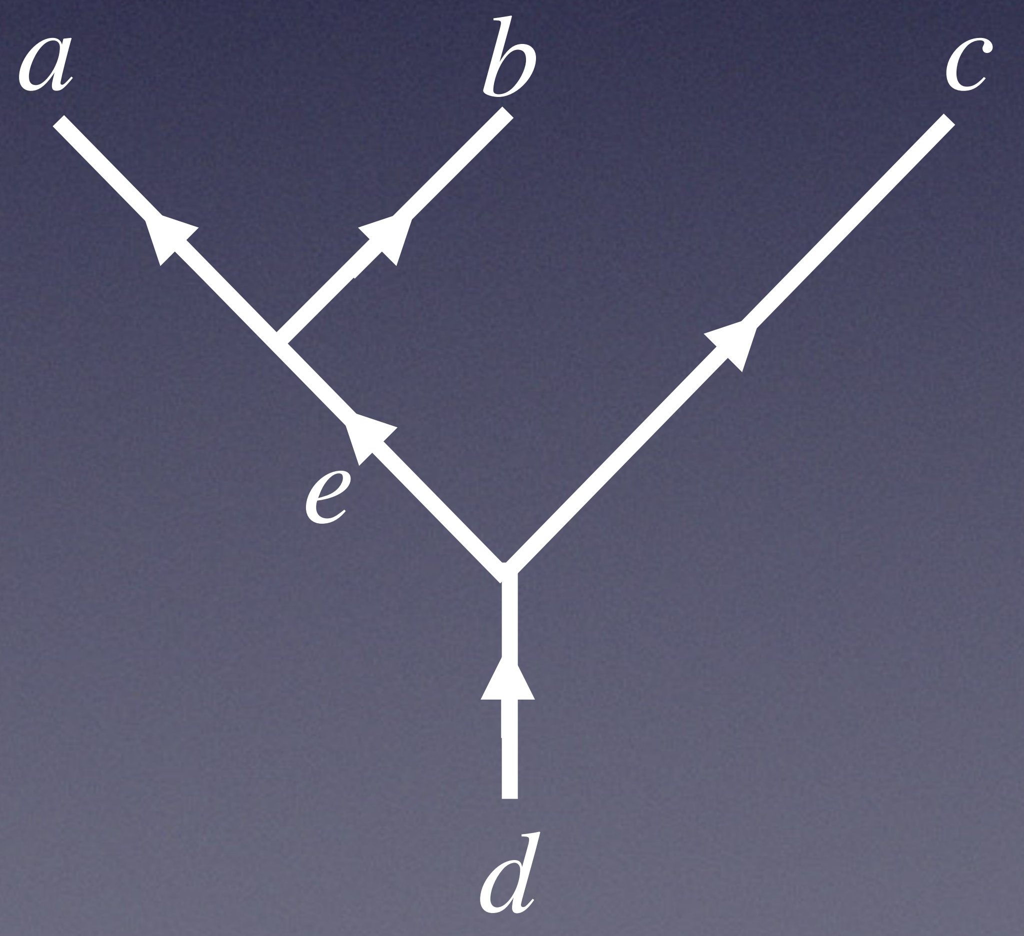
Fusion rule $a \times b = \sum_c N_{ab}^c c \quad N_{ab}^c = \delta_{abc} = \begin{cases} 1 & |j_a - j_b| \le j_c \le j_a + j_b, j_a + j_b + j_c \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$



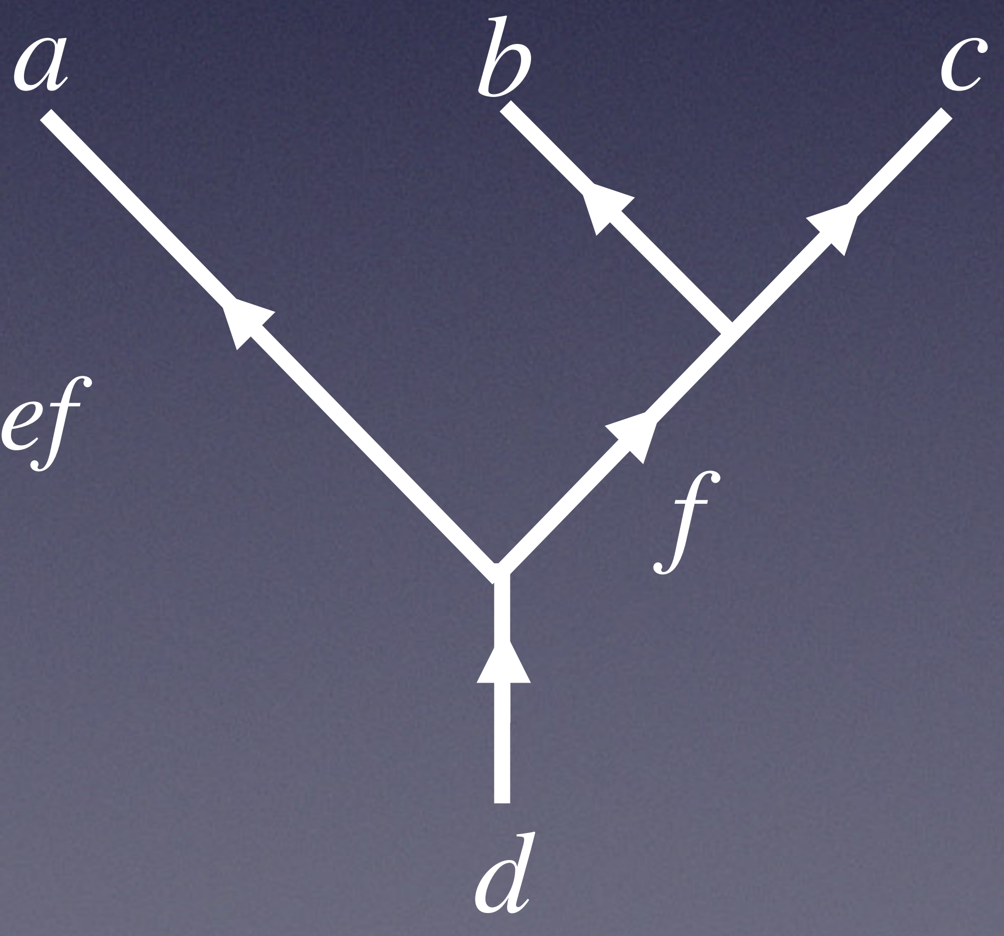
$$= \sum_c \sqrt{\frac{d_c}{d_a d_b}}$$



$$= \delta_c^{c'} \sqrt{\frac{d_a d_b}{d_c}}$$



$$= \sum_f [F_d^{abc}]_{ef}$$



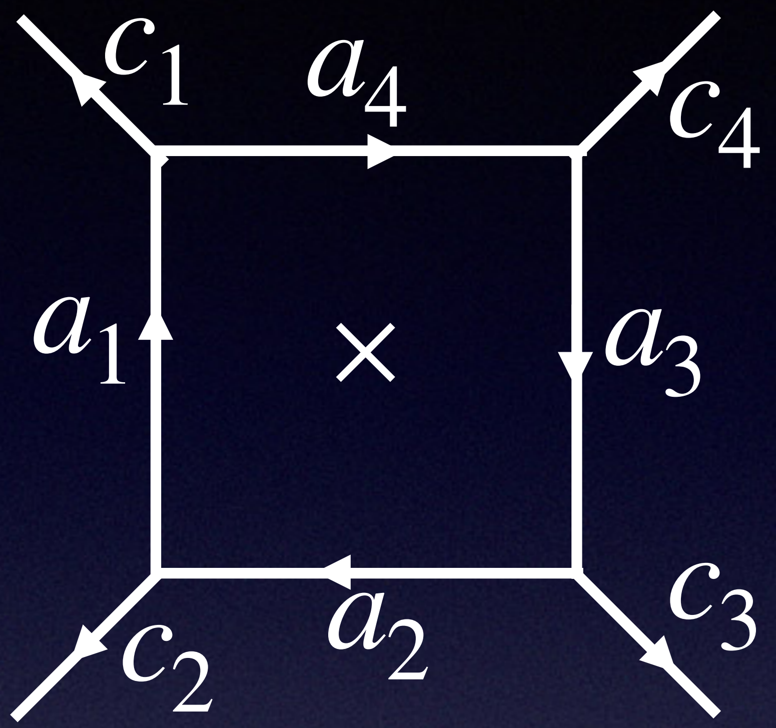
Wigner 6-j symbol

$$[F_d^{abc}]_{ef} = (-1)^{j_a + j_b + j_c + j_d} \sqrt{d_e d_f} \begin{Bmatrix} j_a & j_b & j_e \\ j_c & j_d & j_f \end{Bmatrix}$$

+consistency condition

Action of Wilson loop on network

$\mathrm{tr}U_d$



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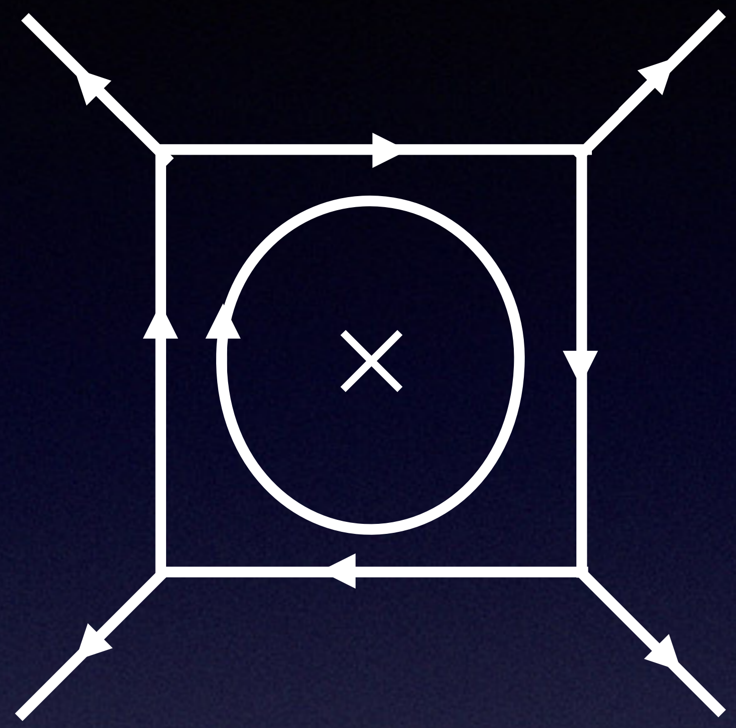
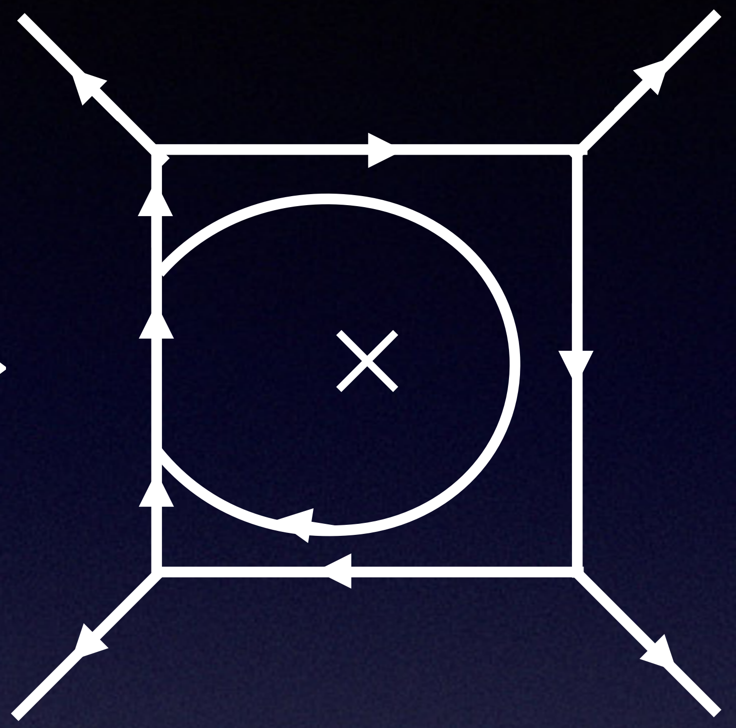
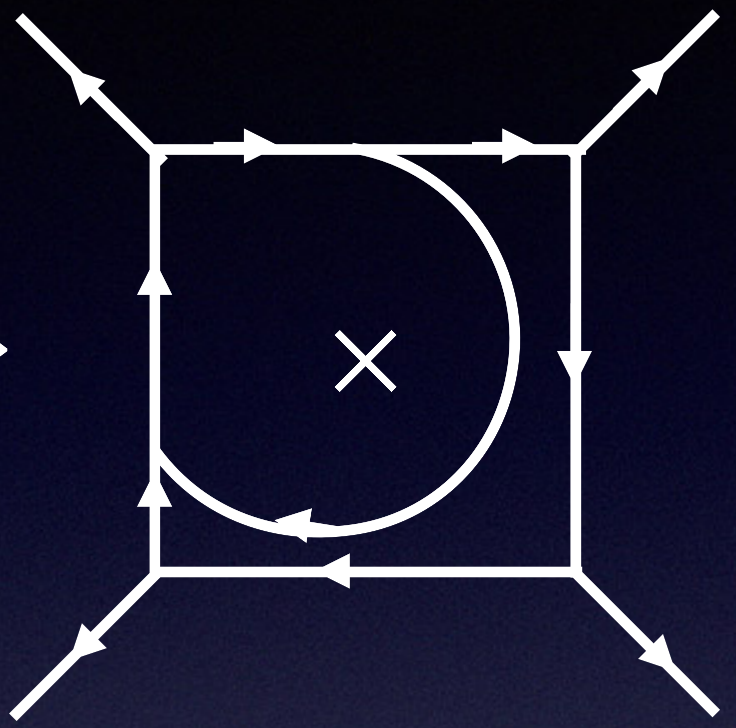


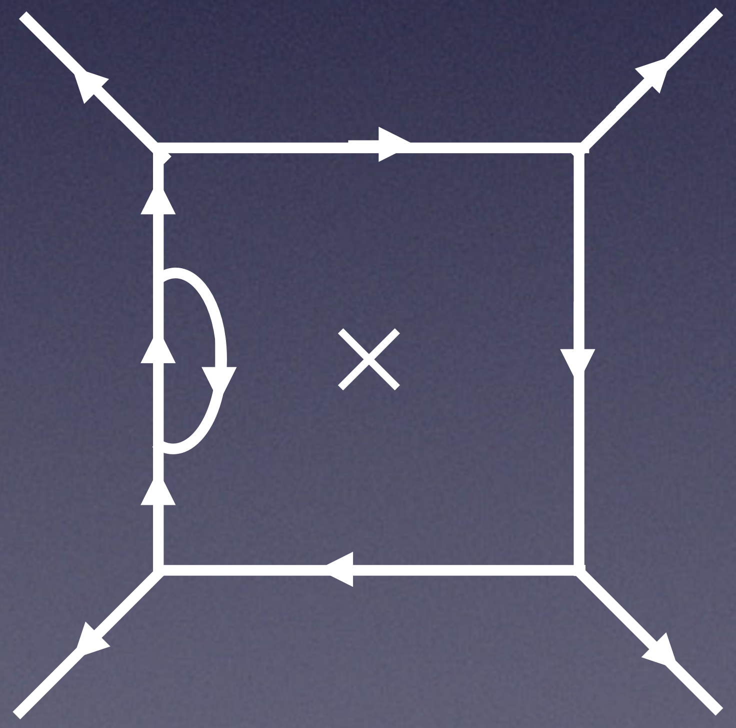
$\Rightarrow$



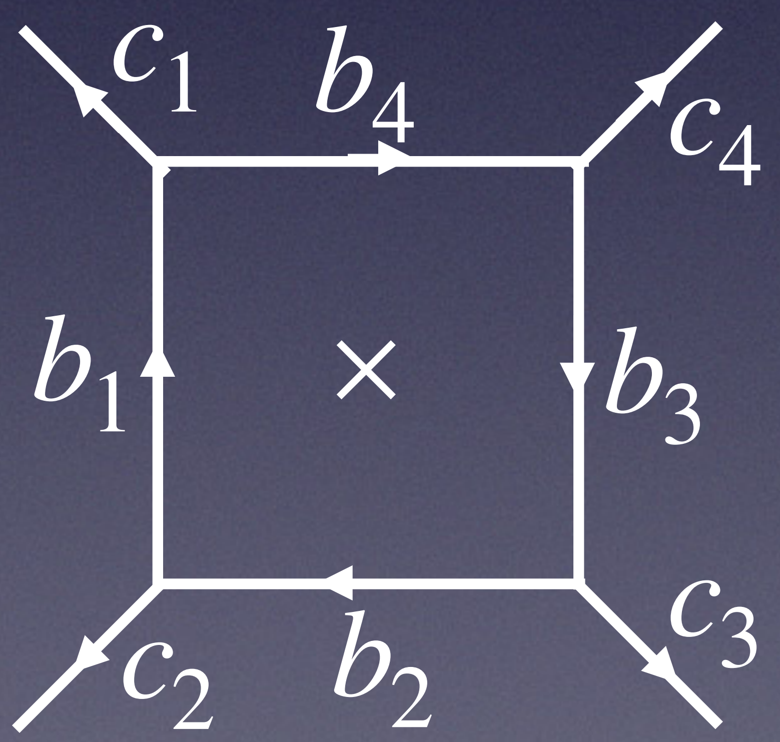
$\Rightarrow$



$\Rightarrow$



$$= \prod_{i=1}^{4} \sum_{b_i} [F^{c_i a_{i-1} d}_{b_i}]_{a_i b_{i-1}}$$



Regularization by representation labels

Cut off the maximum value of the representation $j \leq k/2$ by an integer k to satisfy the composition rule

⇒Quantum group $\mathrm{SU}(2)_k$

Integer n is replaced by $[n] = \dfrac{\sin \frac{\pi}{k+2} n}{\sin \frac{\pi}{k+2}}$

e.g., $C_2(j) = [j][j+1]$

Spin networks are non-local

We do not impose the triangular inequality on the state. Instead, we add a penalty term in the Hamiltonian.

$$\delta_{abc} = \begin{cases} 1 & |j_a - j_b| \le j_c \le j_a + j_b, j_a + j_b + j_c \in \mathbb{Z} \text{ and } j_a + j_b, j_a + j_b + j_c \le k/2 \\ 0 & \text{else} \end{cases}$$

$$Q\,|j_a, j_b, j_c\rangle = \delta_{abc}\,|j_a, j_b, j_c\rangle$$

$$H \to H - t \sum_{v \in C_0} Q_v \quad t \to \infty$$

set of vertices

Spin model of $SU(2)_k$ Yang-Mills theory

d.o.f. on edges : $j = 0, 1/2, \cdots, k/2$

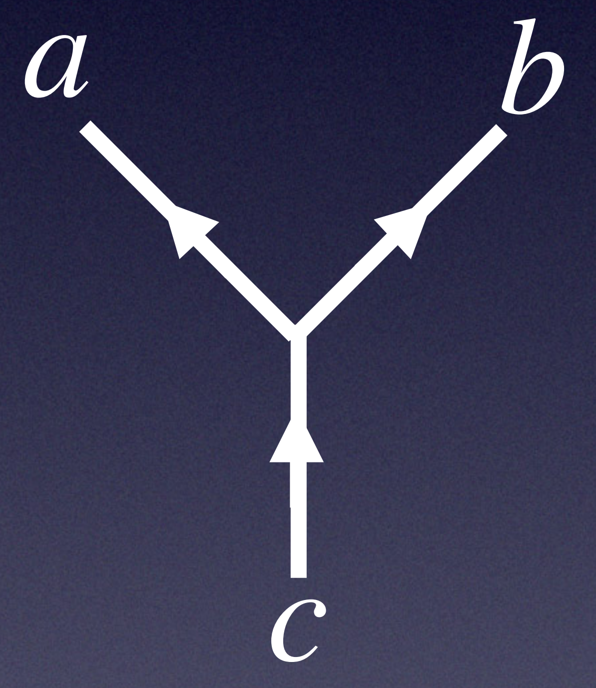
Hamiltonian $$H = \frac{1}{2}\sum_{e \in C_1} (E_i(e))^2 - K \sum_{f \in C_2} \mathrm{tr}U(f) - t \sum_{v \in C_0} Q(v)$$

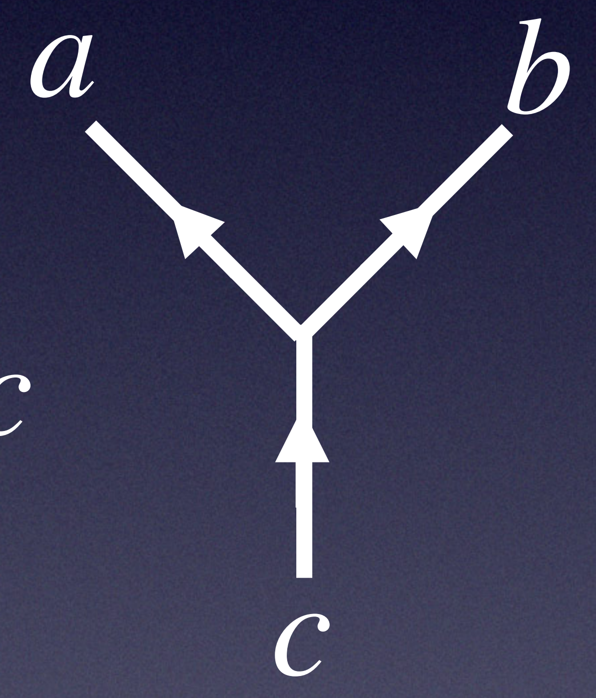
C_1: set of edges; C_2: set of faces; C_0: set of vertices

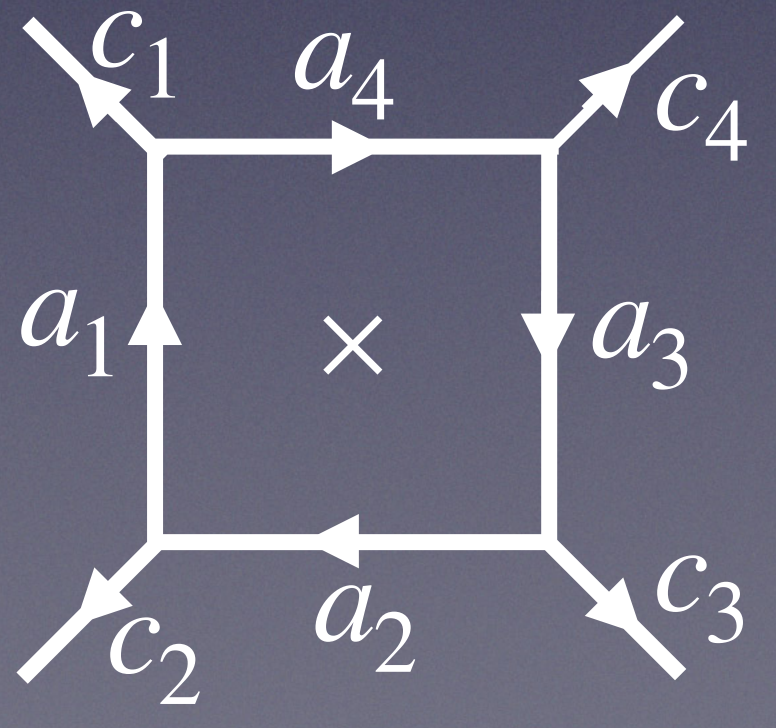
Action on a state

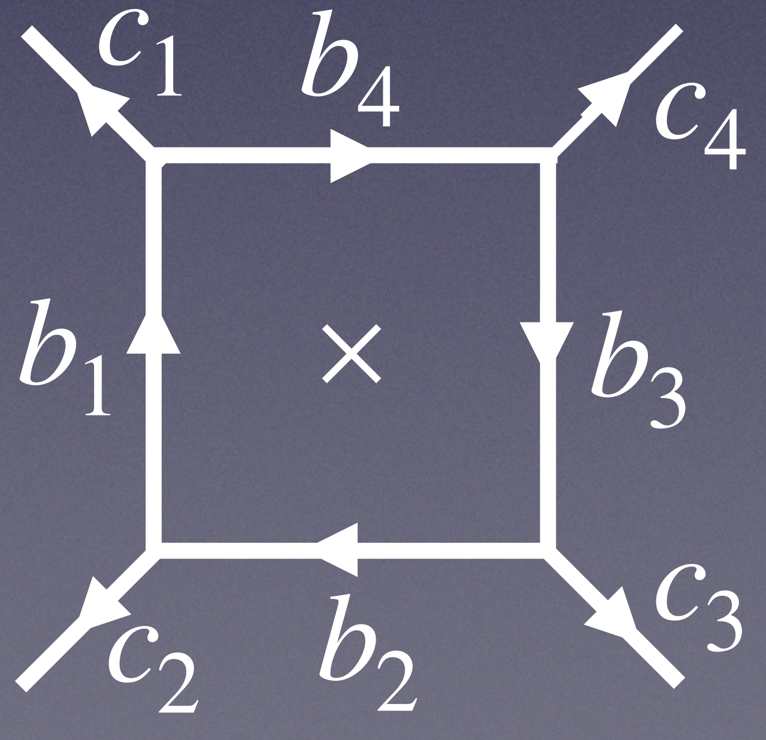
E_i^2 

$= C_2(a)$ 

Q 

$= \delta_{abc}$ 

$\mathrm{tr}U$ 

$= \prod_{i=1}^{4} \sum_{b_i} [F^{c_i a_{i-1} \frac{1}{2}}_{b_i}]_{a_i b_{i-1}}$ 

cf. Levin, Wen, Phys. Rev. B 71 (2005) 045110

$\mathrm{SU}(3)_k$ Yang-Mills theory

A difference is the existence of multiplicity of composition N_{ab}^{c}

Example $\mathrm{SU}(2)_k$ $\mathrm{SU}(3)_k$

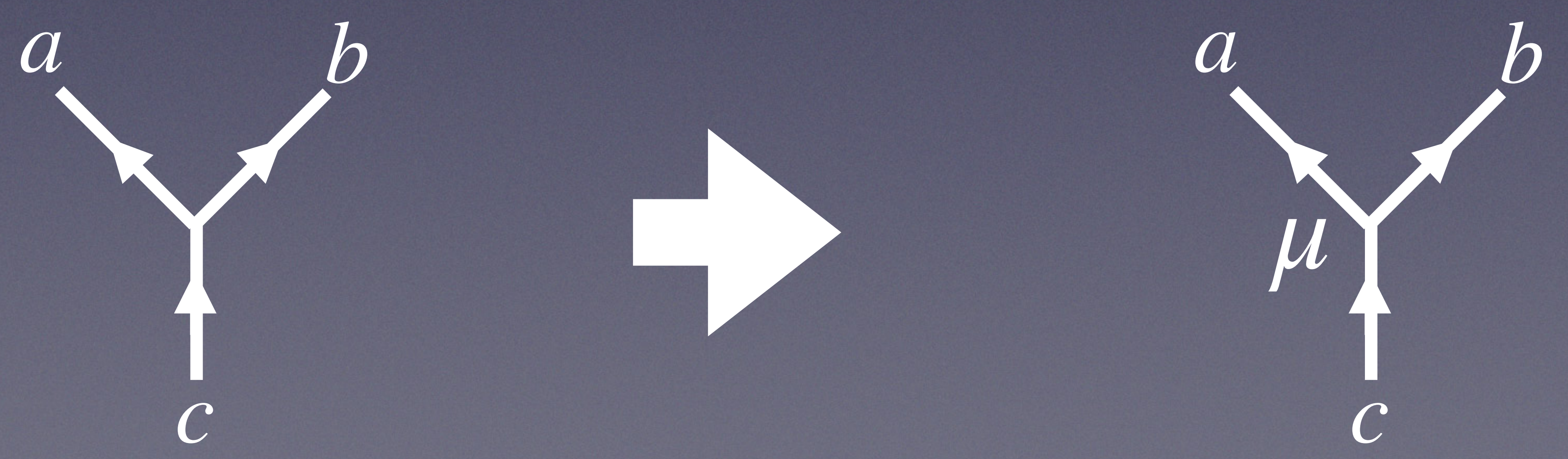
$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$$

$$N_{\mathbf{33}}^{a} = 0 \text{ or } 1$$

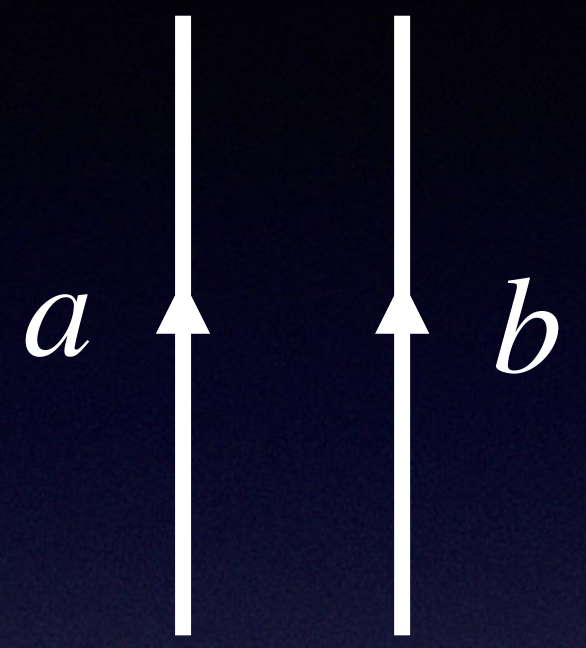
$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \overline{\mathbf{10}} \oplus \mathbf{10} \oplus \mathbf{27}$$

$$\Rightarrow N_{\mathbf{88}}^{\mathbf{8}} = 2$$

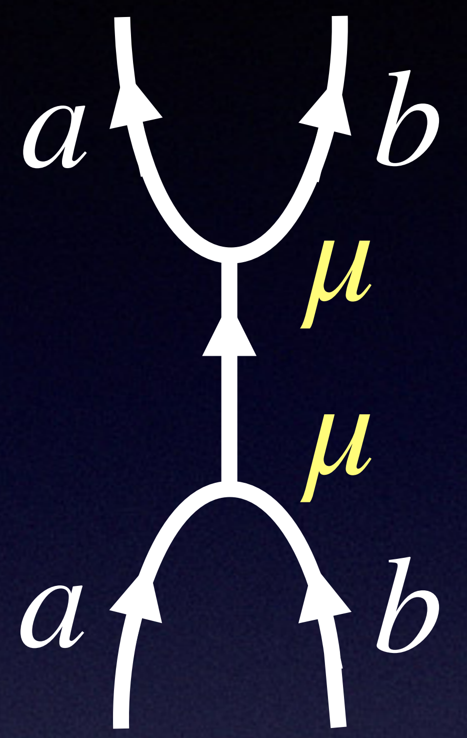
We need an additional label to represent a state of $\mathrm{SU}(3)_k$

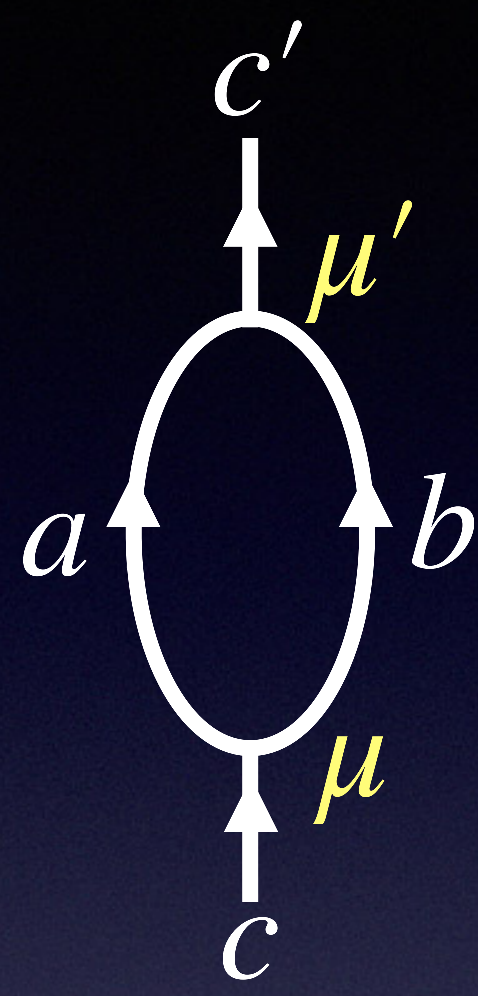


Algebra of Networks for $\mathrm{SU}(3)_k$

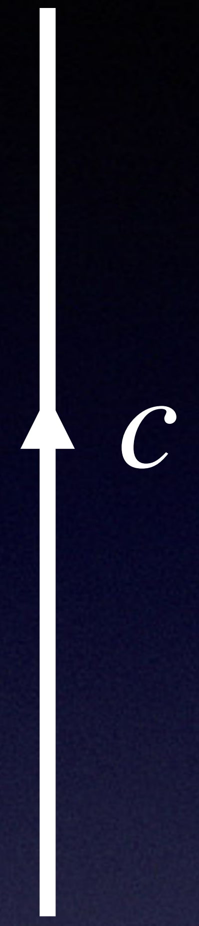


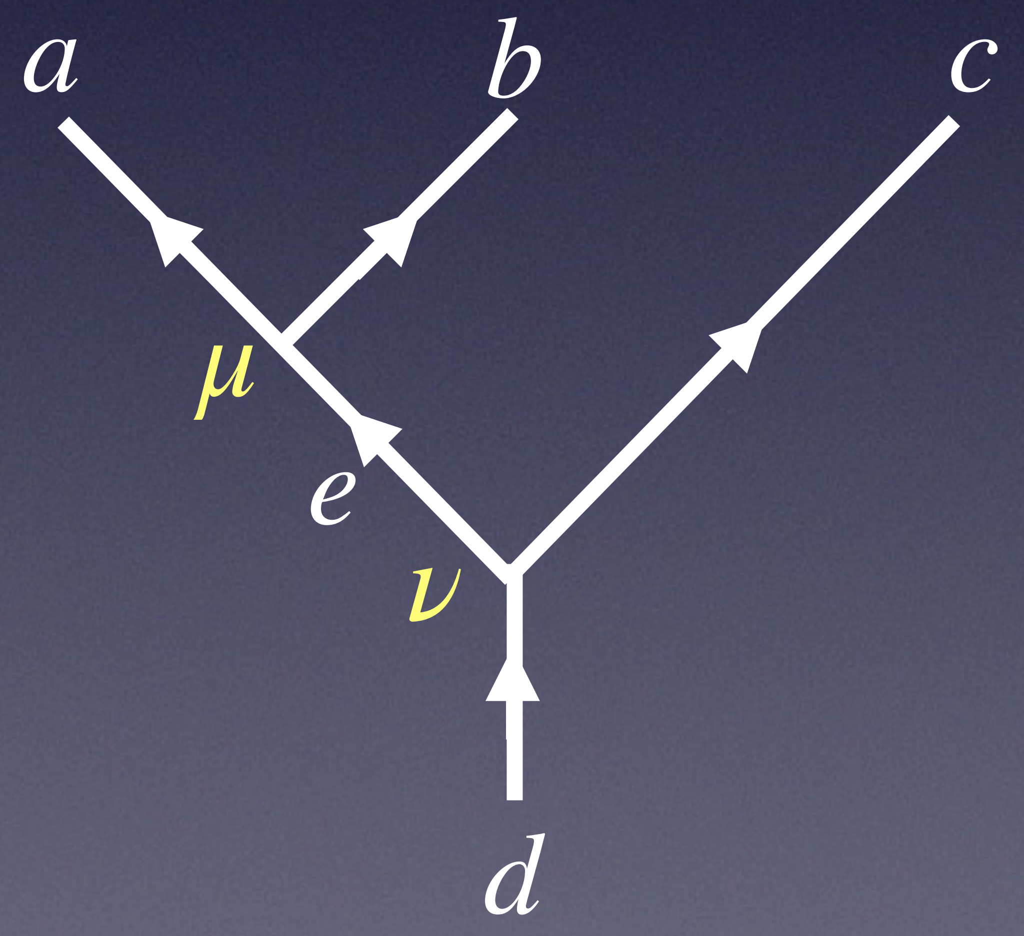
$$=\sum_{c,\mu}\sqrt{\frac{d_c}{d_a d_b}}$$



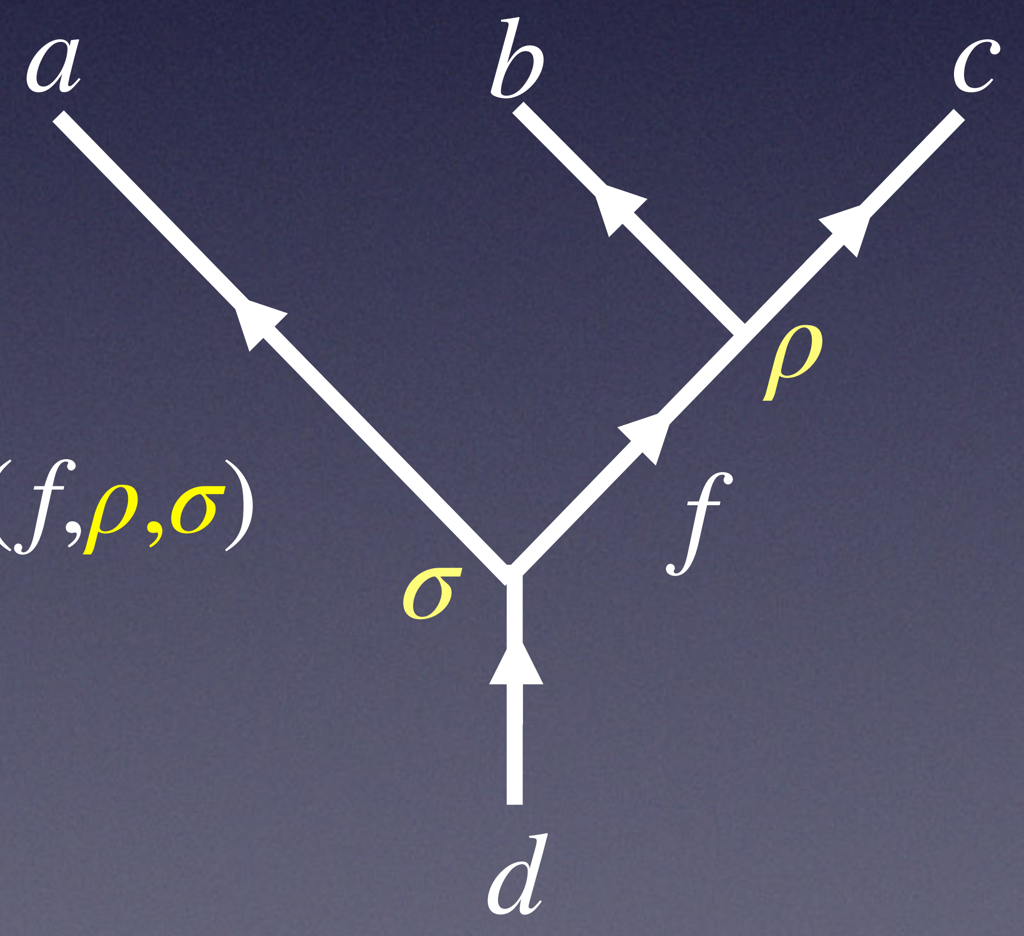


$$=\delta_c^{c'}\delta_\mu^{\mu'}\sqrt{\frac{d_a d_b}{d_c}}$$





$$=\sum_{f,\rho,\sigma}[F^{abc}_d]_{(e,\mu,\nu)(f,\rho,\sigma)}$$



+consistency condition

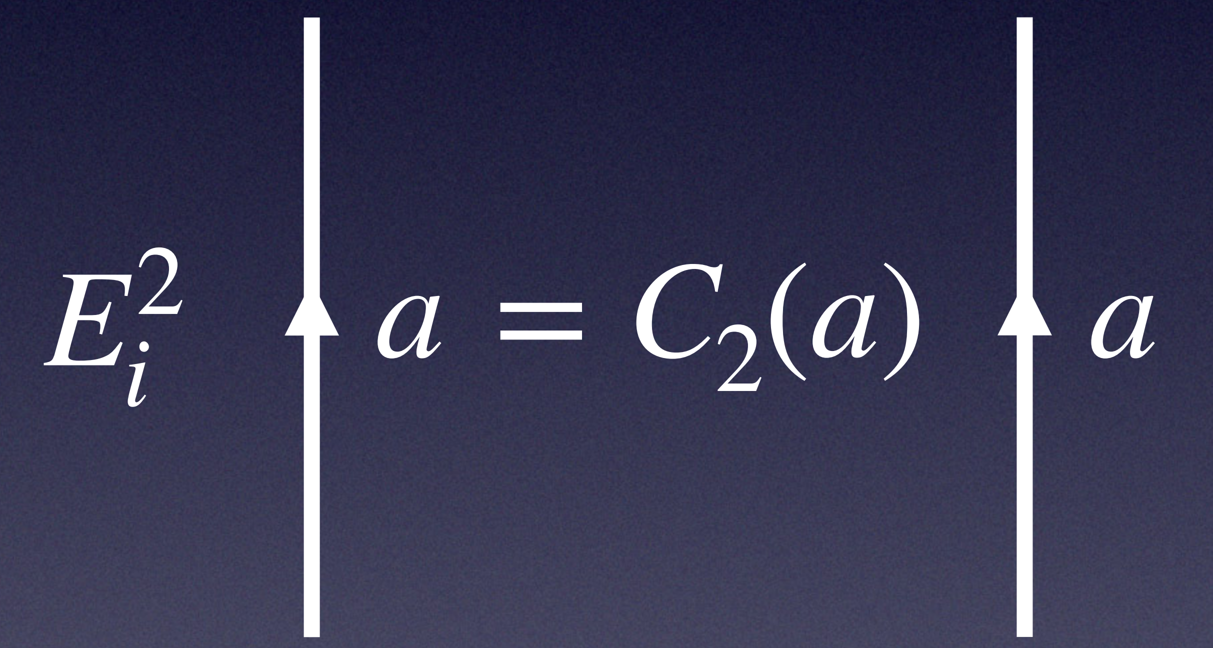
In my best knowledge, general form of F symbol for $\mathrm{SU}(3)_k$ are not known.

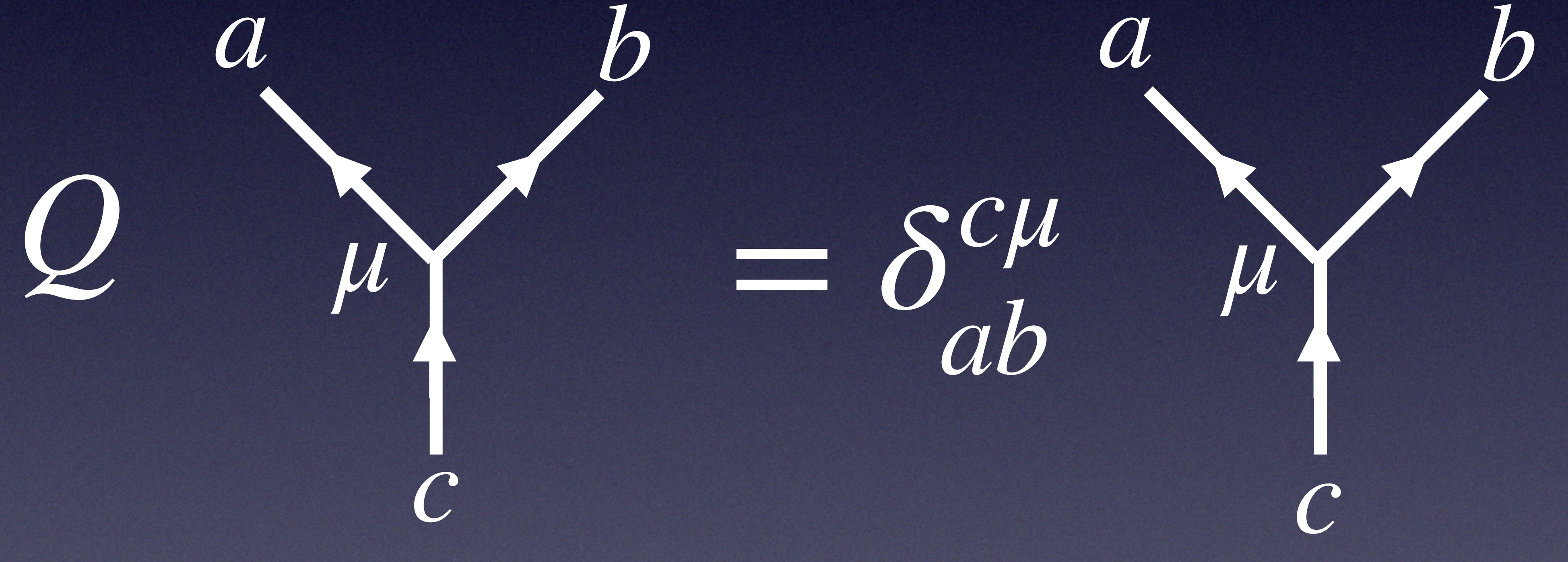
Spin model of $\mathrm{SU}(3)_k$ Yang-Mills theory

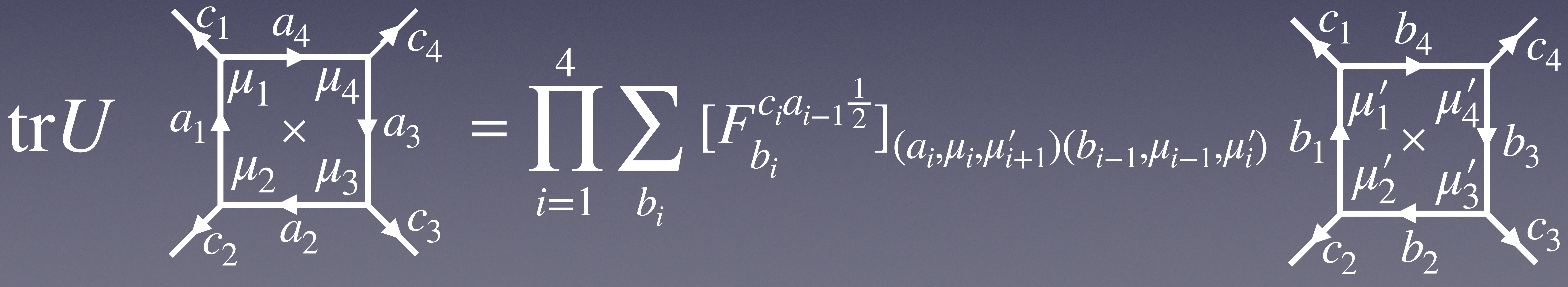
Hamiltonian $$H = \frac{1}{2}\sum_{e\in C_1}(E(e))^2 - \frac{K}{2}\sum_{f\in C_2}(\mathrm{tr}U(f) + \mathrm{tr}U^\dagger(f)) - t\sum_{v\in C_0}Q(v)$$

set of edges — set of faces — set of vertices

Action on a state







- **Application**
  - Confinement-deconfinement phase transition in mean field approximation
  - Thermalization on a small lattice

Confinement-deconfinement phase transition in mean field approximation for $\mathrm{SU}(3)_k$ in (2+1) dimensions

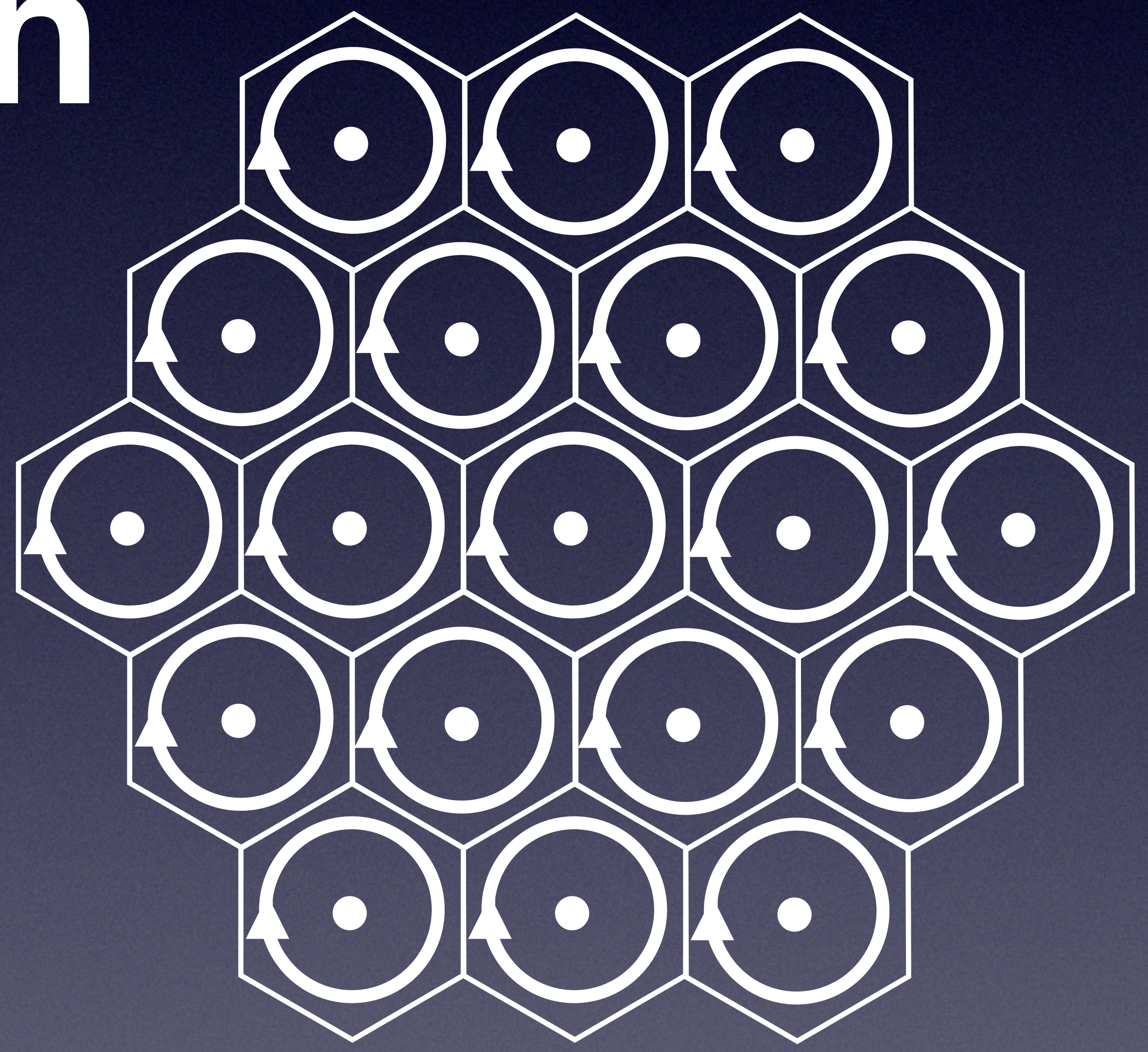
Variational ansatz for wave function

Dusuel, Vidal, Phys. Rev. B 92 (2015) 12, 125150, Zache, González-Cuadra, Zoller, 2304.02527, Hayata, YH, 2306.12324

$$|\Psi\rangle = \prod_{f\in\mathcal{F}} \sum_{a_f} \psi(a_f)\, \mathrm{tr}\, U_{a_f}(f)\, |0\rangle$$

Graphical representation

$$\prod_{f\in\mathcal{F}} \mathrm{tr}\, U_{a_f}(f)\, |0\rangle =$$



We minimize the energy expectation value

open boundary condition, infinite volume limit

$$E = \min_{\psi} \langle\Psi| H |\Psi\rangle$$

We can calculate observables for given wave function

Hayata, YH, 2306.12324

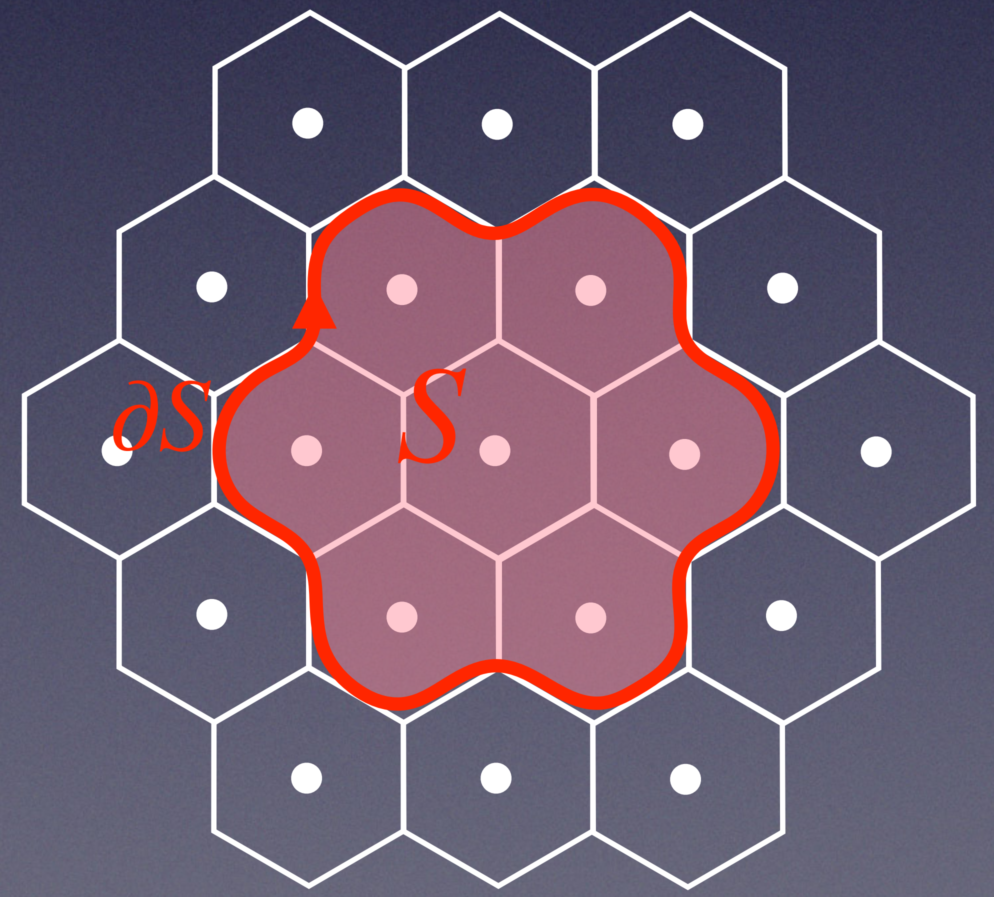
Energy density

$$h = \frac{1}{V}\langle H \rangle = \sum_{a,b,c} C_2(c) N^c_{\bar{a}b} \frac{d_c}{d_a d_b} |\psi(a)|^2 |\psi(b)|^2 - \frac{K}{2} \sum_{a,b} \psi^*(a) \left(N^a_{(1,0)b} + N^a_{(0,1)b} \right) \psi(b)$$

Wilson loop

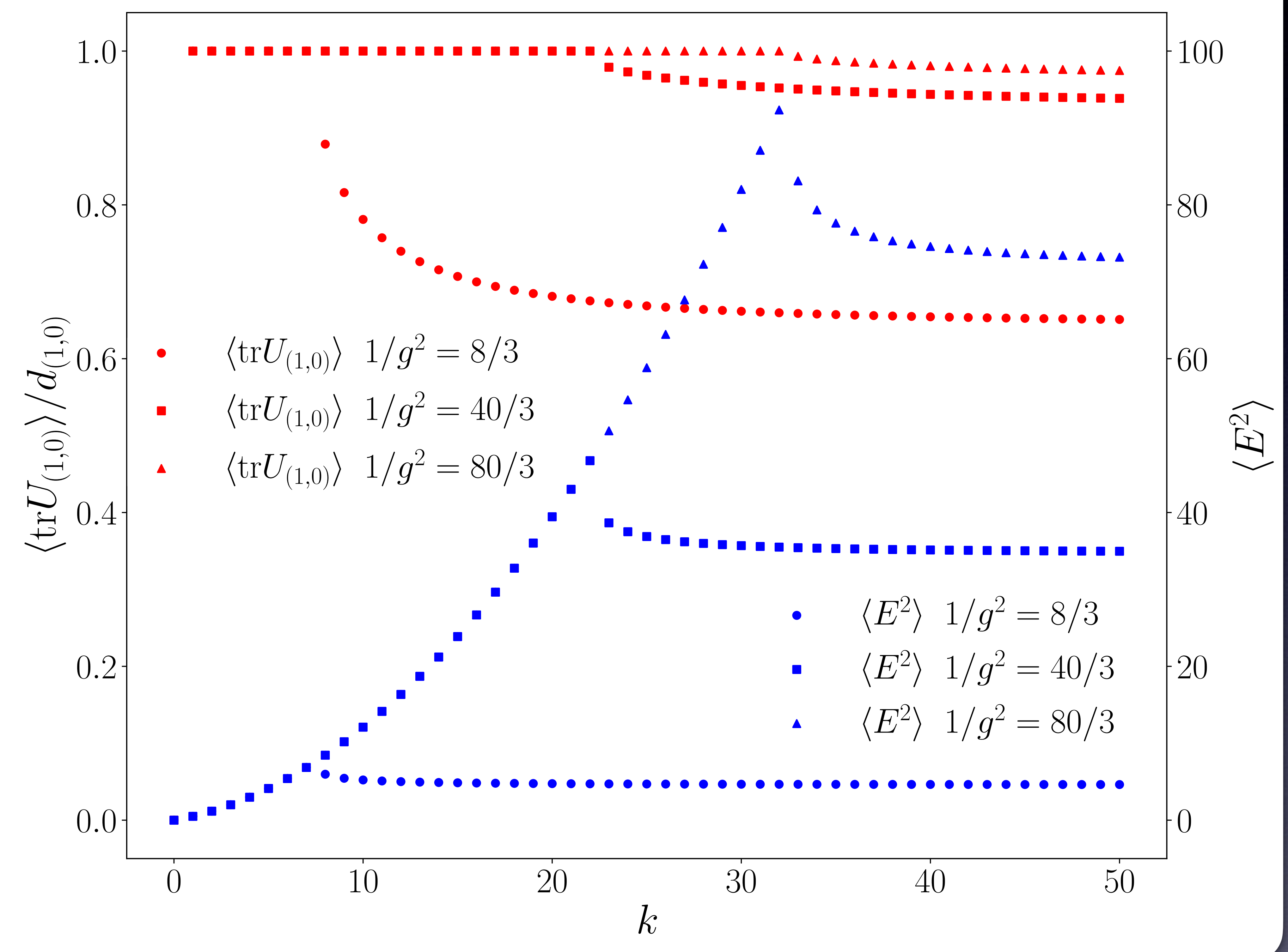
$$\langle \mathrm{tr}\, U_d(\partial S) \rangle = d_d \exp(-|S|\sigma_d)$$

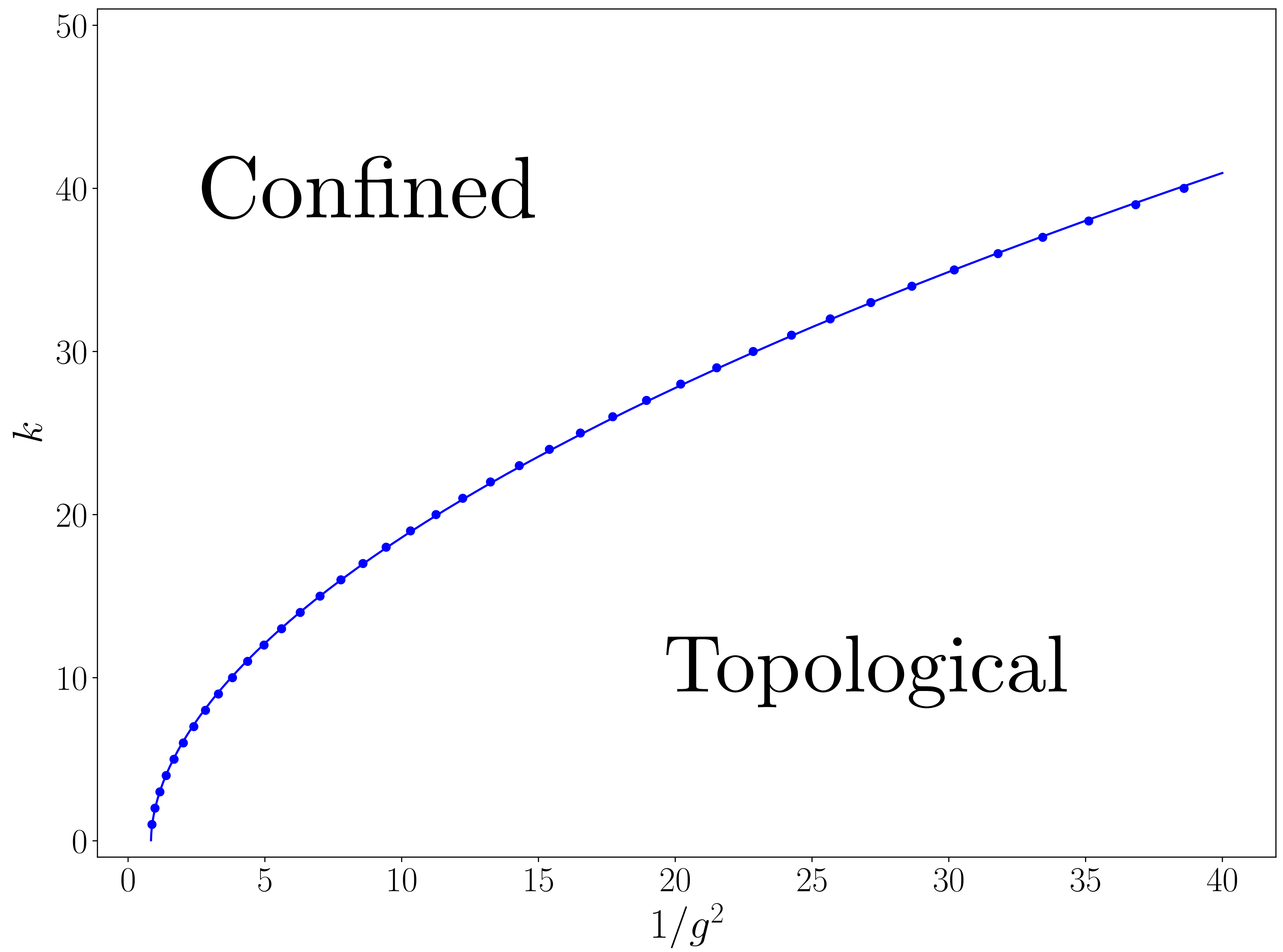
String tension $\sigma_d := \ln \frac{d_d}{\sum_{a,b} N^a_{db} \psi^*(a)\psi(b)}$



Numerical results

Numerical results





Phase transition occurs

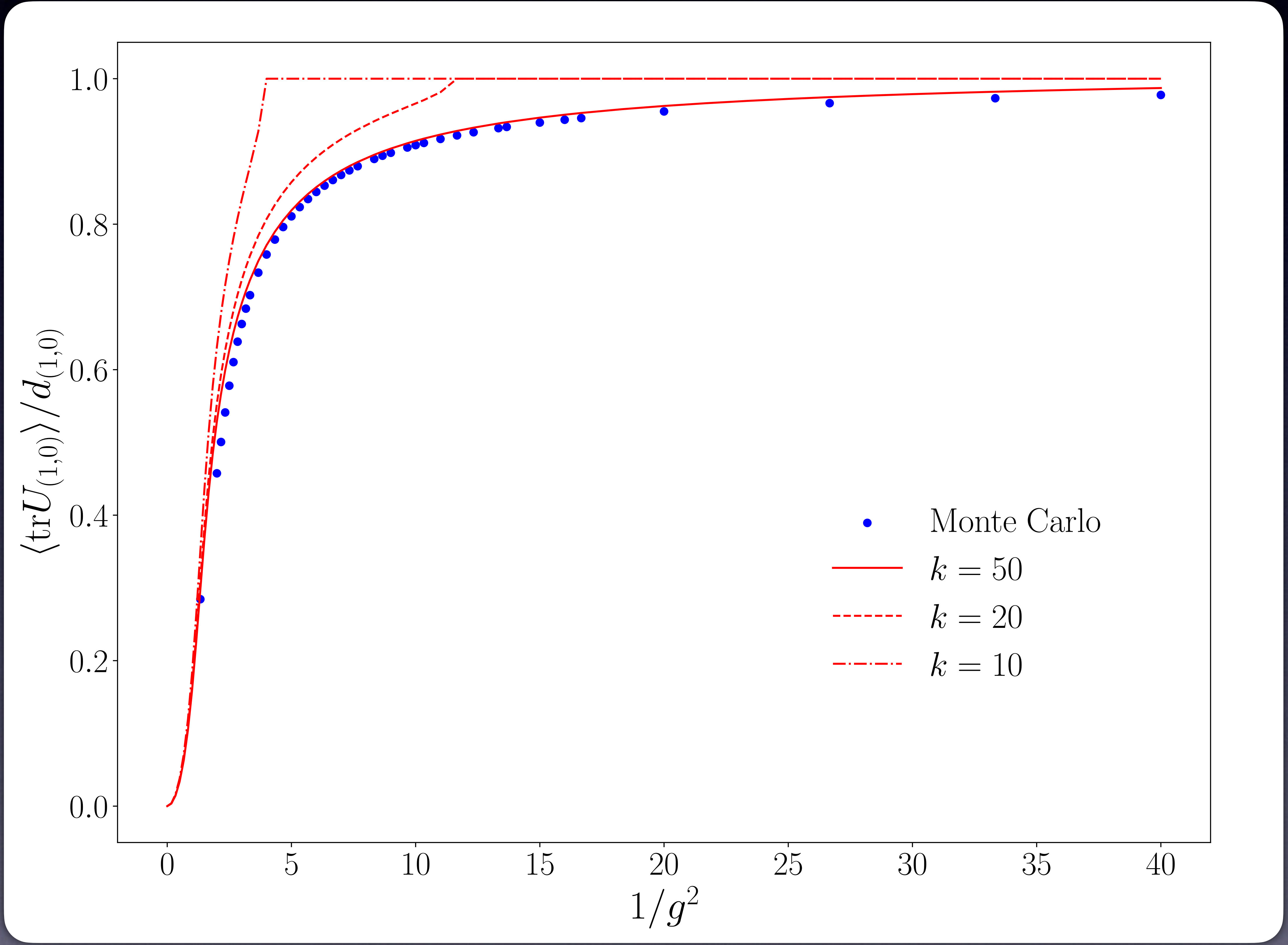
Topological phase:

String-net condensation: $\psi(a) \sim d_a$

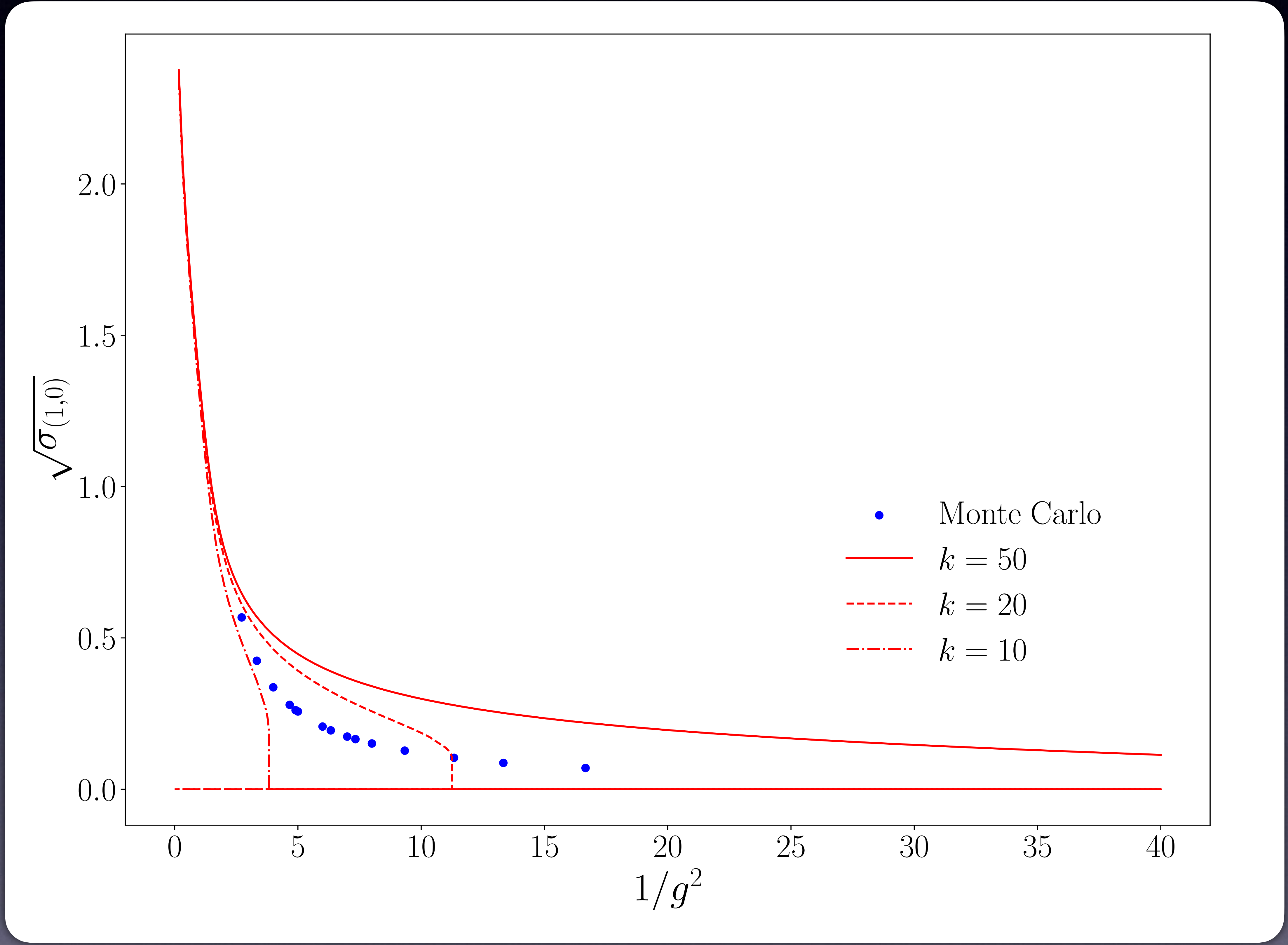
where string tension vanishes

Comparison with Monte-Carlo simulation

Plaquette (small Wilson loop)



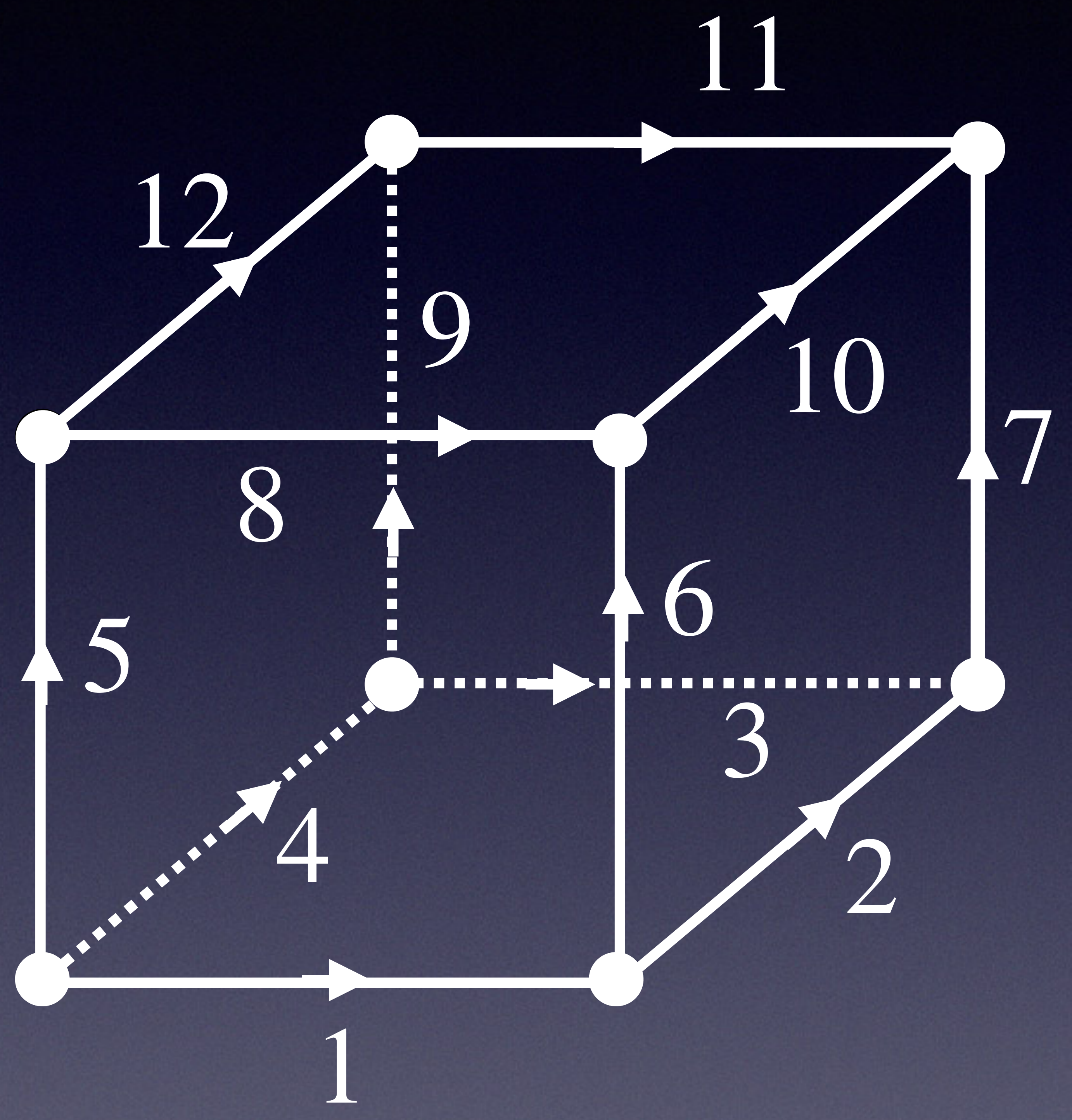
String tension



Good agreement for large k!

Thermalization on a small lattice

Small lattice system

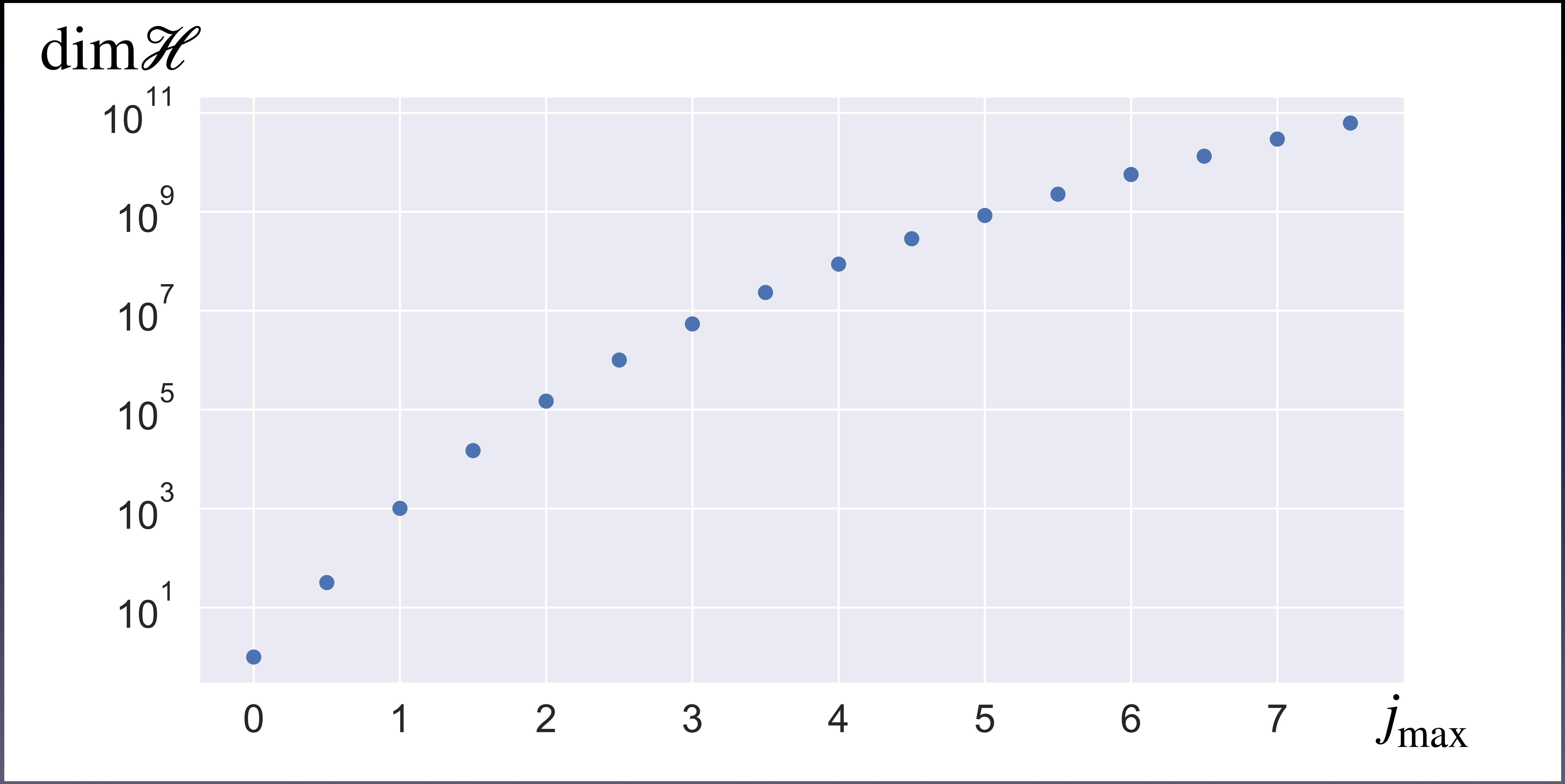


Basis

$$|j_1, \cdots, j_{12}\rangle = |j_1, j_2, j_6\rangle\, |j_2, j_3, j_7\rangle\, |j_3, j_4, j_8\rangle\, |j_1, j_4, j_5\rangle$$
$$|j_6, j_9, j_{10}\rangle\, |j_7, j_{10}, j_{11}\rangle\, |j_8, j_{11}, j_{12}\rangle\, |j_5, j_9, j_{12}\rangle$$

Naive cutoff $j_i \leq j_{\max} = k/2$

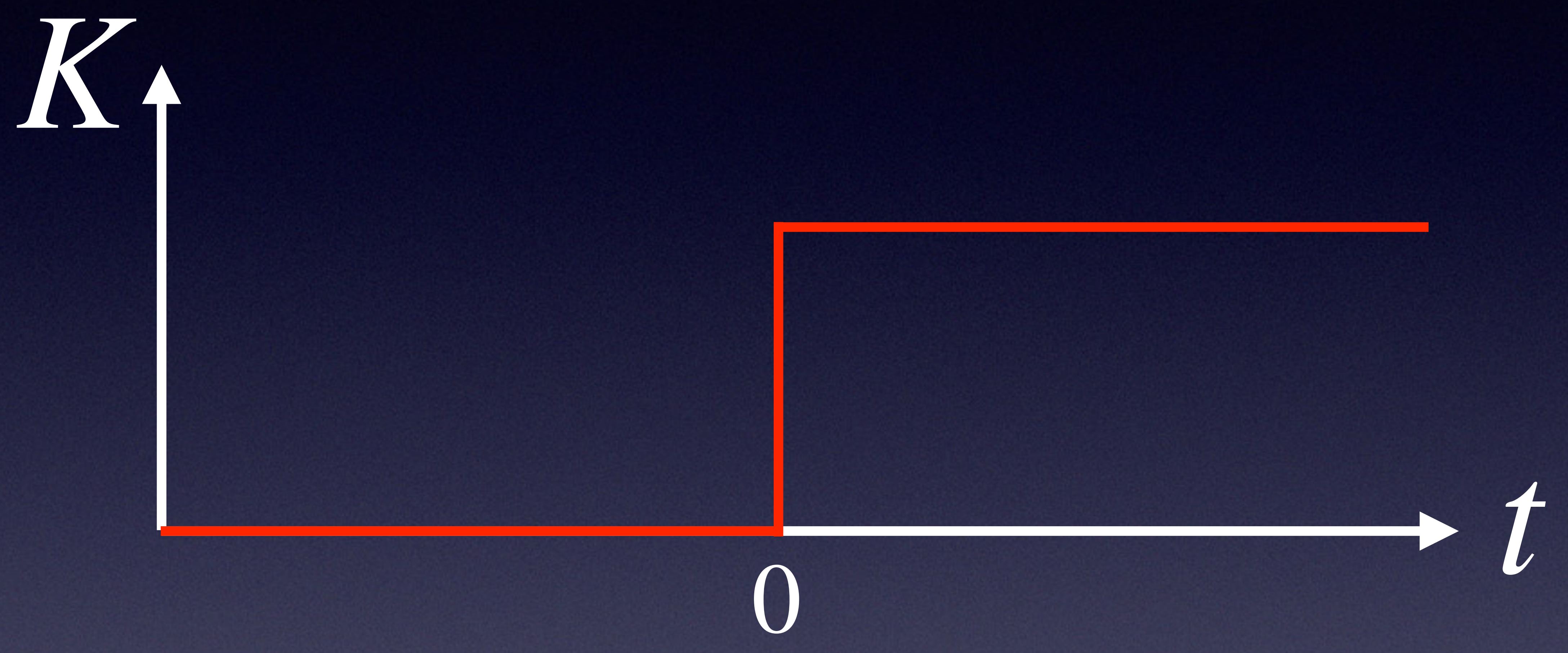
Dimension of Hilbert space



We employ $j_{\max} = 4: \ \dim\mathscr{H} = 87{,}426{,}119$

Setup

In order to mimic heavy ion collision experiments,
the interaction quenching

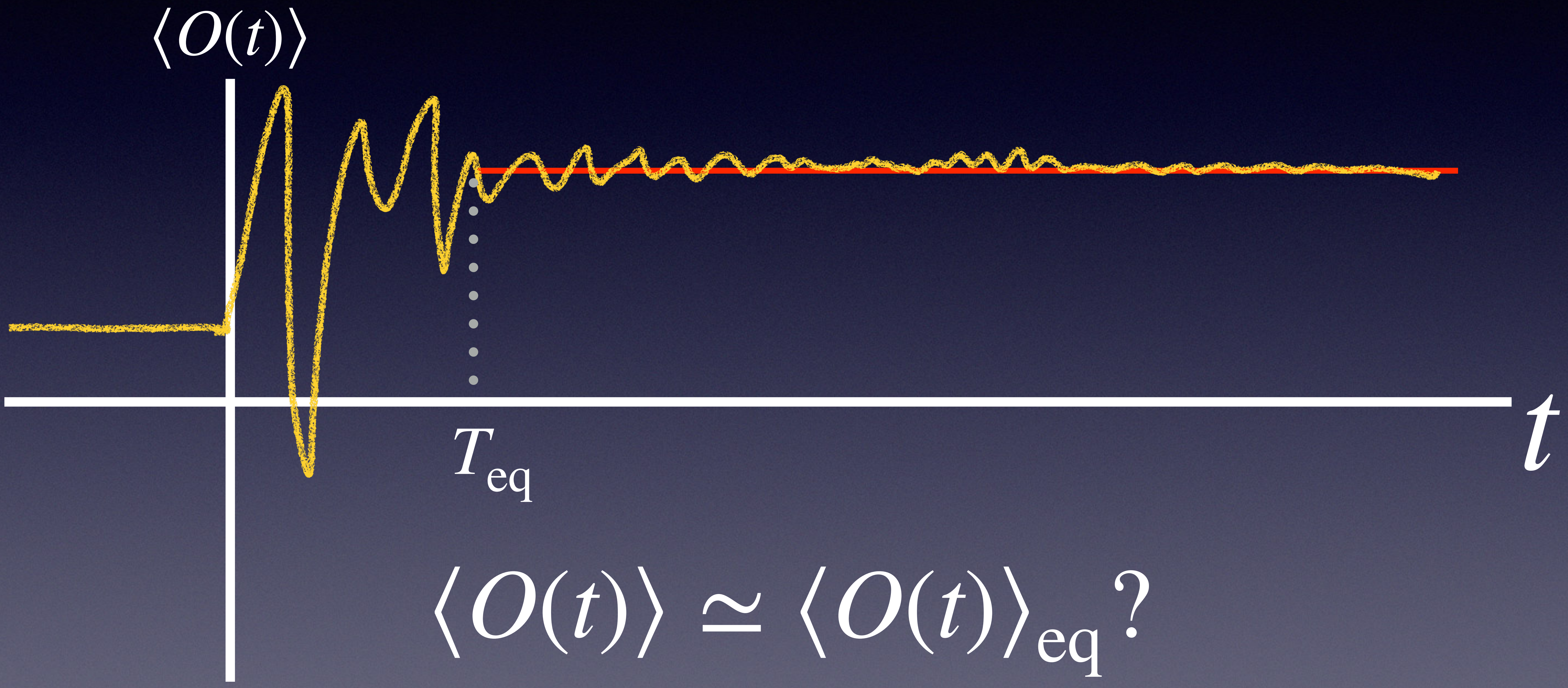


$$t < 0 \ |\mathrm{Vac}\rangle_{K=0}$$

$$t \geq 0 \ |\Psi(t)\rangle = e^{-\mathrm{i}Ht}|\mathrm{Vac}\rangle_{K=0}$$

Expected behavior

for an operator O $\quad \langle O(t) \rangle := \langle \Psi(t) | O | \Psi(t) \rangle$



$$\langle O(t) \rangle \simeq \langle O(t) \rangle_{\text{eq}} ?$$

Temperature and Canonical Ensemble

Energy is fixed by an initial condition

$$E = \langle H \rangle = \langle \Psi(t) \,|\, H \,|\, \Psi(t) \rangle$$

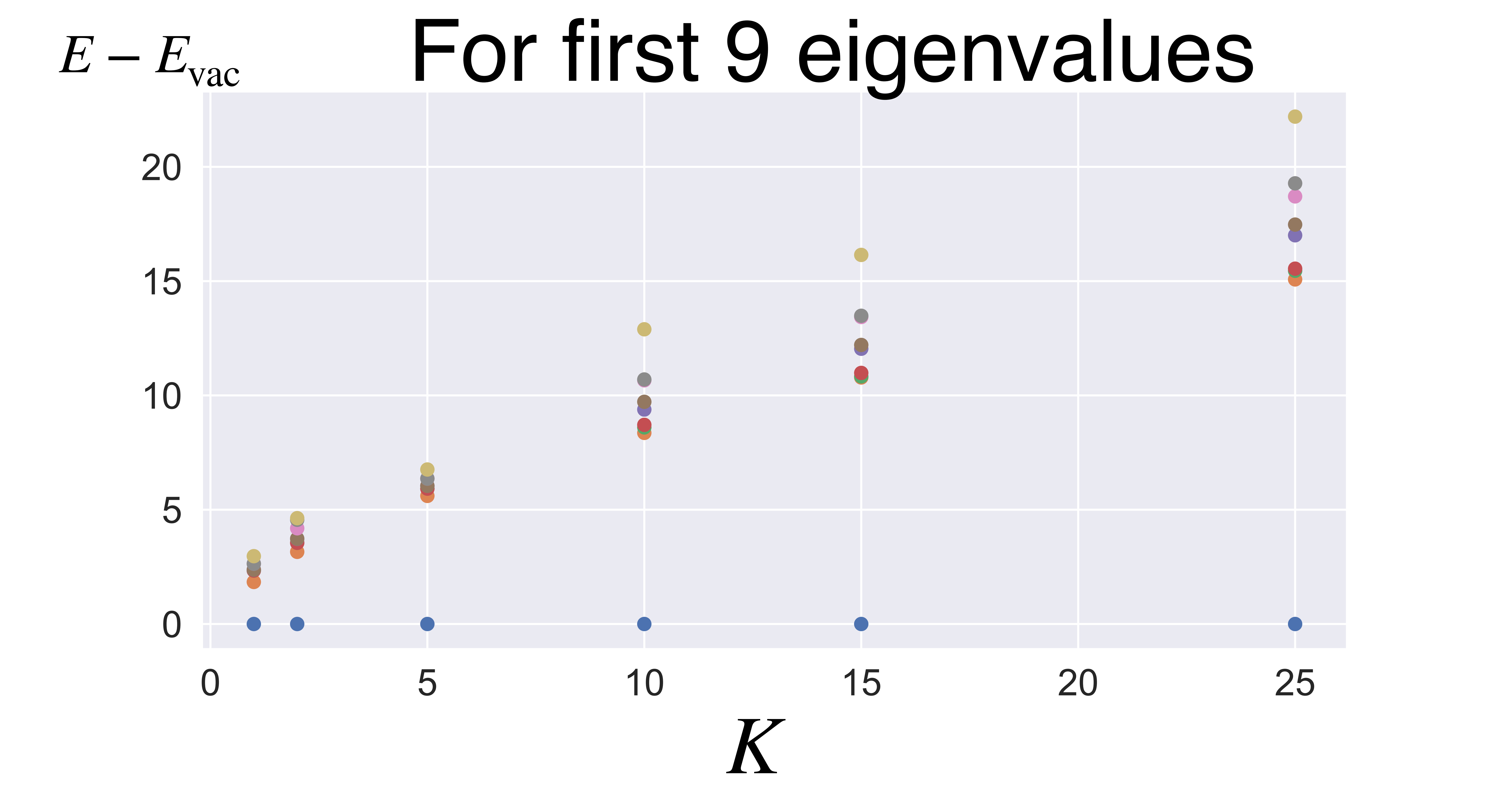
(Independent of time)

For a given energy,
a canonical distribution that reproduces
the expected value can be defined

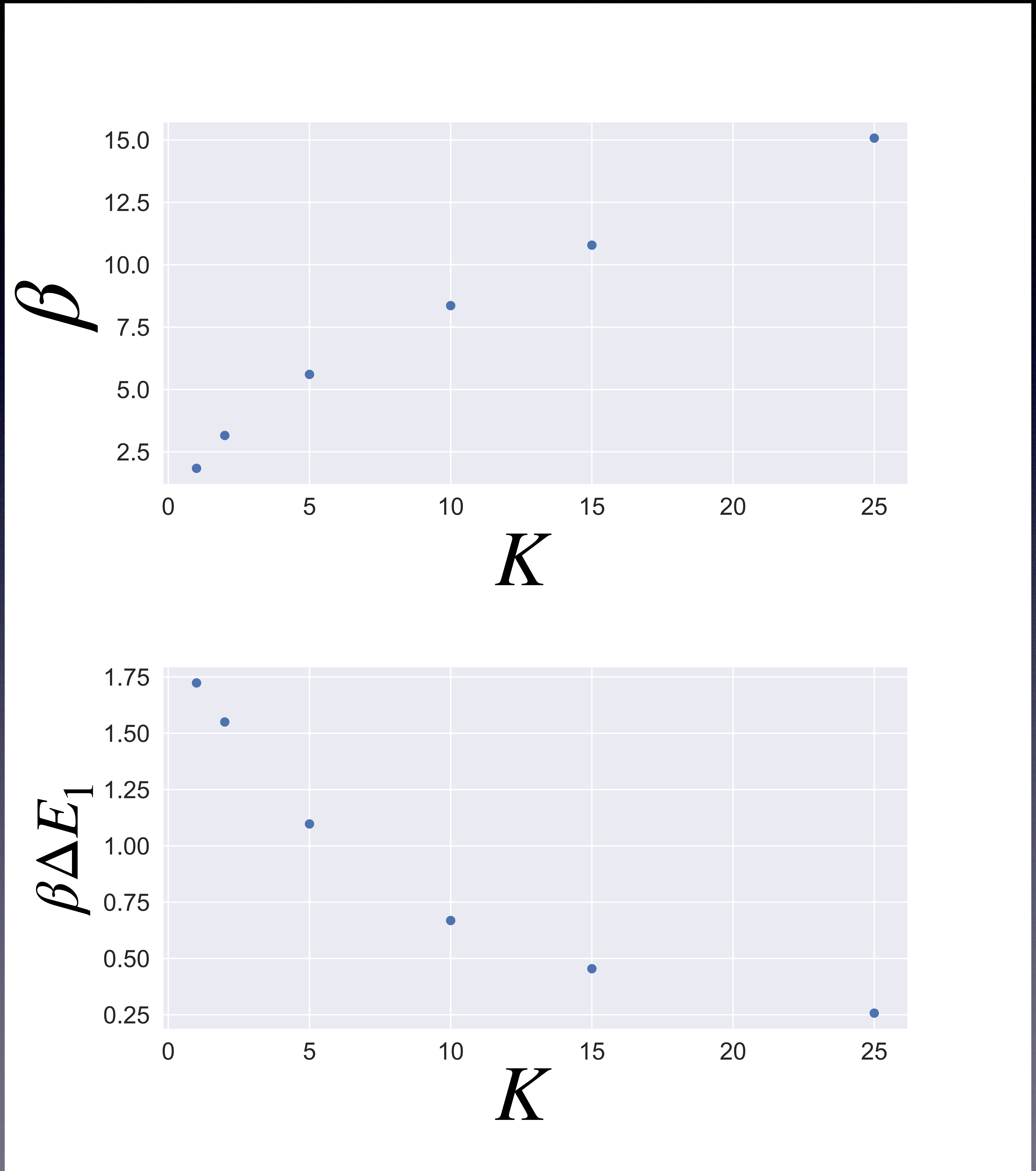
$$E = \langle H \rangle_{\mathrm{eq}} := \mathrm{tr}\rho_{\mathrm{eq}} H \quad \textbf{with} \quad \rho_{\mathrm{eq}} = \frac{e^{-\beta H}}{\mathrm{tr} e^{-\beta H}}$$

Numerical results

Energy eigenvalues for $j_{\max} = 4$



K-dependence of temperature



The first excitation energy

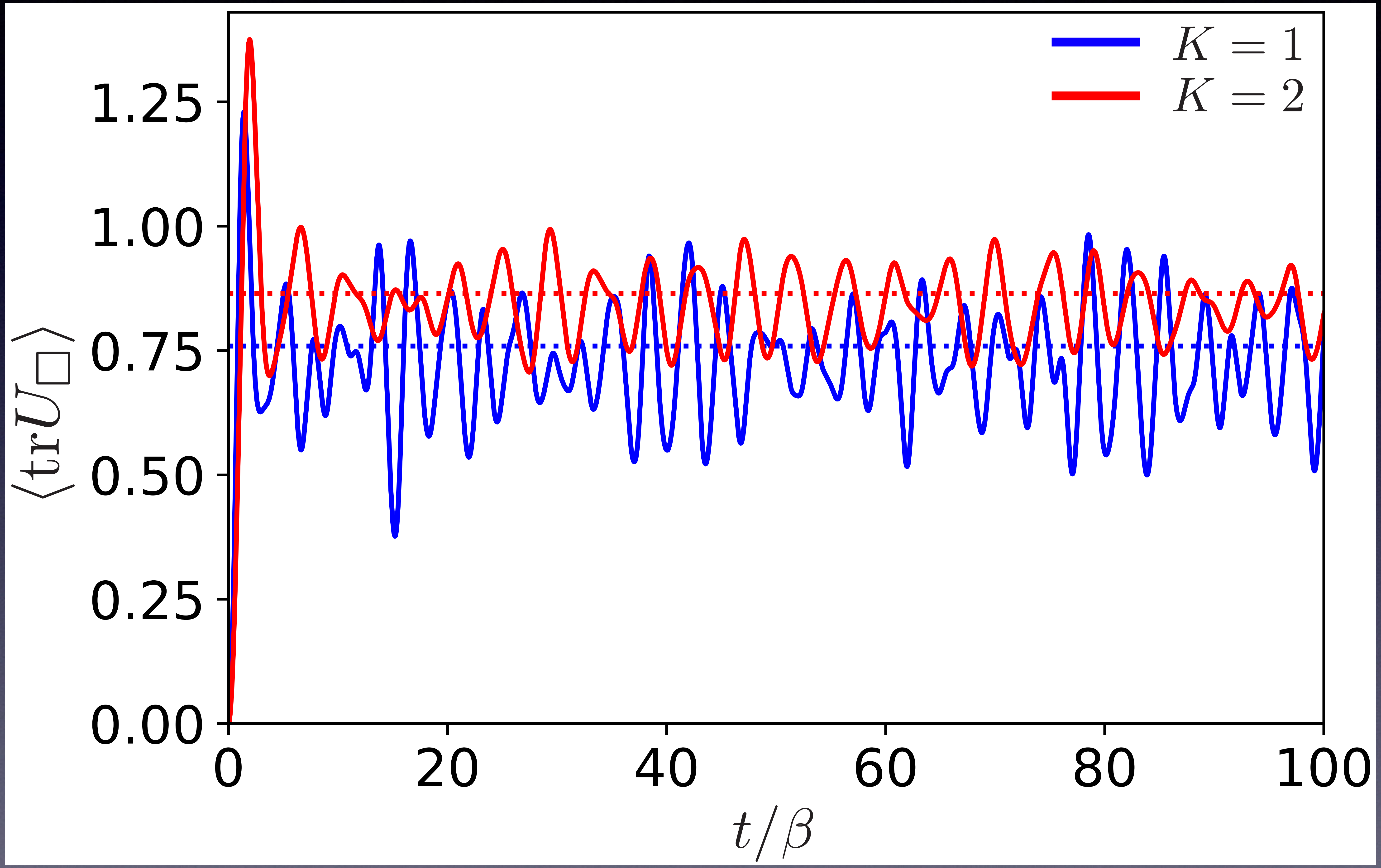
$$\Delta E_1 : E_1 - E_0$$

Typical energy scale

$\beta\Delta E_1 > 1$ **Low T**

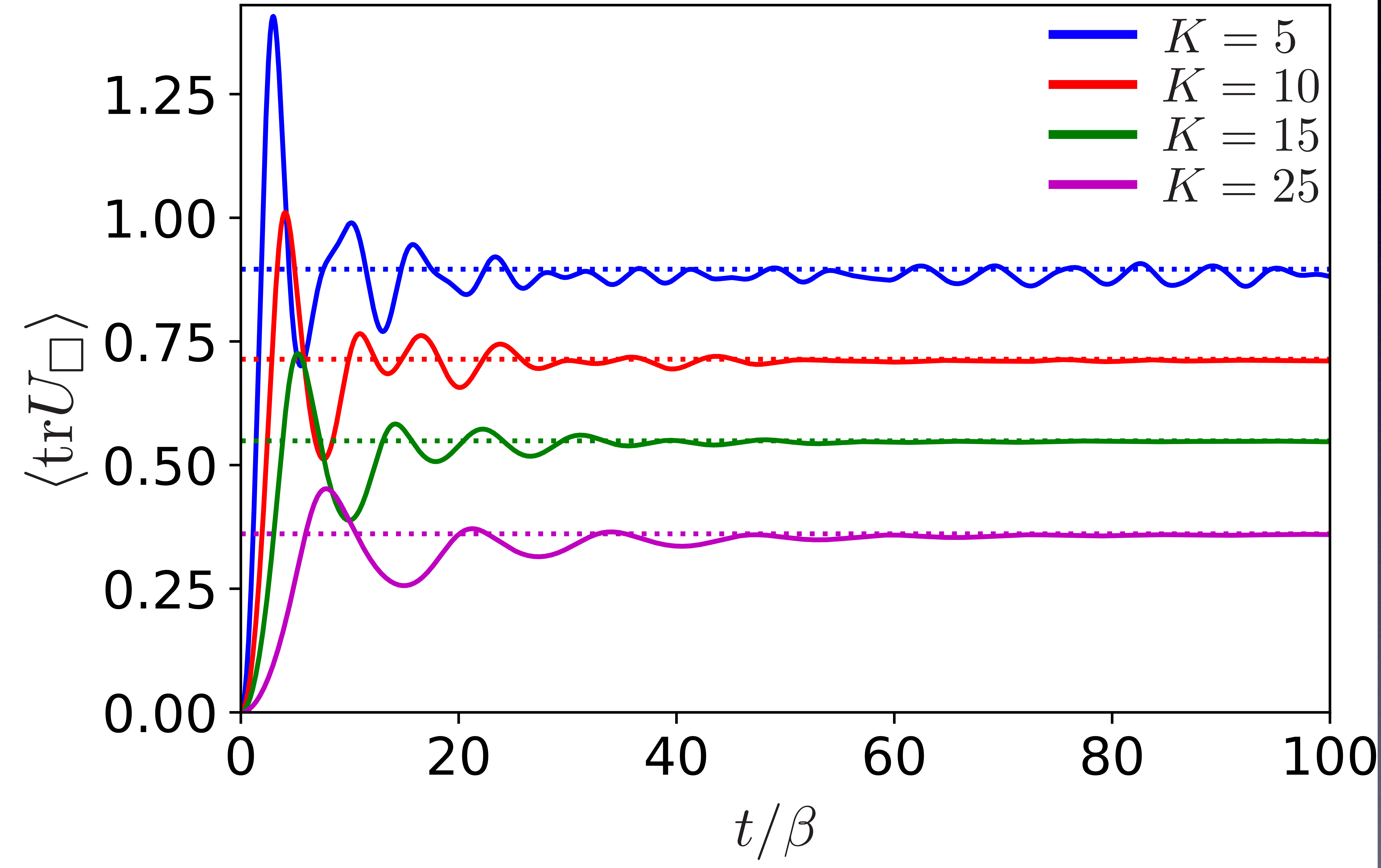
$\beta\Delta E_1 < 1$ **Hight T**

Expected value of Wilson loop
Strong coupling (low T)



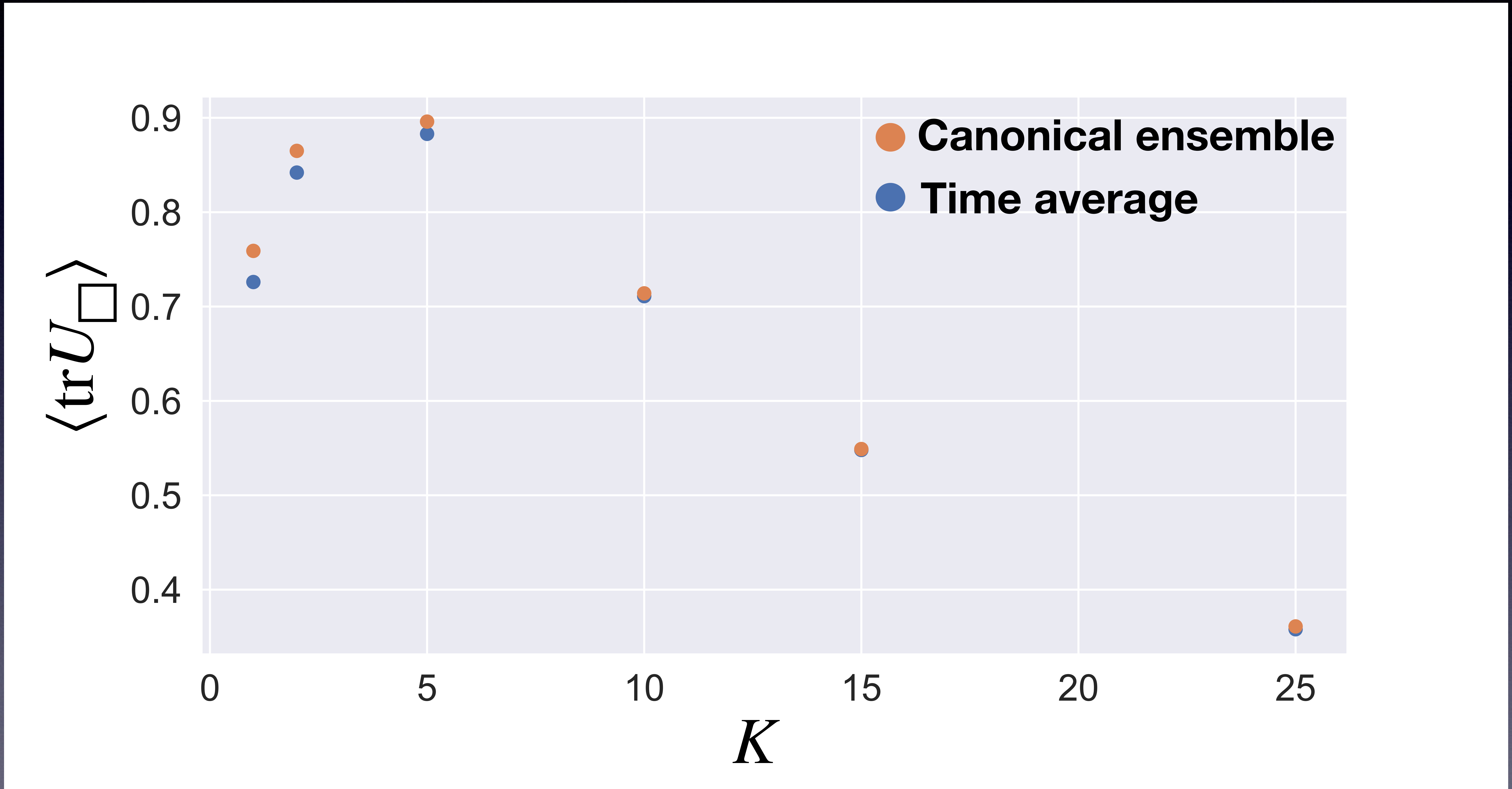
Fluctuations are not small.

Expected value of Wilson loop
Weak coupling (high T)



Steady state observed

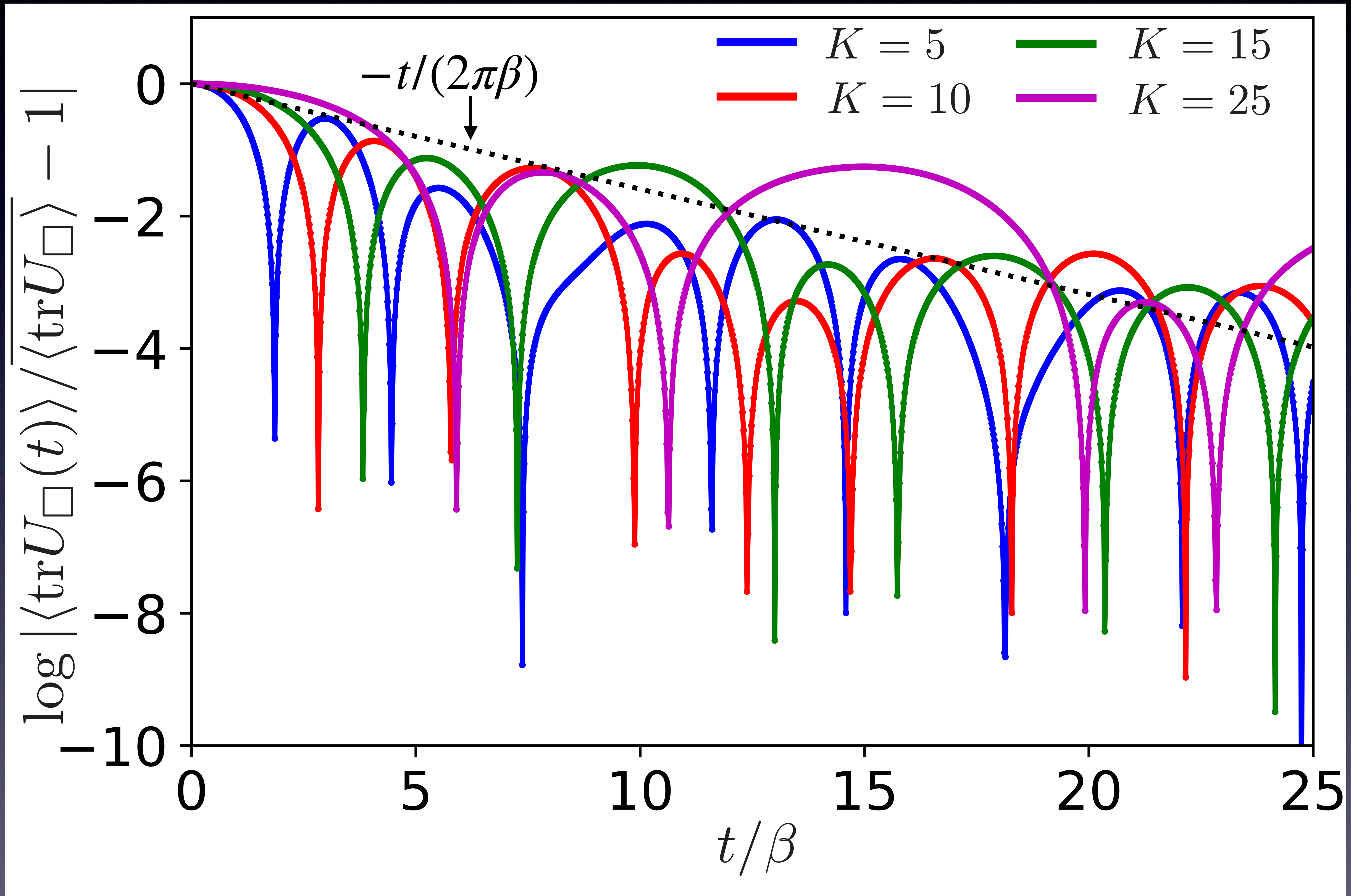
Long-time average vs canonical ensemble



Difference is less than 1% for $K > 5$

Relaxation to equilibrium

$$\langle \mathrm{tr} U_{\square}(t) \rangle - \overline{\langle \mathrm{tr} U_{\square} \rangle} \sim e^{-t/\tau_{\mathrm{eq}}}$$



Close to Boltzmann time $2\pi\beta$.

Goldstein, Hara, Tasaki, New J. Phys. 17 (2015) 045002

Summary

- **Formalism**

  Kogut-Susskind Hamiltonian formalism
  Spin network or stringnet is useful
  in Hamiltonian formalism

- **Application**

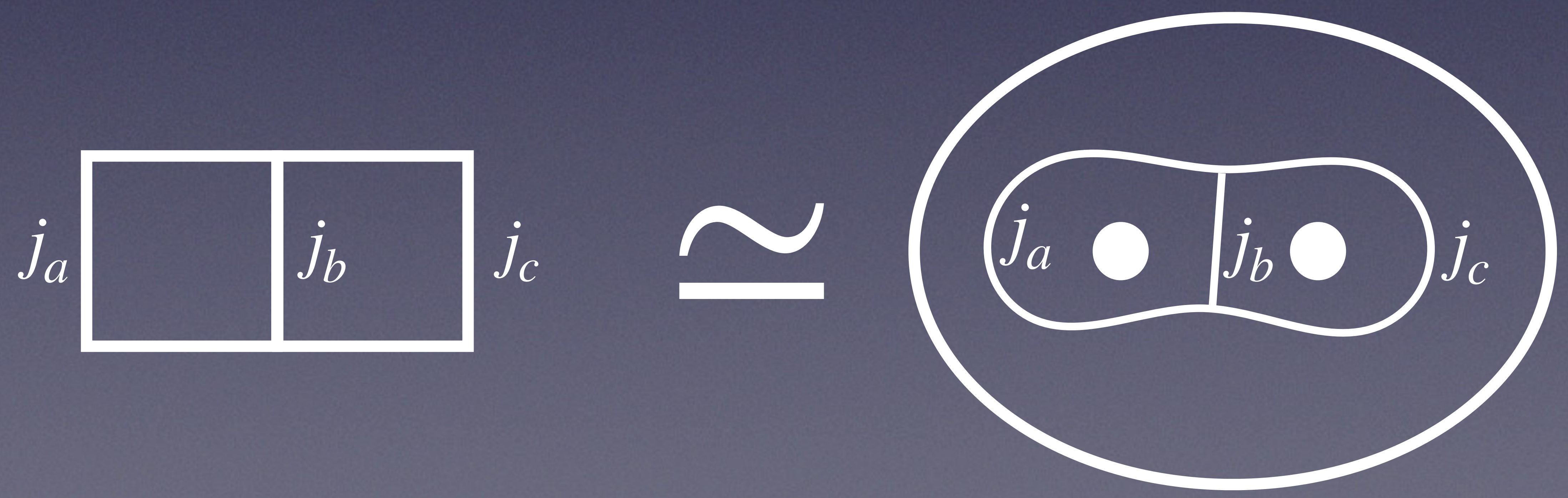
  $\mathrm{SU}(3)_k$ **gauge theory in** $(2+1)$ **dimensions**
  Confinement-topological phase transition

  Thermalization of Yang-Mills theory
  in (3+1)-dimensional small systems
  Relaxation time of thermalization
  $\tau_{\mathrm{eq}} \sim 2\pi/T$ **Boltzmann time**

Outlook

- Large dimension
- Large volume
- quantum simulation
- Calculation of entanglement entropy(EE), negativity(NE) etc.

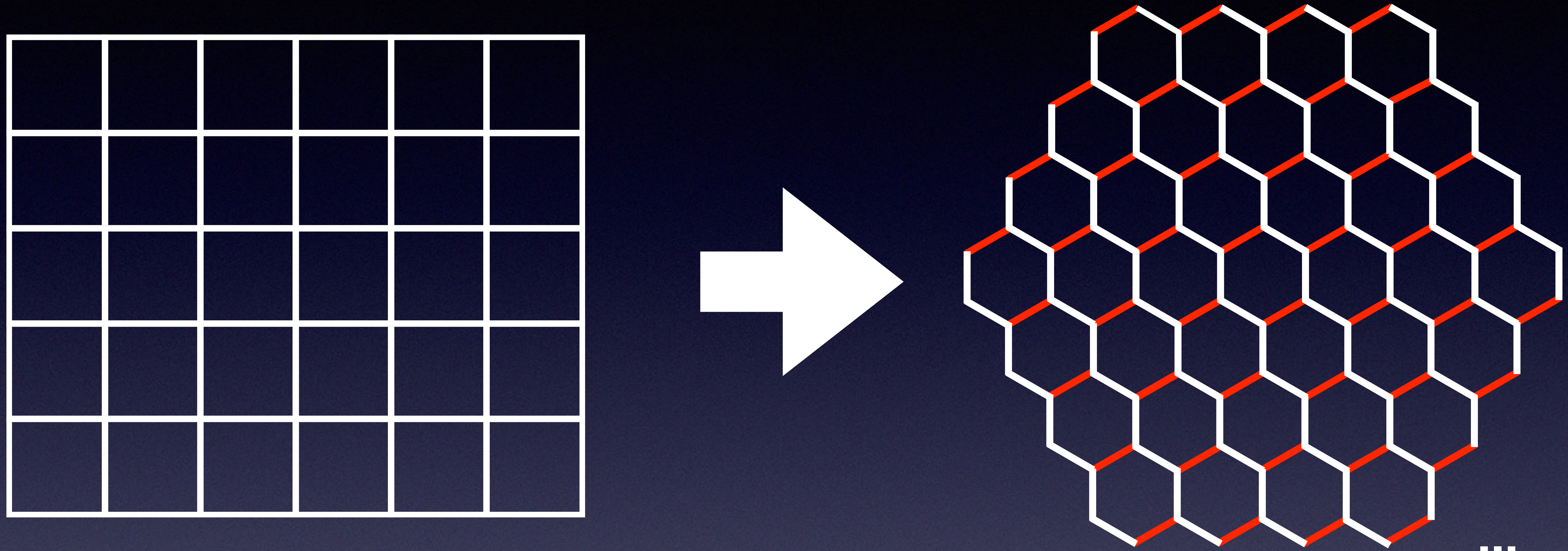
Hilbert space of gauge thoery $\simeq$ Topological field theory with defects

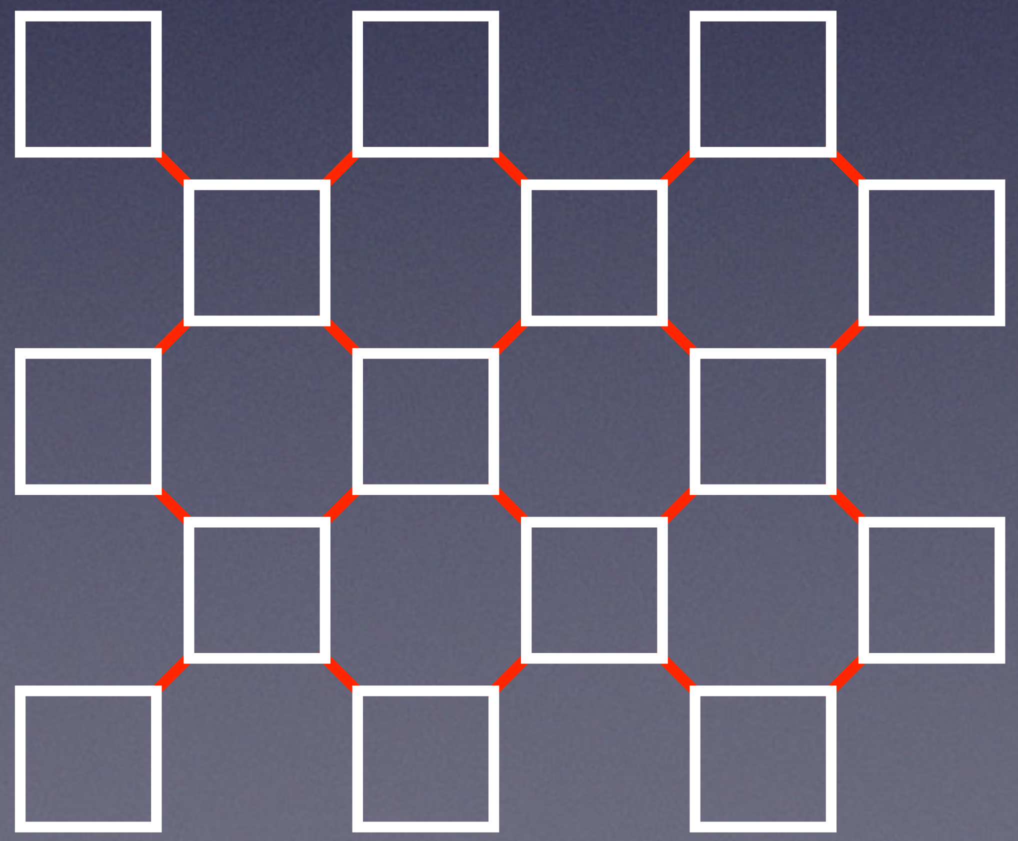


⇒Evaluate EE and NE using TQFT technique

Backup

How to treat square lattice

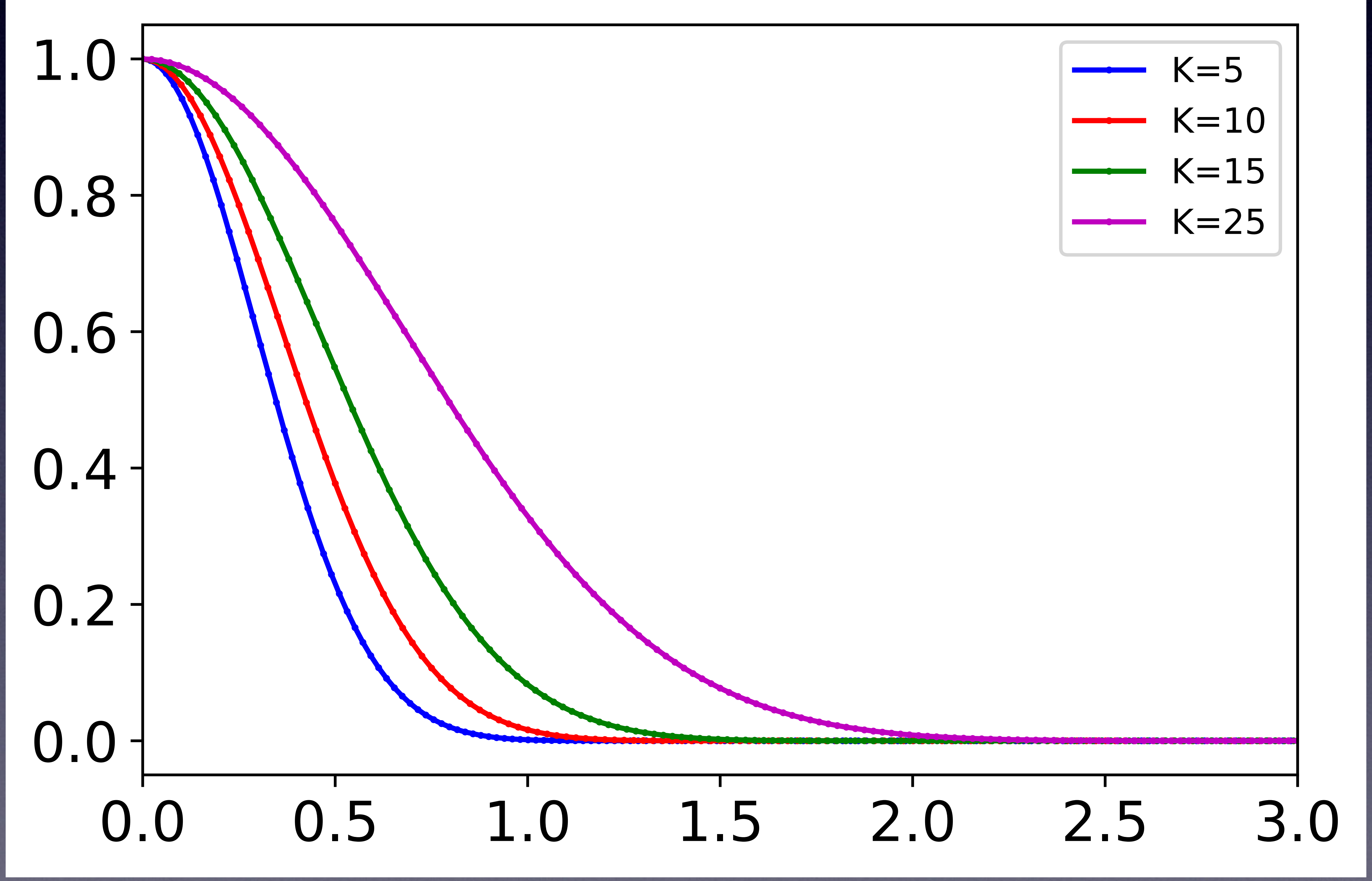




By the composition rule of the network
Matrix elements do not depend
on the inclusion of auxiliary links

Vacuum persistency probability (Loschmidt echo)

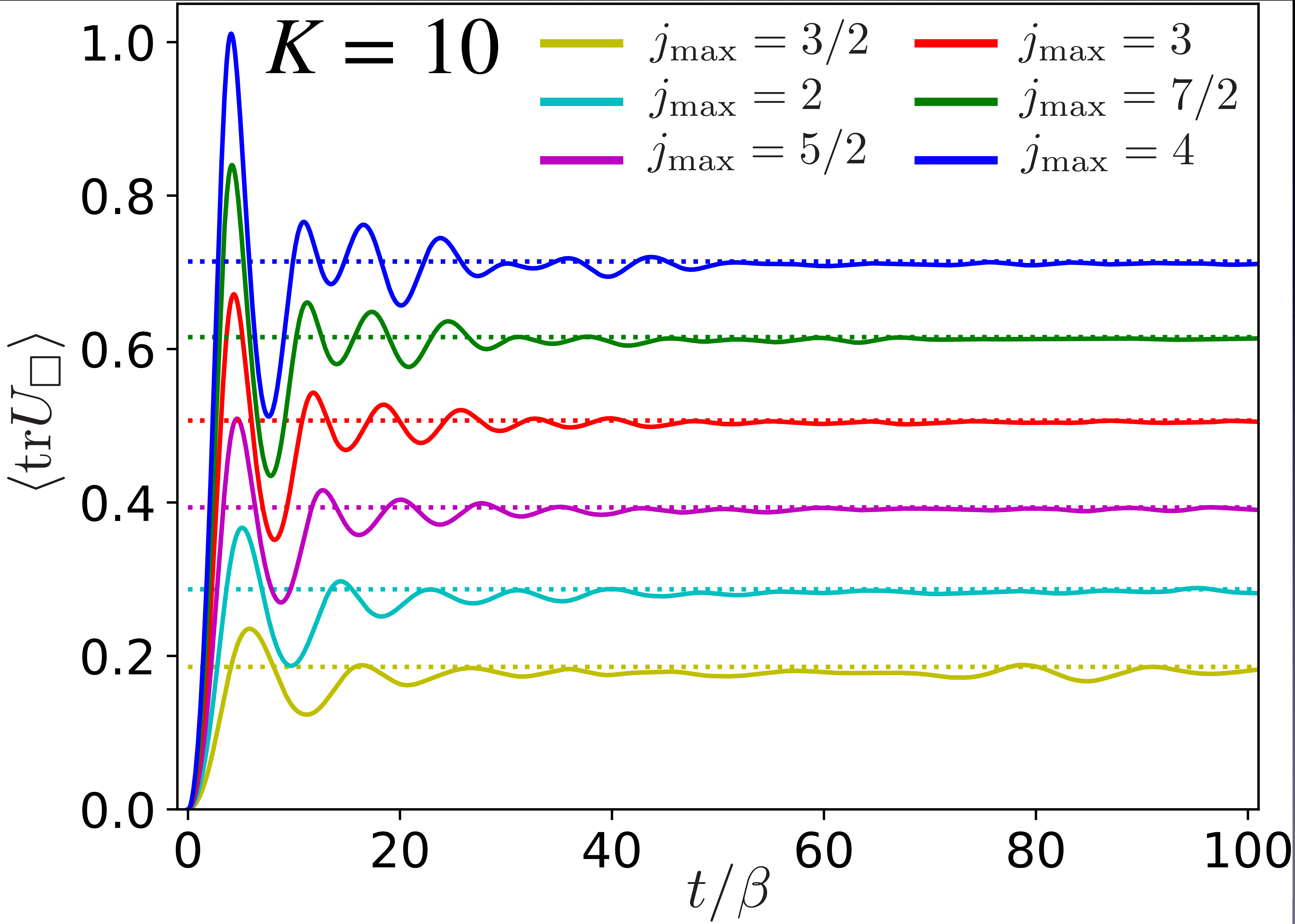
$$P_{\text{vac}} := |\langle \Psi(0) | \Psi(t) \rangle|^2$$



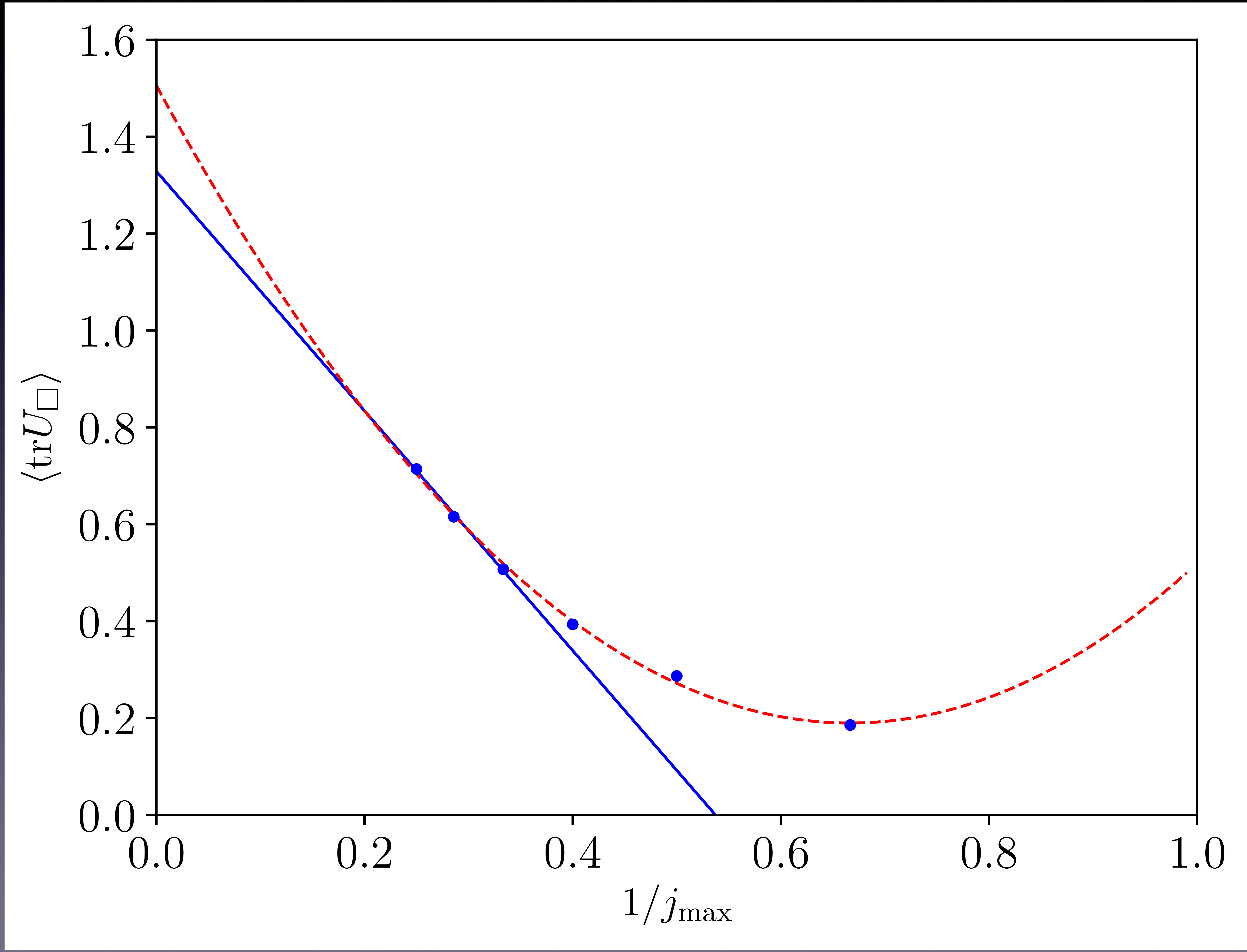
P_{vac}

t/β

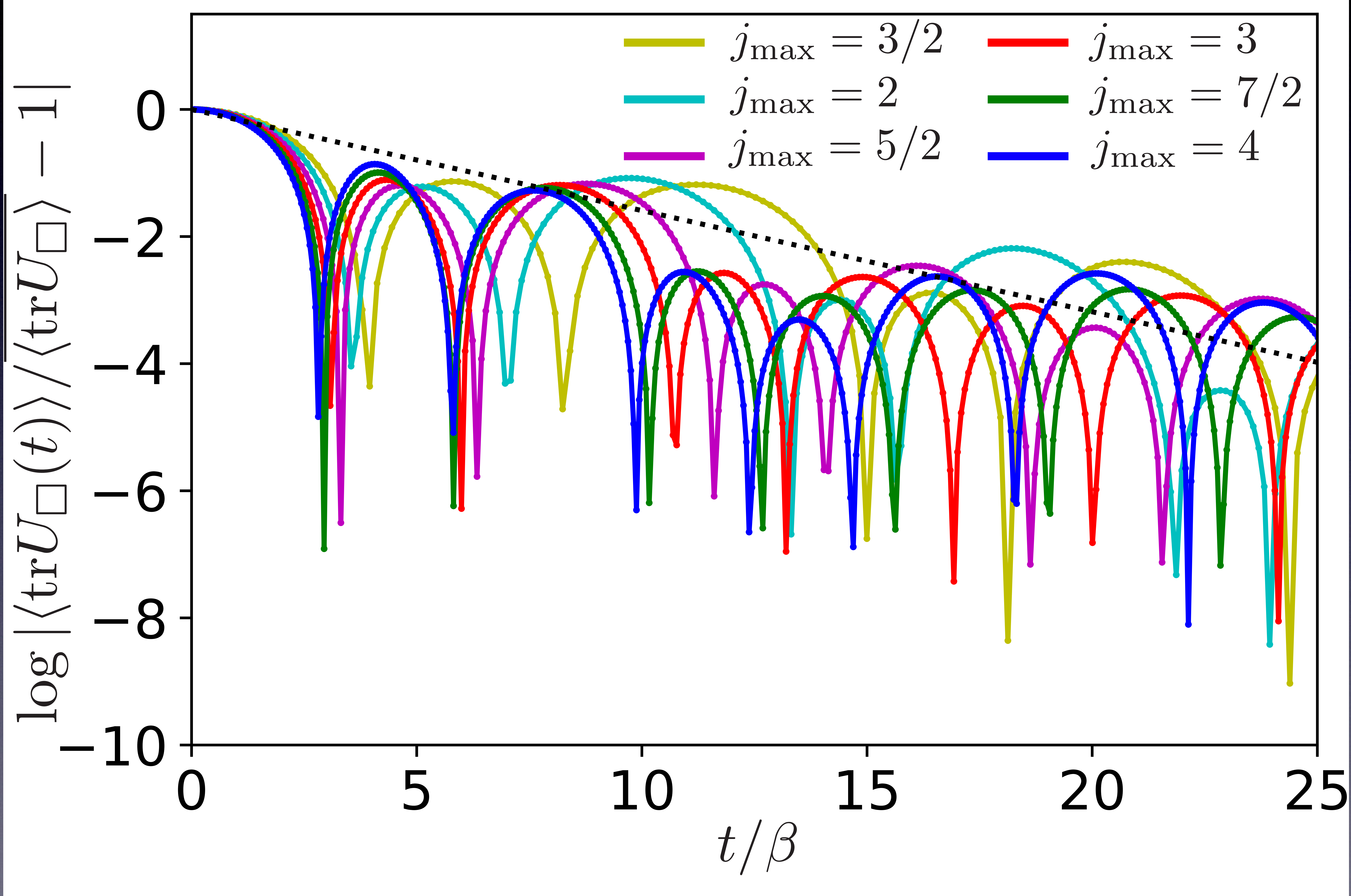
jmax dependence



Extrapolation



$j_{\max}$ dependence for relaxation time



$SU(3)_k$

$\mathrm{SU}(3)_k$ fusion coefficients

Begin, Walton, Mod. Phys. Lett. A 7 (1992) 3255

$$N_{ab}^c = (k_0^{\max} - k_0^{\min} + 1)\delta_{ab}^c$$

$$\begin{aligned}
k_0^{\min} &= \max(p_a + q_a, p_b + q_b, p_c + q_c, \mathcal{A} - \min(p_a, p_b, q_c), \mathcal{B} - \min(q_a, q_b, p_c)), \\
k_0^{\max} &= \min(\mathcal{A}, \mathcal{B}), \\
\mathcal{A} &= \frac{1}{3}(2(p_a + p_b + q_c) + q_a + q_b + p_c), \\
\mathcal{B} &= \frac{1}{3}(p_a + p_b + q_c + 2(q_a + q_b + p_c)), \\
\delta_{ab}^c &= \begin{cases} 1 & \text{if } k_0^{\max} > k_0^{\min} \text{ and } \mathcal{A}, \mathcal{B} \in \mathbb{Z}_+ \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Quantum dimension

e.g., Coquereaux, Hammaoui, Schieber, Tahri, J. Geom. Phys. 57 (2006) 269

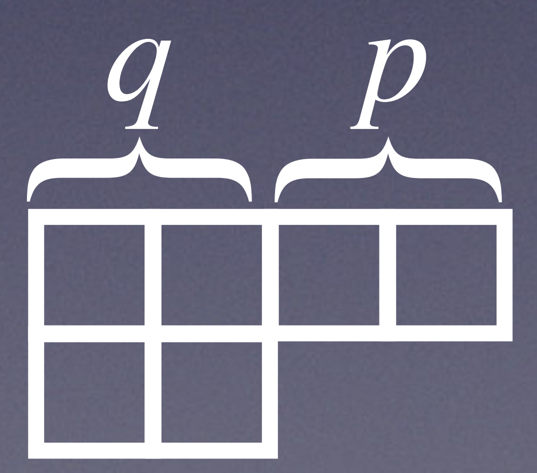
$$d_a = \frac{1}{[2]}[p_a + 1][q_a + 1][p_a + q_a + 2]$$

Casimir invariant

e.g., Bonatsos, Daskaloyannis, Prog. Part. Nucl. Phys. 43 (1999) 537

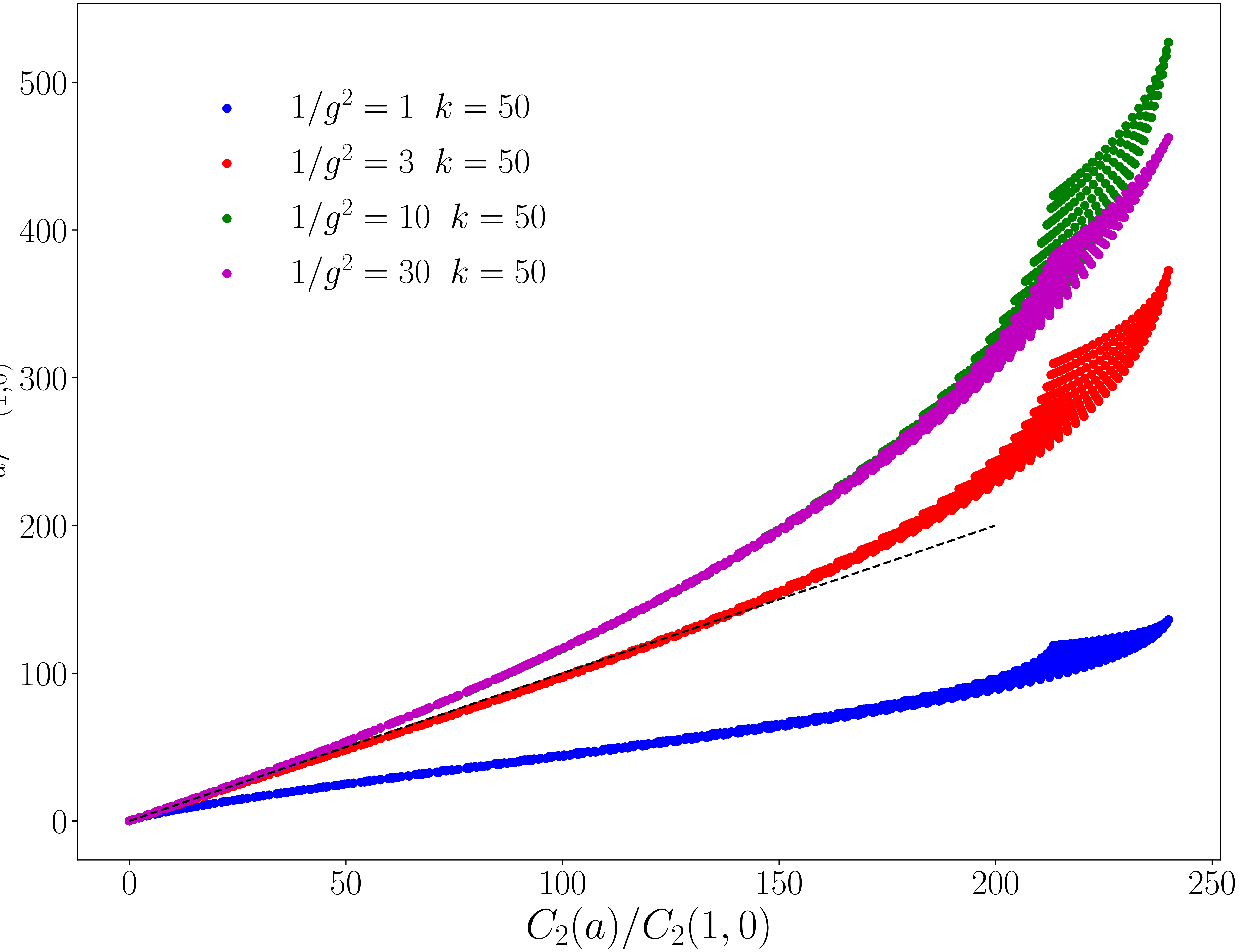
$$C_2(a) = \frac{1}{2}\left(\left[\frac{p_a}{3} - \frac{q_a}{3}\right]^2 + \left[\frac{2p_a}{3} + \frac{q_a}{3} + 1\right]^2 + \left[\frac{p_a}{3} + \frac{2q_a}{3} + 1\right]^2 - 2\right)$$

(p, q)



$$[n] := \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \frac{\sin\left(\frac{\pi}{k+3}n\right)}{\sin\left(\frac{\pi}{k+3}\right)} \qquad q = \exp\left(\mathrm{i}\frac{2\pi}{3+k}\right)$$

Casimir scaling



Casimir scaling