The Holographic Entropy Cone Beyond N=5

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w/ T. He, M. Headrick, S. **Hernández-Cuenca**, F. Jia, M. Rangamani, & M. **Rota**
+ WIP w/ T. He & M. Rota]

Motivation & Context

- Understand the emergence of spacetime
  - ~ Use holography
  - ~ Focus on classical bulk geometry (i.e. $N = \infty, \lambda = \infty$ regime)
- Hints / expectation: "spacetime built from entanglement"
  - ~ Understand entanglement structure of 'geometric' states
  - ~ Focus on entanglement entropy (EE) for spatially-delimited subsystems
- Characterize {EEs} relevant for holography
  - ~ Region in entropy space dubbed *holographic entropy cone* (HEC)
  - ~ Focus on its boundary (delimited by *holographic entropy inequalities*)
- Seek lessons independent of # of subsystems (≡N)
  - ~ Bootstrap from low N
  - ~ Focus on structural relations

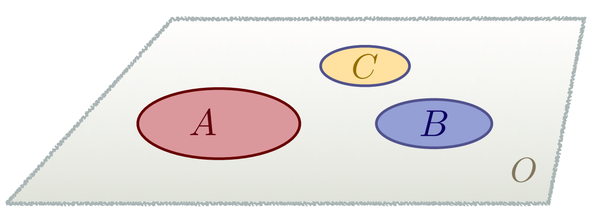
Entanglement entropy

For CFT state $|\psi\rangle$ and bi-partition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$

$\leadsto$ reduced density matrix $\rho_A \equiv \mathrm{Tr}_{\bar{A}} |\psi\rangle\langle\psi|$

EE = $S(A) \equiv -\mathrm{Tr}\, \rho_A \, \log \rho_A$

- Decompose CFT into N elementary subsystems



$$\underbrace{\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \cdots}_{N} \otimes \underbrace{\mathcal{H}_{\overline{ABC\cdots}}}_{\text{"purifier" } O}$$

- $\leadsto$ entropy vector in D = 2^N - 1 dimensional *entropy space*

e.g. for N=3, $\vec{S} = \{S(A), S(B), S(C), S(AB), S(AC), S(BC), S(ABC)\}$

conceptually useful to consider large N…

Entropy relations

- Physically realizable entropy vectors are restricted
- Universal restrictions:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$

    ⇨ Mutual information positivity $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - Strong sub-additivity (SSA) $S(AB) + S(BC) \geq S(B) + S(ABC)$

    ⇨ Mutual information monotonicity $I(A:C|B) \equiv I(A:BC) - I(A:B) \geq 0$
  - … (expect more relations with increasing N)
  - always permutation & purification symmetric

in any pure state, $S(A) = S(A^c)$

⇒ e.g. transforms SA into Araki-Lieb: $S(A) + S(AB) \geq S(B)$

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  - … (expect more relations with increasing N)
  - always permutation & purification symmetric

- Further restrictions, depending on the system

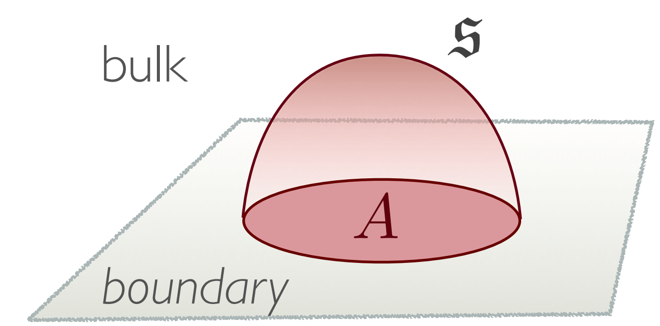
- Our task: understand the full set in holography

Holographic entanglement entropy

Proposal [RT=Ryu & Takayanagi, '06] for static configurations,
covariantized by [HRT=VH, Rangamani, Takayanagi, '07] for time-dependent situations:

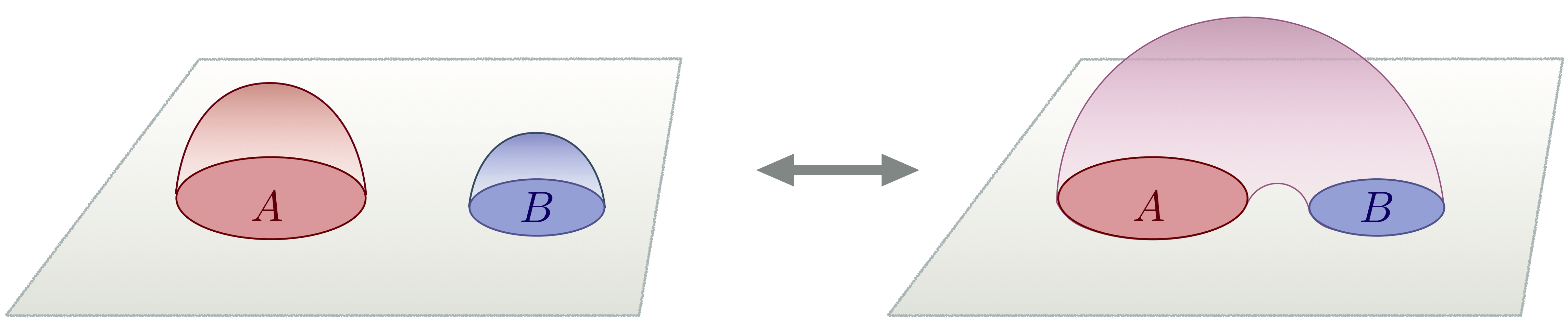
Entanglement entropy $S(A)$ for a boundary region A is captured by the area of a bulk extremal surface $\mathfrak{s}$ homologous to A;

for multiple candidates, choose least area one.



$$S(A) = \min_{\mathfrak{s} \sim A} \frac{\text{Area}(\mathfrak{s})}{4\, G_N}$$

Allows for phase transitions, e.g. jump in surface for S(AB):



Entropy cone

{All physically allowed entropy vectors} = convex cone in entropy space

when restricted to geometric states in holography ⇝ *holographic entropy cone* (HEC)

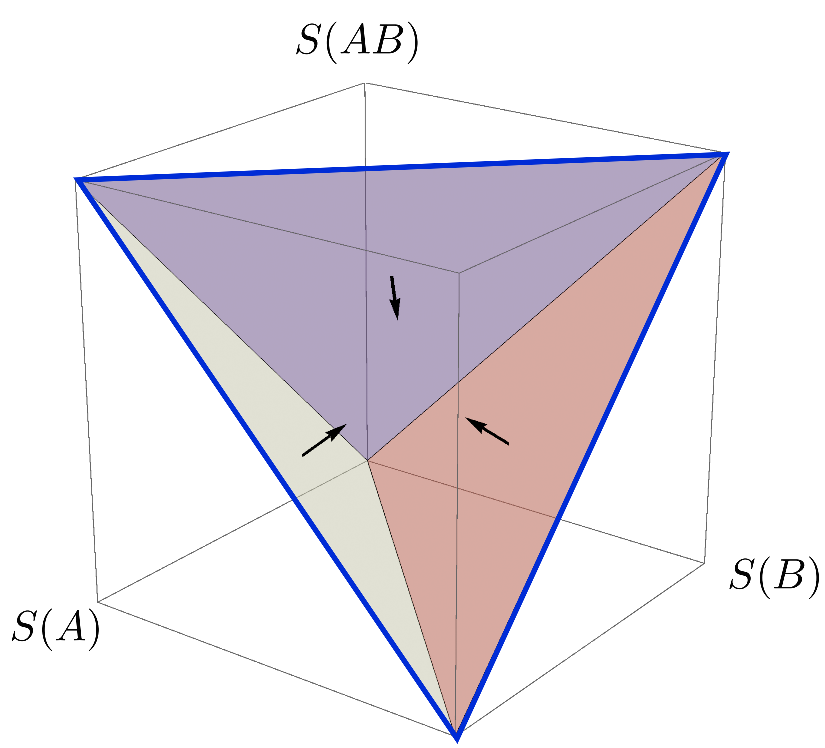
[Bao, Nezami, Ooguri, Stoica, Sully, Walter '15]

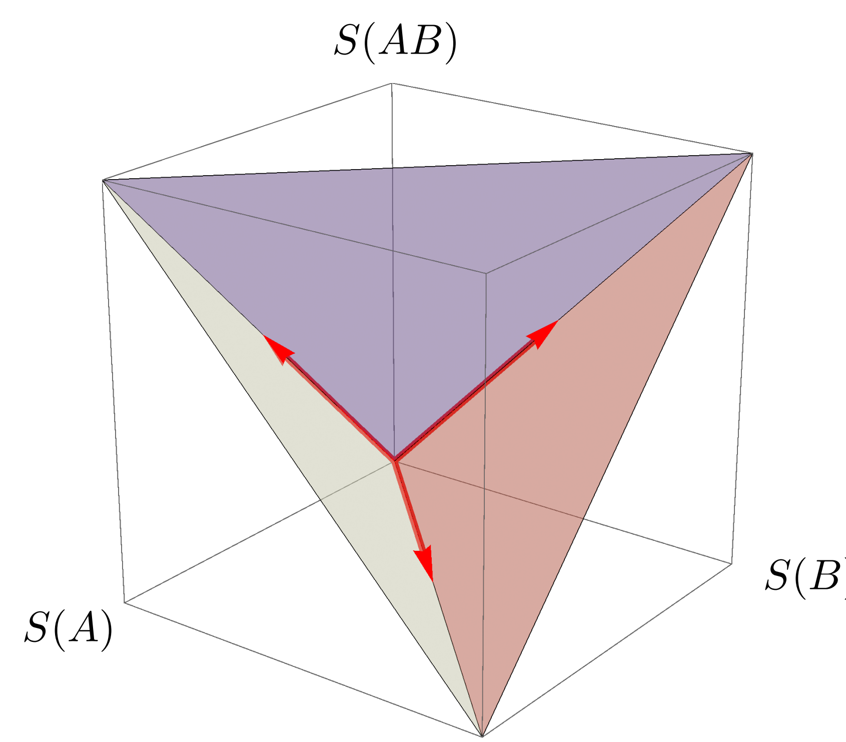
2 useful characterizations of a polyhedral cone:

intersection of half-spaces delineated by entropy inequalities (facets)

convex hull of extreme rays

eg. at N=2:





Hierarchy of cones in entropy space

- Consider entropy space for fixed N
  - holographic entropy cone: HEC = { holographically realizable $\vec{S}$ }
  - quantum entropy cone: QEC = { physically realizable $\vec{S}$ }
  - subadditivity cone: SAC = { $\vec{S}$ compatible with all instances of SA }

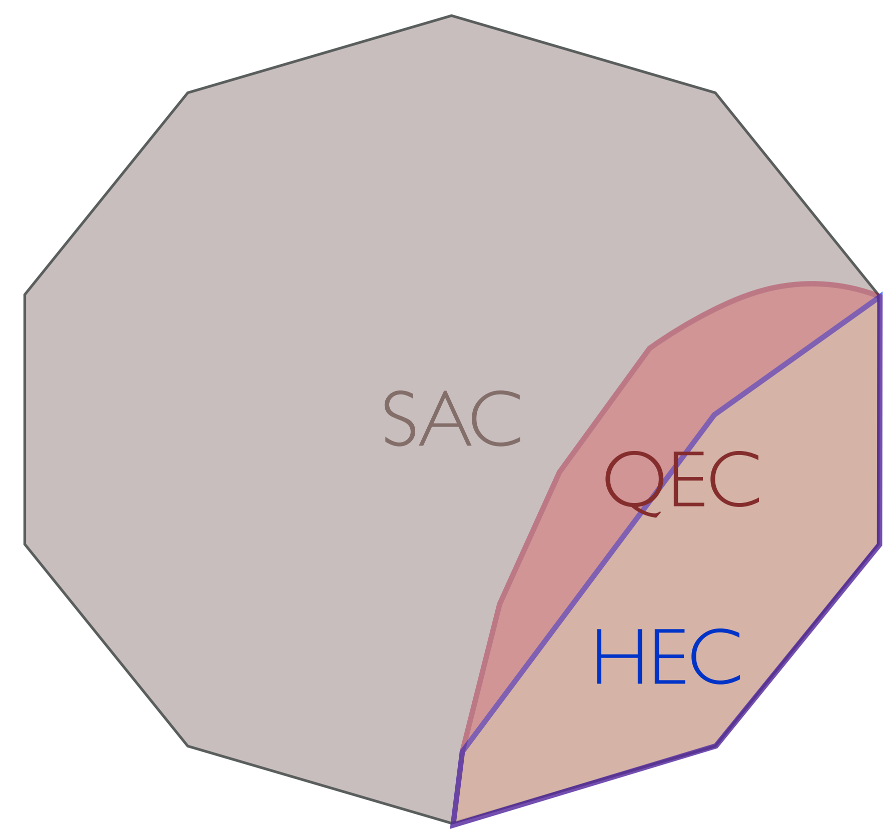
- These are nested convex sets:

  SAC $\supset$ QEC $\supset$ HEC

- Many more...

  { $\vec{S}$ compatible w/ SA & SSA }

  convex hull of SAC ERs in HEC

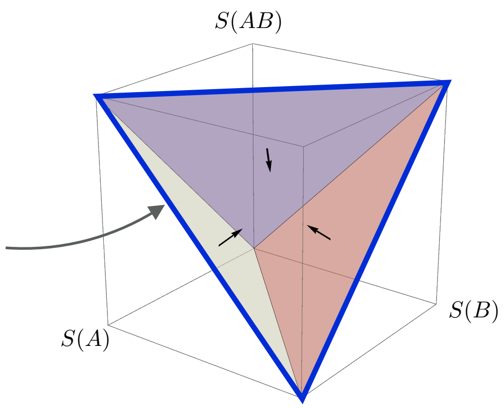


- At N=2 all these cones in $\mathbb{R}^3$ coincide, at N>2 they are strictly nested

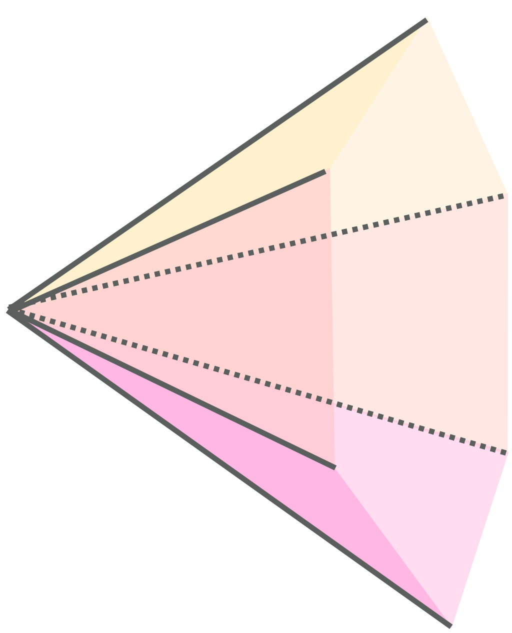
N=3 example

cartoon of 3-d cross-section of $\mathbb{R}^7$ (not including the origin)

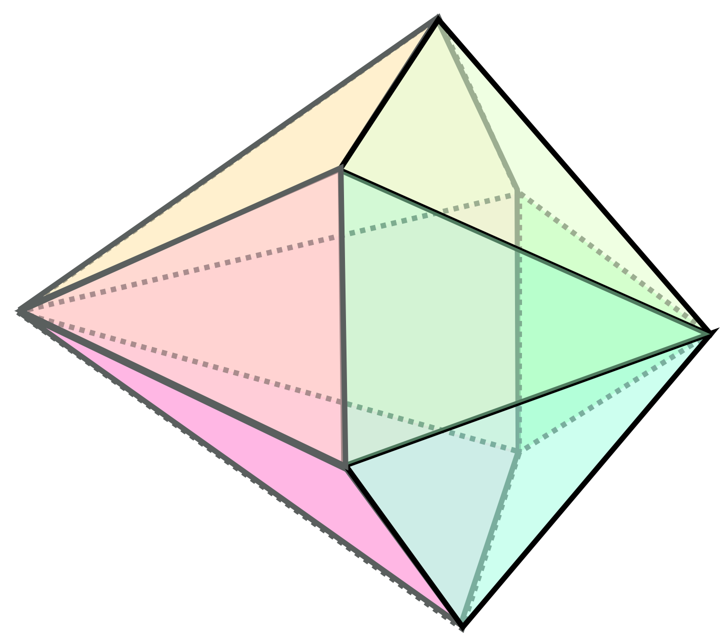
analogous to for N=2



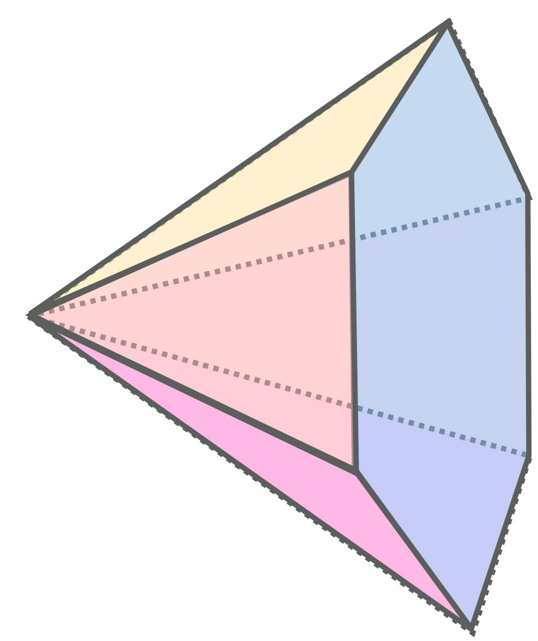
SAC



QEC



HEC



delimited by:

{SA}

{SA, SSA}

{SA, MMI}

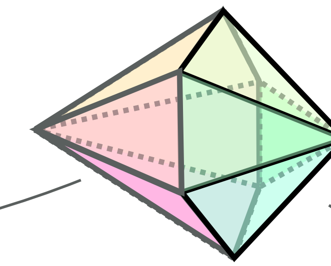
Entropy relations for N=3

Recall:

- Universal:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$

    ⇨ Mutual information positivity $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - Strong sub-additivity (SSA) $S(AC) + S(BC) \geq S(C) + S(ABC)$

    ⇨ Mutual information monotonicity $I(A:B|C) \equiv I(A:BC) - I(A:C) \geq 0$

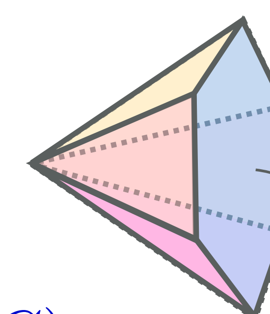


- True in holography:
  - Monogamy of mutual information (MMI)

$$S(AB) + S(BC) + S(CA) \geq S(A) + S(B) + S(C) + S(ABC)$$

  ⇨ Tripartite information $I_3(A:B:C) \equiv I(A:B) + I(A:C) - I(A:BC) \leq 0$

  - Note: SSA becomes redundant since SSA = SA + MMI

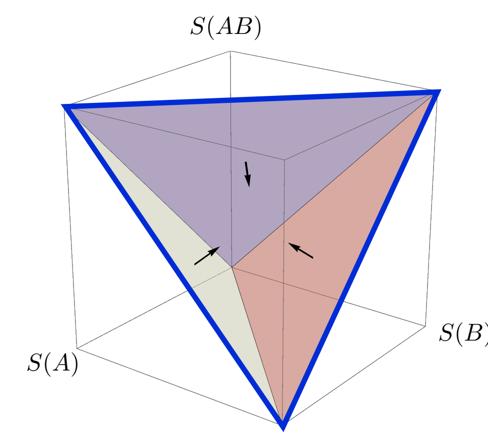


- We want non-redundant holographic entropy inequalities (HEIs)

Facets vs. extreme rays

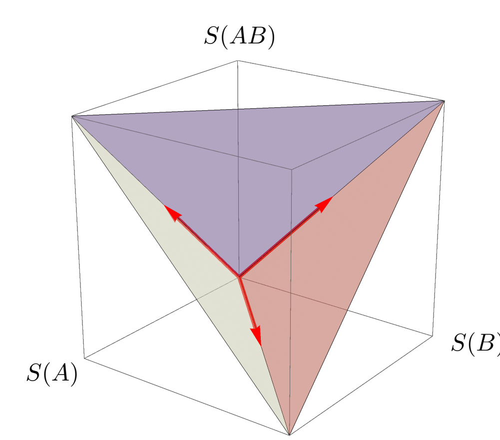
information quantity

- $Q = \sum_{\mathcal{I} \subseteq [\mathsf{N}]} c_{\mathcal{I}} \, S_{\mathcal{I}}$ describes a facet (HEI) if:
  - $Q \geq 0 \quad \forall$ states + configurations (i.e. for all realizable $\vec{S}$)
  - it is non-redundant (i.e. indep. of others): $\exists \quad (\mathsf{D}-1)$ lin. indep. entropy vectors on which $Q = 0$



$\Rightarrow$ {HEIs} = tightest possible outer-bound for {allowed states}

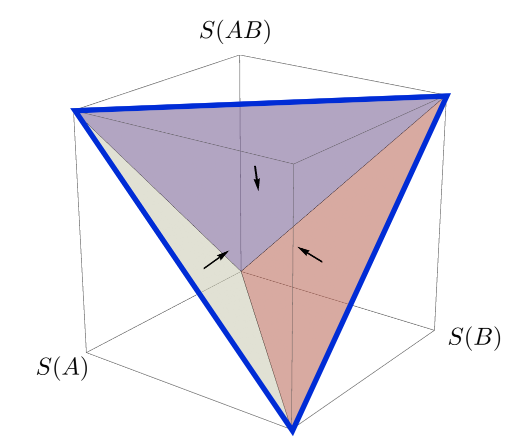
- $\vec{S}$ is an extreme ray if:
  - there exists a state + configuration realizing it
  - $\exists \quad (\mathsf{D}-1)$ lin. indep. HEIs Q which it saturates



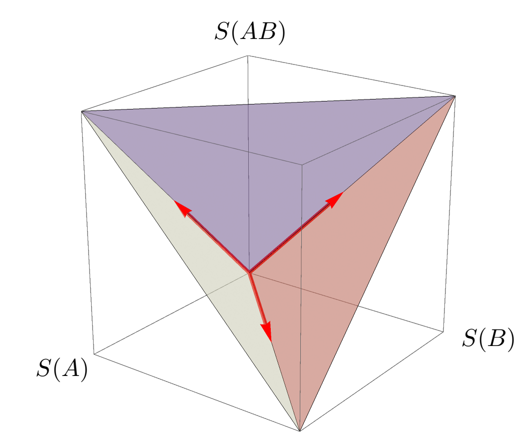
$\Rightarrow$ {ERs} = outermost possible inner bound for {allowed states}

OUTLINE

- HEC in terms of facets
  - previously-known results: N≤5
  - rewriting HEIs in tripartite form
  - new holographic entropy inequalities for N=6
  - no-go for correlation measures



- HEC in terms of extreme rays
  - previously-known results: N≤5
  - HEC from SAC and marginal independence
  - new extreme rays for N=6
  - gap between holographic and quantum SAC ERs



- Summary & future directions

OUTLINE

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  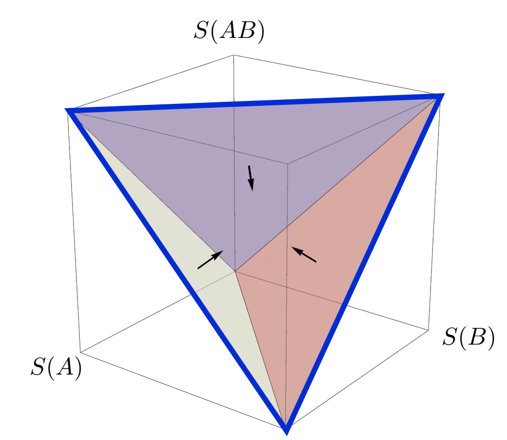

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  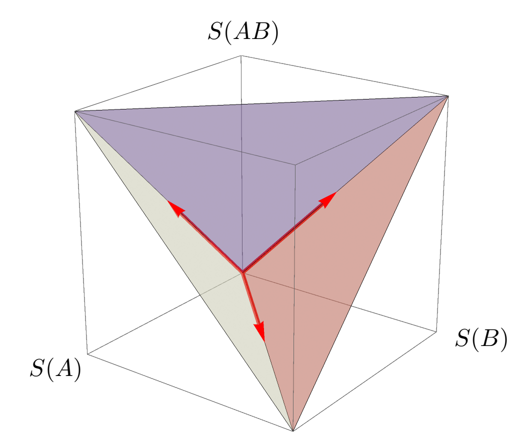

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HEC for N≤5

- While QEC_N is known only up to N=3, HEC_N is known up to N=5.
  ⤳ Important open problem in Quantum Information Science: QEC_N

- N=4 HEC still consists of only {SA, MMI} (gives 20 independent inequalities)

- N=5 HEC has 5 further inequalities

  specified in [Bao, Nezami, Ooguri, Stoica, Sully, Walter, '15]

  & proved to be the complete set in [Hernández-Cuenca, '19]

⤳ N=5 HEC has 8 orbits of facets (total 372)

e.g.: $0 \leq -S_{AB} - S_{BC} - S_{CD} - S_{DE} - S_{EA} - S_{ABCDE} + S_{ABC} + S_{BCD} + S_{CDE} + S_{DEA} + S_{EAB}$

$$0 \leq -2S_{ABCD} - S_{ABCE} - 2S_{ABDE} - S_{ACDE} - 2S_{AB} - S_{AC} - 2S_{AD} - S_{AE} - 2S_{BC} - S_{BD} - S_{CE} - 2S_{DE}$$
$$3S_{ABC} + 3S_{ABD} + S_{ABE} + S_{ACD} + S_{ACE} + 3S_{ADE} + S_{BCD} + S_{BCE} + S_{BDE} + S_{CDE}$$

?: How do we find HEC systematically & understand its meaning / implications?

HEI in terms of I_n

- Re-write in terms of more compact expressions
  - Multipartite informations with 'singleton' arguments form a basis

$$\{I_1(A), \ldots, I_2(A:B), \ldots, I_3(A:B:C), \ldots, I_{\mathsf{N}}(A:B:\ldots)\}$$

w/ tripartite information:

$$I_3(\text{A:B:C}) = \mathtt{S}_{\text{A}} + \mathtt{S}_{\text{B}} + \mathtt{S}_{\text{C}} - \mathtt{S}_{\text{AB}} - \mathtt{S}_{\text{AC}} - \mathtt{S}_{\text{BC}} + \mathtt{S}_{\text{ABC}}$$

& more generally, multipartite information:

$$\mathtt{I}_{\mathcal{J}} = \sum_{\mathcal{K} \subseteq \mathcal{J}} (-1)^{|\mathcal{K}|+1} \, \mathtt{S}_{\mathcal{K}}$$

  - Multipartite informations with composite arguments:

e.g. $I_3(\text{A:B:CD}) = \mathtt{S}_{\text{A}} + \mathtt{S}_{\text{B}} + \mathtt{S}_{\text{CD}} - \mathtt{S}_{\text{AB}} - \mathtt{S}_{\text{ACD}} - \mathtt{S}_{\text{BCD}} + \mathtt{S}_{\text{ABCD}}$

  - Conditional multipartite informations with composite arguments:

$$I_n(\text{B} : \underset{\cdot}{\mathcal{J}} | \text{A}) = I_n(\text{AB} : \underset{\cdot}{\mathcal{J}}) - I_n(\text{A} : \underset{\cdot}{\mathcal{J}})$$

shorthand for remaining arguments

Tripartite form of HEIs

- In the I-basis, HEIs are simpler [He, Headrick, VH, '19], but not simple enough...

e.g.: $0 \leq Q = -S_{ABCD} - S_{BCDE} - S_{ABE} - S_{BC} - S_{BD} - S_A - S_C - S_D - S_E + S_{ABC} + S_{ABD} + S_{BCD} + S_{BCE} + S_{BDE} + S_{AE} + S_{CD}$

vs. $Q = I_{ABCD} + I_{BCDE} - I_{ABE} - I_{ACD} - I_{BCD} - I_{CDE}$

- However, written in terms of I_3 and its conditional form, w/ composite arguments, and negative coeffs ≡ *tripartite form*:

$$Q = \sum_i -I_3(X_i : Y_i : Z_i \,|\, W_i) \qquad \text{w/} \quad I_3(X_i : Y_i : Z_i \,|\, \varnothing) := I_3(X_i : Y_i : Z_i)$$

HEIs become more compact [Hernández-Cuenca, VH, Jia, '23]:

now $Q = -I_3(AB{:}C{:}D) - I_3(B{:}D{:}E|C) - I_3(A{:}C{:}E|D)$

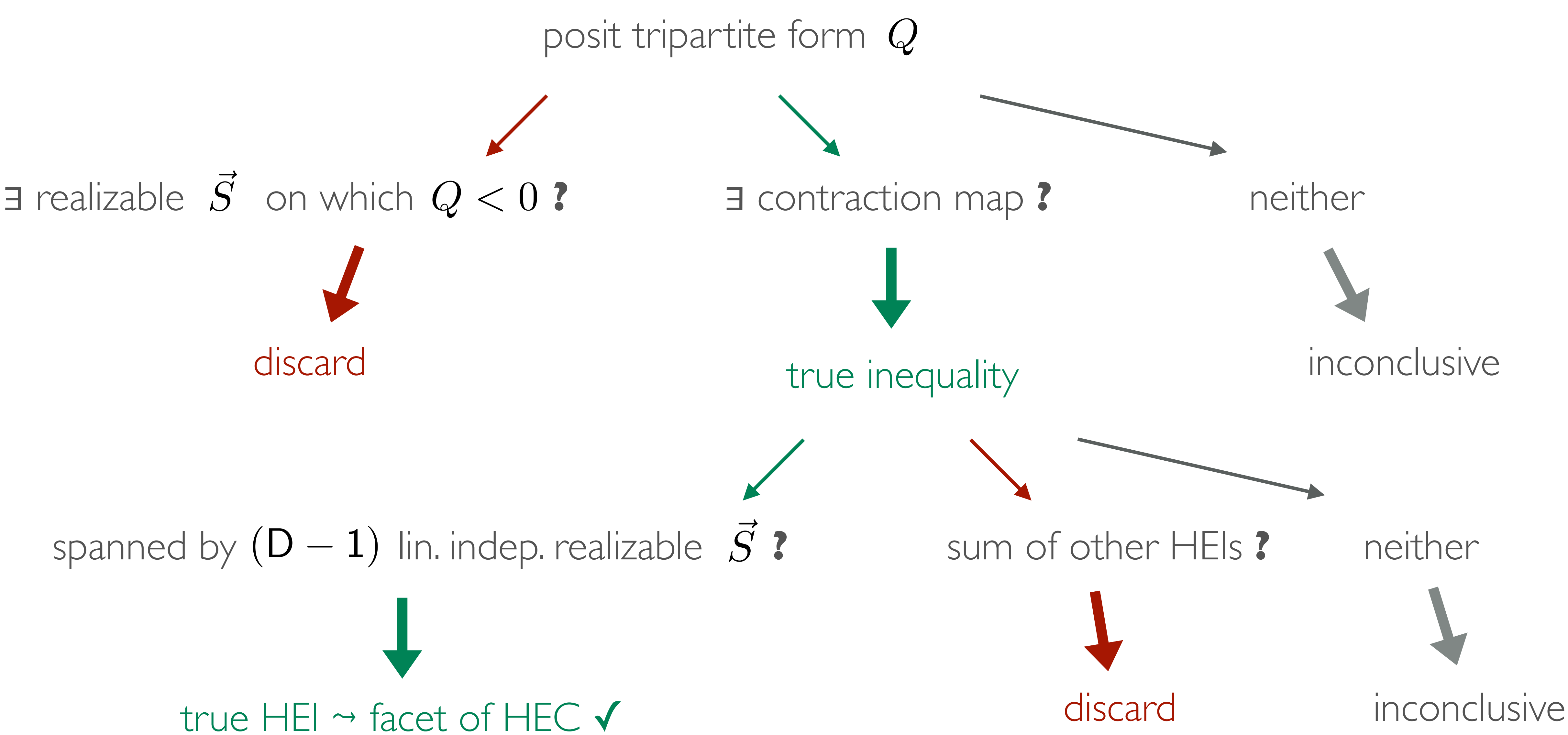
- $\rightsquigarrow$ (so far) typical reduction in # of terms by factor of 5, sometimes >10

HEIs for N=5

$\mathsf{N} = 5$ HEI information quantities
$-\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{ACDE}} - \mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{AD}} - \mathtt{S}_{\mathrm{DE}} - \mathtt{S}_{\mathrm{C}}$
$\mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{ACD}} + \mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{CDE}}$
$\mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{ACDE}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{ACE}} - \mathtt{I}_{\mathrm{BCD}}$
$-I_3(\mathrm{AB{:}C{:}D}) - I_3(\mathrm{A{:}C{:}E}\vert\mathrm{D})$
$-\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{BCDE}} - \mathtt{S}_{\mathrm{ABE}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BD}} - \mathtt{S}_{\mathrm{A}} - \mathtt{S}_{\mathrm{C}} - \mathtt{S}_{\mathrm{D}} - \mathtt{S}_{\mathrm{E}}$
$\mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{AE}} + \mathtt{S}_{\mathrm{CD}}$
$\mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{BCDE}} - \mathtt{I}_{\mathrm{ABE}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{CDE}}$
$-I_3(\mathrm{AB{:}C{:}D}) - I_3(\mathrm{A{:}B{:}E}) - I_3(\mathrm{C{:}D{:}E}\vert\mathrm{B})$
$-\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{ACDE}} - \mathtt{S}_{\mathrm{BCDE}} - \mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{AD}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{CD}} - \mathtt{S}_{\mathrm{CE}} - \mathtt{S}_{\mathrm{DE}}$
$\mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{ACD}} + \mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + 2\mathtt{S}_{\mathrm{CDE}}$
$\mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{ACDE}} + \mathtt{I}_{\mathrm{BCDE}} - \mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{ACE}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{BDE}}$
$-I_3(\mathrm{AB{:}C{:}D}) - I_3(\mathrm{B{:}D{:}E}\vert\mathrm{C}) - I_3(\mathrm{A{:}C{:}E}\vert\mathrm{D})$
$-\mathtt{S}_{\mathrm{ABCE}} - \mathtt{S}_{\mathrm{ACDE}} - \mathtt{S}_{\mathrm{BCDE}} - \mathtt{S}_{\mathrm{ABD}} - \mathtt{S}_{\mathrm{AC}} - \mathtt{S}_{\mathrm{AE}} - \mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BE}} - \mathtt{S}_{\mathrm{CD}} - \mathtt{S}_{\mathrm{DE}}$
$\mathtt{S}_{\mathrm{ABC}} + \mathtt{S}_{\mathrm{ABE}} + \mathtt{S}_{\mathrm{ACD}} + \mathtt{S}_{\mathrm{ACE}} + \mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{CDE}}$
$\mathtt{I}_{\mathrm{ABCE}} + \mathtt{I}_{\mathrm{ACDE}} + \mathtt{I}_{\mathrm{BCDE}} - \mathtt{I}_{\mathrm{ABD}} - \mathtt{I}_{\mathrm{ACE}} - \mathtt{I}_{\mathrm{BCE}} - \mathtt{I}_{\mathrm{CDE}}$
$-I_3(\mathrm{A{:}B{:}D}) - I_3(\mathrm{B{:}C{:}E}\vert\mathrm{A}) - I_3(\mathrm{C{:}D{:}E}\vert\mathrm{B}) - I_3(\mathrm{A{:}C{:}E}\vert\mathrm{D})$
$-2\mathtt{S}_{\mathrm{ABCD}} - \mathtt{S}_{\mathrm{ABCE}} - 2\mathtt{S}_{\mathrm{ABDE}} - \mathtt{S}_{\mathrm{ACDE}} - 2\mathtt{S}_{\mathrm{AB}} - \mathtt{S}_{\mathrm{AC}} - 2\mathtt{S}_{\mathrm{AD}} - \mathtt{S}_{\mathrm{AE}} - 2\mathtt{S}_{\mathrm{BC}} - \mathtt{S}_{\mathrm{BD}} - \mathtt{S}_{\mathrm{CE}} - 2\mathtt{S}_{\mathrm{DE}}$
$3\mathtt{S}_{\mathrm{ABC}} + 3\mathtt{S}_{\mathrm{ABD}} + \mathtt{S}_{\mathrm{ABE}} + \mathtt{S}_{\mathrm{ACD}} + \mathtt{S}_{\mathrm{ACE}} + 3\mathtt{S}_{\mathrm{ADE}} + \mathtt{S}_{\mathrm{BCD}} + \mathtt{S}_{\mathrm{BCE}} + \mathtt{S}_{\mathrm{BDE}} + \mathtt{S}_{\mathrm{CDE}}$
$2\mathtt{I}_{\mathrm{ABCD}} + \mathtt{I}_{\mathrm{ABCE}} + 2\mathtt{I}_{\mathrm{ABDE}} + \mathtt{I}_{\mathrm{ACDE}} - \mathtt{I}_{\mathrm{ABD}} - 2\mathtt{I}_{\mathrm{ABE}} - 2\mathtt{I}_{\mathrm{ACD}} - \mathtt{I}_{\mathrm{ACE}} - \mathtt{I}_{\mathrm{BCD}} - \mathtt{I}_{\mathrm{BDE}}$
$-I_3(\mathrm{AB{:}C{:}D}) - I_3(\mathrm{AE{:}B{:}D}) - I_3(\mathrm{A{:}B{:}E}\vert\mathrm{C}) - I_3(\mathrm{A{:}B{:}E}\vert\mathrm{D}) - I_3(\mathrm{A{:}C{:}D}\vert\mathrm{B}) - I_3(\mathrm{A{:}C{:}E}\vert\mathrm{D})$

Finding new HEIs

- This provides a useful HEI generating technique: [Hernández-Cuenca, VH, Jia, '23]



check that not in perm/purif. orbit of any other HEI (permute to convenient instance)

New HEIs for N=6

- For N=6 we obtained >370 new **orbits** of HEIs
  - 330 exhaustively for tripartite form w/ length ≤ 4 (I, IC, IIC, ICC, IIIC, IICC, ICCC);

Structure	I	IC	IIC	ICC	IIIC	IICC	ICCC
Facets	3	3	13	3	92 (+17?)	215 (+17?)	1
Lifts	3	2	2	1	0	0	1

  - e.g.:

form	Selected $\mathsf{N}=6$ HEI information quantities
IC	$-S_{ABDEF}-S_{ABCF}-S_{AD}-S_{AF}-S_{BE}-S_{BF}-S_{CF}$ $+S_{ABDE}+S_{ABF}+S_{ACF}+S_{ADF}+S_{BCF}+S_{BEF}$ $-I_{ABDEF}+I_{ABCF}+I_{ABDF}+I_{ABEF}+I_{ADEF}+I_{BDEF}-I_{ABC}-I_{ABF}-I_{AEF}-I_{BDF}-I_{DEF}$ $-I_3(AD{:}BE{:}F)-I_3(A{:}B{:}C\|F)$
ICC	$-S_{ABCF}-S_{ABDF}-S_{ABEF}-S_{AB}-S_{AD}-S_{AF}-S_{BE}-S_{BF}-S_{CF}$ $+S_{ABD}+S_{ABE}+2S_{ABF}+S_{ACF}+S_{ADF}+S_{BCF}+S_{BEF}$ $I_{ABCF}+I_{ABDF}+I_{ABEF}-I_{ABC}-I_{ABF}-I_{AEF}-I_{BDF}$ $-I_3(AD{:}B{:}F)-I_3(A{:}E{:}F\|B)-I_3(A{:}B{:}C\|F)$
IIC	$-S_{ABCF}-S_{ABEF}-S_{CDEF}-S_{AF}-S_{BF}-S_{CD}-S_{CF}-S_{A}-S_{B}-S_{E}$ $+S_{ABF}+S_{ACF}+S_{AEF}+S_{BCF}+S_{BEF}+S_{CDE}+S_{CDF}+S_{AB}$ $I_{ABCF}+I_{ABEF}+I_{CDEF}-I_{ABC}-I_{ABE}-I_{ABF}-I_{CEF}-I_{DEF}$ $-I_3(CD{:}E{:}F)-I_3(A{:}B{:}EF)-I_3(A{:}B{:}C\|F)$
IIIC	$-2S_{ABCDF}-S_{ACEF}-S_{BDEF}-S_{CDF}-S_{AF}-S_{DE}-S_{EF}-S_{A}-S_{B}-2S_{C}-S_{D}$ $+S_{ABCF}+S_{ABDF}+S_{ACDF}+S_{BCDF}+S_{AEF}+S_{BDE}+S_{CEF}+S_{DEF}+S_{AC}+S_{CD}$ $-2I_{ABCDF}+2I_{ABCD}+I_{ABCF}+I_{ABDF}+I_{ACDF}+I_{ACEF}+I_{BCDF}+I_{BDEF}$ $-I_{ABC}-I_{ABD}-I_{ACD}-I_{ACE}-I_{ACF}-I_{BCD}-I_{BDF}-I_{BEF}-I_{CDF}$ $-I_3(ABF{:}C{:}D)-I_3(B{:}DE{:}F)-I_3(A{:}C{:}EF)-I_3(A{:}B{:}CD\|F)$

Structural relations

- Advantage of tripartite form
  - much more compact
  - easier to see structural relations
  - may suggest how to "bootstrap" to higher N

e.g.: HEI #

$$\begin{aligned}
Q^{[6]}_{\{4,2\}} &= -I_3(\mathrm{C{:}E{:}F}) - I_3(\mathrm{A{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F}) \\
Q^{[13]}_{\{5,3\}} &= -I_3(\mathrm{CD{:}E{:}F}) - I_3(\mathrm{A{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F}) \\
Q^{[14]}_{\{6,5,1\}} &= -I_3(\mathrm{C{:}E{:}F}) - I_3(\mathrm{AD{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F}) \\
Q^{[18]}_{\{7,6,1\}} &= -I_3(\mathrm{CD{:}E{:}F}) - I_3(\mathrm{AD{:}B{:}EF}) - I_3(\mathrm{A{:}B{:}C|F})
\end{aligned}$$

$\{\#I_3, \#I_4, \#I_5\}$

or:

$$\begin{aligned}
Q^{[27]}_{\{8,6,1\}} &= -I_3(\mathrm{AF{:}C{:}D}) - I_3(\mathrm{B{:}DE{:}F}) - I_3(\mathrm{A{:}C{:}EF}) - I_3(\mathrm{A{:}B{:}CD|F}) \\
Q^{[28]}_{\{8,6,1\}} &= -I_3(\mathrm{BF{:}C{:}D}) - I_3(\mathrm{B{:}DE{:}F}) - I_3(\mathrm{A{:}C{:}EF}) - I_3(\mathrm{A{:}B{:}CD|F}) \\
Q^{[34]}_{\{9,8,2\}} &= -I_3(\mathrm{ABF{:}C{:}D}) - I_3(\mathrm{B{:}DE{:}F}) - I_3(\mathrm{A{:}C{:}EF}) - I_3(\mathrm{A{:}B{:}CD|F})
\end{aligned}$$

Building up HEIs?

- Add successive terms
  - need conditional I_3 at each step (s.t. not sign-definite)

$$Q^{[3]}_{\{2,1\}} = -I_3(\mathrm{A{:}BE{:}D})$$
$$Q^{[5]}_{\{3,2\}} = -I_3(\mathrm{A{:}BE{:}D}) \; -I_3(\mathrm{A{:}B{:}C|D})$$
$$Q^{[8]}_{\{4,3\}} = -I_3(\mathrm{A{:}BE{:}D}) \; -I_3(\mathrm{A{:}B{:}C|D}) \; -I_3(\mathrm{C{:}D{:}E|A})$$
$$Q^{[10]}_{\{5,5,1\}} = -I_3(\mathrm{A{:}BE{:}D}) \; -I_3(\mathrm{A{:}B{:}C|D}) \; -I_3(\mathrm{C{:}D{:}E|A}) \; -I_3(\mathrm{B{:}C{:}E|AD})$$

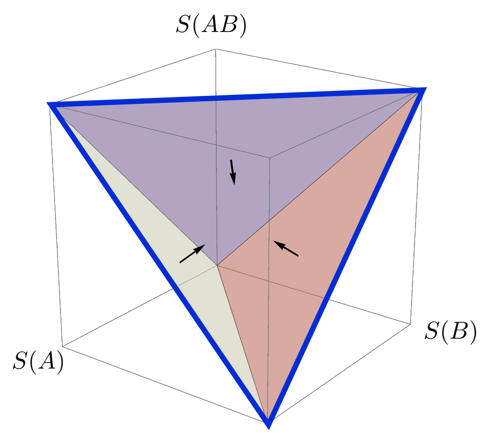
- Successive / partial augmentations
  - start w/ MMI: $Q^{[2]}_{\{1\}} = -\mathtt{I}_{\mathrm{A:B:C}}$
  - uplift to N=4: $Q^{[3]}_{\{2,1\}} = -I_3(\mathrm{A{:}B{:}CD}) = -I_3(\mathrm{A{:}B{:}D}) - I_3(\mathrm{A{:}B{:}C|D})$
  - augment 2nd argument in 1st term: $Q^{[5]}_{\{3,2\}} = -I_3(\mathrm{A{:}BE{:}D}) - I_3(\mathrm{A{:}B{:}C|D})$
  - augment 1st & 2nd arguments in 1st term: $Q^{[12]}_{\{5,5,1\}} = -I_3(\mathrm{AF{:}BE{:}D}) - I_3(\mathrm{A{:}B{:}C|D})$

Higher HEIs & correlation measures

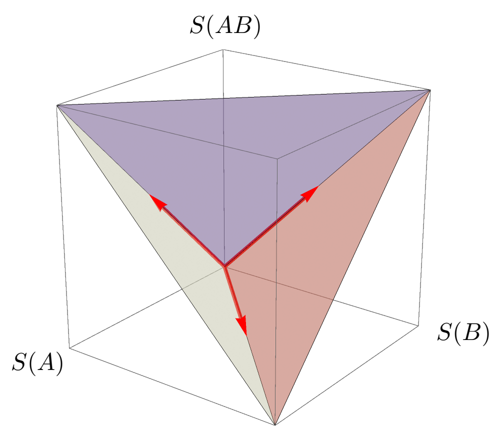
- Correlation measure:
  - Non-negative
  - Monotonically increasing under inclusion
- cf. mutual information:
  - SA: $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - SSA: $I(A:B|C) \equiv I(A:BC) - I(A:C) \geq 0$
- Higher HEIs are *superbalanced* [He, VH, Rangamani, '20]
  - finite on any configuration
  - in I-basis, contain only I_n for $n \geq 3$
  - $\Rightarrow$ not monotonic under inclusion [Hernández-Cuenca, VH, Jia, '23]
- Higher HEI quantities are <u>not</u> correlation measures!
  - What are they?

OUTLINE

- HEC in terms of facets
  - previously-known results: N≤5
  - rewriting HEIs in tripartite form
  - new holographic entropy inequalities for N=6
  - no-go for correlation measures



- HEC in terms of extreme rays
  - previously-known results: N≤5
  - HEC from SAC and marginal independence
  - new extreme rays for N=6
  - gap between holographic and quantum SAC ERs



- Summary & future directions

HEC in terms of ERs

- ERs correspond to special/extreme states
  - HEC = convex hull of ERs
  - Simultaneously saturate (D-1) independent HEIs
  - Holographically correspond to multi-boundary wormholes

- Useful toolkit:
  - Holographic graph model ← $\vec{S}$ from spacetime
  - Pattern of marginal independence (PMI) ← entanglement structure from $\vec{S}$

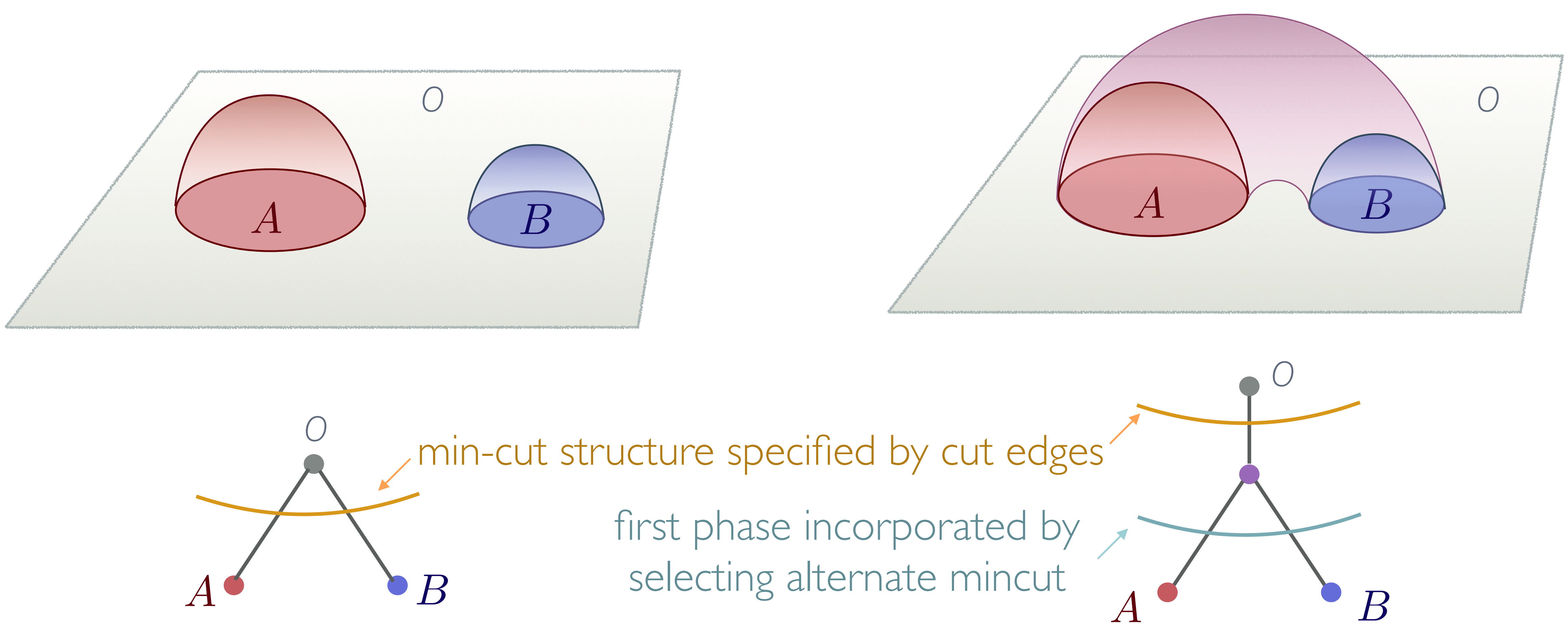
distills information content:

These will lead us to study the ERs of the SAC.

Holographic graph models

cf. [Bao, Nezami, Ooguri, Stoica, Sully, Walter]

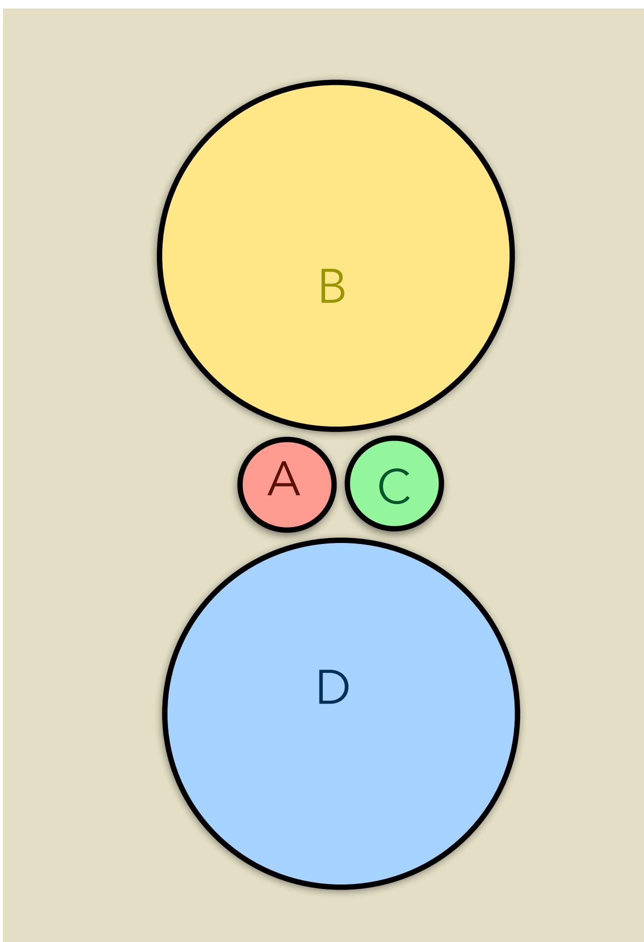
- Distill the discrete elements from a holographic configuration
  - vertices = cells in RT surface network
  - edges = pieces of RT surfaces separating neighboring regions, with weight = corresponding area
  - EE = min-cut

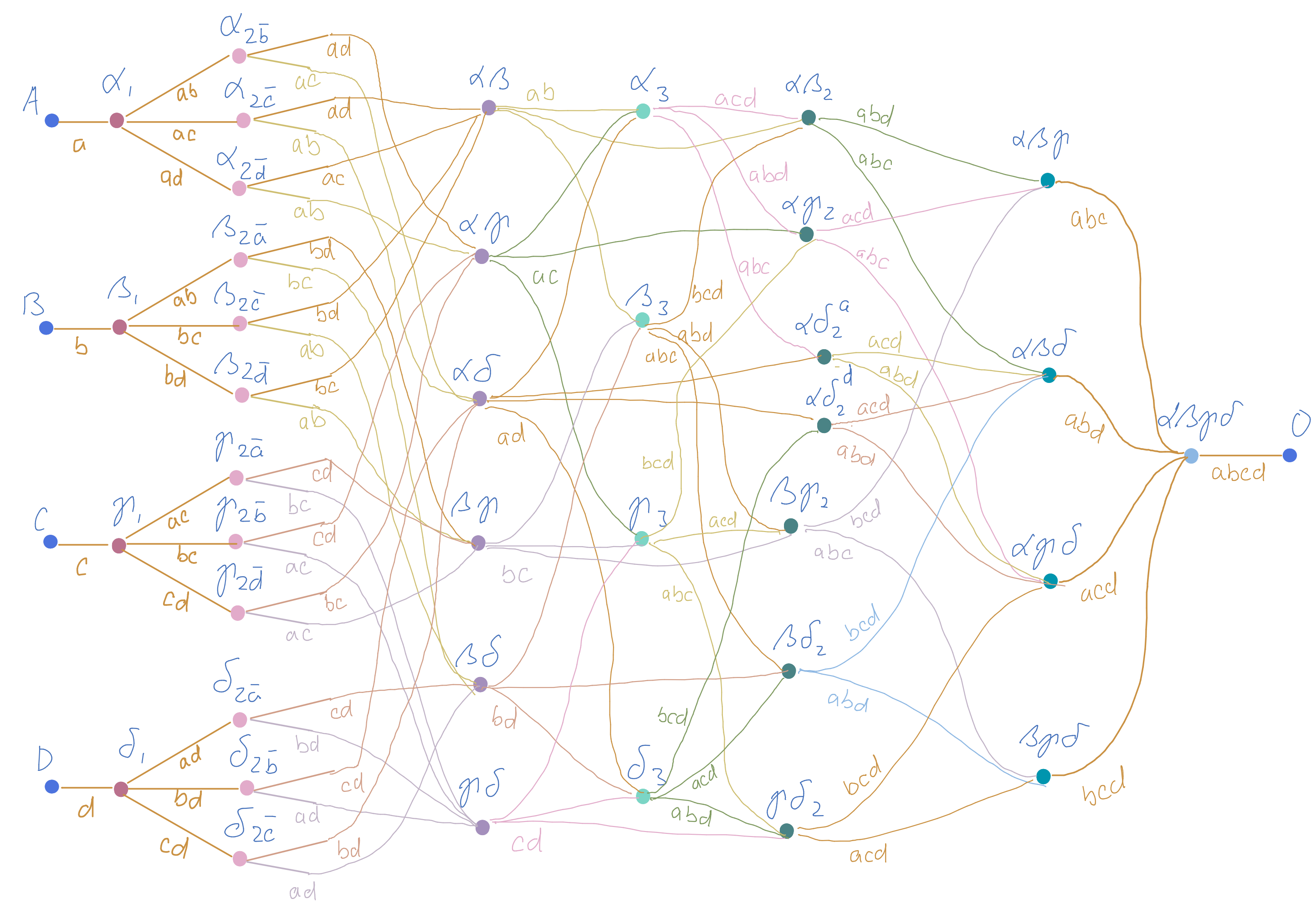


Holographic graph models

- Graphs can get complicated...

e.g. graph model for





has 43 vertices & 88 edges

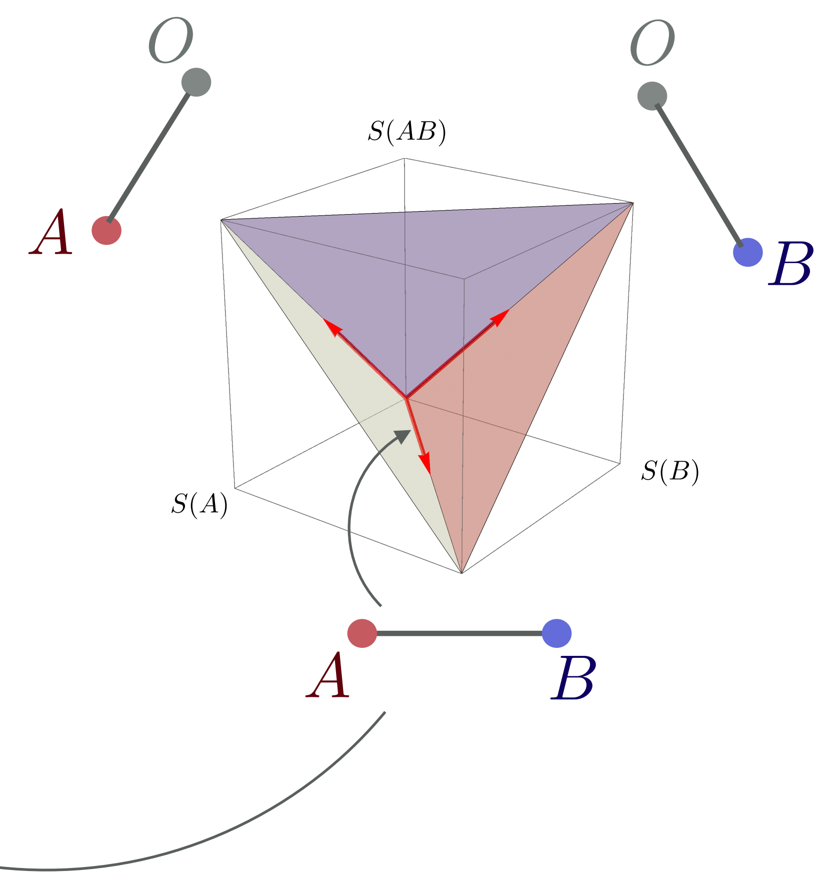
- But graph cone = HEC

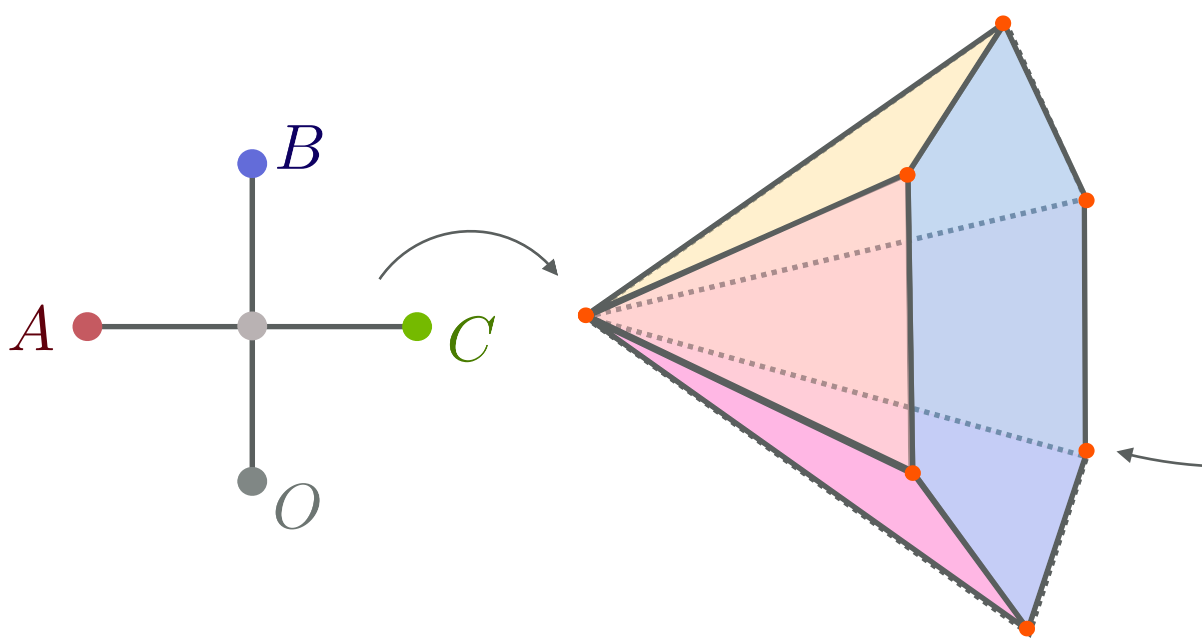
[Bao, Nezami, Ooguri, Stoica, Sully, Walter]

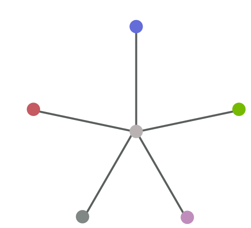
Graph representation of HEC ERs

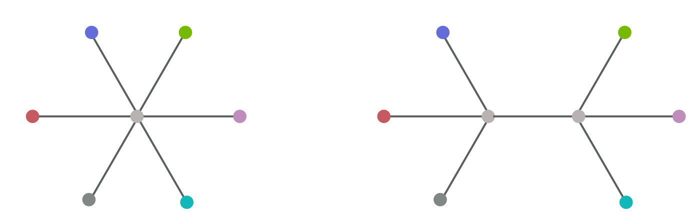
- N=2 HEC has 3 extreme rays (1 orbit)

- N=3 HEC has 7 extreme rays (2 orbits)





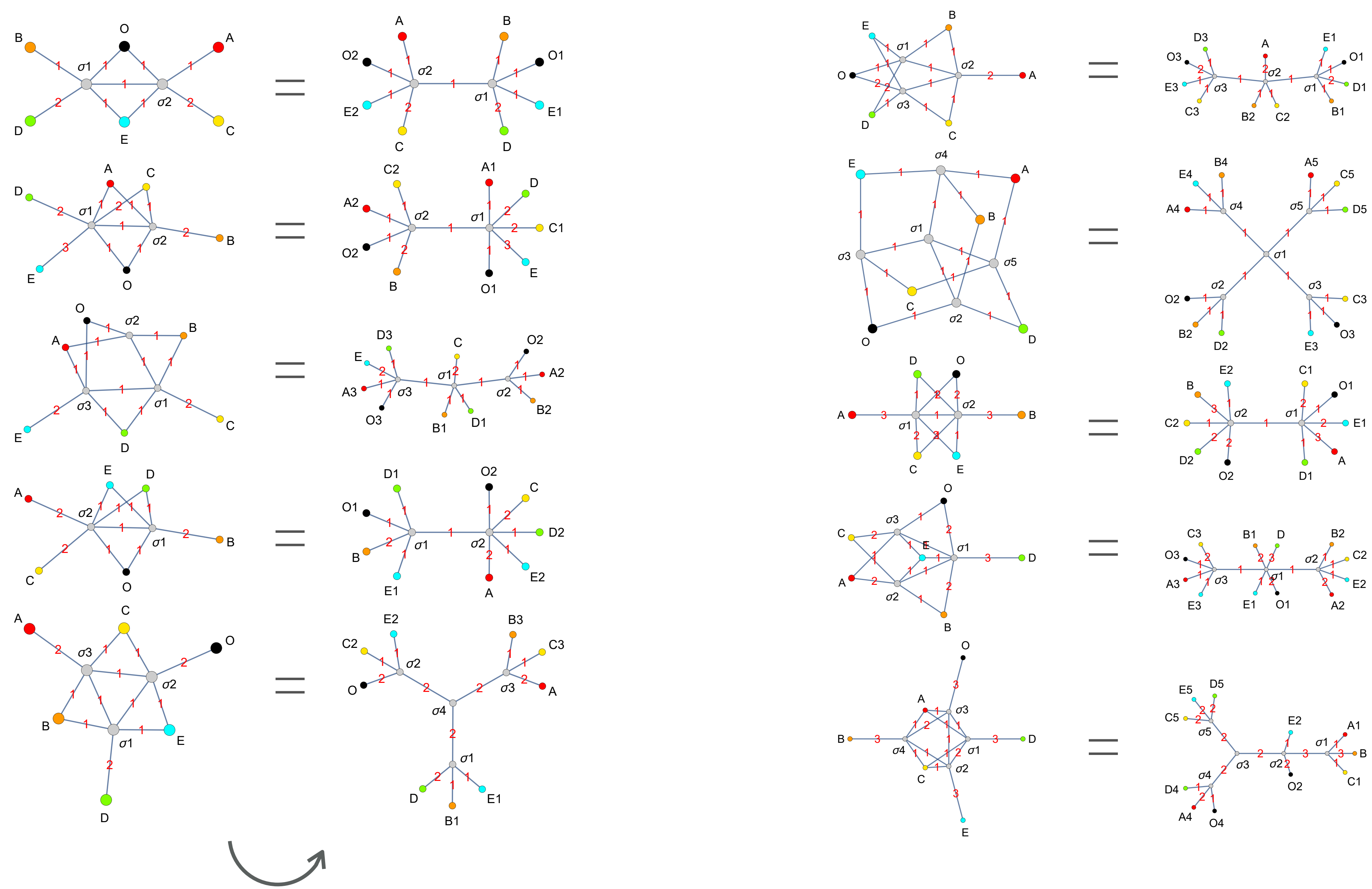
- N=4 HEC has 20 extreme rays (3 orbits)   adds 

- N=5 HEC has 2267 extreme rays (19 orbits)   adds 

and:

Graph representation of HEC_5 ERs

[Hernández-Cuenca, '19]



- All deformable to tree graphs (though with multiple vertices of same color)

[Hernández-Cuenca, VH, Rota '22]

PMI

- **Def**: *Pattern of Marginal Independence* (PMI) is a specification of full set of subsystems $\{ X , Y \}$ for which $I(X{:}Y) = 0$.
  - *marginals* = reduced density matrices
  - *independent* if factorized structure

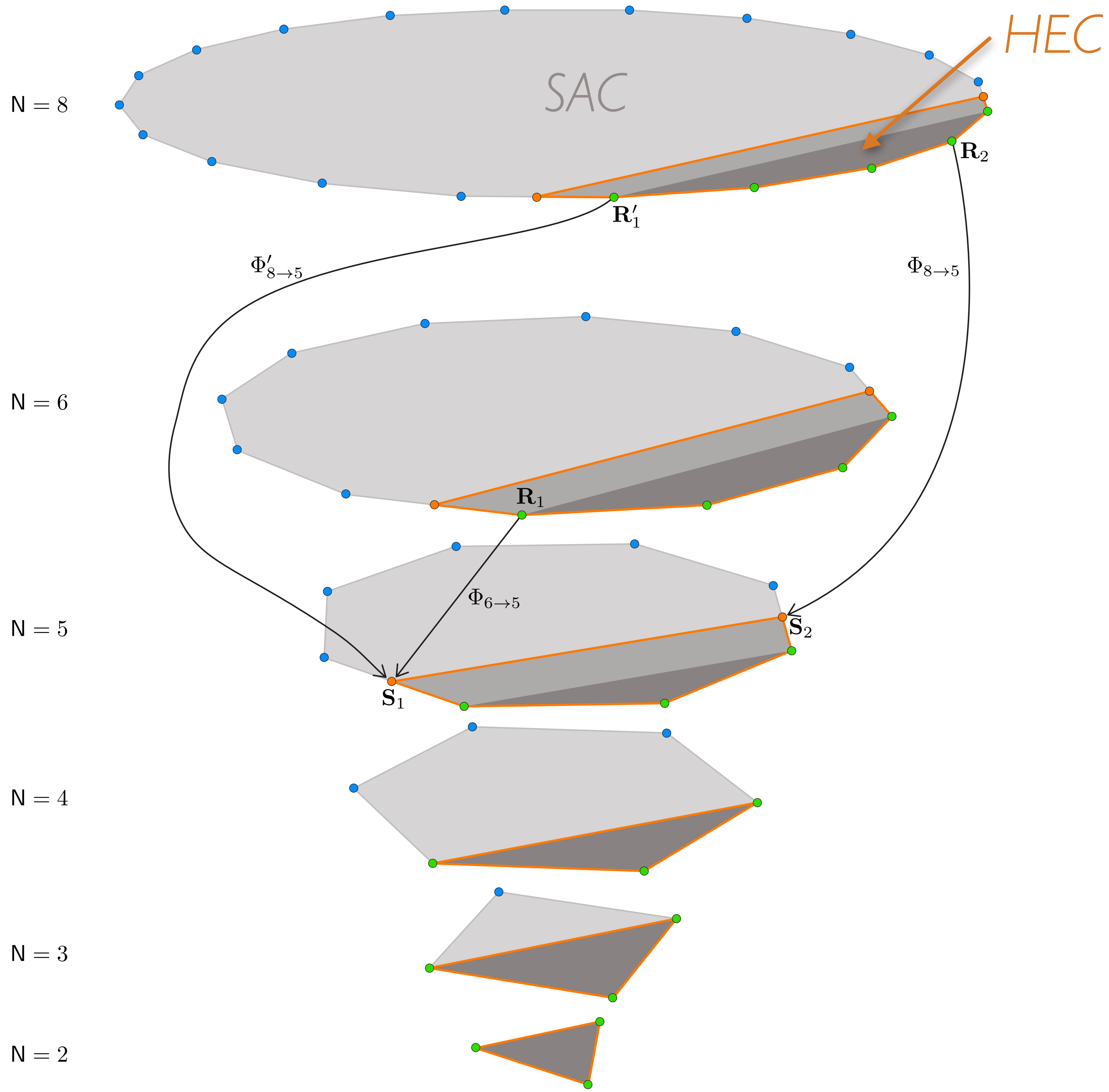
- Meaning & Utility
  - = intersection of all saturated SA or AL hyperplanes
  - linear subspace of entropy space
  - (out of finite arrangement) ⇒ discrete structure
  - holographically: bdy region pairs, s.t. the joint entanglement wedge is disconnected (since connected XY ent. wedge $\Longrightarrow$ $I(X{:}Y) > 0$)
  - every entropy vector $\vec{S}$ has a unique PMI
  - for physical $\vec{S}$, PMI = span of a face of the SAC

HEC from SAC

- Utilize graph model
  - ER graph has maximal min-cut degeneracy (cf. phase transition of RT surfaces)
  - Conjecture: ERs can be rendered as tree graphs (has strong evidence)
- **Thm**: Assuming Conjecture, every ER of HEC_N is obtained as a projection of ER of a subadditivity cone $SAC_{N'}$ (for some $N' \geq N$)

[Hernández-Cuenca, VH, Rota, '22]

Cartoon of HEC vs. SAC



HEC from SAC

- Utilize graph model
  - ER graph has maximal min-cut degeneracy (cf. phase transition of RT surfaces)
  - Conjecture: ERs can be rendered as tree graphs (has large evidence)
- **Thm**: Assuming Conjecture, every ER of HEC_N is obtained as a projection of ER of a subadditivity cone $SAC_{N'}$ (for some $N' \geq N$)

  [Hernández-Cuenca, VH, Rota]

- This in principle allows us to construct the full HEC_N for any N
  - In practice complicated: requires correct set of ERs of $SAC_{N'}$ for all N', projecting, taking convex hull, extracting ERs, and constructing facets...
  - But conceptually demystifies the HEC (and entanglement structure of holographic states)
- Crux: how can we characterize the requisite set of SAC ERs?
  - Formulate in terms of PMIs

Marginal Independence Problem

- Not all marginal independence bipartitions are possible, due to:
  - mathematical inconsistency

    e.g. violates the identity $I_2(A:BC) + I_2(B:C) = I_2(B:AC) + I_2(A:C)$

  - physical inconsistency

    violates entropy inequality, e.g. SA ↗

    or SSA $\Rightarrow$ $I_2(A:BC) = 0 \quad \Rightarrow \quad I_2(A:B) = 0$

    ↑ Klein's condition (KC) [He, VH, Rota]

- *Marginal Independence Problem* (MIP): what PMIs are realizable?
  - QMIP: what PMIs are realizable in QM?

    considered in [Hernández-Cuenca, VH, Rangamani, Rota]

  - HMIP: what PMIs are realizable by geometric states in holography?

    [He, VH, Rota] & WIP

Power of restrictions

- Linear dependence: e.g. $I_2(A:BC) + I_2(B:C) = I_2(B:AC) + I_2(A:C)$
  - Implemented geometrically by PMI = span of face of SAC
- KC: i.e. $I_2(A:BC) = 0 \quad \Rightarrow \quad I_2(A:B) = 0$
  - Implemented as a down-set in MI poset
  - Reduces the # of potentially viable PMIs drastically:

N=3: # of SAC faces (PMIs) of a given dimension d ≤ D:

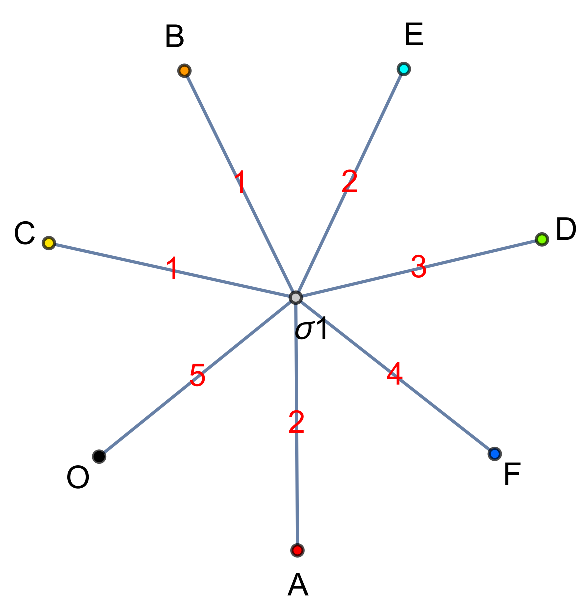
d	0	1	2	3	4	5	6	7
total	1	11	48	107	127	75	18	1
KC	1	7	21	35	32	15	6	1

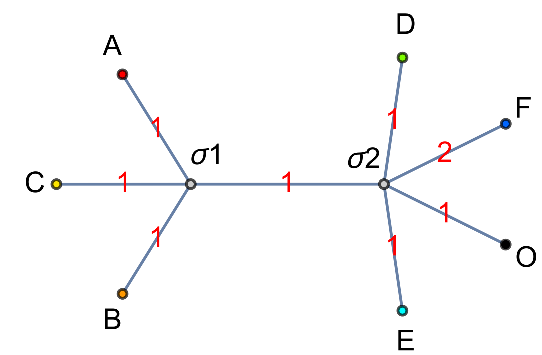
N=4:

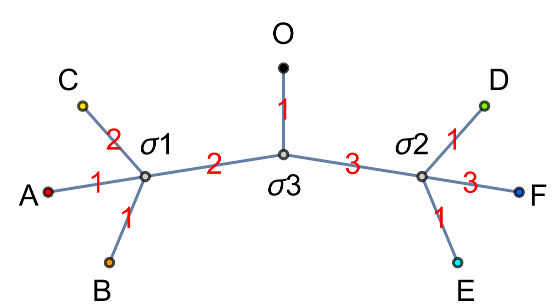
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3085	66005	532585	2254005	5719656	9301825	10032200	7275805	3541900	1138826	234470	29455	2100	75	1
1	20	175	840	2465	4843	6345	5875	4100	2300	1072	430	150	45	10	1

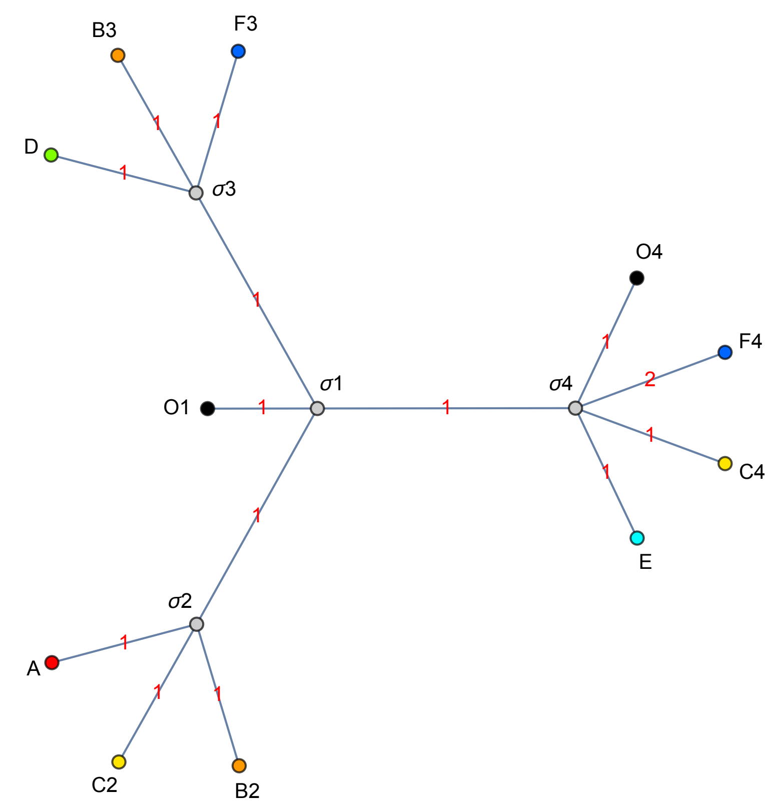
New ERs for N=6

- Hernández-Cuenca obtained >3700 orbits of HEC_6 ERs
- [WIP: He, VH, Rota] solve {I(X:Y)=0} ↝ SAC ERs; then find graph for $\vec{S}$
  - So far (WIP) se obtained >50 orbits of holographic SAC_6 ERs (some new)
  - e.g.:









Classes of ERs of SAC

Let $\mathcal{R} := \{\mathbb{P} \in \mathfrak{L}^{\mathsf{N}}_{\mathrm{PMI}} : \dim(\mathbb{P}) = 1\}$ = set of all extreme rays of the SAC

$\mathcal{R}_{\mathrm{KC}} := \{\mathbb{P} \in \mathfrak{L}^{\mathsf{N}}_{\mathrm{KC}} : \dim(\mathbb{P}) = 1\}$ = set of KC-compatible ERs of the SAC

$$\mathcal{R}_{\mathrm{SSA}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is SSA-compatible}\}$$

$$\mathcal{R}_{\mathrm{Q}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is realizable by a quantum state}\}$$

$$\mathcal{R}_{\mathrm{H}} := \{\mathbb{P} \in \mathcal{R} : \mathbb{P} \text{ is realizable by a graph model}\}\,,$$

$$\Rightarrow \qquad \mathcal{R}_{\mathrm{H}} \subseteq \mathcal{R}_{\mathrm{Q}} \subseteq \mathcal{R}_{\mathrm{SSA}} \subseteq \mathcal{R}_{\mathrm{KC}} \subseteq \mathcal{R}$$

- Observations:
  - For N=2, $\mathcal{R}_{\mathrm{SSA}} = \mathcal{R}$; but otherwise $|\mathcal{R}_{\mathrm{SSA}}| \ll |\mathcal{R}|$
  - For N ≤ 5, $\mathcal{R}_{\mathrm{H}} = \mathcal{R}_{\mathrm{SSA}}$
  - Original hope: this prevails ∀ N, which would give $\mathcal{R}_{\mathrm{H}} = \mathcal{R}_{\mathrm{Q}}$
  - (establishing this would be useful since little known about QEC)

Holographic - quantum gap

- However, ∃ a counter-example to $\mathcal{R}_{\mathrm{H}} \overset{?}{=} \mathcal{R}_{\mathrm{Q}}$ at N=6: [He, VH, Rota, '23]

The following ER of SAC₆ : $$\mathsf{R}_6 = (2,1,1,1,2,2;3,3,3,4,4,2,2,3,3,2,3,3,3,3,4;2,4,5,5,4,5,5,3,5,4,3,4,4,4,4,5,4,4,5,3;3,4,4,4,4,3,4,4,3,3,3,5,4,4,4;3,3,2,2,2,3;1)$$

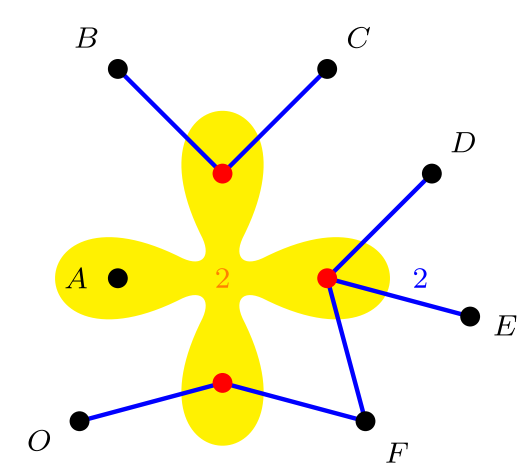
w/ components ordered as $(A,\ldots,F;AB,AC,\ldots,EF;ABC,\ldots;ABCDEF)$

violates MMI, since $$-\mathsf{I}_3\,(A:BC:DE) = -2 \not\geq 0$$

$$\Longrightarrow \mathsf{R}_6 \notin \mathcal{R}_{\mathrm{H}}$$

But, ∃ a hypergraph which realizes R_6, so it describes a stabilizer state.

[Walter & Witteveen]



$$\Longrightarrow \mathsf{R}_6 \in \mathcal{R}_{\mathrm{Q}}$$

$\Longrightarrow \mathcal{R}_{\mathrm{H}} \neq \mathcal{R}_{\mathrm{Q}}$ i.e. ∃ a gap between holographic and quantum ERs of SAC.

Summary & future directions

- Holographic entropy inequalities
  - Can be packaged efficiently using the tripartite form
  - Constructed 384 orbits of HEIs for N=6
  - These manifest rich structural relations
  - How can we bootstrap these to generate new HEIs for higher N?
  - Are all HEIs guaranteed to admit the tripartite form?
  - Is there an even better packaging?
- Interpretation?
  - Not correlation measures (since not monotonic under inclusion)
  - More generalized multipartite correlation?
  - Operational meaning?

Summary & future directions

- HEC from extreme rays:
  - HEC_N can be fully reconstructed from far simpler structure, at finer N':

    Holographically realizable ERs of $\text{SAC}_{N'}$ (← solution to HMIP)

    ↓

    describe using PMIs, specified by { I(X:Y) }
- Implications:
  - { HEC_N ∀ N } ⟸ { HMIP_N ∀ N } ("Holographic entropy cone from marginal independence")
  - Any fixed N contaminated by structural (combinatorial) artifacts
  - Seemingly minimal dependence on holography (SAC universal)

    ↓

    - arbitrarily refined partition (N)
    - classicality: admits phase transitions

Summary & future directions

- HEC from extreme rays:
  - HEC_N can be fully reconstructed from far simpler structure, at finer N':

    Holographically realizable ERs of $SAC_{N'}$ (↜ solution to HMIP)

    ↓

    describe using PMIs, specified by { I(X:Y) }
- Implications:
  - { HEC_N ∀ N } ⟸ { $HMIP_N$ ∀ N } ("Holographic entropy cone from marginal independence")
  - Any fixed N contaminated by structural (combinatorial) artifacts
  - Seemingly minimal dependence on holography (SAC universal)
- Nevertheless
  - HEC is not merely the largest possible polyhedral cone compatible with QM and reconstruction from SAC, since $\mathcal{R}_H \neq \mathcal{R}_Q$
  - Genuinely holographic input is needed to determine $\mathcal{R}_H$

Future directions

- Within the present context:
  - Complete soln. to HMIP
  - Explain HEIs
  - Bootstrap ERs and HEIs to higher N
  - Relate the two (ER & HEI) descriptions
  - Internal structure of HEC

- Beyond the present context:
  - Beyond classical bulk (quantum and stringy corrections)
  - Other QI quantities
  - …

Thank you