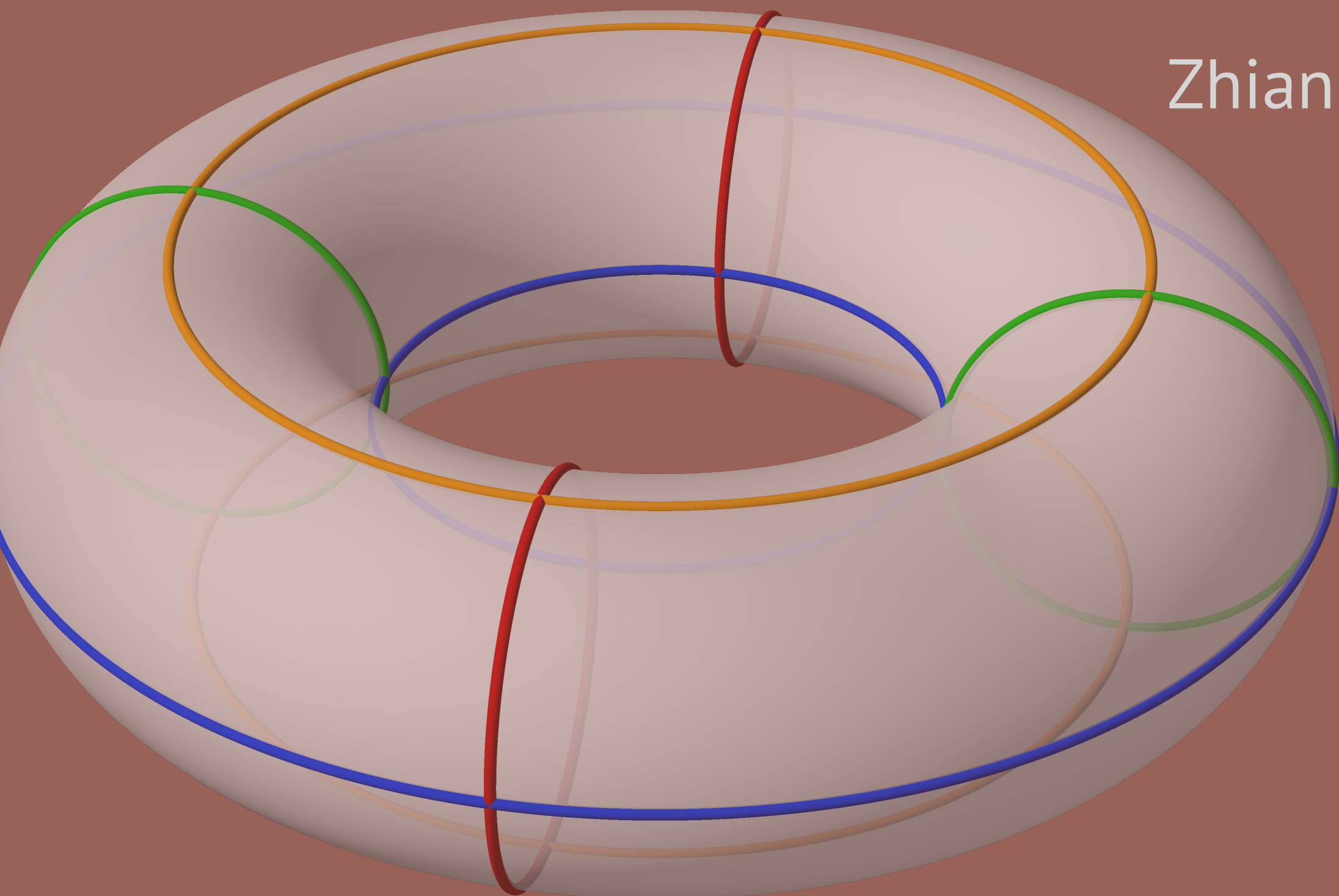


# Gapped boundary theory of Hopf and weak Hopf quantum double model

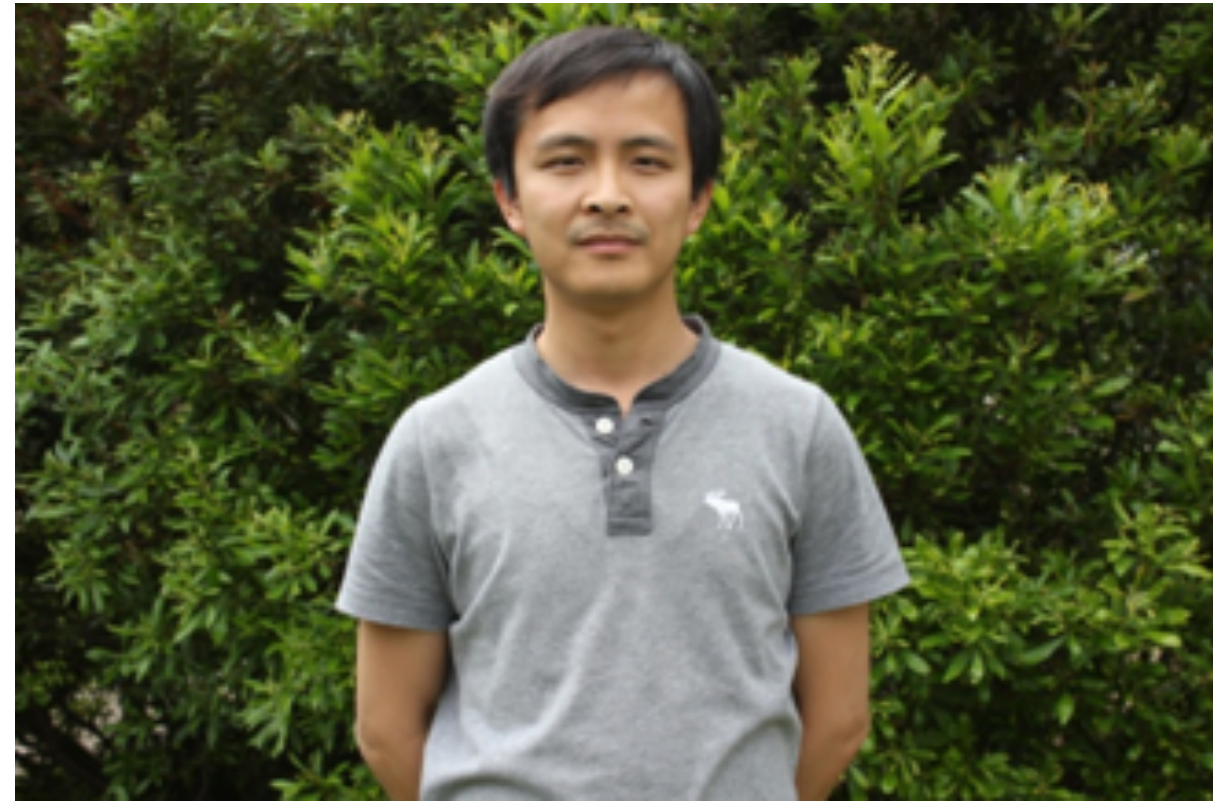
Zhian Jia (贾治安) | CQT, National University of Singapore

09/22/2023

YITP, Kyoto University



# Collaborators and references:



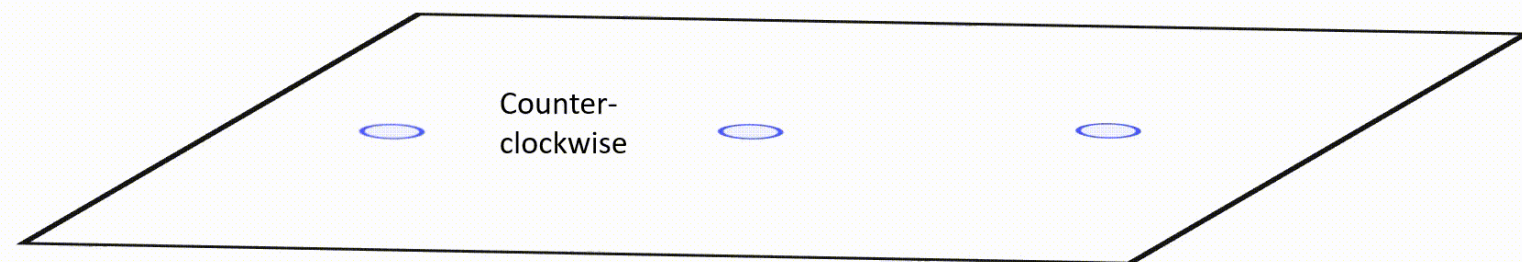
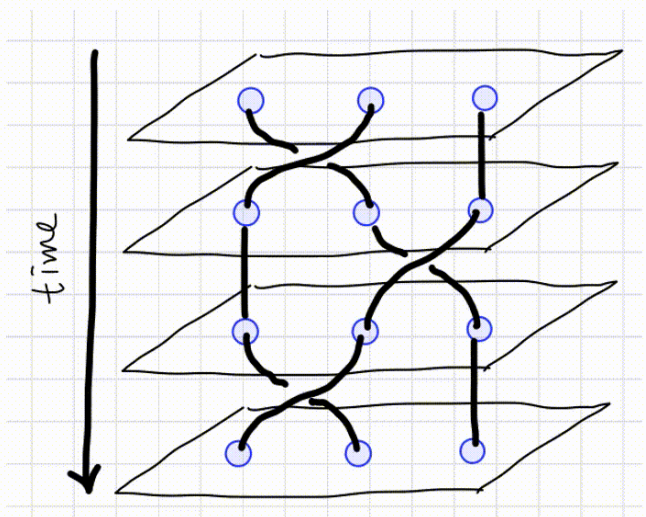
**Liang Chang**  
Chern Institute, Nankai University



**Dagomir Kaszlikowski**  
CQT, National University of Singapore



**Sheng Tan**  
Purdue University, BIMSA



Electric-magnetic duality and  $Z_2$  symmetry enriched Abelian lattice gauge theory  
Z Jia, D Kaszlikowski, S Tan,  
arXiv preprint arXiv:2201.12361

Boundary and domain wall theories of 2d generalized quantum double model  
Z Jia, D Kaszlikowski, S Tan  
JHEP 2023 (7-160), 1-78; arXiv:2207.03970

On weak Hopf symmetry and weak Hopf quantum double model  
Z Jia, S Tan, D Kaszlikowski, L Chang  
Comm. Math. Phys. 402, 3045–3107; arXiv:2302.08131

Quantum double model: from toric code to weak Hopf lattice gauge theory, Z Jia, S Tan , to be published

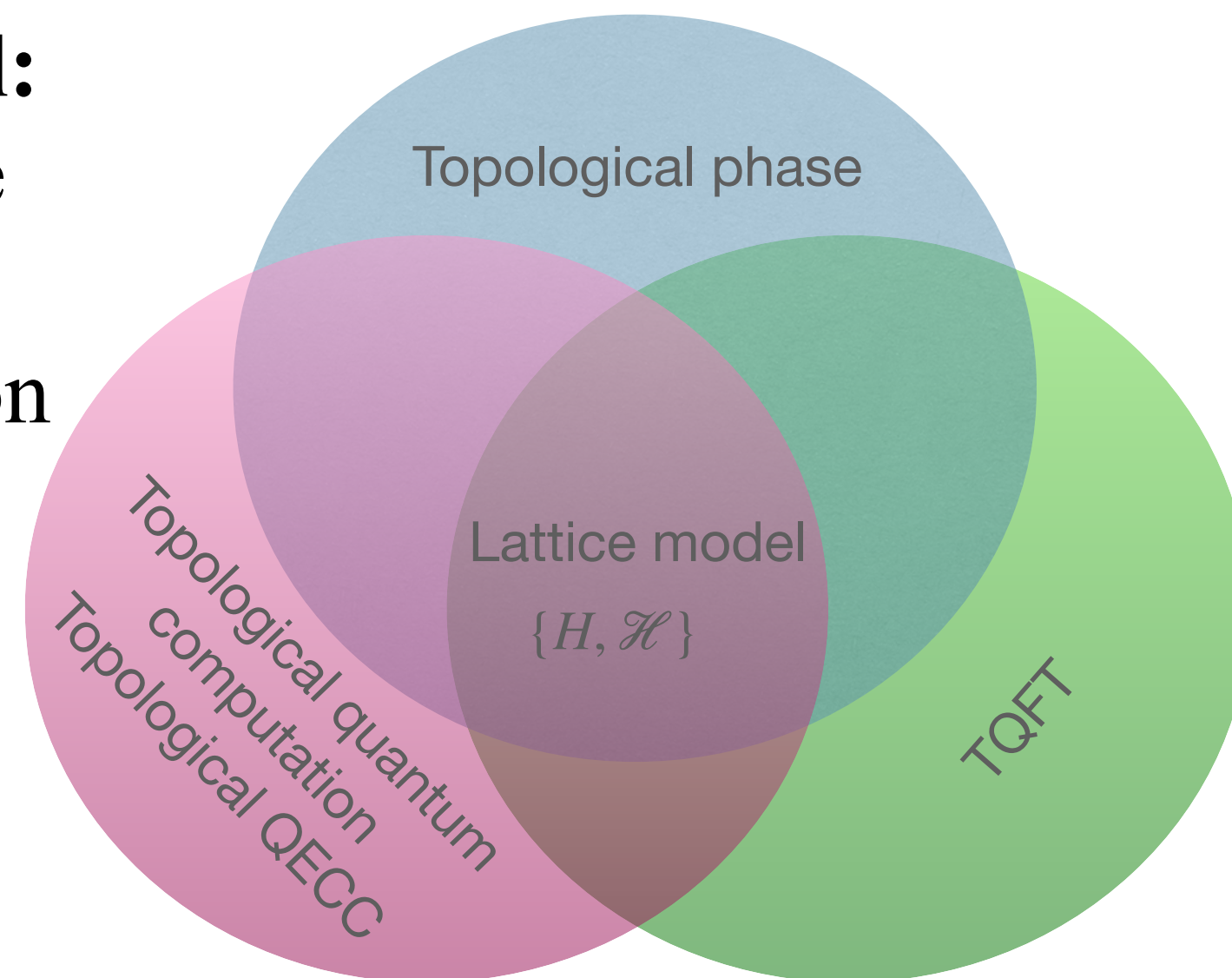
# Topological order

## Definition

Equivalence classes of gapped local Hamiltonian whose low-energy effective theories realize given TQFT

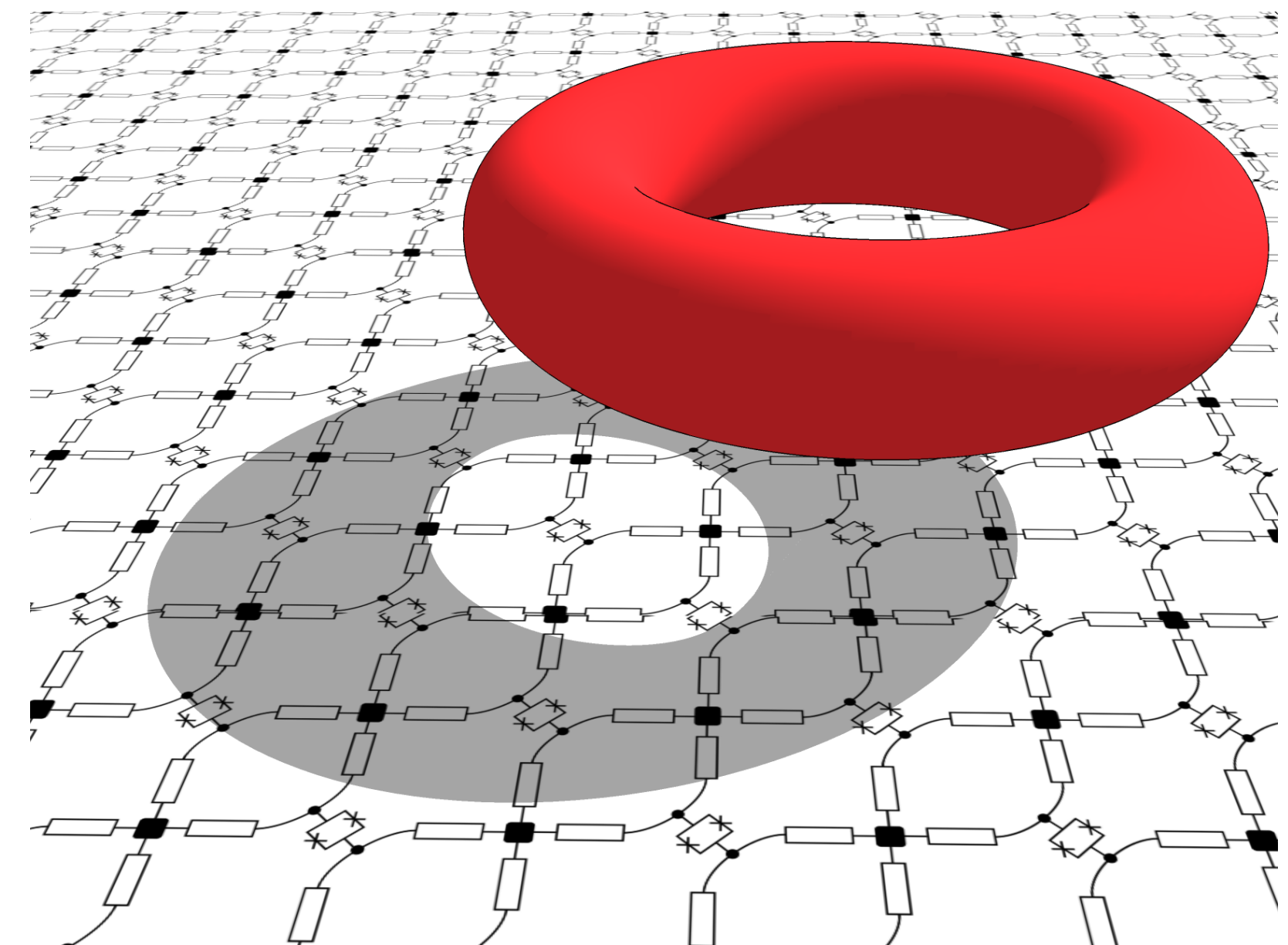
## (2+1)D anyon model:

- Topological charge
- Antiparticle
- Quantum dimension
- Fusion rule
- Braiding
- Topological spin
- S-matrix, T-matrix
- Verlinde formula



## Physical properties:

- Topologically protected GSD
- Long-range entanglement
- Topological entanglement entropy
- Fractional statistics



<https://www.quantumtheory.nat.fau.eu/2017/04/21/superconducting-quantum-simulator-for-topological-order-and-the-toric-code/>

# Topological order: lattice model

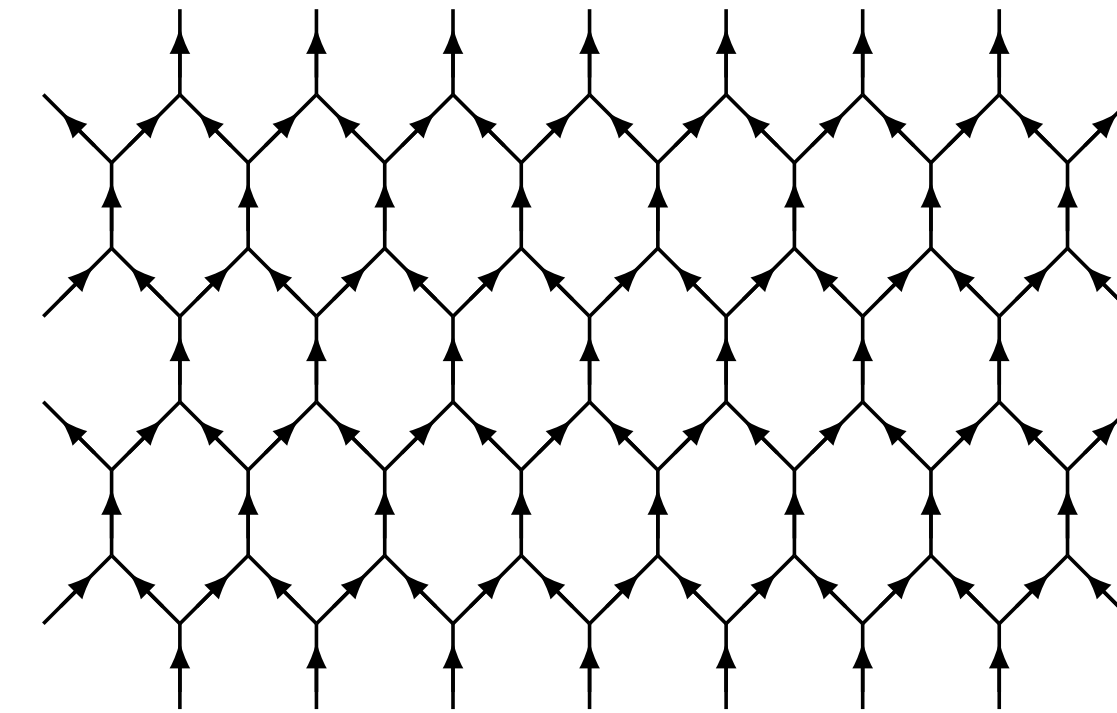
## Definition

Equivalence classes of gapped local Hamiltonian whose low-energy effective theories realize given TQFT

The mathematical structure behind the topological order is **tensor category theory**, for the 2d case, the topological excitation is described by a unitary modular tensor category (UMTC)

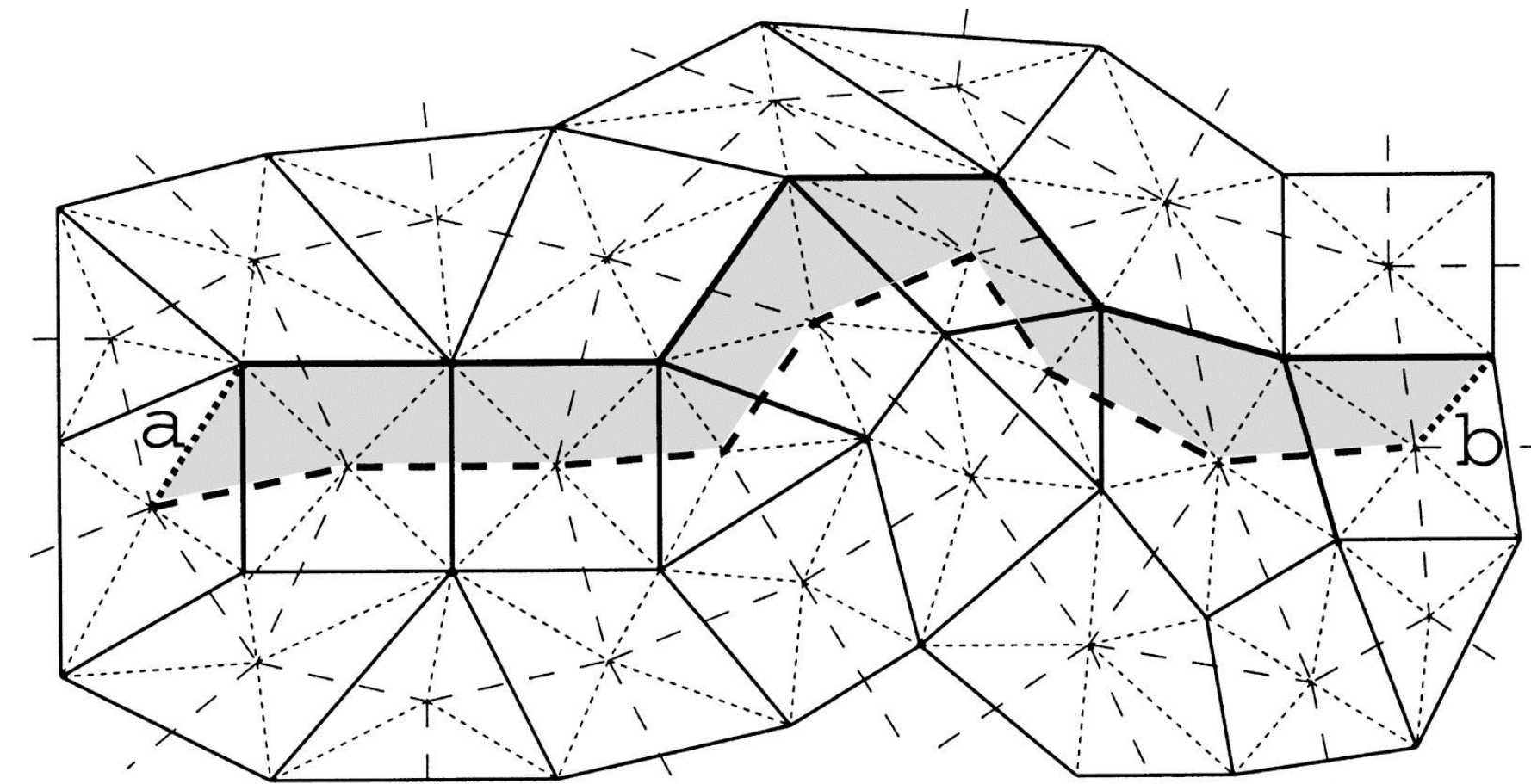
- anomaly-free
- non-chiral

Higher category for higher dimensional cases!



[Levin-Wen 2005]

Levin-Wen string-net model  $\leftrightarrow$  Turaev-Viro-Barrett-Westbury



[Kitaev 2003]

Kitaev quantum double model  $\leftrightarrow$  Dijkgraaf-Witten

**Main topic of This talk**

# Hopf QD model: definition of Hopf algebra

A (complex) Hopf algebra is a complex vector space  $H$  equipped with several structure morphisms: multiplication  $\mu : H \otimes H \rightarrow H$ , unit  $\eta : \mathbb{C} \rightarrow H$ , comultiplication  $\Delta : H \rightarrow H \otimes H$ , counit  $\varepsilon : H \rightarrow \mathbb{C}$  and antipode  $S : H \rightarrow H$ , for which some consistency conditions are satisfied:

1.  $(H, \mu, \eta)$  is an algebra:  $\mu \circ (\mu \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu)$ , and  $\mu \circ (\eta \otimes \text{id}) = \text{id} = \mu \circ (\text{id} \otimes \eta)$ .
2.  $(H, \Delta, \varepsilon)$  is a coalgebra:  $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ , and  $(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta$ .
3.  $(H, \mu, \eta, \Delta, \varepsilon)$  is a bialgebra:  $\Delta$  and  $\varepsilon$  are algebra homomorphisms (equivalently  $\mu$  and  $\eta$  are coalgebra homomorphisms).
4. The antipode  $S$  satisfies:  $\mu \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\text{id} \otimes S) \circ \Delta$ .

$$\text{Sweedler's Notation: } \Delta(h) = \sum_{(h)} h^{(1)} \otimes h^{(2)}; f \rightharpoonup h = \sum_{(h)} h^{(1)} f(h^{(2)})$$

# Hopf QD model: quantum double of Hopf algebra

The quantum double of  $H$  is the vector space  $(H^\vee)^{\text{cop}} \otimes H$  equipped with a Hopf algebra structure; we denote it as  $D(H) = (H^\vee)^{\text{cop}} \bowtie H$ . The multiplication is given by

$$(\varphi \otimes x)(\psi \otimes y) := \sum_{(x)} \varphi\psi(S^{-1}(x^{(3)}) \bullet x^{(1)}) \otimes x^{(2)}y,$$

where “ $\bullet$ ” denotes the argument of the function. The other data are given by

$$1_{D(H)} = \hat{1} \otimes 1,$$

$$\Delta_{D(H)}(\varphi \otimes x) = \sum_{(\varphi)} \sum_{(x)} (\varphi^{(2)} \otimes x^{(1)}) \otimes (\varphi^{(1)} \otimes x^{(2)}),$$

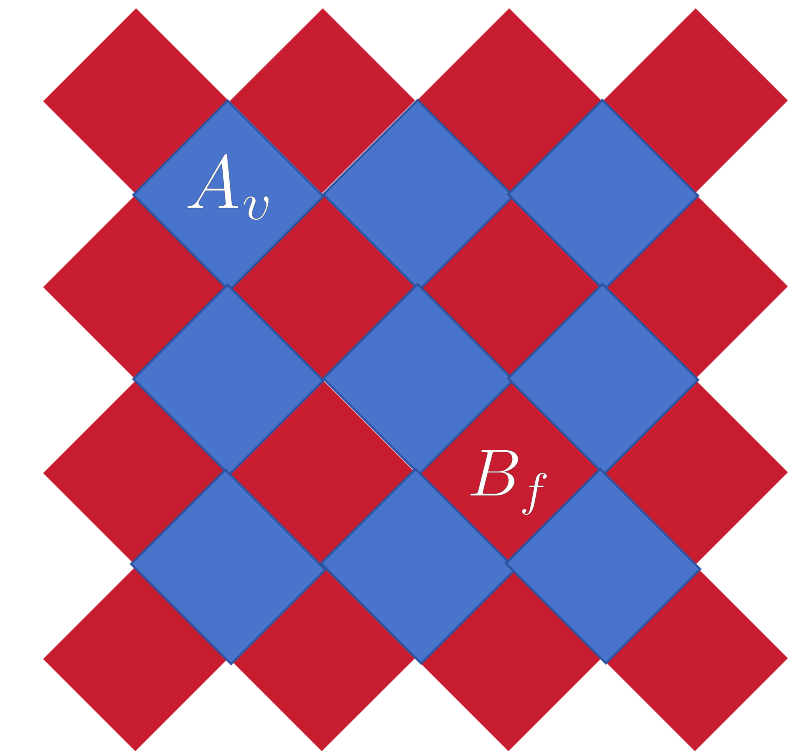
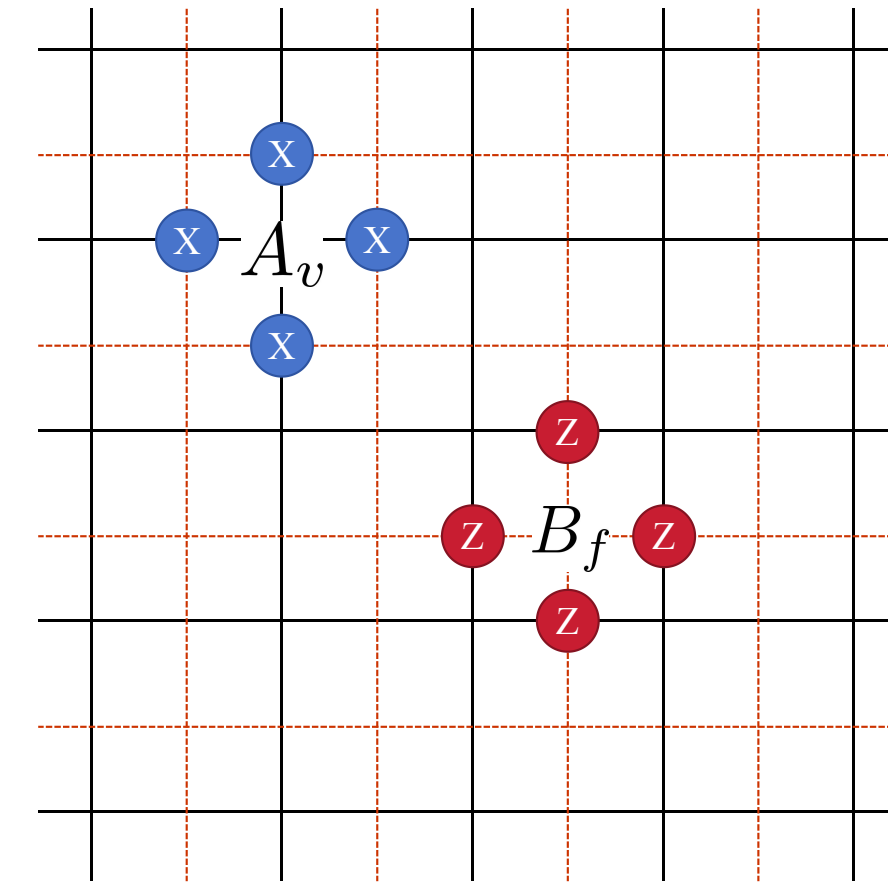
$$\varepsilon_{D(H)}(\varphi \otimes x) = \varepsilon(x)\varphi(1_H),$$

$$S_{D(H)}(\varphi \otimes x) = \sum_{(\varphi)} \sum_{(x)} \langle \varphi^{(1)} \otimes \varphi^{(3)}, h^{(3)} \otimes S^{-1}(h^{(1)}) \rangle \hat{S}^{-1}(\varphi^{(2)}) \otimes S(h^{(2)}).$$

# Hopf QD model: finite group algebra

Let  $G$  be a finite group, the linear span  $\mathbb{C}[G]$  is a Hopf algebra

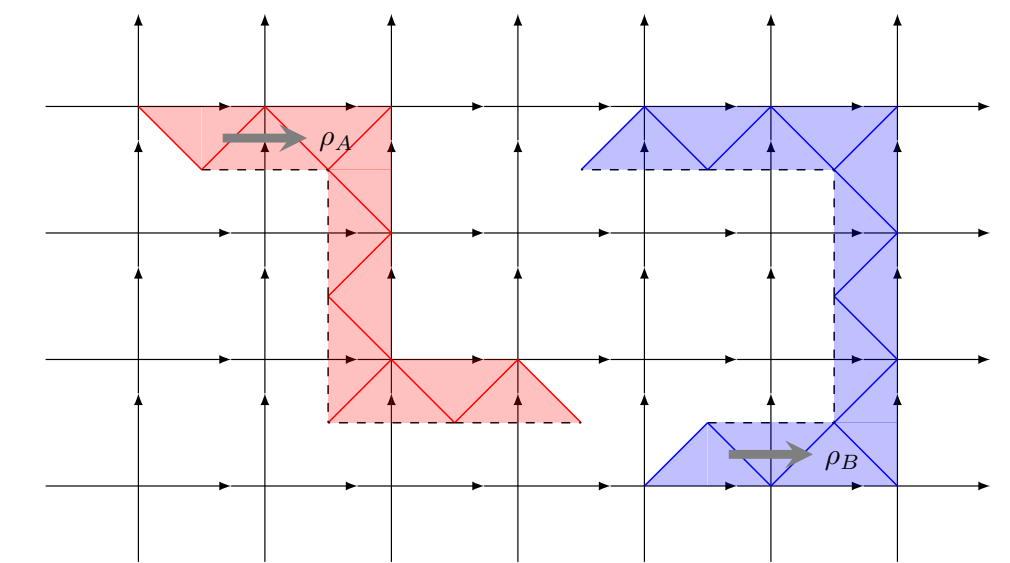
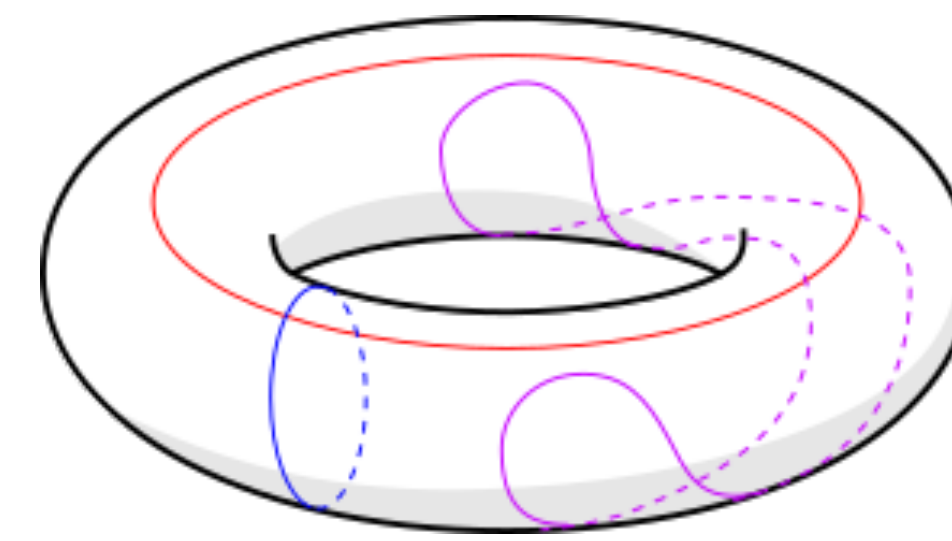
- Multiplication:  $g \cdot h$
- Unit:  $1_G$
- Comultiplication:  $\Delta(g) = g \otimes g, \forall g$
- Coint:  $\varepsilon(g) = 1, \forall g$
- Antipode:  $S(g) = g^{-1}, \forall g$



## Kitaev quantum double model

$$A_v \left| \begin{array}{c} x_2 \\ \uparrow \\ x_3 \rightarrow \bullet \rightarrow x_1 \\ \downarrow \\ x_4 \end{array} \right\rangle = \frac{1}{|G|} \sum_{g \in G} \left| \begin{array}{c} x_2 g^{-1} \\ \uparrow \\ g x_3 \rightarrow \bullet \rightarrow x_1 g^{-1} \\ \downarrow \\ g x_4 \end{array} \right\rangle.$$

$$B_f \left| \begin{array}{c} x_2 \\ \uparrow \\ x_3 \rightarrow \bullet \rightarrow x_1 \\ \downarrow \\ x_4 \end{array} \right\rangle = \delta_{x_1^{-1} x_2 x_3 x_4^{-1}, e} \left| \begin{array}{c} x_2 \\ \uparrow \\ x_3 \rightarrow \bullet \rightarrow x_1 \\ \downarrow \\ x_4 \end{array} \right\rangle.$$



## Many applications

- Topological QC
- Topological quantum memory
- Topological order
- Lattice gauge theory

# Hopf QD model: general C\* Hopf algebra

- Edge space: Hopf algebra  $H$

- Vertex:  $H$  as a left  $H$ -module:

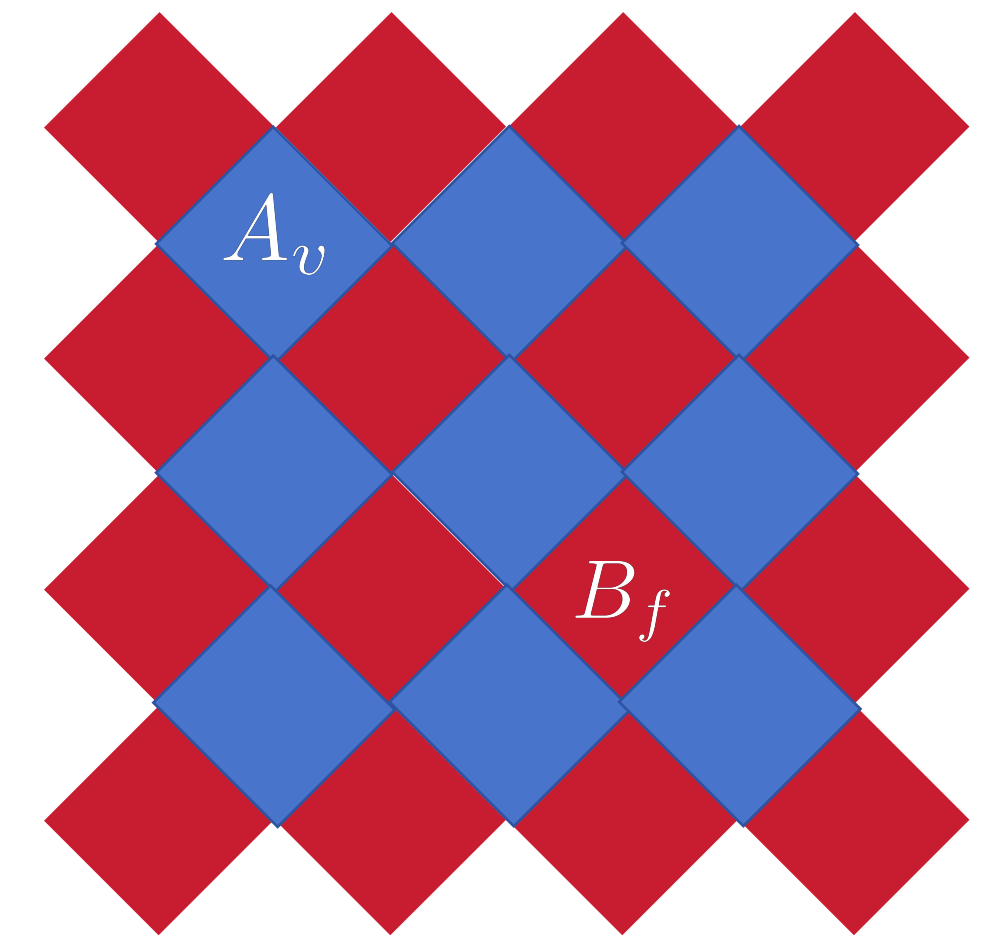
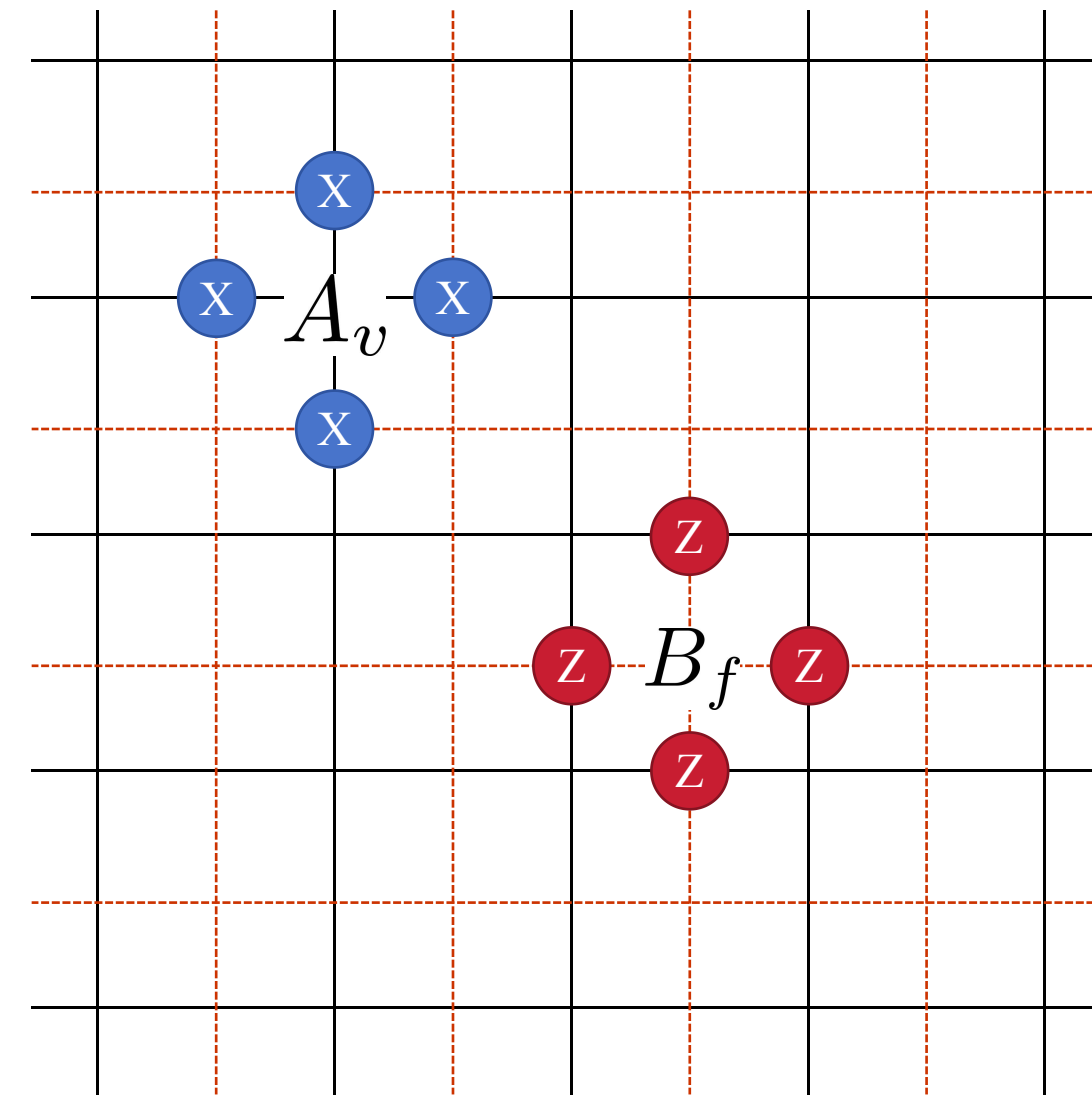
$$L_+^h|x\rangle = |h \triangleright x\rangle = |hx\rangle,$$

$$L_-^h|x\rangle = |x \triangleleft S(h)\rangle = |xS(h)\rangle,$$

- Face:  $H$  as a left  $\hat{H}$ -module

$$T_+^\varphi|x\rangle = |\varphi \rightarrow x\rangle = \left| \sum_{(x)} \langle \varphi, x^{(2)} \rangle x^{(1)} \right\rangle,$$

$$T_-^\varphi|x\rangle = |x \leftarrow \hat{S}(\varphi)\rangle = \left| \sum_{(x)} \langle \hat{S}(\varphi), x^{(1)} \rangle x^{(2)} \right\rangle = \left| \sum_{(x)} \langle \varphi, S(x^{(1)}) \rangle x^{(2)} \right\rangle$$



To construct the face and vertex operators, we need a new concept: **Haar integral of Hopf algebra**



# Hopf QD model: general $C^*$ Hopf algebra

## Haar integral

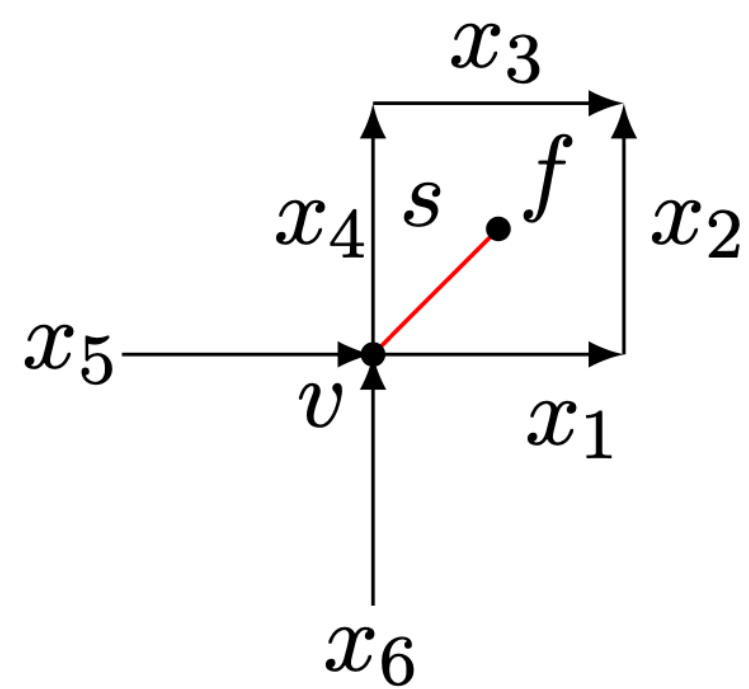
Let  $H$  be a Hopf algebra. A left (resp. right) integral of  $H$  is an element  $\ell \in H$  (resp.  $r \in H$ ) such that  $x\ell = \varepsilon(x)\ell$  (resp.  $rx = r\varepsilon(x)$ ) for all  $x \in H$ . They are normalized if  $\varepsilon(\ell) = \varepsilon(r) = 1$ . A two-sided integral is an element that is simultaneously a left and a right integral. A Haar integral is a normalized two-sided integral.

The Haar integral for the group algebra  $\mathbb{C}[G]$  is  $h_{\mathbb{C}[G]} = \frac{1}{|G|} \sum_{g \in G} g$ , and the Haar measure is  $\delta_{1_G}$ , where for any  $g \in G$  the function  $\delta_g : \mathbb{C}[G] \rightarrow \mathbb{C}$  is given by  $\delta_g(h) = \delta_{g,h} = 1$  if  $g = h$  and 0 otherwise. Hence the inner product on  $\mathbb{C}[G]$  induced by  $\delta_{1_G}$  is  $\langle g, h \rangle = \delta_{g,h}$ , for  $\forall g, h \in G$ .

**Every  $C^*$  Hopf algebra has a unique Haar integral !**

# Hopf QD model: general C\* Hopf algebra

## Local stabilizers



$$A^h(s) = \sum_{(h)} L_-^{h^{(1)}}(j_4) \otimes L_+^{h^{(2)}}(j_5) \otimes L_+^{h^{(3)}}(j_6) \otimes L_-^{h^{(4)}}(j_1),$$

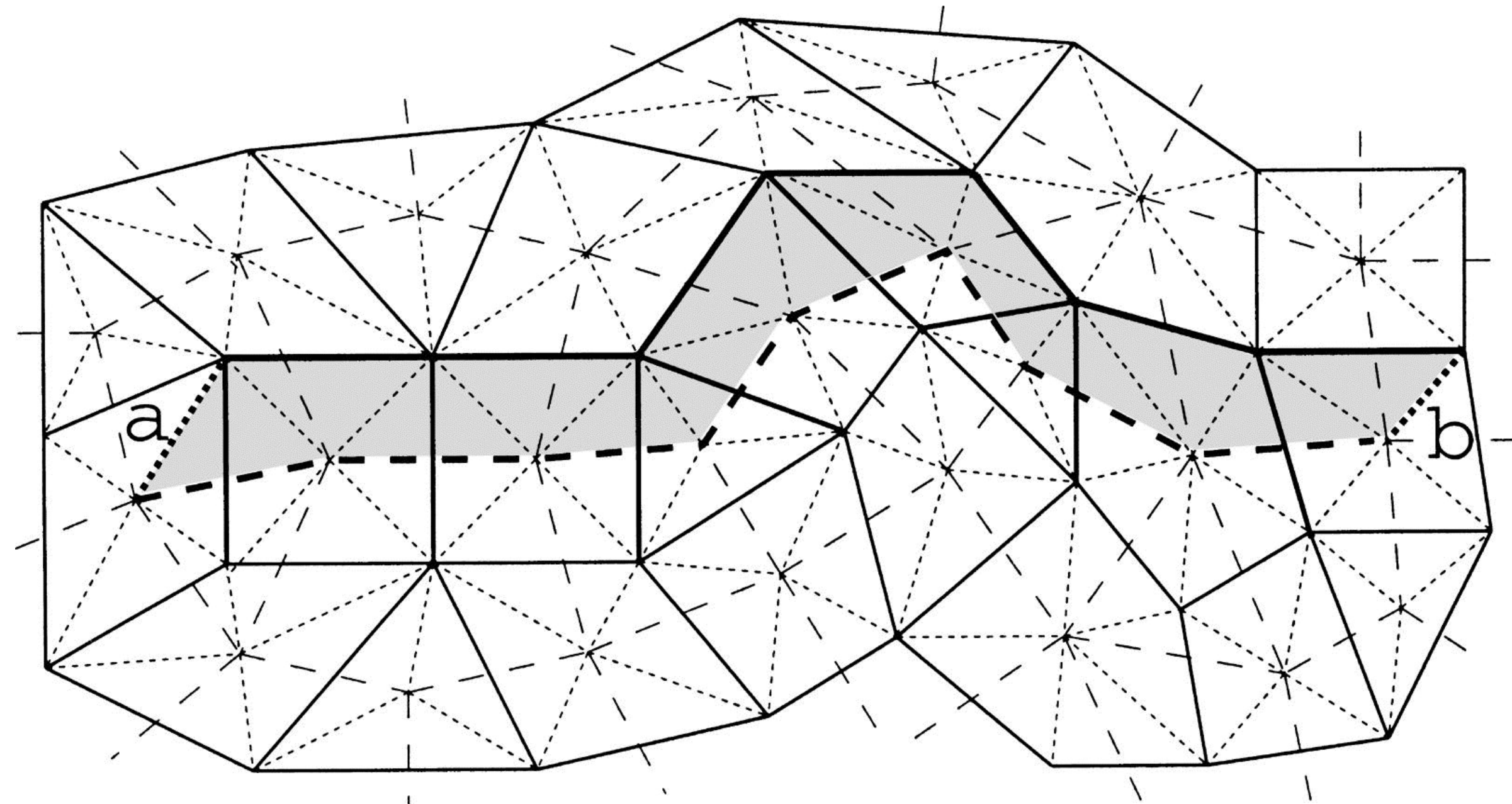
$$B^\varphi(s) = \sum_{(\varphi)} T_-^{\varphi^{(1)}}(j_1) \otimes T_-^{\varphi^{(2)}}(j_2) \otimes T_+^{\varphi^{(3)}}(j_3) \otimes T_+^{\varphi^{(4)}}(j_4),$$

$$A^h(s)B^\varphi(s) = \sum_{(h)} B^{\varphi(S^{-1}(h^{(3)}) \bullet h^{(1)})}(s)A^{h^{(2)}}(s),$$

## Hamiltonian

Set  $h$  and  $\varphi$  as Haar integrals of  $H$  and  $\hat{H}$

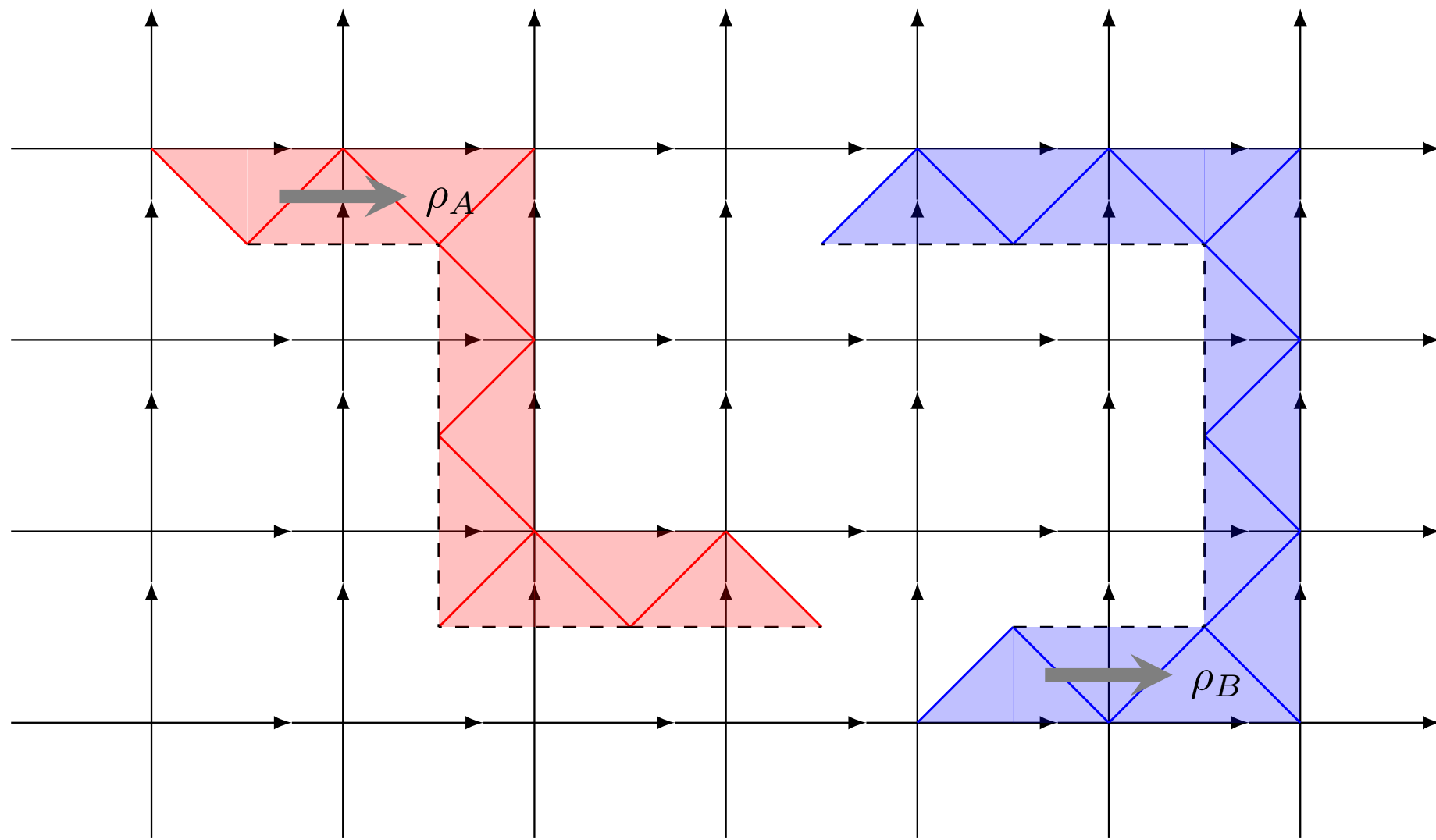
$$H = \sum_v (I - A_v) + \sum_f (I - B_f)$$



[Buerschaper, Mombelli, Christandl, Aguado, 2013]

# Ribbon operator and topological excitation

## Two types of ribbon



type-A:  $D(H)^{\vee, \text{op}}$

type-B:  $D(H)^{\vee}$

[Jia, Kaszlikowski, Tan, 2023]

## Example: type-B $D(H)^{\vee}$

Dual Hopf algebra  $D(H)^{\vee} = D_B(H)^{\vee} = H^{\text{op}} \otimes \hat{H}$ , with

$$(h \otimes \varphi)(g \otimes \psi) = gh \otimes \varphi\psi,$$

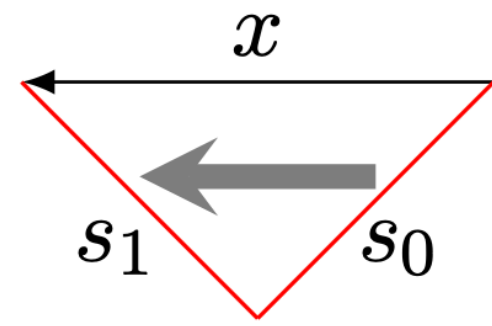
$$\Delta_{D(H)^{\vee}}(h \otimes \alpha) = \sum_{k, (k), (h)} (h^{(1)} \otimes \hat{k}) \otimes (S(k^{(3)})h^{(2)}k^{(1)} \otimes \alpha(k^{(2)} \bullet)),$$

$$1_{D(H)^{\vee}} = 1_H \otimes \varepsilon_H, \quad \varepsilon_{D(H)^{\vee}}(h \otimes \alpha) = \varepsilon(h)\alpha(1),$$

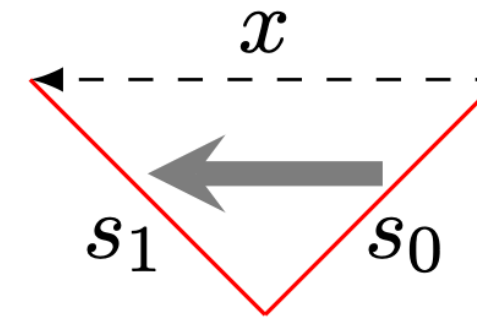
$$\begin{aligned} F^{h, \varphi}(\rho) &= \sum_{(h \otimes \varphi)} F^{(h \otimes \varphi)^{(1)}}(\tau_1) F^{(h \otimes \varphi)^{(2)}}(\tau_2) \\ &= \sum_k \sum_{(k), (h)} F^{h^{(1)}, \hat{k}}(\tau_1) F^{S(k^{(3)})h^{(2)}k^{(1)}, \varphi(k^{(2)} \bullet)}(\tau_2). \end{aligned}$$

# Ribbon operator and topological excitation

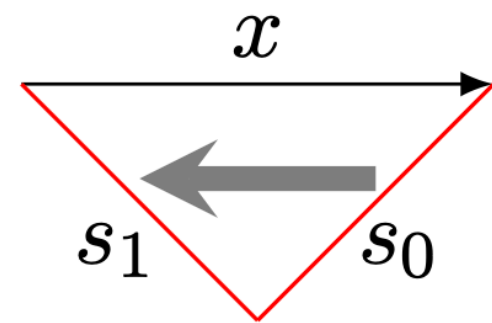
## Triangle operators



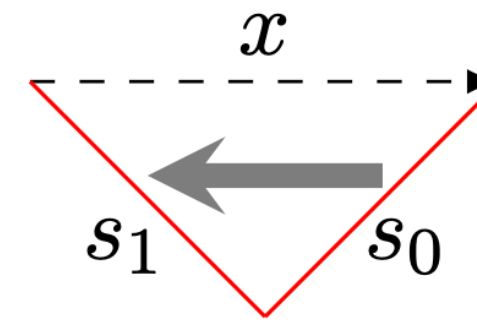
$$F^{h,\varphi}(\tau_R)|x\rangle = \varepsilon(h)T_-^\varphi|x\rangle = \varepsilon(h)|x \leftarrow \hat{S}(\varphi)\rangle,$$



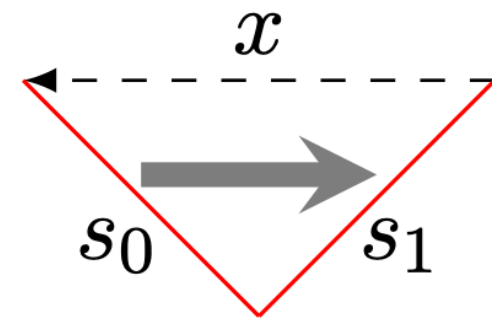
$$F^{h,\varphi}(\tilde{\tau}_R)|x\rangle = \hat{\varepsilon}(\varphi)L_-^h|x\rangle = \hat{\varepsilon}(\varphi)|x \triangleleft S(h)\rangle,$$



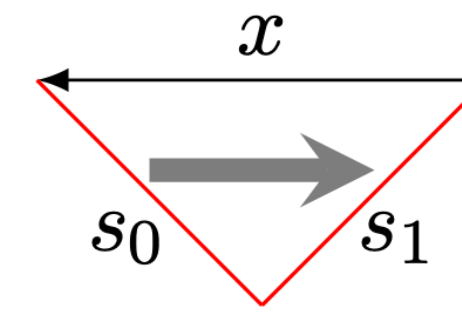
$$F^{h,\varphi}(\tau_R)|x\rangle = \varepsilon(h)T_+^\varphi|x\rangle = \varepsilon(h)|\varphi \rightarrow x\rangle.$$



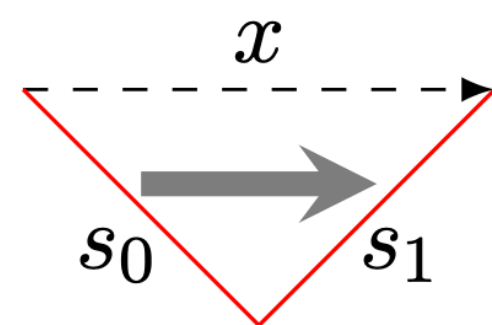
$$F^{h,\varphi}(\tilde{\tau}_R)|x\rangle = \hat{\varepsilon}(\varphi)L_+^h|x\rangle = \hat{\varepsilon}(\varphi)|h \triangleright x\rangle.$$



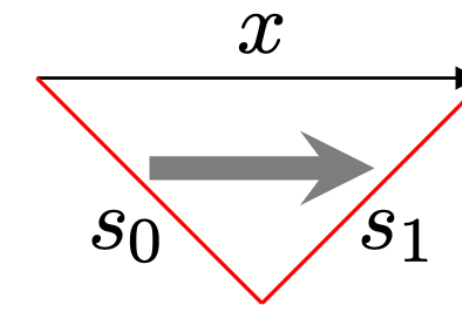
$$F^{h,\varphi}(\tilde{\tau}_L)|x\rangle = \hat{\varepsilon}(\varphi)\tilde{L}_-^h|x\rangle = \hat{\varepsilon}(\varphi)|x \triangleleft h\rangle.$$



$$F^{h,\varphi}(\tau_L)|x\rangle = \varepsilon(h)\tilde{T}_-^\varphi|x\rangle = \varepsilon(h)|x \leftarrow \varphi\rangle,$$



$$F^{h,\varphi}(\tilde{\tau}_L)|x\rangle = \hat{\varepsilon}(\varphi)\tilde{L}_+^h|x\rangle = \hat{\varepsilon}(\varphi)|S(h) \triangleright x\rangle.$$



$$F^{h,\varphi}(\tau_L)|x\rangle = \varepsilon(h)\tilde{T}_+^\varphi|x\rangle = \varepsilon(h)|\hat{S}(\varphi) \rightarrow x\rangle.$$

[Jia, Kaszlikowski, Tan, 2023]

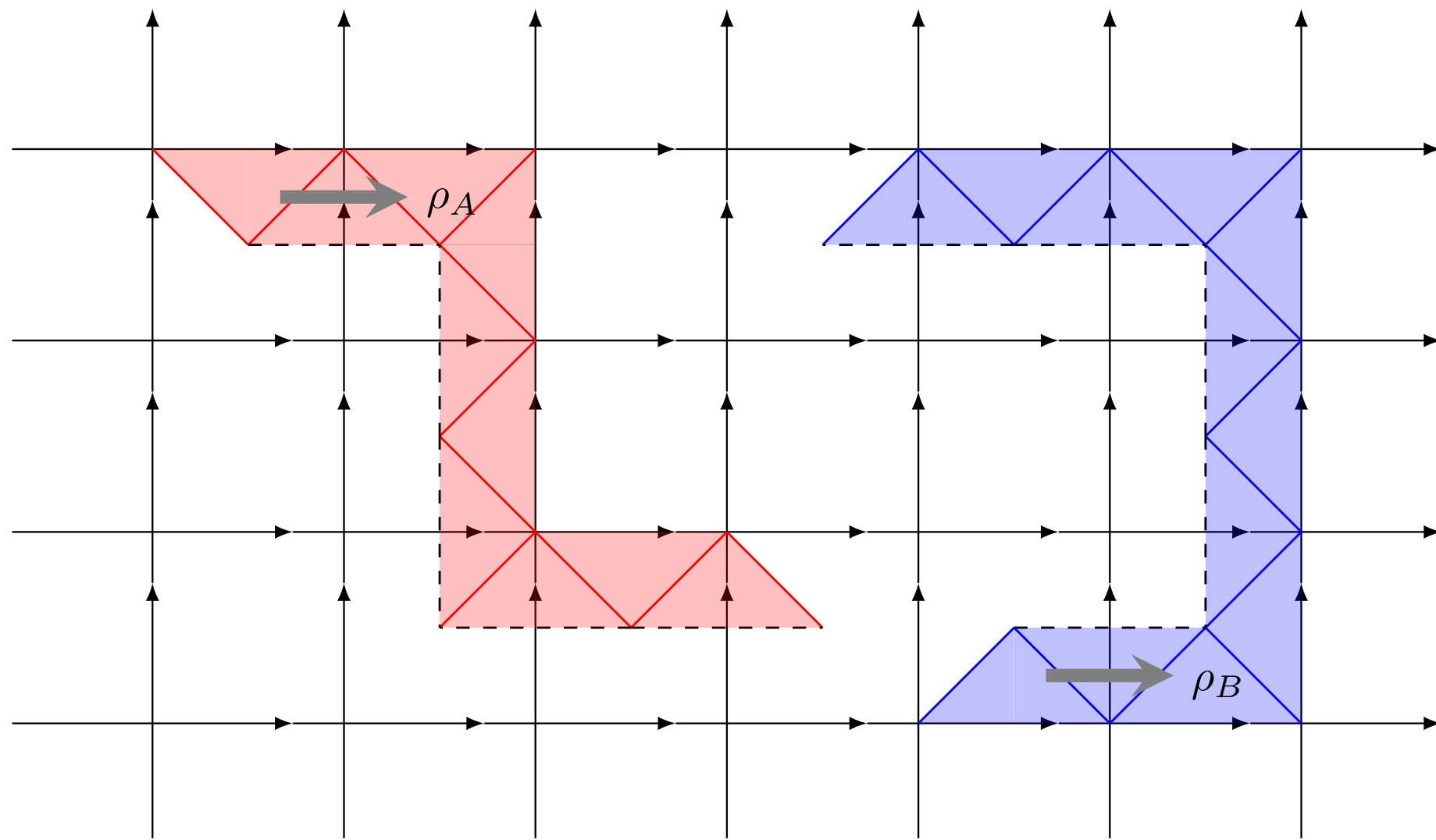
# Ribbon operator and topological excitation

## Topological excitations: finite group

A topological charge is an irrep of  $D(H)$

$$a_{[g],\pi} = \mathbb{C}[G] \otimes_{\mathbb{C}C_G(g)} M_\pi$$

- Electric charge  $g=1$
- Magnetic charge: trivial rep
- Dyonic charge: electric + magnetic



## Hopf quantum double excitation

For general Hopf case  $D(H)$

$$a_{g,M_g} = H \otimes_{H_g} M_g$$

$$\text{FPdim } a_{g,M_g} = \frac{|G|}{\dim H_g} \dim M_g.$$

- There is a universal grading group  $G = U(\text{Rep}(H))$  for any semisimple Hopf algebra  $H$ .
- Consider the largest central Hopf subalgebra  $K(\hat{H}^{\text{cop}})$  of  $\hat{H}^{\text{cop}}$ , we have  $K(\hat{H}^{\text{cop}}) = \mathbb{C}G^\vee$ , which is commutative, and  $\text{Rep}(\hat{H}^{\text{cop}}) = \bigoplus_{g \in G} \text{Rep}(\hat{H}^{\text{cop}})_g$ . Suppose that  $H_g$  is a Hopf subalgebra of  $H$  such that  $\text{Rep}(\hat{H}_g) = \bigoplus_{x \in C_G(g)} \text{Rep}(\hat{H}^{\text{cop}})_x$ . It is proved that  $K(\hat{H}^{\text{cop}}) = \mathbb{C}G^\vee$  is a normal Hopf subalgebra of  $D(H)$ .
- $\mathcal{I}_g = \{M_g\}$  is the set of all irreducible representations of  $\hat{H}^{\text{cop}} \bowtie H_g$  (here “ $\bowtie$ ” denotes bicrossed product) such that the character  $\chi_{M_g}$ , when restricted on  $K(\hat{H}^{\text{cop}})$ , satisfies  $\chi_{M_g}|_{K(\hat{H}^{\text{cop}})} = g \dim M_g$ .

[Jia, Kaszlikowski, Tan, 2023]

# Gapped boundary: algebraic theory

	Hopf quantum double model	String-net model
Bulk	Hopf algebra $H$	UFC $\mathcal{C} = \text{Rep}(H)$
Bulk phase	$\mathcal{D} = \text{Rep}(D(H))$	$\text{Fun}_{\text{Rep}(H) \text{Rep}(H)}(\text{Rep}(H), \text{Rep}(H))$
Boundary	$H$ -comodule algebra $\mathfrak{A}$	$\text{Rep}(H)$ -module category ${}_{\mathfrak{A}}\mathcal{M} = {}_{\mathfrak{A}}\text{Mod}$
Boundary phase	$\mathcal{B} \simeq {}_H^{\mathfrak{A}}\text{Mod}_{\mathfrak{A}}$	$\text{Fun}_{\text{Rep}(H)}({}_{\mathfrak{A}}\mathcal{M}, {}_{\mathfrak{A}}\mathcal{M})$
Boundary defect	${}_{\mathfrak{B}}^H\text{Mod}_{\mathfrak{A}}$	$\text{Fun}_{\text{Rep}(H)}({}_{\mathfrak{A}}\mathcal{M}, {}_{\mathfrak{B}}\mathcal{M})$

	Hopf quantum double model	String-net model
Bulk	Hopf algebra $H$	UFC $\mathcal{C} = \text{Rep}(H)$
Bulk phase	$\mathcal{D} = \text{Rep}(D(H))$	$\text{Fun}_{\text{Rep}(H) \text{Rep}(H)}(\text{Rep}(H), \text{Rep}(H))$
Boundary	$H$ -module algebra $\mathfrak{M}$	$\mathcal{C}_{\mathfrak{M}}$
Boundary phase	$\mathcal{B} \simeq {}_{\mathfrak{M}}\mathcal{C}_{\mathfrak{M}}$	$\text{Fun}_{\mathcal{C}}(\mathcal{C}_{\mathfrak{M}}, \mathcal{C}_{\mathfrak{M}})$
Boundary defect	${}_{\mathfrak{N}}\mathcal{C}_{\mathfrak{M}}$	$\text{Fun}_{\mathcal{C}}(\mathcal{C}_{\mathfrak{M}}, \mathcal{C}_{\mathfrak{N}})$

[Jia, Kaszlikowski, Tan, 2023]

# Gapped boundary: algebraic theory

**Definition** *Let  $H$  be a Hopf algebra and  $(\mathfrak{A}, \mu_{\mathfrak{A}}, \eta_{\mathfrak{A}})$  an algebra. If  $\mathfrak{A}$  is a left  $H$ -comodule with left coaction  $\beta_{\mathfrak{A}} : \mathfrak{A} \rightarrow H \otimes \mathfrak{A}$  such that  $\beta_{\mathfrak{A}}(xy) = \beta_{\mathfrak{A}}(x)\beta_{\mathfrak{A}}(y)$  and  $\beta_{\mathfrak{A}}(1_{\mathfrak{A}}) = 1_H \otimes 1_{\mathfrak{A}}$ , then  $\mathfrak{A}$  is called a left  $H$ -comodule algebra. A right  $H$ -comodule algebra can be defined similarly.*

**Definition** *Let  $H$  be a Hopf algebra and  $\mathfrak{M}$  an algebra. If  $\mathfrak{M}$  is a left  $H$ -module such that  $h \triangleright (xy) = \sum_{(h)} (h^{(1)} \triangleright x)(h^{(2)} \triangleright y)$  and  $h \triangleright 1_{\mathfrak{M}} = \varepsilon(h)1_{\mathfrak{M}}$ , then  $\mathfrak{M}$  is called a left  $H$ -module algebra. A right  $H$ -module algebra can be defined similarly.*

# Gapped boundary: Hamiltonian theory

**Definition** A symmetric separability idempotent of an algebra  $\mathfrak{A}$  is an element  $\lambda = \sum_{\langle \lambda \rangle} \lambda^{(1)} \otimes \lambda^{(2)} \in \mathfrak{A} \otimes \mathfrak{A}$  that satisfies the following conditions:

$$(1) \sum_{\langle \lambda \rangle} x \lambda^{(1)} \otimes \lambda^{(2)} = \sum_{\langle \lambda \rangle} \lambda^{(1)} \otimes \lambda^{(2)} x, \text{ for all } x \in \mathfrak{A}.$$

$$(2) \sum_{\langle \lambda \rangle} \lambda^{(1)} \lambda^{(2)} = 1.$$

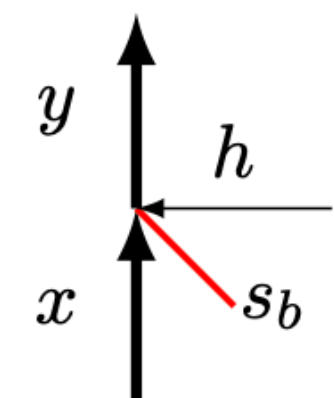
$$(3) \sum_{\langle \lambda \rangle} \lambda^{(1)} \otimes \lambda^{(2)} = \sum_{\langle \lambda \rangle} \lambda^{(2)} \otimes \lambda^{(1)}.$$

**Always exist!**

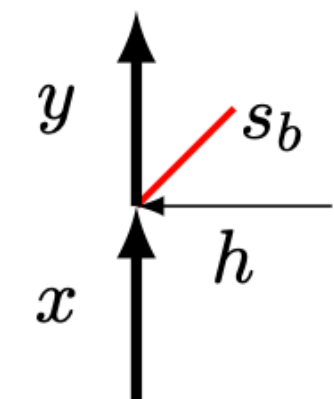
## Boundary stabilizers

$$B^\varphi(s_b) | \begin{array}{c} \text{square with } l \text{ top, } h \text{ bottom, } x \text{ left, } k \text{ right} \\ \text{red diagonal from top-left to bottom-right} \end{array} \rangle = \sum_{(\varphi)} | \begin{array}{c} \bar{T}_+^{\varphi(1)} x \\ \text{square with } T_+^{\varphi(4)} l \text{ top, } T_-^{\varphi(2)} h \text{ bottom} \\ \text{red diagonal from top-left to bottom-right} \\ T_-^{\varphi(3)} k \end{array} \rangle$$

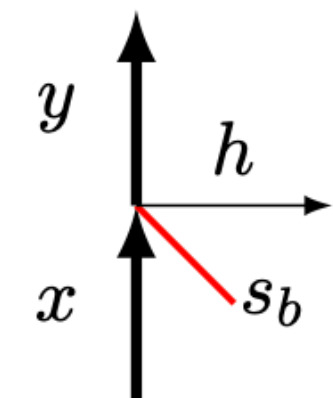
[Jia, Kaszlikowski, Tan, 2023]



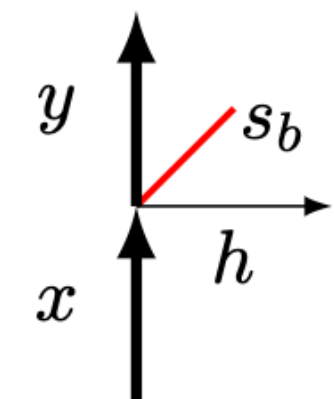
$$A^{a \otimes b}(s_b) |x, y, h\rangle = \sum_{[b]} |ax, yb^{[0]}, S(b^{[1]})h\rangle,$$



$$A^{a \otimes b}(s_b) |x, y, h\rangle = \sum_{[a]} |a^{[0]}x, yb, a^{[1]}h\rangle,$$



$$A^{a \otimes b}(s_b) |x, y, h\rangle = \sum_{[b]} |ax, yb^{[0]}, hb^{[1]}\rangle,$$



$$A^{a \otimes b}(s_b) |x, y, h\rangle = \sum_{[a]} |a^{[0]}x, yb, hS(a^{[1]})\rangle.$$



# Gapped boundary: Hamiltonian theory

We introduce the crossed product  $(\mathfrak{A} \otimes \mathfrak{A}^{\text{op}}) \star \hat{H}$  between  $\mathfrak{A} \otimes \mathfrak{A}^{\text{op}}$  and  $\hat{H}$  as follows: the underlying vector space is  $(\mathfrak{A} \otimes \mathfrak{A}) \otimes \hat{H}$ , the multiplication is given by

$$((a \otimes b) \star \varphi) \cdot ((c \otimes d) \star \psi) = \sum (ac^{[0]} \otimes d^{[0]}b) \star \varphi(c^{[1]} \bullet d^{[1]})\psi,$$

and the unit is  $(1_{\mathfrak{A}} \otimes 1_{\mathfrak{A}}) \star \varepsilon$ . Clearly,  $((1_{\mathfrak{A}} \otimes 1_{\mathfrak{A}}) \star \varepsilon) \cdot ((a \otimes b) \star \varphi) = ((a \otimes b) \star \varphi) \cdot ((1_{\mathfrak{A}} \otimes 1_{\mathfrak{A}}) \star \varepsilon) = (a \otimes b) \star \varphi$ . The multiplication is associative:

$$\begin{aligned} & [((a \otimes b) \star \varphi) \cdot ((c \otimes d) \star \psi)] \cdot ((e \otimes f) \star \theta) \\ &= \sum ((ac^{[0]} \otimes d^{[0]}b) \star \varphi(c^{[1]} \bullet d^{[1]})\psi) \cdot ((e \otimes f) \star \theta) \\ &= \sum (ac^{[0]}e^{[0]} \otimes f^{[0]}d^{[0]}b) \star \varphi(c^{[1]}e^{[1]} \bullet f^{[1]}d^{[1]})\psi(e^{[2]} \bullet f^{[2]})\theta \\ &= \sum ((a \otimes b) \star \varphi) \cdot ((ce^{[0]} \otimes f^{[0]}d) \star \psi(e^{[1]} \bullet f^{[1]})\theta) \\ &= ((a \otimes b) \star \varphi) \cdot [((c \otimes d) \star \psi) \cdot ((e \otimes f) \star \theta)]. \end{aligned}$$

$$\varphi(a \otimes b) = \sum (a^{[0]} \otimes b^{[0]})\varphi(a^{[1]} \bullet b^{[1]})$$

# Gapped boundary: Hamiltonian theory

**Proposition** At a boundary site  $s_b$ , the boundary face and vertex operators generate the algebra  $(\mathfrak{A} \otimes \mathfrak{A}^{\text{op}}) \star \hat{H}$ , with the straightening relation

$$B^\varphi(s_b)A^{a \otimes b}(s_b) = \sum_{[a],[b]} A^{a^{[0]} \otimes b^{[0]}}(s_b) B^{\varphi(a^{[1]} \bullet b^{[1]})}(s_b),$$

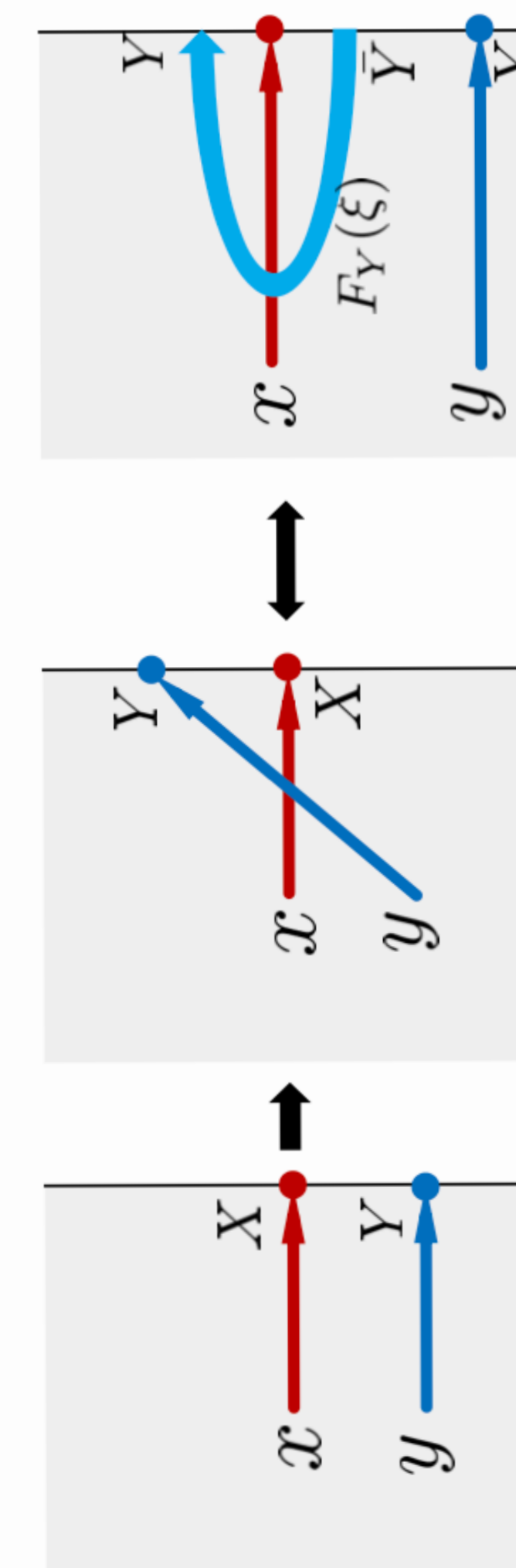
where  $a \otimes b \in \mathfrak{A} \otimes \mathfrak{A}^{\text{op}}$  and  $\varphi \in \hat{H}$ . Therefore, the map

$$\Psi_{s_b} : (\mathfrak{A} \otimes \mathfrak{A}^{\text{op}}) \star \hat{H} \rightarrow \text{End}(\mathcal{H}(s_b)), \quad (a \otimes b) \star \varphi \mapsto A^{a \otimes b}(s_b) B^\varphi(s_b)$$

is an algebra homomorphism. That is, every boundary site supports a representation of  $(\mathfrak{A} \otimes \mathfrak{A}^{\text{op}}) \star \hat{H}$ .

**Proposition** Consider a gapped boundary determined by an  $H$ -comodule algebra  $\mathfrak{A}$ :

1. The boundary local operator algebra is fusion-categorical Morita equivalent to  $\mathcal{D}(s_b) \simeq (\mathfrak{A} \otimes \mathfrak{A}^{\text{op}}) \star \hat{H}$ .
2. The bulk-to-boundary ribbon operator algebra  $\mathcal{A}_{\rho \downarrow}$  is the dual of the boundary local operator algebra  $\mathcal{A}_{\rho \downarrow} \simeq \mathcal{D}(s_b)^\vee$ . Or equivalently, the dual of the bulk-to-boundary ribbon operator algebra is fusion-categorical Morita equivalent to the boundary local operator algebra  $\mathcal{A}_{\rho \downarrow}^\vee \simeq (\mathfrak{A} \otimes \mathfrak{A}^{\text{op}}) \star \hat{H}$ .



**Boundary-bulk duality**  
**Holographic principle**

# Gapped boundary: Solve the boundary model

## Hopf Tensor Network

$$\langle \phi, x \rangle = \text{Diagram} = \text{Diagram} S(x),$$

$$\langle \phi, x \rangle = \text{Diagram} = \text{Diagram} S(\phi).$$

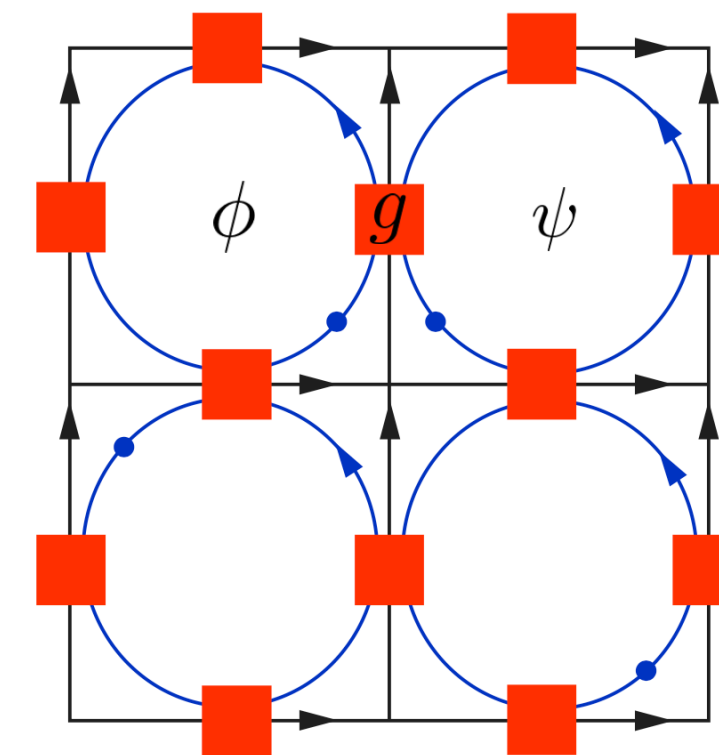
$$\begin{aligned} \text{Diagram} & := \sum_{(x)} \text{Diagram} = \sum_{(x)} \langle \phi, x^{(2)} \rangle \langle \psi, S(x^{(1)}) \rangle. \\ \text{Diagram} & := \sum_{(\phi)} \text{Diagram} = \sum_{(\phi)} \langle \phi^{(1)}, x \rangle \langle \phi^{(2)}, y \rangle. \end{aligned}$$

**Proposition** (Ground state on a disk). *Suppose that  $\Sigma$  is a disk, then the ground state is unique. Then the ground state of the boundary lattice model determined by an  $H$ -comodule algebra  $\mathfrak{A}$  is the Hopf tensor network state*

$$|\Psi_{GS}\rangle = |\Psi_{C(\Sigma \setminus \partial\Sigma); C(\partial\Sigma)}(\{h_{H,e}\}, \{\phi_{\hat{H},f}\}; \{h_{\mathfrak{A},e_b}\}, \{\varphi_{\hat{H},f_b}\})\rangle.$$

For each bulk site  $s$ , we have

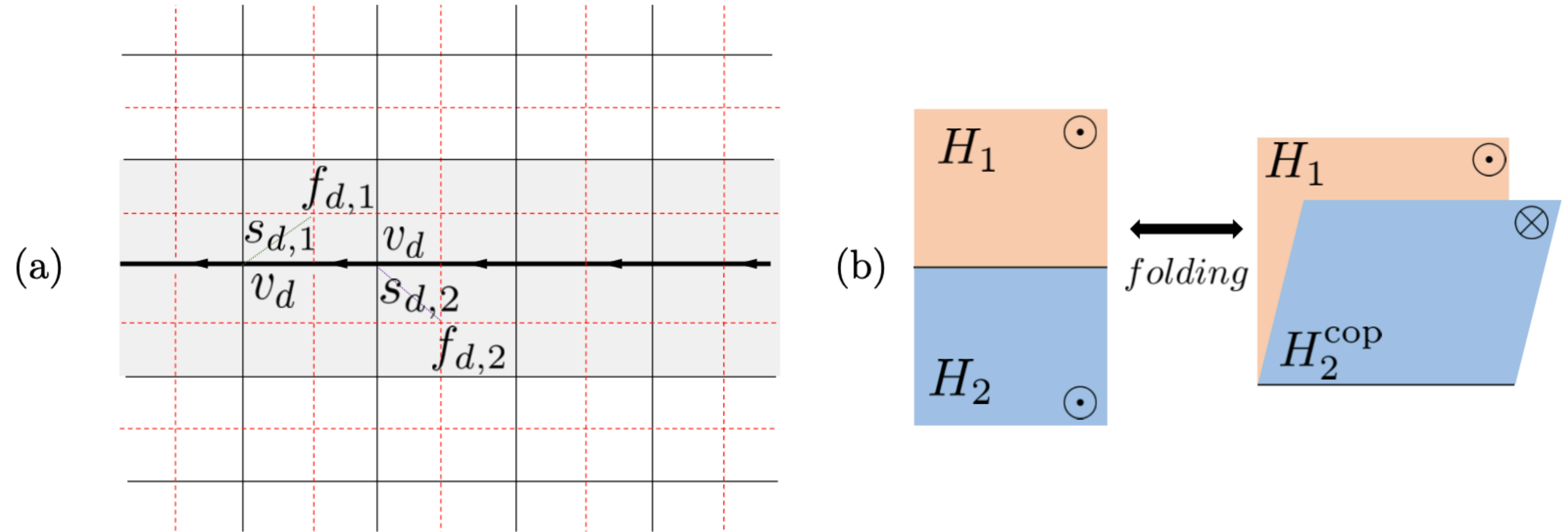
$$B^\varphi(s)A^g(s)|\Psi_{GS}\rangle = \varepsilon(g)\varphi(1_H)|\Psi_{GS}\rangle.$$



# Domain Wall: Algebraic Theory

## Folding Trick

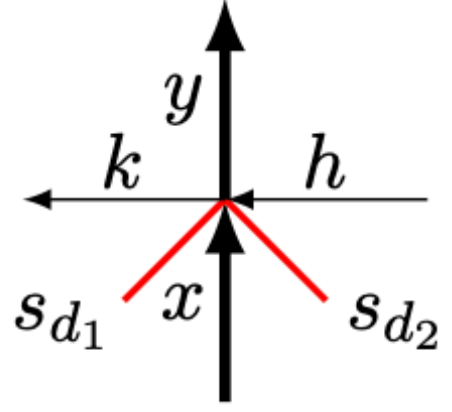
- Domain wall is a boundary
- Boundary is a domain wall



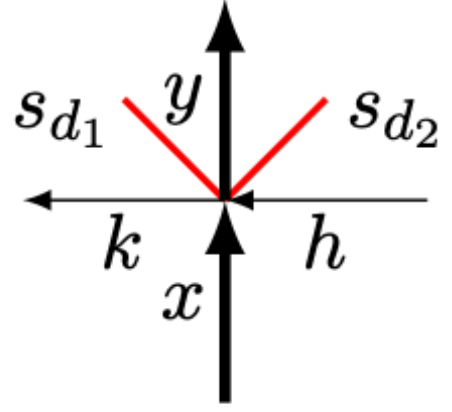
	Hopf quantum double model	String-net model
Bulks ( $i = 1, 2$ )	Hopf algebras $H_i$	UFC $\mathcal{C}_i = \text{Rep}(H_i)$
Bulk phases	$\mathcal{D}_i = \text{Rep}(D(H_i))$	$\text{Fun}_{\text{Rep}(H_i) \text{Rep}(H_i)}(\text{Rep}(H_i), \text{Rep}(H_i))$
Domain wall	$H_1 H_2$ -bicomodule algebra $\mathfrak{B}$	$\mathcal{C}_1 \mathcal{C}_2$ -bimodule category $\mathfrak{B}\text{Mod}$
Domain wall phase	$\mathfrak{B}\text{Mod}_{\mathfrak{B}}$	$\text{Fun}_{\mathcal{C}_1 \mathcal{C}_2}(\mathfrak{B}\mathcal{M}, \mathfrak{B}\mathcal{M})$
Domain wall defect	$\mathfrak{B}\text{Mod}_{\mathfrak{A}}$	$\text{Fun}_{\text{Rep}(H_1) \text{Rep}(H_2)}(\mathfrak{A}\mathcal{M}, \mathfrak{B}\mathcal{M})$

# Domain Wall: Hamiltonian Theory

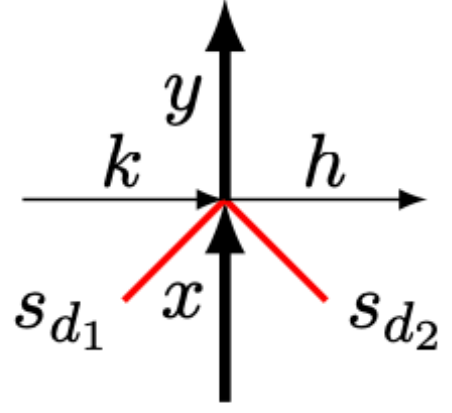
## Domain wall lattice



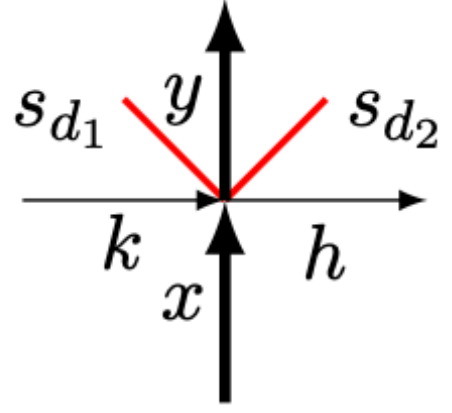
$$A^{a \otimes b}(s_{d_i})|x, k, y, h\rangle = \sum_{[b]} |ax, kb^{[-1]}, yb^{[0]}, S(b^{[1]})h\rangle,$$



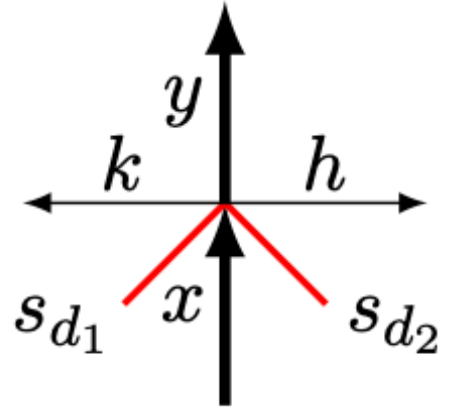
$$A^{a \otimes b}(s_{d_i})|x, k, y, h\rangle = \sum_{[a]} |a^{[0]}x, kS(a^{[-1]}), yb^{[0]}, a^{[1]}h\rangle,$$



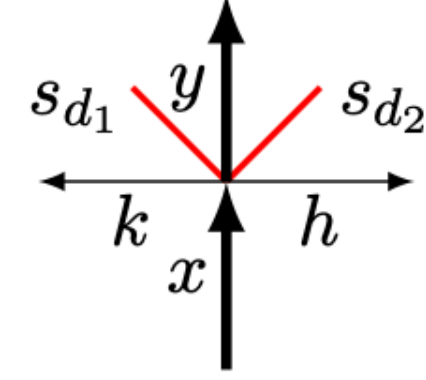
$$A^{a \otimes b}(s_{d_i})|x, k, y, h\rangle = \sum_{[b]} |ax, S(b^{[-1]})k, yb^{[0]}, hb^{[1]}\rangle,$$



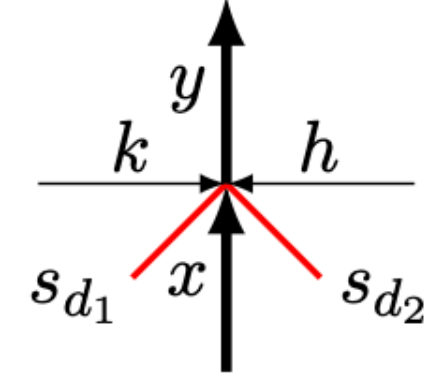
$$A^{a \otimes b}(s_{d_i})|x, k, y, h\rangle = \sum_{[a]} |a^{[0]}x, a^{[-1]}k, yb, hS(a^{[1]})\rangle,$$



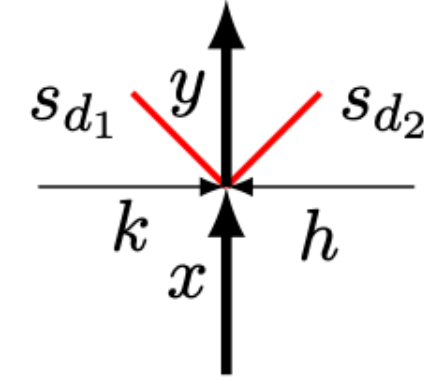
$$A^{a \otimes b}(s_{d_i})|x, k, y, h\rangle = \sum_{[b]} |ax, kb^{[-1]}, yb^{[0]}, hb^{[1]}\rangle,$$



$$A^{a \otimes b}(s_{d_i})|x, k, y, h\rangle = \sum_{[a]} |a^{[0]}x, kS(a^{[-1]}), yb, hS(a^{[1]})\rangle,$$



$$A^{a \otimes b}(s_{d_i})|x, k, y, h\rangle = \sum_{[b]} |ax, S(b^{[-1]})k, yb^{[0]}, S(b^{[1]})h\rangle,$$



$$A^{a \otimes b}(s_{d_i})|x, k, y, h\rangle = \sum_{[a]} |a^{[0]}x, a^{[-1]}k, yb, a^{[1]}h\rangle.$$

$$H[C(\Sigma_d)] = \sum_{i=1,2} \sum_{d_i} (I - A^\lambda(s_{d_i})) + \sum_{i=1,2} \sum_{d_i} (I - B^{\varphi \hat{H}_i}(s_{d_i})).$$

[Jia, Kaszlikowski, Tan, 2023]

# Discussion

- A complete theory of boundary and domain wall theory of Hopf quantum double
- Weak Hopf case, weak Hopf symmetry ([Jia, Tan, Kaszlikowski, Chang, CMP 2023])
- Higher dimensional case
- Symmetry-enriched case (largely remains open!)
- Entangle entropy is sensitive to defect and boundary
- Operator algebra perspective: stability, Haag duality, infinite-volume sector, etc.
- Weak Hopf quantum double  $\leftrightarrow$  extended string-net model

**Thank you for you attention!**