Fusion Surface Models:

2+1d Lattice Models from Fusion 2-Categories

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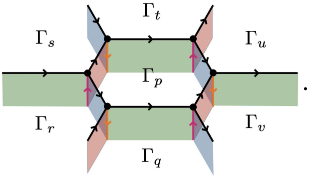
Based on joint work with Kantaro Ohmori [arXiv:2305.05774].

Take home message

- We can construct 2+1d lattice models from fusion 2-categories, which we call fusion surface models.

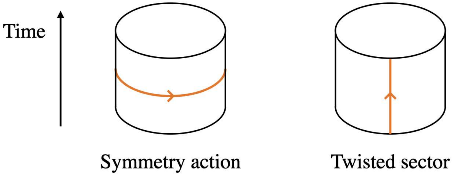


- The symmetry of the model is described by the input fusion 2-cat.
- Symmetries of anyons ⇒ candidates for chiral topological phases?

Introduction: generalized symmetry

From a modern point of view, [Gaiotto-Kapustin-Seiberg-Willett]

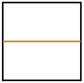
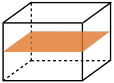
Symmetry = Algebraic structure of topological defects.



- A topological defect extended in space defines a symmetry op.
- A topological defect extended in time defines a twisted bc.

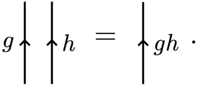
Conventional symmetry

- Topological defects are n-dimensional objects in $(n+1)$d spacetime:

  e.g.,  in 1+1d,  in 2+1d.

  Namely, topological defects have codimension 1.

- They form a group under the fusion:

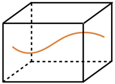


  In particular, topological defects are invertible.

conventional symmetry $\longleftrightarrow$ codim 1 invertible topological defects.

Generalized symmetry

- Topological defects can have various codimensions.

  e.g.,  in 2+1d.

  Higher codim $\Rightarrow$ Higher-form symmetry.

- Topological defects can be non-invertible, in particular, non-group-like:

$$a \uparrow \; \uparrow b \;=\; \bigoplus_{c} N_{ab}^{c} \; \uparrow c \;.$$

$\rightsquigarrow$ mathematically described by a higher category.

Fusion 1-category symmetry

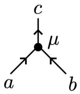
In 1+1 dimensions,

- Symmetry = Topological lines and their junctions.
- Finitely many topological defects in 1+1d form a fusion 1-category.

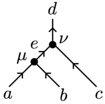
Finite symmetries in 1+1d are called fusion 1-category symmetries.
[Bhardwaj, Tachikawa], [Chang, et al.], [Thorngren, Wang]

Basic data of a fusion 1-category: [Etingof, Nikshych, Ostrik]

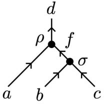
- topological lines $\{a, b, c, \cdots\}$, (objects)
- topological junctions $\{\mu, \nu, \rho, \cdots\}$ between lines, (morphisms)



- F-symbols $(F^{abc}_d)_{(e;\mu,\nu),(f;\rho,\sigma)}$ that describe the crossing relation:



$$= \sum_f (F^{abc}_d)_{(e;\mu,\nu),(f;\rho,\sigma)}$$



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The F-symbols must satisfy the pentagon eq. [Moore, Seiberg]

Fusion 2-category symmetry

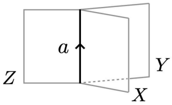
In 2+1 dimensions,

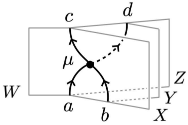
- Symmetry = Topological surfaces, lines, and their junctions
- Finitely many topological defects form a fusion 2-category.

Finite symmetries in 2+1d are called fusion 2-category symmetries.

Basic data of a fusion 2-category: [Douglas, Reutter]

- topological surfaces $\{X, Y, Z, \cdots\}$, (objects)
- topological interfaces $\{a, b, c, \cdots\}$ between surfaces, (1-morphisms)
- topological junctions $\{\mu, \nu, \rho, \cdots\}$ of interfaces, (2-morphisms)





- 10-j symbols (generalization of F-symbols to 2+1d).

Goal: construct 2+1d lattice models with fusion 2-category sym.

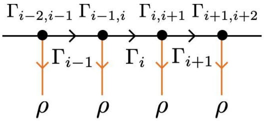
Warmup: 1+1d anyon chain model

Input: a fusion 1-cat $\mathcal{C}$ and its object ρ.

State space

- The state space $\mathcal{H}$ is spanned by the following fusion diagrams:

$$|\{\Gamma\}\rangle =$$



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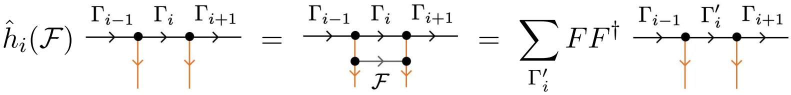
  Dynamical variables are

  - objects $\{\Gamma_i\}$ on horizontal edges,
  - morphisms $\{\Gamma_{i,i+1}\}$ on vertices.

Hamiltonian

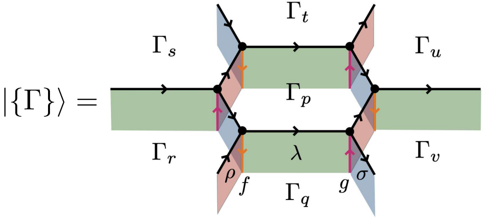
- The Hamiltonian is given by $H = \sum_i h_{i-1,i,i+1}$.
- The local term is $h_{i-1,i,i+1} = \sum_{\mathcal{F}\in\mathcal{C}} w(\mathcal{F})\hat{h}_i(\mathcal{F})$, where



The above model is known as the anyon chain [Feiguin, et al.].

Fusion surface model: state space

- Input: a fusion 2-cat $\mathcal{C}$, objects ρ, σ, λ, and 1-morphisms f, g.
- Let $\mathcal{H}$ be the vector space spanned by the following fusion diagrams:

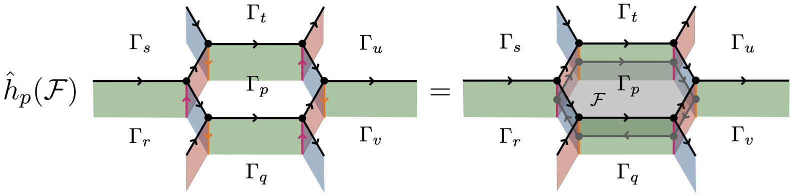


- The state space $\mathcal{H}_0$ is defined by

$$\mathcal{H}_0 := \Big(\prod_{\text{plaquettes}} B_p\Big)\mathcal{H}, \quad B_p: \text{ Levin-Wen plaquette op.}$$

Fusion surface model: Hamiltonian

- The Hamiltonian is given by $H = \sum_{p:\text{plaquettes}} \hat{h}_p$.
- The plaquette Hamiltonian is $\hat{h}_p = \sum_{\mathcal{F}} w(\mathcal{F}) \hat{h}_p(\mathcal{F})$, where



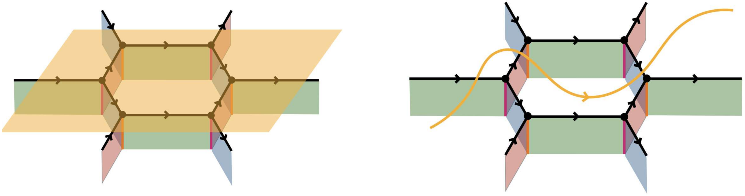
This can be expressed in terms of 10-j symbols explicitly.

We call the above model fusion surface model.

Examples: Levin-Wen models, Kitaev honeycomb model w/o magnetic field, etc.

Fusion surface model: symmetry

The fusion 2-cat $\mathcal{C}$ acts on states by the fusion of top. surfaces/lines:



This commutes with the Hamiltonian $\rightsquigarrow$ fusion 2-cat symmetry.

- E.g., $\mathcal{C} = \mathrm{Mod}(\mathcal{B})$ ($\mathcal{B}$: MTC) describes the symmetry of anyons.
- SSBing this symmetry leads to chiral topological order $\mathcal{B}$.

Summary

- We constructed 2+1d lattice models dubbed fusion surface models.
- Fusion surface models have general fusion 2-category symmetries.
- Candidates for chiral topological phases? Further study is needed.