Parisi’s hypercube, double-scaled SYK, Fock-space fluxes and NAdS$_2$/NCFT$_1$ duality

Yiyang Jia
Weizmann Institute of Science
(QIMG 2023, Kyoto)

Based on:
Berkooz-YJ-Silberstein 2023 [arxiv:2303.18182]
Outline

• Double-scaled Sachdev-Ye-Kitaev model and near-AdS$_2$/near-CFT$_1$
• Parisi’s hypercube
• Chord diagrams
• Characterization of the NAdS$_2$/NCFT$_1$ microscopics
(Double-scaled) SYK and NAdS$_2$/NCFT$_1$

- Sachdev-Ye-Kitaev:
  \[
  H_{SYK} = i^{p/2} \sum_{i_1,\ldots,i_p}^{N} J_{i_1,\ldots,i_p} \psi_{i_1} \cdots \psi_{i_p}, \quad \{\psi_i, \psi_j\} = 2\delta_{ij}
  \]

  \[p \ll N \text{ (} p \text{-locality)}\]

- Two limits: 1. \( p = \text{fixed}, N \to \infty. \)
  \[
  2. \lambda = \frac{p^2}{N} = \text{fixed}, N \to \infty \text{ (double scaled SYK), then } \lambda \to 0 \text{ (“triple scaling”).}
  \]

Both limits give nearly-conformal QM (NCFT$_1$) at low temperatures.
(Double-scaled) SYK and NAdS$_2$/NCFT$_1$

- NAdS$_2$: Jackiw-Teitelboim + matter (dim reduction from higher-dim black holes). Some characteristic behaviours:
  1. $\sim \sinh \sqrt{E}$ density of states
  2. Conformal correlation functions
  3. OTOC $\langle O(t)O(0)O(t)O(0)\rangle \sim \exp\left(\frac{2\pi}{\beta} t\right)$ (maximal chaos/fast scrambling)

- NCFT$_1$ from SYK reproduces all the above: NAdS$_2$/NCFT$_1$ duality

- Puzzle: NAdS$_2$ is a very ubiquitous solution in GR, but microscopic constructions of NCFT$_1$ are comparably rare. Essentially all SYK-like models so far.

What’s the general characterization of NCFT$_1$ microscopics?
Parisi’s hypercube

• A useful stepping stone: Parisi’s hypercube (Parisi 1994):
  1) d-dim hypercube, $d \to \infty$
  2) Single particle hopping, random uniform background fluxes $F_{\mu\nu}$, i.i.d with $\langle \sin F_{\mu\nu} \rangle = 0$, $\langle \cos F_{\mu\nu} \rangle \equiv q$
• It’s a (continuous-time) quantum random walk model.
• Lattice gauge Hamiltonian (gauge-covariant Laplacian):
  
  $$ H_{\vec{x},\vec{y}} = -\frac{1}{\sqrt{d}} \sum_{\mu=1}^{d} U_{\mu}(\vec{x}) \delta_{\vec{x},\vec{y}+\hat{e}_{\mu}} + h. c. $$

  Holonomy on a plaquette: 
  
  $$ U_{\nu}^{-1}(\vec{x})U_{\mu}^{-1}(\vec{x} + \hat{e}_{\nu})U_{\nu}(\vec{x} + \hat{e}_{\mu})U_{\mu}(\vec{x}) = e^{-i F_{\mu\nu}} $$
Parisi’s hypercube

• Hamiltonian in qubit form:

\[ H = -\frac{1}{\sqrt{d}} \sum_{\mu=1}^{d} (T^{+}_\mu + T^{-}_\mu), \quad T^{+}_\mu \equiv (\prod_{\nu, \nu \neq \mu} e^{i F_{\mu \nu} \sigma^3_\nu}) \sigma^+ \mu \cdot \text{NOT p-local!} \]

Holonomy \[ T_v^{-} T_\mu^{-} T_v^{+} T_\mu^{+} \propto e^{-i F_{\mu \nu}} \]

• Superficially looks nothing like an SYK, but will give identical phenomenology.

• Goal: pinpoint what is actually in common, use it as the more general characterization for NCFT\(_1\) microscopics.
Parisi’s hypercube

Alternative interpretation: hypercube as a Fock-space graph

1. Take the qubit Hamiltonian as the starting point, many-body (but NOT p-local!).

2. Represent each basis vector as a point, connect two points if there is a nonzero transition amplitude.

3. Gives back the hypercube picture. Hypercube as a graphical representation of Fock space evolution (instead of a real-space hypercube)

4. Fluxes are defined in the Fock space.
Holonomy and moments

- A more convenient expression for holonomies:

\[ D_\mu \equiv T_\mu^+ + T_\mu^-, \quad W_{\mu\nu} \equiv D_\nu D_\mu D_\nu D_\mu = \cos F_{\mu\nu} - i \sin F_{\mu\nu} \sigma_\mu^3 \sigma_\nu^3, \]

\[ \langle W_{\mu\nu} \rangle = \langle \cos F_{\mu\nu} \rangle = q. \]

- Moments

\[ \langle Tr \ H^{2k} \rangle = \frac{1}{d^k} \sum_{\mu_1, ..., \mu_{2k}} \langle Tr \ D_{\mu_1} \cdots D_{\mu_{2k}} \rangle \]

- Trace  \rightarrow  Loops in the Fock space  \rightarrow  a forward hopping must be matched with a backward hopping in the same direction  \rightarrow  \{\mu_1, ..., \mu_{2k}\} form k pairs of indices

- Further coincidence among the  k pairs  \rightarrow  1/d suppressed
Chord diagrams

• Represent trace as a circle, draw subscripts on the circle, paired indices as chords.

Example:

\[ \frac{1}{a^3} \sum_{\mu \neq \nu \neq \rho} \langle TrD_\rho D_\nu D_\rho D_\mu D_\nu D_\mu \rangle \]

• Apply $W_{\mu \nu}$ formula repeatedly (and that $D_\mu^2 = 1$), each nontrivial holonomy (interlaced ordering) appears as a chord intersection.

Example:

\[ \frac{1}{a^3} \sum_{\mu \neq \nu \neq \rho} \langle \cos F_{\mu \nu} \rangle \langle \cos F_{\nu \rho} \rangle = q^2 \]
Moments and chords

• In general \([\text{Parisi 1994, Marinari-Parisi-Ritort 1995, Cappelli-Colomo 1998, in a different language}]

\[
\langle Tr H^{2k} \rangle = \sum_{\text{chord diagrams}} q^{\text{number of chord intersections}}
\]


\[\sim \sinh \sqrt{E}\] density of states
Correlation functions

• Probes:

\[ O = -\frac{1}{\sqrt{d}} \sum_{\mu} (\tilde{T}_{\mu}^+ + \tilde{T}_{\mu}^-), \quad \tilde{T}_{\mu}^+ \equiv (\prod_{\nu,\nu\neq\mu} e^{iF_{\mu\nu}\sigma^3_\nu}) \sigma^+_\mu , \]

recall

\[ T_{\mu}^+ \equiv (\prod_{\nu,\nu\neq\mu} e^{iF_{\mu\nu}\sigma^3_\nu}) \sigma^+_\mu \]

• Two-point:

\[ \langle \text{Tr} \, H^{k_2} \, O \, H^{k_1} \, O \rangle = \sum q^\# \text{H-H inters.} \, \tilde{q}^\# \text{O-H inters.} , \quad \tilde{q} \equiv \langle \cos \frac{F + \tilde{F}}{2} \rangle. \]

Same as DSSYK [Berkooz-Narayan-Simon 2018, Berkooz-Isachenkov-Narovlansky-Torrents 2018]

• True for arbitrary n-point function, all identical to DSSYK.
Correlation functions

• Consequences of such chord rules [Berkooz-Narayan-Simon 2018, Berkooz-Isachenkov-Narovlansky-Torrents 2018] in triple scaling limit:

  1. Conformal correlation functions

  2. OTOC \langle O(t)O(0)O(t)O(0) \rangle \sim \exp(\frac{2\pi}{\beta} t) \quad \text{(maximal chaos)}.

• Parisi model is at least as good a NCFT$_1$ microscopic construction as DSSYK
Comparison with the (DS)SYK

• Recall

\[ H_{\text{SYK}} = \sum_I J_I \Psi_I, \quad I = \{i_1, i_2, \ldots, i_p\}, \quad \Psi_I = i^{p/2} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_p} \]

• \( \Psi_I \) is a hopping operator in the Fock space, \( I \) specifies the hopping direction (like the \( \mu \) in the hypercube model).

• Fock-space holonomy \( W_{IK} \equiv \Psi_K \Psi_I \Psi_K \Psi_I = (-1)^{|I \cap K|} \).

• Compare with the hypercube \( D_\nu D_\mu D_\nu D_\mu = \cos F_{\mu \nu} - i \sin F_{\mu \nu} \sigma_\mu^3 \sigma_\nu^3 \), we see the SYK holonomies are generated by uniform random fluxes of 0 or \( \pi \) on all plaquettes.
Comparison with the (DS)SYK

• From the Fock-space flux picture SYK is very similar to Parisi, however to achieve a complete analogy we still need
  1) fluxes on different plaquettes to be independent,
  2) the average holonomy to be a tunable parameter.

• These are achieved by going to the double scaled SYK (DSSYK) limit:
  \[ \frac{p^2}{N} = \text{fixed}, \quad N \to \infty \]

• In the DS limit, set intersections |I \cap K| becomes i.i.d., and average holonomy is
  \[ \langle (-1)^{|I \cap K|} \rangle = \exp \left( -\frac{2p^2}{N} \right) \equiv q. \]

• This analogy also extends to probes.
• This is essentially how you also obtain DSSYK chord diagrams.
Characterization of NCFT$_1$

- We now have a set of sufficient (not necessary) conditions for the microscopics that give rise to chord combinatorics and hence NAdS$_2$/NCFT$_1$ physics. All we need is a Fock-space flux that is
  1) uniform and quench-disordered, and
  2) i.i.d on different plaquettes, with a tunable average holonomy.
- In operator language, if $H = \sum_I J_I \hat{M}_I$ ($\hat{M}_I$ needs not be p-local):
  1) $[\hat{M}_I,\hat{M}_K\hat{M}_L\hat{M}_K\hat{M}_L] = 0$ almost always.
  2) $Tr\hat{M}_K\hat{M}_L\hat{M}_K\hat{M}_L = i.i.d$, $\langle Tr\hat{M}_K\hat{M}_L\hat{M}_K\hat{M}_L \rangle = q$. 
Characterization of NCFT$_1$

- (double-scaled) p-local approach is an effective way to generate such fluxes, but it’s not the only way.
- Larger tool box for model building.
- One may wonder where the fluxes come from, I speculate they could arise as Berry curvatures.
Thank you!