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Quench Dynamics of Rényi Entanglement Entropy in Non-Interacting and Strongly-Interacting Bosons

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Ref) Phys. Rev. A **107**, 033305 (2023), arXiv:2209.13340 (accepted in Phys. Rev. Res.).

Outline

1. **Introduction**
   -Brief review of entanglement
   -Measurement of entanglement entropy
   -Motivation of this study

2. Entanglement dynamics of non-interacting Bosons
   -Model and quench protocol
   -Analytical result of Rényi entanglement entropy
   -Numerical results

   Ref) Phys. Rev. A **107**, 033305 (2023)

3. Entanglement dynamics of strongly-interacting Bosons
   -Model and low-energy effective theory
   -Analytical result of Rényi entanglement entropy

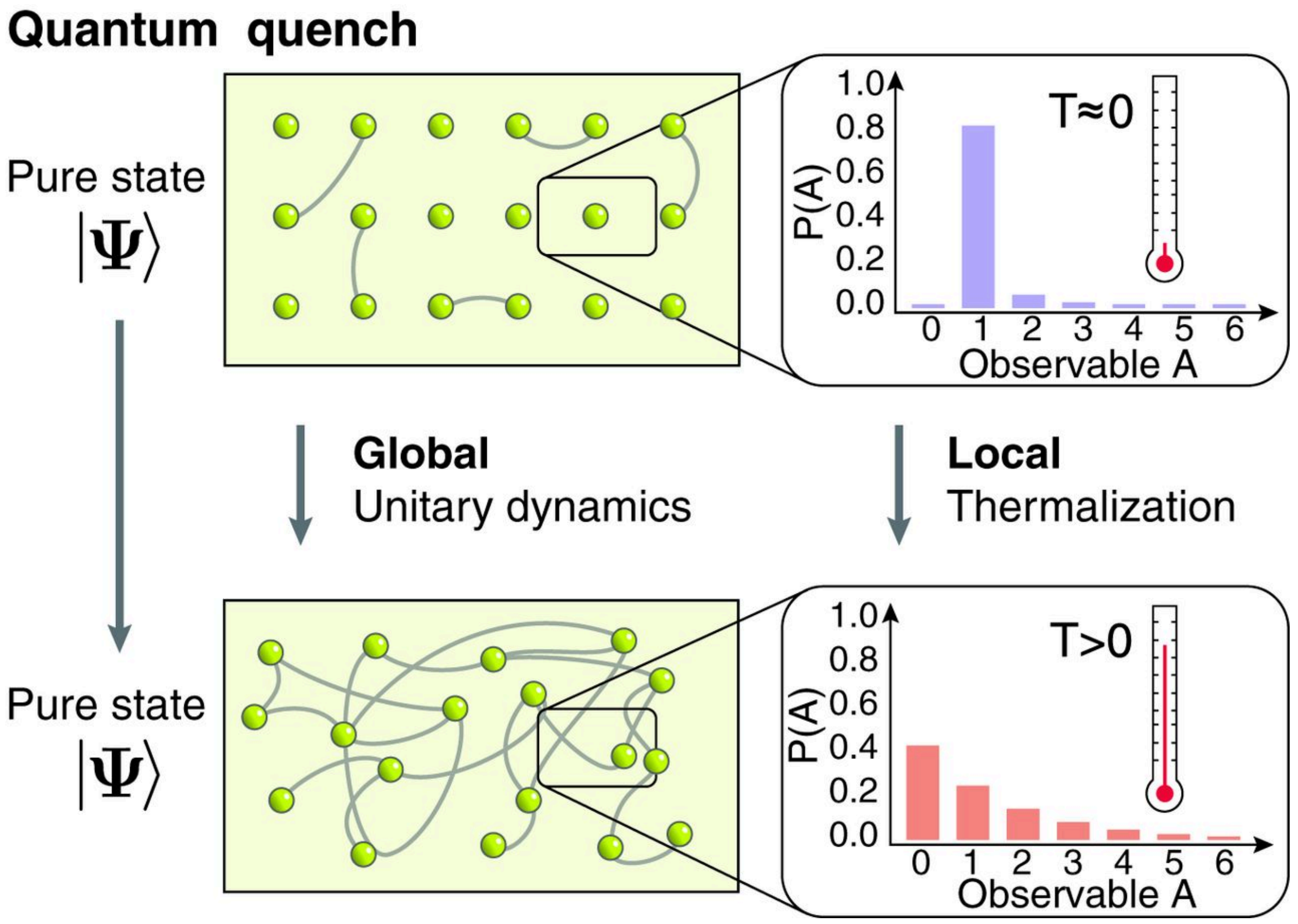
   Ref) arXiv:2209.13340.

4. Summary

Entanglement in quantum many-body physics

Entanglement: non-local correlations in quantum mechanics

Thermalization of isolated quantum systems



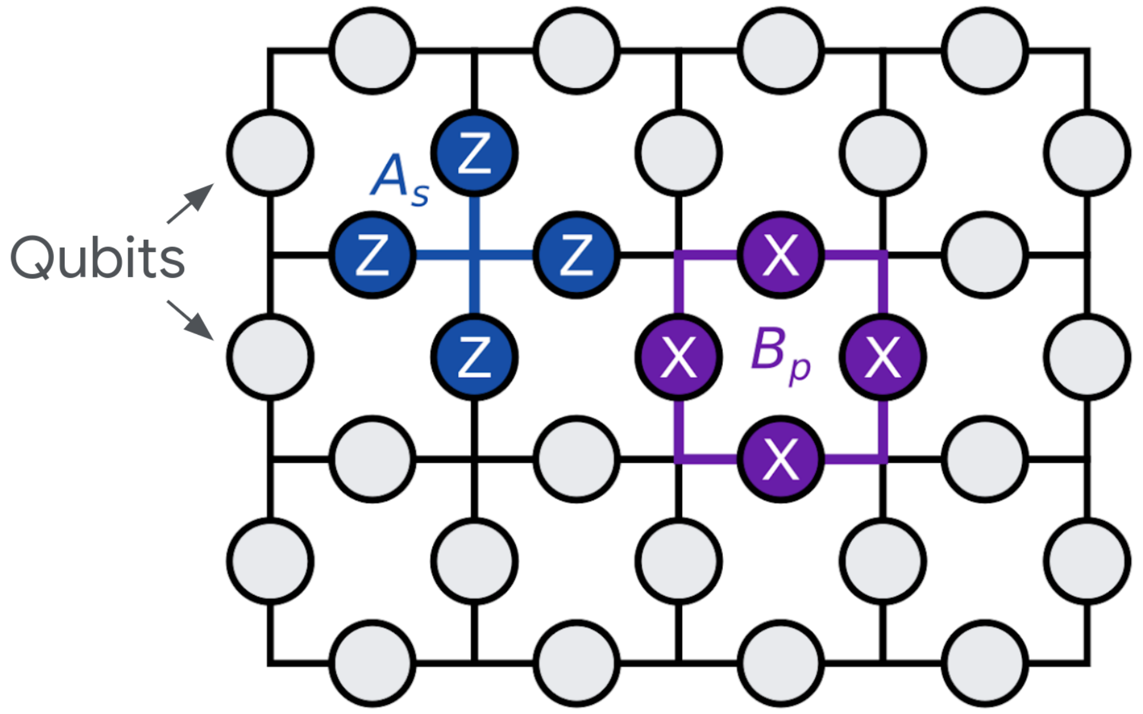
A. M. Kaufman *et al.*, Science (2016)

Entanglement is used as an indicator of thermalization

Characterization of quantum states

E.g.) Thermal states should have extensive entanglement

E.g.) Topological ordered state

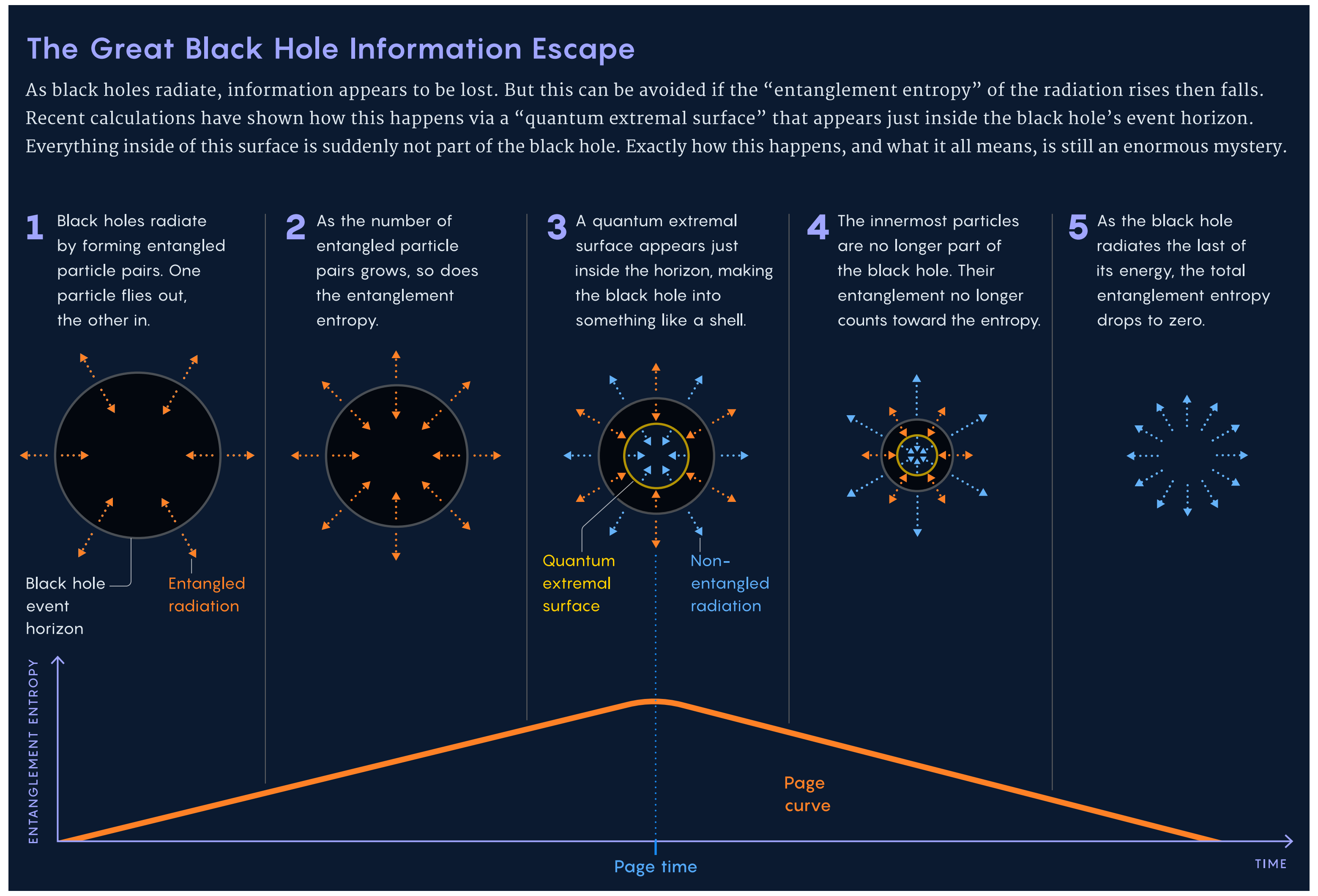


Toric Code

From Google AI website

Entanglement characterizes some quantum states

Information paradox of black hole



From quantamagazine website

Entanglement is related to an information paradox

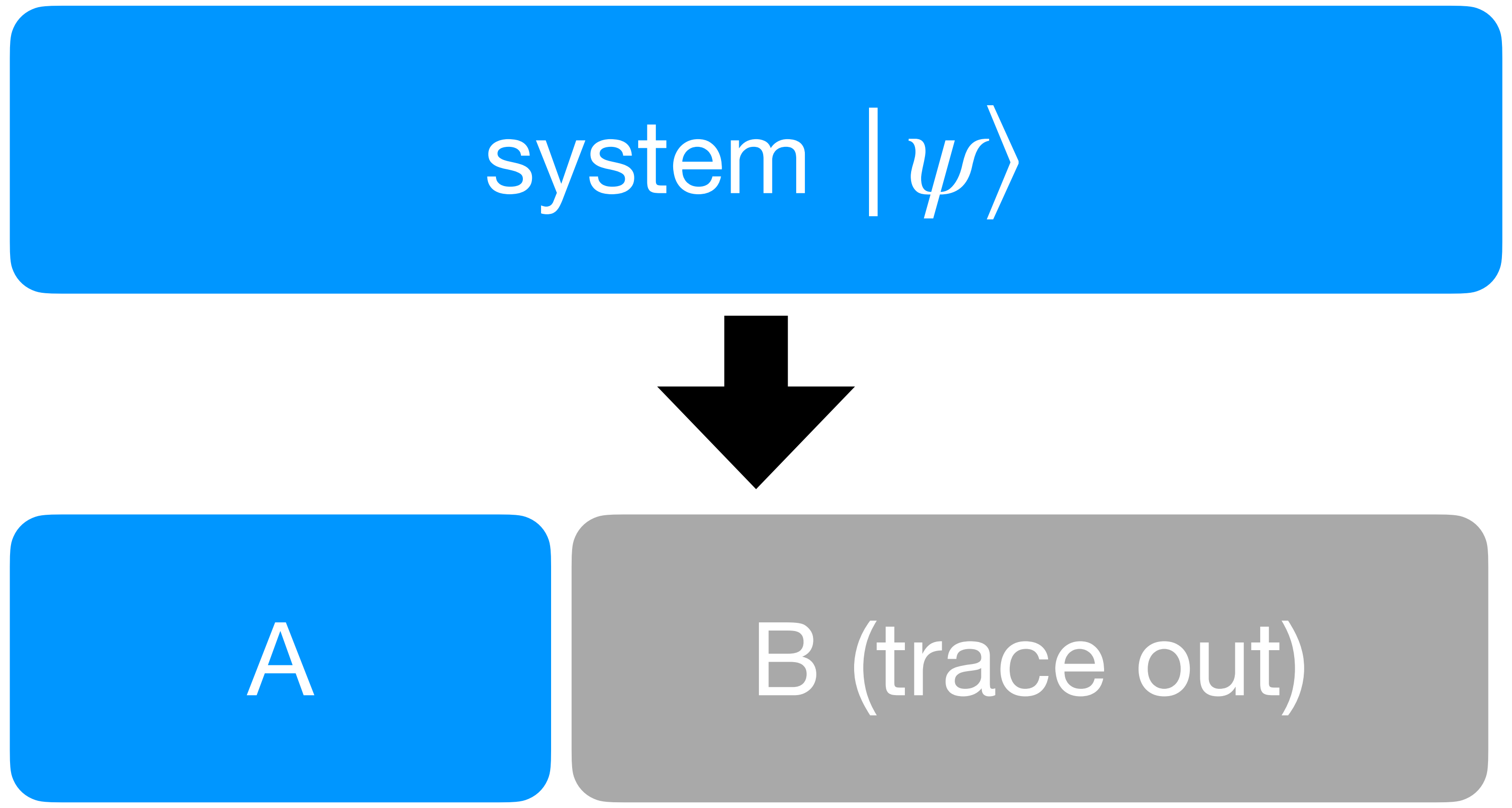
Entanglement entropy

<u>Entanglement is often quantified by entropies</u>

von Neumann entropy: $S_{\mathrm{vN}} = -\mathrm{Tr}_{\mathrm{A}}\hat{\rho}_{\mathrm{A}} \ln \hat{\rho}_{\mathrm{A}}$

(αth) Rényi entropy: $S_\alpha = \dfrac{\ln \mathrm{Tr}_{\mathrm{A}}\hat{\rho}_{\mathrm{A}}^{\alpha}}{1-\alpha}$

$S_{\mathrm{vN}} \geq S_2 \geq S_3 \geq \ldots, \quad \lim_{\alpha\to 1} S_\alpha = S_{\mathrm{vN}}$



$\hat{\rho}_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}|\psi\rangle\langle\psi|$: reduced density matrix

<u>Application: classification of quantum states</u>

Size dependence of S in D dimensions

gapped ground states	$S \sim L^{D-1}$
Critical states, Fermi liquids	$S \sim L^{D-1}\ln L$
Thermal states	$S \sim L^{D}$

Topologically ordered state $S = \alpha L - \gamma + \ldots$

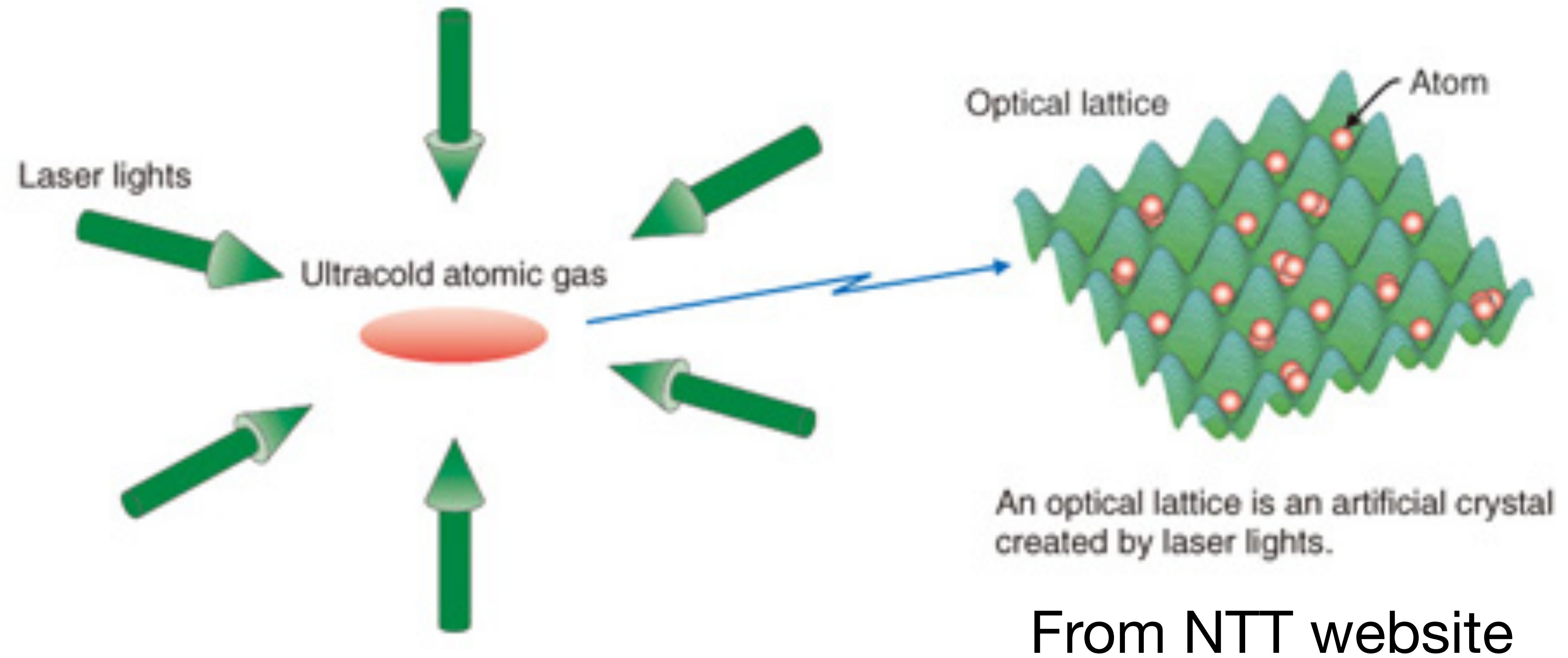
γ: topological entanglement entropy

A. Kitaev and J. Preskill, PRL (2006)
M. Levin and X.-G. Wen, PRL (2006)

Bosons in an optical lattice

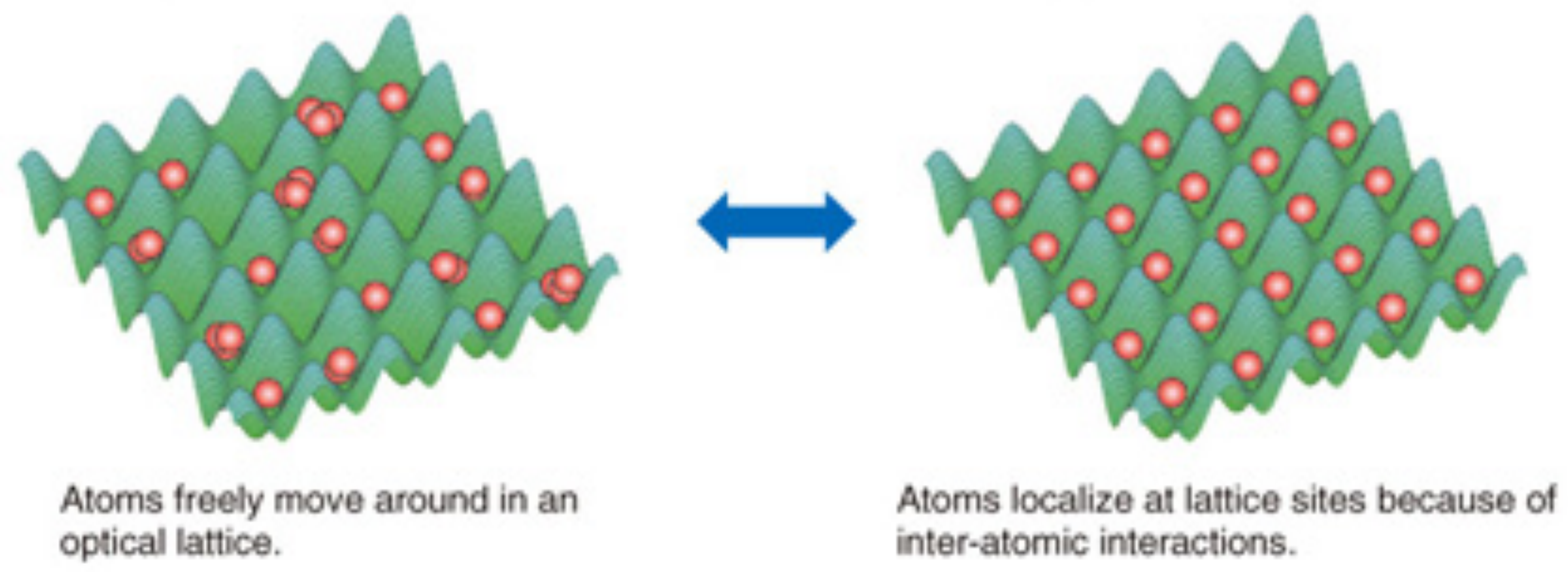
Measuring entanglement in highly-controllable quantum systems

- **Bosons in an optical lattice** R. Islam *et al.*, Nature (2015); A. M. Kaufman *et al.*, Science (2016)
- Trapped ions T. Brydges, *et al.*, Science (2019)
- Rydberg atoms D. Bluvstein, *et al.*, Nature (2022)



From NTT website

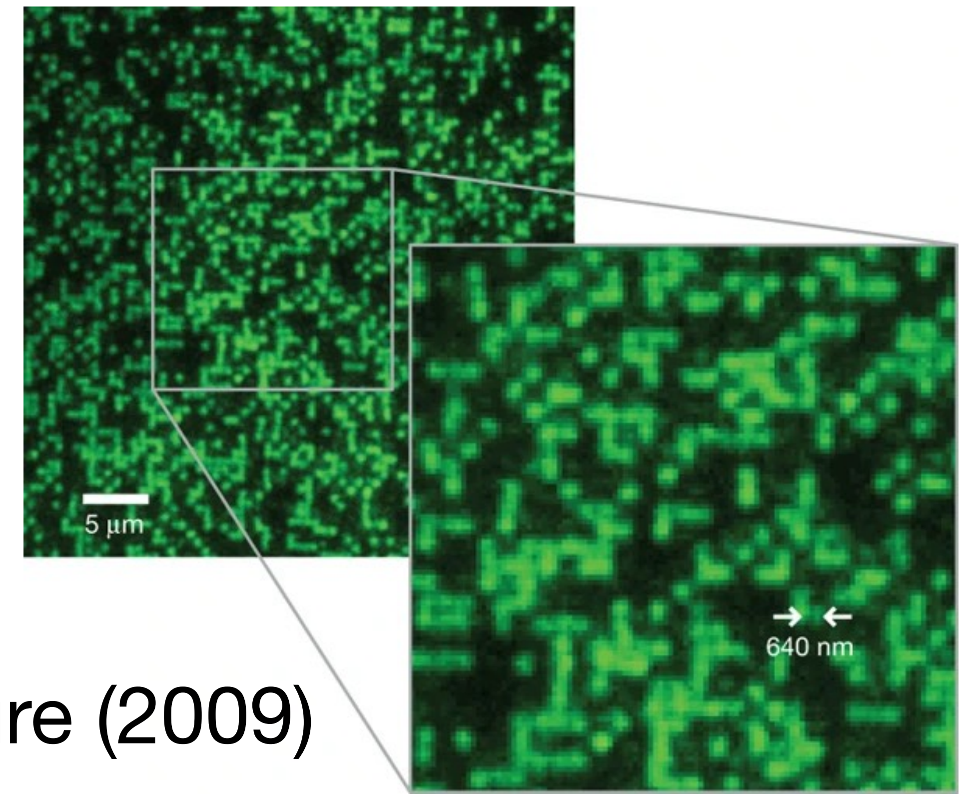
Superfluid: $J \gg U$ Mott insulator: $J \ll U$



Bose-Hubbard model

$$\hat{H} = -J\sum_{j=1}^{L-1}\left(\hat{b}_j^{\dagger}\hat{b}_{j+1} + \hat{b}_{j+1}^{\dagger}\hat{b}_j\right) + \frac{U}{2}\sum_{j=1}^{L}\hat{n}_j(\hat{n}_j - 1)$$

- Real-time control of parameters
- Single-site measurement by microscope



W. S. Bakr *al.*, Nature (2009)

Measurement of Rényi entanglement entropy

<u>Measuring entanglement</u>

A. J. Daley *et al.*, PRL (2012),
D. A. Abanin and E. Demler, PRL (2012)

2nd Rényi entropy: $S_2 = -\ln \mathrm{Tr}_A \hat{\rho}_A^2$

Purity: $\mathrm{Tr}_A \hat{\rho}_A^2$

1. Two copies of the same state

$$|\Psi\rangle = |\psi\rangle \otimes |\psi\rangle$$

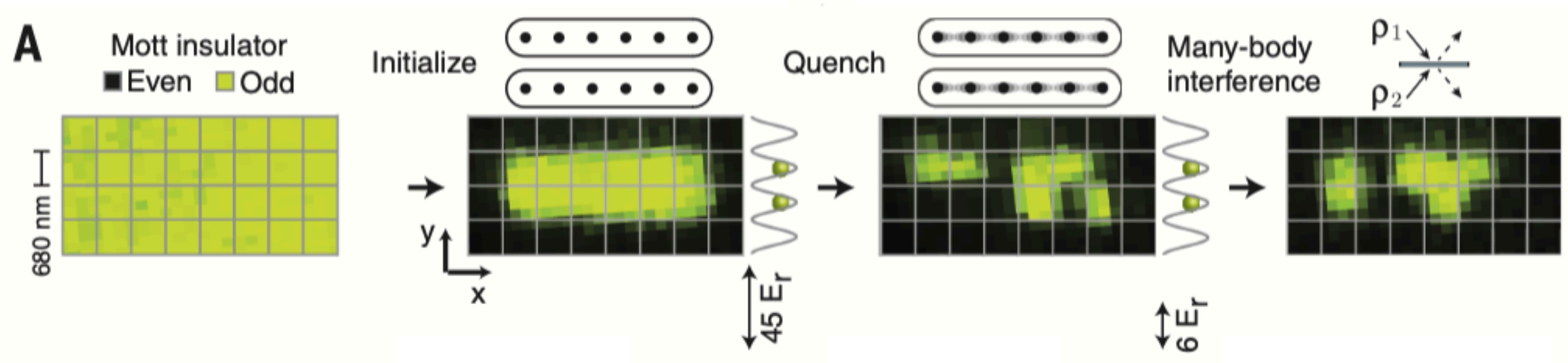
2. SWAP operator: $\hat{V}|\psi_1\rangle|\psi_2\rangle = |\psi_2\rangle|\psi_1\rangle$

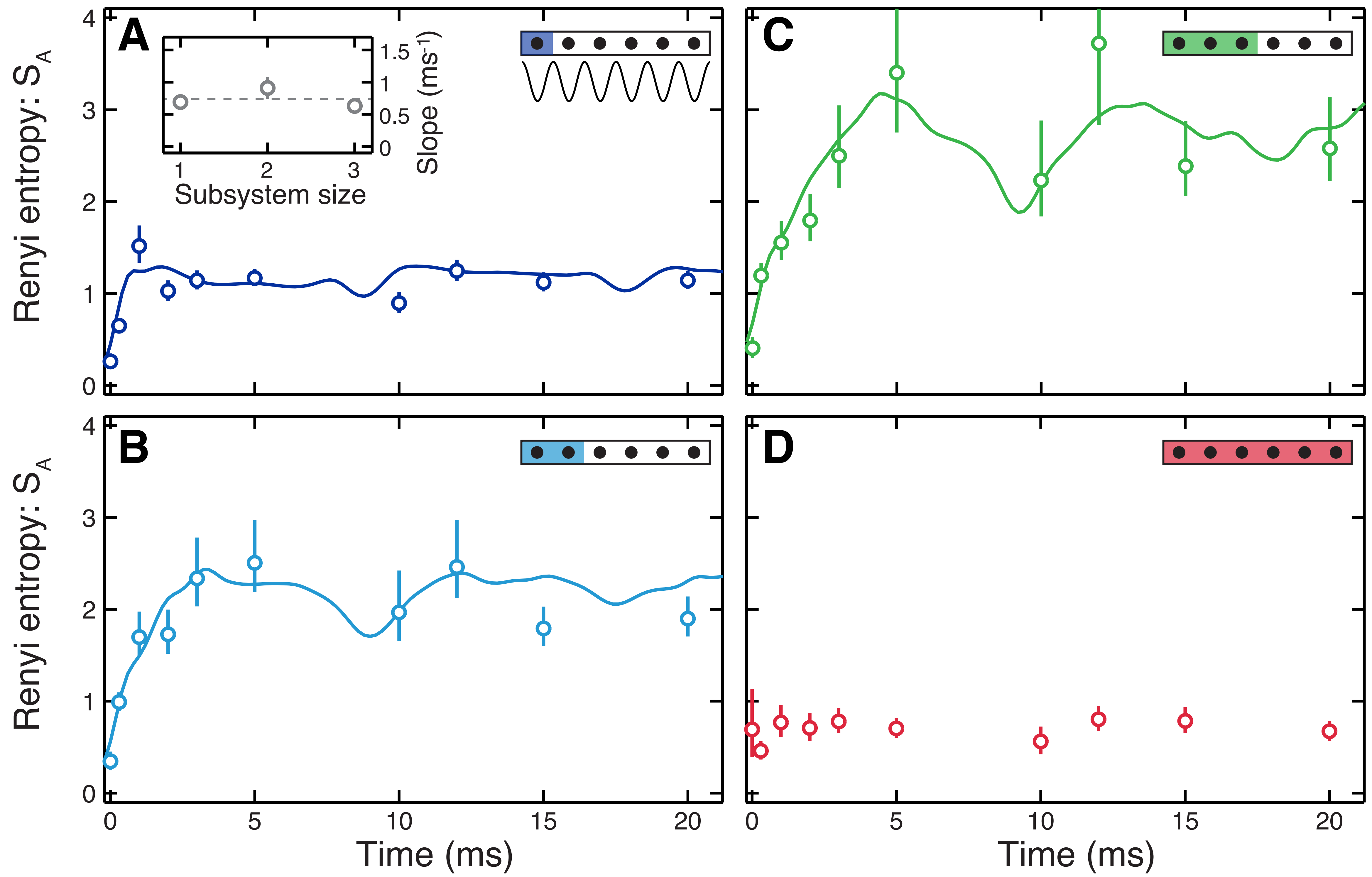
$$\langle\Psi|\hat{V}|\Psi\rangle = \mathrm{Tr}\hat{\rho}^2$$

$$\langle\Psi|\hat{V}_A \otimes \hat{I}_B|\Psi\rangle = \mathrm{Tr}_A \hat{\rho}_A^2$$

3. Measurement of particle parity

R. Islam *et al.*, Nature (2015); A. M. Kaufman *et al.*, Science (2016)





Motivation of this work

Bosons in an optical lattice

✔ Entanglement is experimentally accessible   Cf) Ion trap, Rydberg atoms: spin-1/2

✖ High computational cost: Few studies on entanglement dynamics[1]

Can we analytically solve the entanglement dynamics in this system?

1. **Non-interacting case** [Phys. Rev. A **107**, 033305 (2023)]

   This is the simplest but not so easy to solve, contrary to free fermions[2]

   (For typical states of Bose systems, an efficient technique for Gaussian states cannot be applied.)

   I. Peschel, J. Phys. A: Math. Gen. (2003)

2. **Strongly-interacting case** [arXiv:2209.13340]

   Low-energy excitations ≈ quasi-free fermions   M. Cheneau, *et al.,* Nature (2012)

[1] A.Flesch et al., PRA ('08), S.Goto and I. Danshita, PRB ('19), R. Yao and J.Zakrzewski, PRB ('20), M.Kunimi and I. Danshita, PRA ('21) etc.

[2] P. Calabrese, J. Cardy, JStatMech ('05), M.Fagotti, P.Calabrese, PRA ('08), I.Frerot, T. Roscilde, PRB ('15), V. Alba, P. Calabrese, SciPostPhys ('18), etc.

What we do in this work

Bosons in an optical lattice

described by Bose-Hubbard model
$$\hat{H} = -J\sum_{j=1}^{L-1}\left(\hat{b}_j^\dagger \hat{b}_{j+1} + \hat{b}_{j+1}^\dagger \hat{b}_j\right) + \frac{U}{2}\sum_{j=1}^{L}\hat{n}_j(\hat{n}_j - 1)$$

Initial states Ground state when $U/J \gg 1$, the Mott-insulating (MI) state and the charge-density-wave (CDW) state

※These are not Gaussian states for bosons

Quench protocol

1. **Non-interacting case** [Phys. Rev. A **107**, 033305 (2023)]

At $t = 0$,
We suddenly change
Hamiltonian parameters

2. **Strongly-interacting case** [arXiv:2209.13340]

We demonstrate our results in these simple setups.

Future plan: Application to interesting systems
e.g.) non-unitary dynamics, dissipative dynamics, etc…

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   -Model and quench protocol
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   Ref) Phys. Rev. A **107**, 033305 (2023)

3. Entanglement dynamics of strongly-interacting Bosons
   -Model and low-energy effective theory
   -Analytical result of Rényi entanglement entropy

   Ref) arXiv:2209.13340.

4. Summary

Model: Bosons in optical lattices

One-dimensional Bose-Hubbard model (open boundary)

$$\hat{H} = -J\sum_{j=1}^{L-1}\left(\hat{b}_j^\dagger \hat{b}_{j+1} + \hat{b}_{j+1}^\dagger \hat{b}_j\right) + \frac{U}{2}\sum_{j=1}^{L}\hat{n}_j(\hat{n}_j - 1) + \sum_{j=1}^{L}\Delta\varepsilon(-1)^{j+1}\hat{n}_j$$

$\hat{b}_j$: annihilation operator of boson

$\hat{n}_j$: number operator of boson

J: hopping amplitude

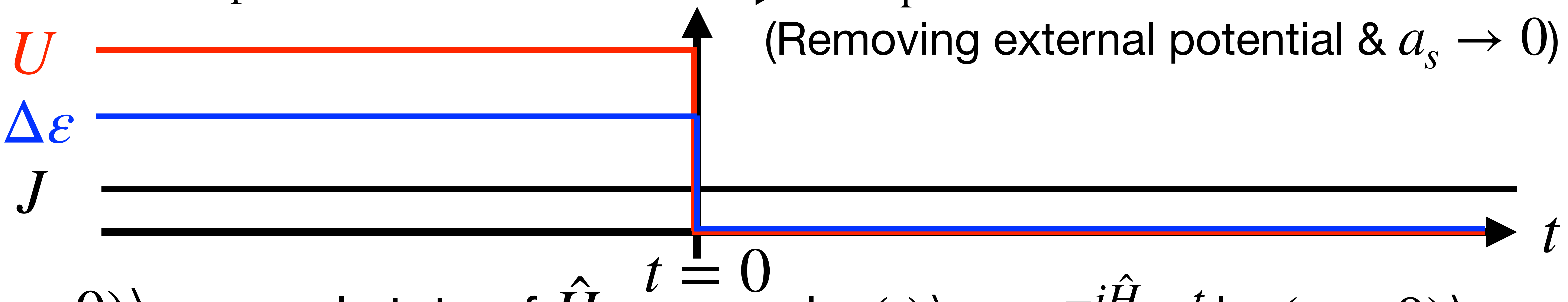
$\Delta\varepsilon$: one-body potential (external potential)

U: interaction strength (tunable via Feshbash resonance)

Quench protocol (zero-temperature)

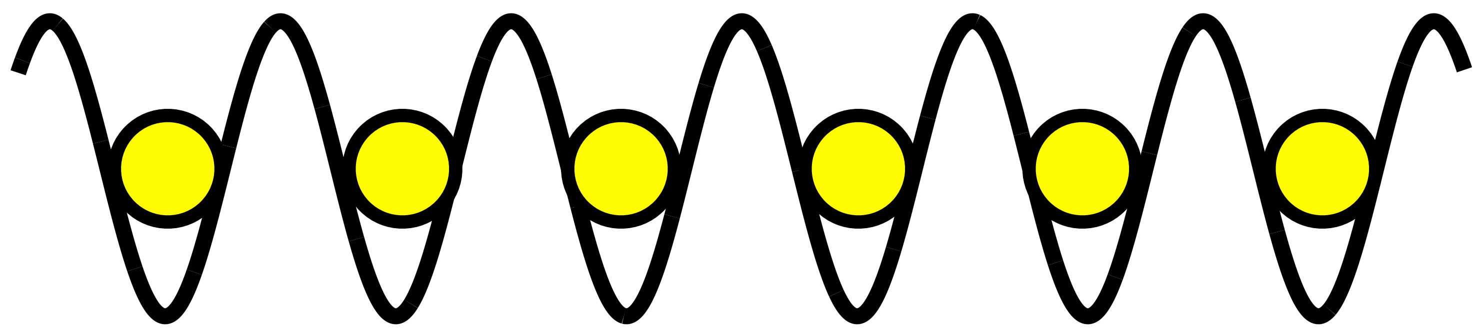
Hamiltonian: $\hat{H}_{\mathrm{pre}}(J, U, \Delta\varepsilon)$ → $\hat{H}_{\mathrm{post}}(J, U = 0, \Delta\varepsilon = 0)$

(Removing external potential & $a_s \to 0$)



state: $|\psi(t=0)\rangle$: ground state of $\hat{H}_{\mathrm{pre}}$ $|\psi(t)\rangle = e^{-i\hat{H}_{\mathrm{post}}t}|\psi(t=0)\rangle$

Quench dynamics

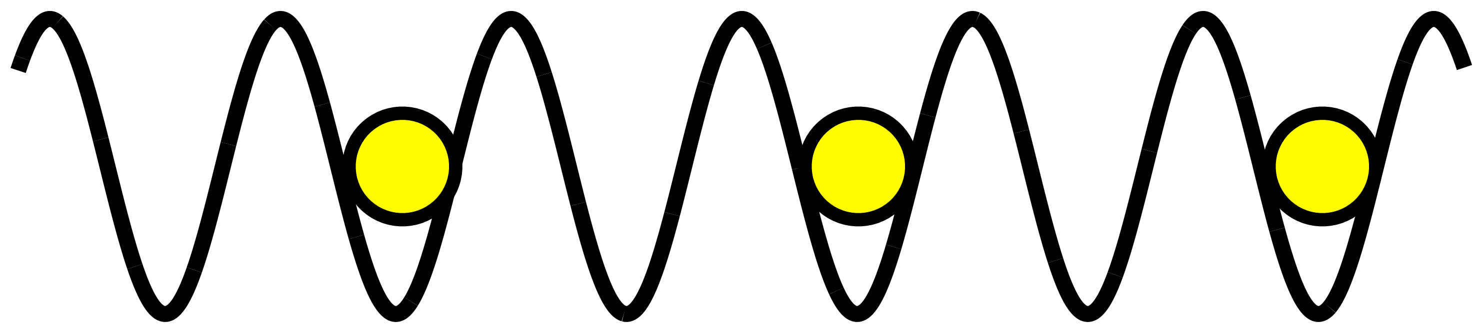


Initial states

1. Mott-insulating (MI) state
   [ground state when $U/J \gg 1,\ \Delta\varepsilon = 0$ and unit filling]

$$|\psi_{\rm MI}(t=0)\rangle = \prod_{j=1} \hat{b}_j^\dagger |0\rangle$$

2. 0101… charge-density-wave (CDW) state
   [ground state when $\Delta\varepsilon/J \gg 1$, $U/J \gg 1$, and half filling]



$$|\psi_{\rm CDW}(t=0)\rangle = \prod_{j=2,4,\dots} \hat{b}_j^\dagger |0\rangle$$

※These are **not** Gaussian states.
An efficient method for Gaussian states cannot be applied.
I. Peschel, J. Phys. A: Math. Gen. (2003)

Post-quench wave function

$$|\psi_{\rm MI}(t)\rangle = e^{-i\hat{H}_{\rm post}t}\prod_{j=1}\hat{b}_j^\dagger|0\rangle = \prod_{j=1}\left[e^{-i\hat{H}_{\rm post}t}\hat{b}_j^\dagger e^{i\hat{H}_{\rm post}t}\right]|0\rangle = \prod_{j=1}\left[\sum_{l=1}^{L} y_{j,l}(t)\hat{b}_l^\dagger\right]|0\rangle$$

$y_{j,l}(t)$: single-particle wave function at lth site from the state initially localized at jth site

Rényi entanglement entropy

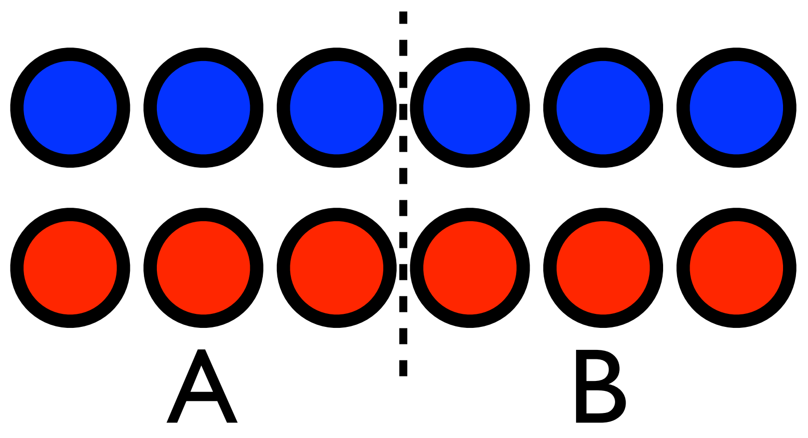
Rényi entanglement entropy in terms of shift operator

$$S_2 = -\ln \mathrm{Tr}_A \hat{\rho}_A^2 \quad \hat{\rho}_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$$

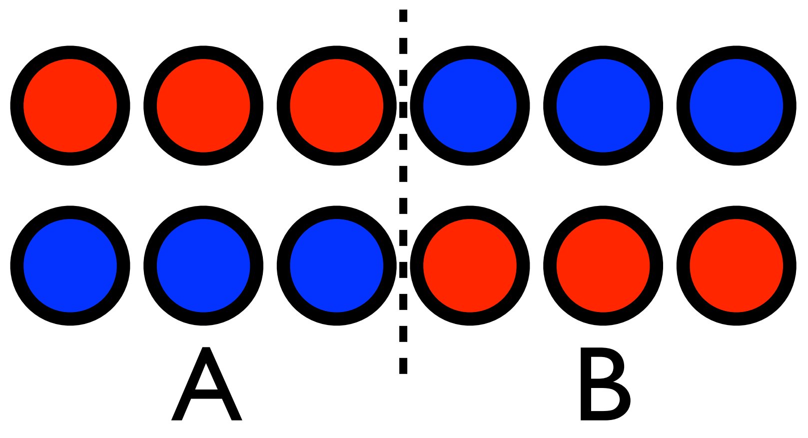
$$|\Psi_{\mathrm{copy}}(t)\rangle = |\psi(t)\rangle \otimes |\psi(t)\rangle$$

$\hat{V}_A$: swap wave functions in subsystem A

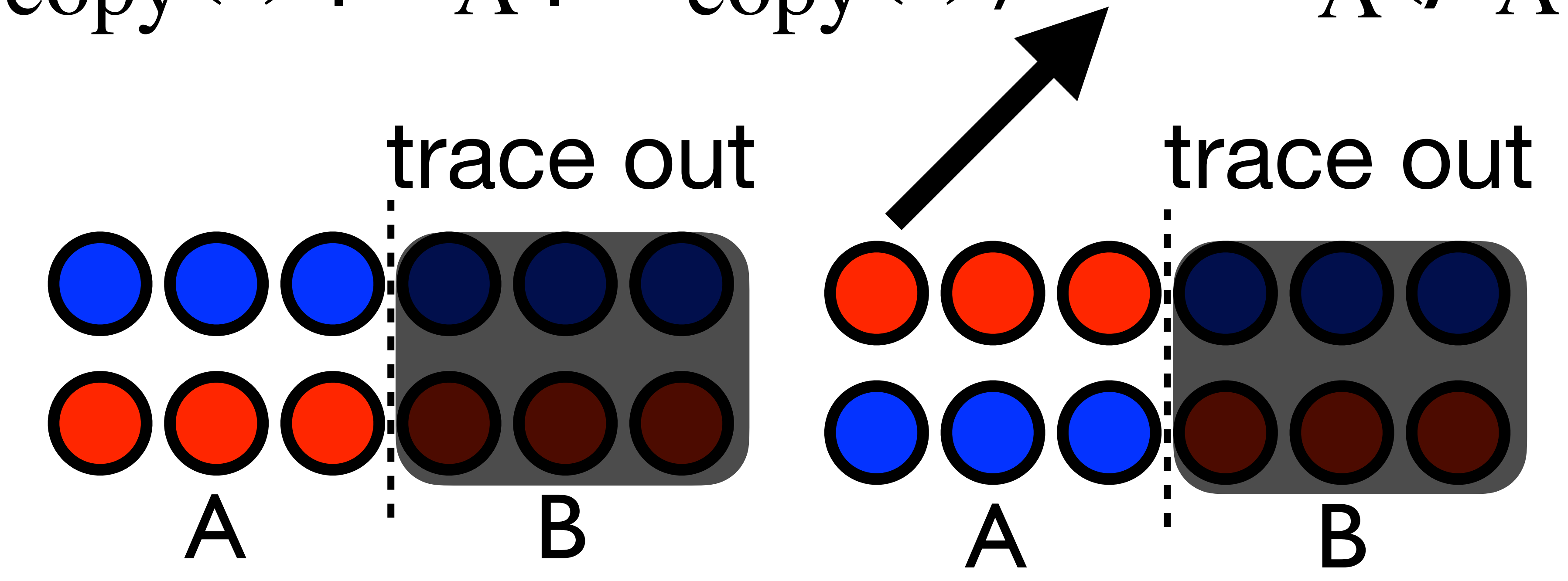
$|\Psi_{\mathrm{copy}}(t)\rangle =$

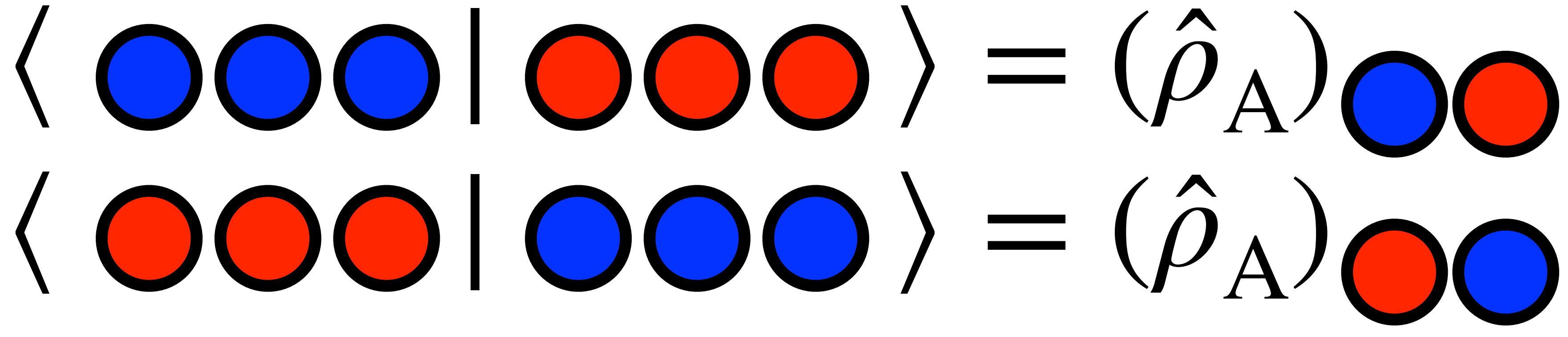


$\hat{V}_A |\Psi_{\mathrm{copy}}(t)\rangle =$



Key relation: $\langle\Psi_{\mathrm{copy}}(t)|\hat{V}_A|\Psi_{\mathrm{copy}}(t)\rangle = \mathrm{Tr}'_A(\hat{\rho}_A \otimes \hat{\rho}_A \hat{V}_A) = \mathrm{Tr}_A \hat{\rho}_A(t)^2$





※Tr'_A: trace over the basis of subsystem A of two copies

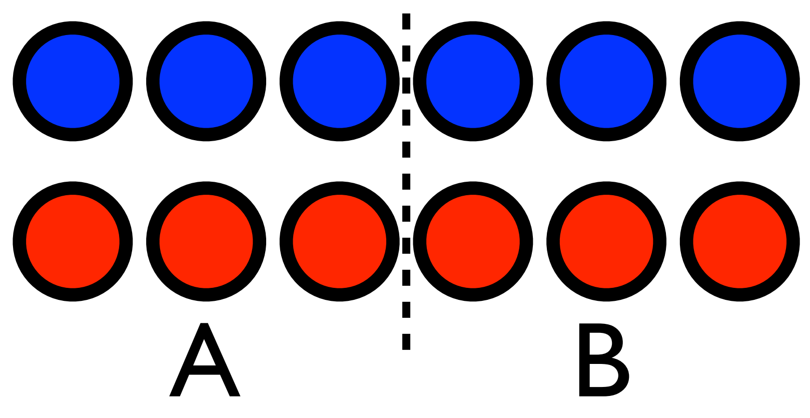
R. Islam *et al.*, Nature (2015)
A. M. Kaufman *et al.*, Science (2016)

Rényi entanglement entropy

Rényi entanglement entropy in terms of shift operator

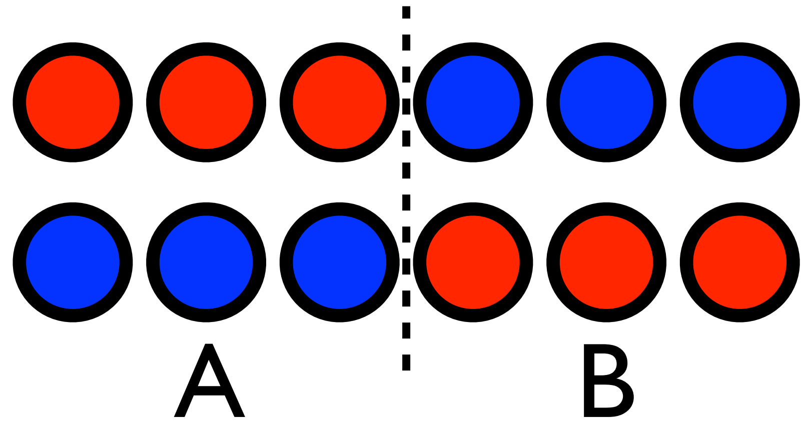
$$S_2 = -\ln\langle\Psi_{\rm copy}(t)\,|\,\hat{V}_{\rm A}\,|\,\Psi_{\rm copy}(t)\rangle$$

$$|\Psi_{\rm copy}(t)\rangle =$$



$$|\Psi^{\rm MI}_{\rm copy}(t)\rangle = \prod_{j=1}\left[\sum_{l=1}^{L} y_{j,l}(t)\hat{b}_l^\dagger\right]\left[\sum_{l=1}^{L} y_{j,l}(t)\hat{c}_l^\dagger\right]|0\rangle^{\otimes 2}$$

$$\hat{V}_{\rm A}\,|\,\Psi_{\rm copy}(t)\rangle =$$



Action of Shift operator: $\hat{V}_{\rm A}\hat{b}_l^\dagger\hat{V}_{\rm A}^{-1} = \begin{cases}\hat{c}_l^\dagger & (l\in{\rm A})\\ \hat{b}_l^\dagger & (l\in{\rm B})\end{cases}$, $\hat{V}_{\rm A}\hat{c}_l^\dagger\hat{V}_{\rm A}^{-1} = \begin{cases}\hat{b}_l^\dagger & (l\in{\rm A})\\ \hat{c}_l^\dagger & (l\in{\rm B})\end{cases}$

$$\hat{V}_{\rm A}\,|\,\Psi^{\rm MI}_{\rm copy}(t)\rangle = \prod_{j=1}\left[\sum_{l\in{\rm A}} y_{j,l}(t)\hat{c}_l^\dagger + \sum_{l\in{\rm B}} y_{j,l}(t)\hat{b}_l^\dagger\right]\left[\sum_{l\in{\rm A}} y_{j,l}(t)\hat{b}_l^\dagger + \sum_{l\in{\rm B}} y_{j,l}(t)\hat{c}_l^\dagger\right]|0\rangle^{\otimes 2}$$

Rényi entanglement entropy

Rényi entanglement entropy in terms of shift operator

$$S_2 = -\ln\langle\Psi_{\mathrm{copy}}(t)|\hat{V}_{\mathrm{A}}|\Psi_{\mathrm{copy}}(t)\rangle$$

$$|\Psi^{\mathrm{MI}}_{\mathrm{copy}}(t)\rangle = \prod_{j=1}\left[\sum_{l=1}^{L} y_{j,l}(t)\hat{b}^{\dagger}_{l}\right]\left[\sum_{l=1}^{L} y_{j,l}(t)\hat{c}^{\dagger}_{l}\right]|0\rangle^{\otimes 2}$$

$$\hat{V}_{\mathrm{A}}|\Psi^{\mathrm{MI}}_{\mathrm{copy}}(t)\rangle = \prod_{j=1}\left[\sum_{l\in\mathrm{A}} y_{j,l}(t)\hat{c}^{\dagger}_{l} + \sum_{l\in\mathrm{B}} y_{j,l}(t)\hat{b}^{\dagger}_{l}\right]\left[\sum_{l\in\mathrm{A}} y_{j,l}(t)\hat{b}^{\dagger}_{l} + \sum_{l\in\mathrm{B}} y_{j,l}(t)\hat{c}^{\dagger}_{l}\right]|0\rangle^{\otimes 2}$$

$|\Psi^{\mathrm{MI}}_{\mathrm{copy}}(t)\rangle$, $\hat{V}_{\mathrm{A}}|\Psi^{\mathrm{MI}}_{\mathrm{copy}}(t)\rangle$: many-boson states → overlap = matrix permanent

$$\langle\Psi(t)|\hat{V}_{\mathrm{A}}|\Psi(t)\rangle = \mathrm{perm}\begin{pmatrix} Z' & Z \\ Z & Z' \end{pmatrix}$$

$$Z'_{j,j'} = \langle 0|^{\otimes 2}\left[\sum_{l=1}^{L} y^{*}_{j,l}(t)\hat{b}_{l}\right]\left[\sum_{l\in\mathrm{A}} y_{j',l}(t)\hat{c}^{\dagger}_{l} + \sum_{l\in\mathrm{B}} y_{j',l}(t)\hat{b}^{\dagger}_{l}\right]|0\rangle^{\otimes 2}$$

$$Z_{j,j'} = \langle 0|^{\otimes 2}\left[\sum_{l=1}^{L} y^{*}_{j,l}(t)\hat{b}_{l}\right]\left[\sum_{l\in\mathrm{A}} y_{j',l}(t)\hat{b}^{\dagger}_{l} + \sum_{l\in\mathrm{B}} y_{j',l}(t)\hat{c}^{\dagger}_{l}\right]|0\rangle^{\otimes 2}$$

e.g.) permanent of 2×2 matrix $\mathrm{perm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + bc$

Similar to the determinant but no minus signs

Rényi entanglement entropy

Rényi entanglement entropy in terms of shift operator

$$S_2 = -\ln\langle\Psi_{\rm copy}(t)|\hat{V}_{\rm A}|\Psi_{\rm copy}(t)\rangle$$

$$|\Psi^{\rm MI}_{\rm copy}(t)\rangle = \prod_{j=1}\left[\sum_{l=1}^{L} y_{j,l}(t)\hat{b}^\dagger_l\right]\left[\sum_{l=1}^{L} y_{j,l}(t)\hat{c}^\dagger_l\right]|0\rangle^{\otimes 2}$$

$$\hat{V}_{\rm A}|\Psi^{\rm MI}_{\rm copy}(t)\rangle = \prod_{j=1}\left[\sum_{l\in{\rm A}} y_{j,l}(t)\hat{c}^\dagger_l + \sum_{l\in{\rm B}} y_{j,l}(t)\hat{b}^\dagger_l\right]\left[\sum_{l\in{\rm A}} y_{j,l}(t)\hat{b}^\dagger_l + \sum_{l\in{\rm B}} y_{j,l}(t)\hat{c}^\dagger_l\right]|0\rangle^{\otimes 2}$$

$|\Psi^{\rm MI}_{\rm copy}(t)\rangle$, $\hat{V}_{\rm A}|\Psi^{\rm MI}_{\rm copy}(t)\rangle$: many-boson states ➡ overlap = matrix permanent

Main Result

$$S_2 = -\ln {\rm perm} A_Z,\ A_Z = \begin{pmatrix} I-Z & Z \\ Z & I-Z \end{pmatrix},\ z_{j,l} = \sum_{m\in{\rm A}} y^*_{j,m}(t)y_{l,m}(t)$$

This analytical expression offers many useful information.

Volume-law scaling condition

$$S_2 = -\ln \mathrm{perm} A_Z,\ A_Z = \begin{pmatrix} I-Z & Z \\ Z & I-Z \end{pmatrix},\ z_{j,l} = \sum_{m \in \mathrm{A}} y^*_{j,m}(t) y_{l,m}(t)$$

<u>Condition for the volume-law entanglement scaling</u>

※ We consider half-chain entanglement

Permanent inequality

R. Berkowitz *et al.*, Israel J. Math (2018)

M: matrix size of A

$$|\mathrm{perm} A| \leq \exp\left[10^{-5} \times [1 - g_A(M)]^2 \times M\right]$$

$$g_A(M) = \frac{1}{M} \sum_{j=1}^{M} \max_{l=1}^{M} \left|A_{j,l}\right|$$

Combined with our results

$$S_2 \geq 10^{-5} \times \left[1 - g_{A_Z}(2N)\right]^2 \times (2N)$$

$M = 2N$; N: particle number

If $\lim_{L \to \infty} [1 - g_{A_Z}(2N)] \neq 0$, $S_2 \propto N \propto L$

volume-law scaling condition

Volume-law scaling condition: examples

Condition for the volume-law entanglement scaling

$$\lim_{L\to\infty}[1 - g_{A_Z}(2N)] \neq 0 \qquad g_A(M) = \frac{1}{M}\sum_{j=1}^{M}\max_{l=1}^{M}\left|A_{j,l}\right|$$

M: matrix size of A

Example 1: $|\psi_{\mathrm{MI}}\rangle = \prod_{j=1} \hat{b}_j^\dagger |0\rangle$

$(L=6) \qquad = |1,1,1,1,1,1\rangle$

$$A_Z = \begin{pmatrix} 0_{N/2} & 0_{N/2} & I_{N/2} & 0_{N/2} \\ 0_{N/2} & I_{N/2} & 0_{N/2} & 0_{N/2} \\ I_{N/2} & 0_{N/2} & 0_{N/2} & 0_{N/2} \\ 0_{N/2} & 0_{N/2} & 0_{N/2} & I_{N/2} \end{pmatrix}$$

$1 - g_{A_Z}(2N) = 0 \qquad S_2 = 0$

(Violate the condition)

Example 2: $|\psi\rangle = \prod_{j=1}^{L/2} \frac{\hat{b}_j^\dagger + \hat{b}_{L-j+1}^\dagger}{\sqrt{2}} |0\rangle$

$$(L=6) \qquad = \frac{1}{\sqrt{8}}\begin{pmatrix} |1,1,1,0,0,0\rangle + |1,1,0,1,0,0\rangle + |1,0,1,0,1,0\rangle \\ +|1,0,0,1,1,0\rangle + |0,1,1,0,0,1\rangle + |0,1,0,1,0,1\rangle \\ +|0,0,1,0,1,1\rangle + |0,0,0,1,1,1\rangle \end{pmatrix}$$

$A_Z = \frac{1}{2} I_{2N} \qquad 1 - g_{A_Z}(2N) = \frac{1}{2}$

(satisfy the condition)

$$S_2 = \frac{L}{2}\ln(2) \propto L$$

Advantage of numerical computation

$$S_2 = -\ln \mathrm{perm} A_Z,\ A_Z = \begin{pmatrix} I-Z & Z \\ Z & I-Z \end{pmatrix},\ z_{j,l} = \sum_{m\in \mathrm{A}} y^*_{j,m}(t) y_{l,m}(t)$$

	Our approach		Exact diagonalization (ED)
Memory cost	$\mathcal{O}(NL)$	$\ll$	$\mathcal{O}({}_{L+N-1}C_N) \sim \mathcal{O}(2^{2L}/\sqrt{L})$
Computational cost	$\mathcal{O}((2N)2^{2N-1})$	$\lesssim$	$\mathcal{O}((\text{size of } \hat{\rho}_{\mathrm{A}})^2) \sim \mathcal{O}(2^{2.75L}/L)$

$(N = L)$

Large systems could be handled?

Balasubramanian-Bax-Fradklin-Glynn (BBFG) Formula

$$\mathrm{perm} A = \frac{1}{2^{n-1}} \sum_{\vec{\delta}} \left(\prod_{k=1}^{M} \delta_k \right) \prod_{j=1}^{M} \sum_{k=1}^{M} \delta_k a_{k,j}$$

$$\vec{\delta} = (\delta_1, \delta_2, \ldots, \delta_M),\ \delta_1 = 1,\ \delta_j = \pm 1\ (j = 2,3\ldots, M)$$

We use the (naive) BBFG formula.

It takes ~30 hours for $N = 20$.

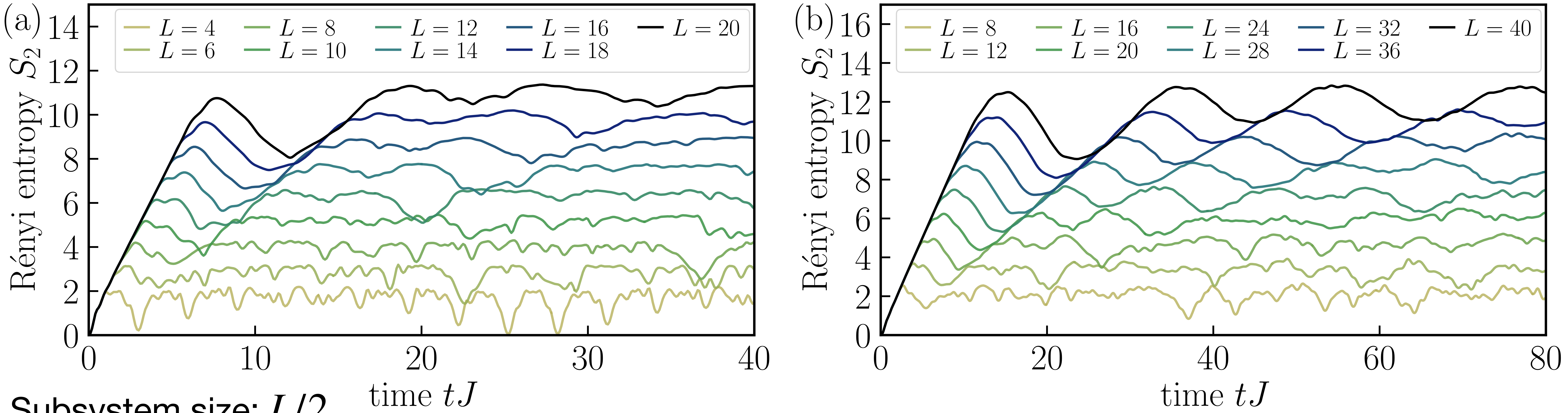
There would be more efficient ways.

E.g.) Utilizing symmetry

Parallel computation, etc.

Time dependence of Rényi entanglement entropy

MI initial state | CDW initial state



Subsystem size: $L/2$

Larger systems can be handled!

cf. $L = 14$ by Exact diagonalization S. Goto and I. Danshita, PRB (2019)
for a $U = 3.01$ quench from the MI state

Dimension of Hilbert space D

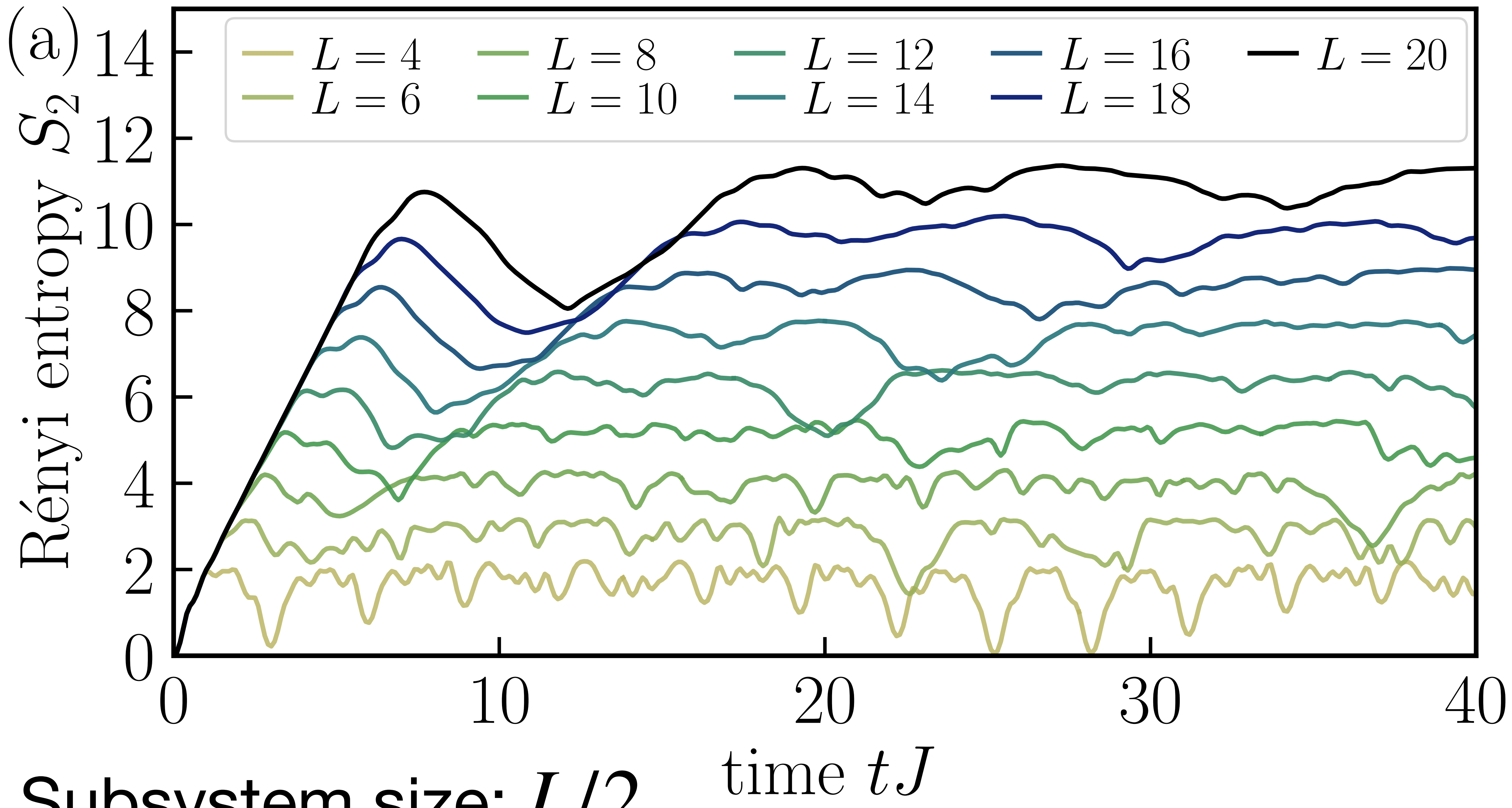
MI state, $L = 14,\ D \approx 2^{24}$

$L = 20,\ D \approx 2^{36}$

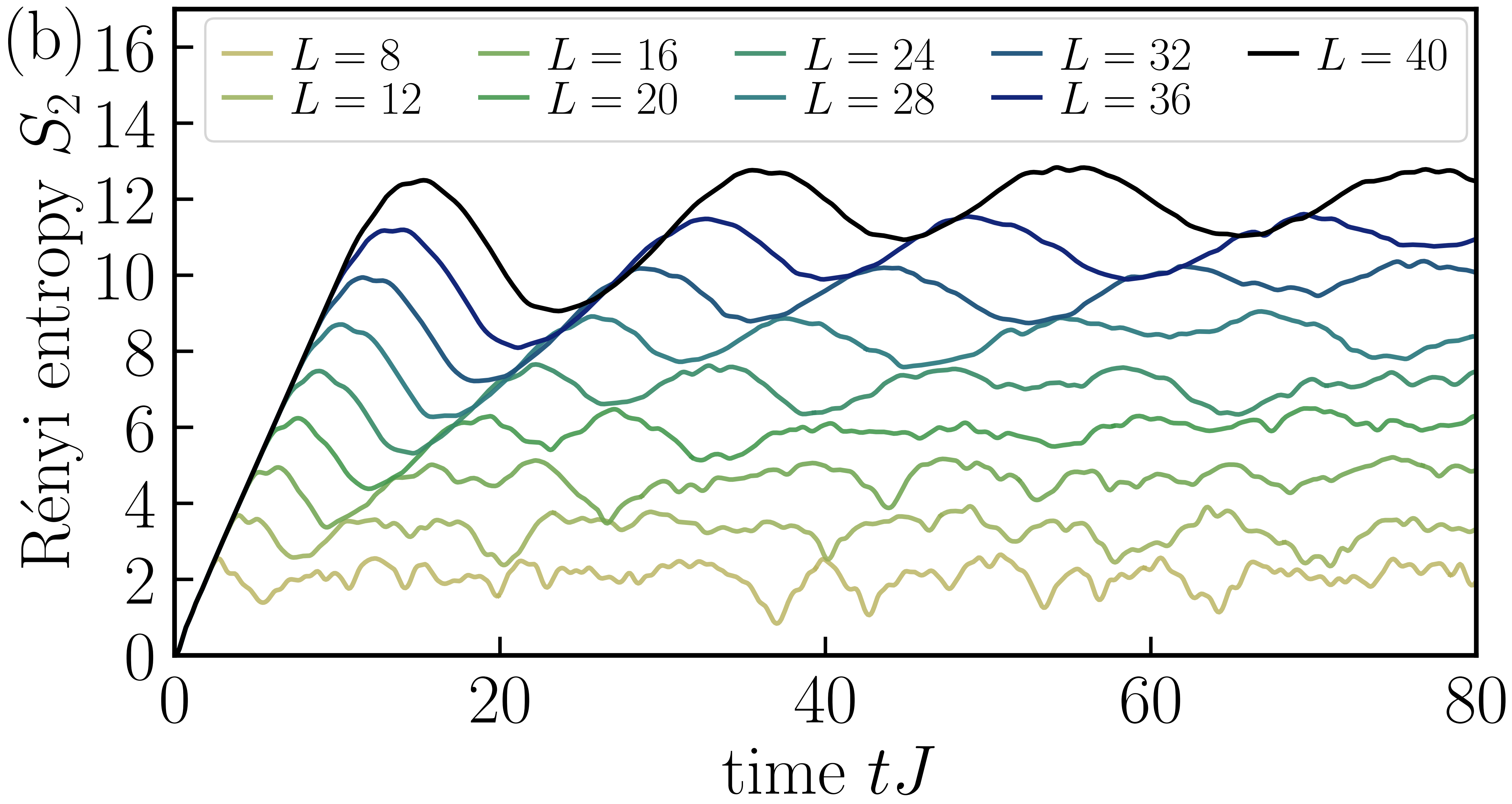
CDW state, $L = 40,\ D \approx 2^{51}$

Time dependence of Rényi entanglement entropy

MI initial state



CDW initial state



Subsystem size: $L/2$

1. $tJ \lesssim L/4$, t-linear growth
2. $tJ \gtrsim L/4$, nearly saturation at an $\mathcal{O}(L)$ value

Typical Behavior of quench dynamics

Generally, t-linear growth terminates at $t \sim L/v_{\mathrm{max}}$ and $v_{\mathrm{max}} = 2J$ in the present case.

(v_{max}: maximal quasiparticle velocity)

V. Alba and P. Calabrese, PNAS (2017)

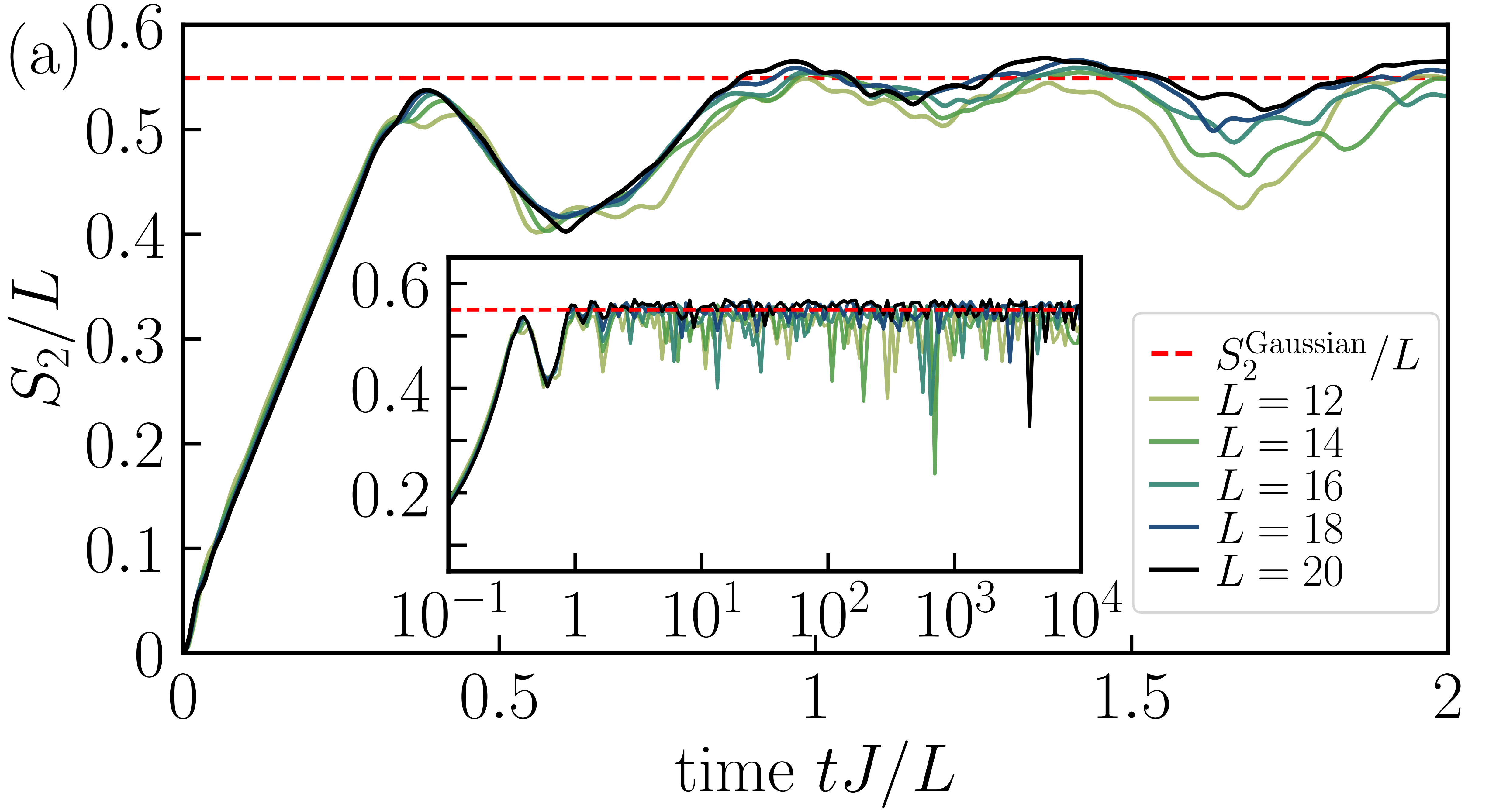
Comparison with Gaussian states

M. Cramer *et al.*, PRL (2008)

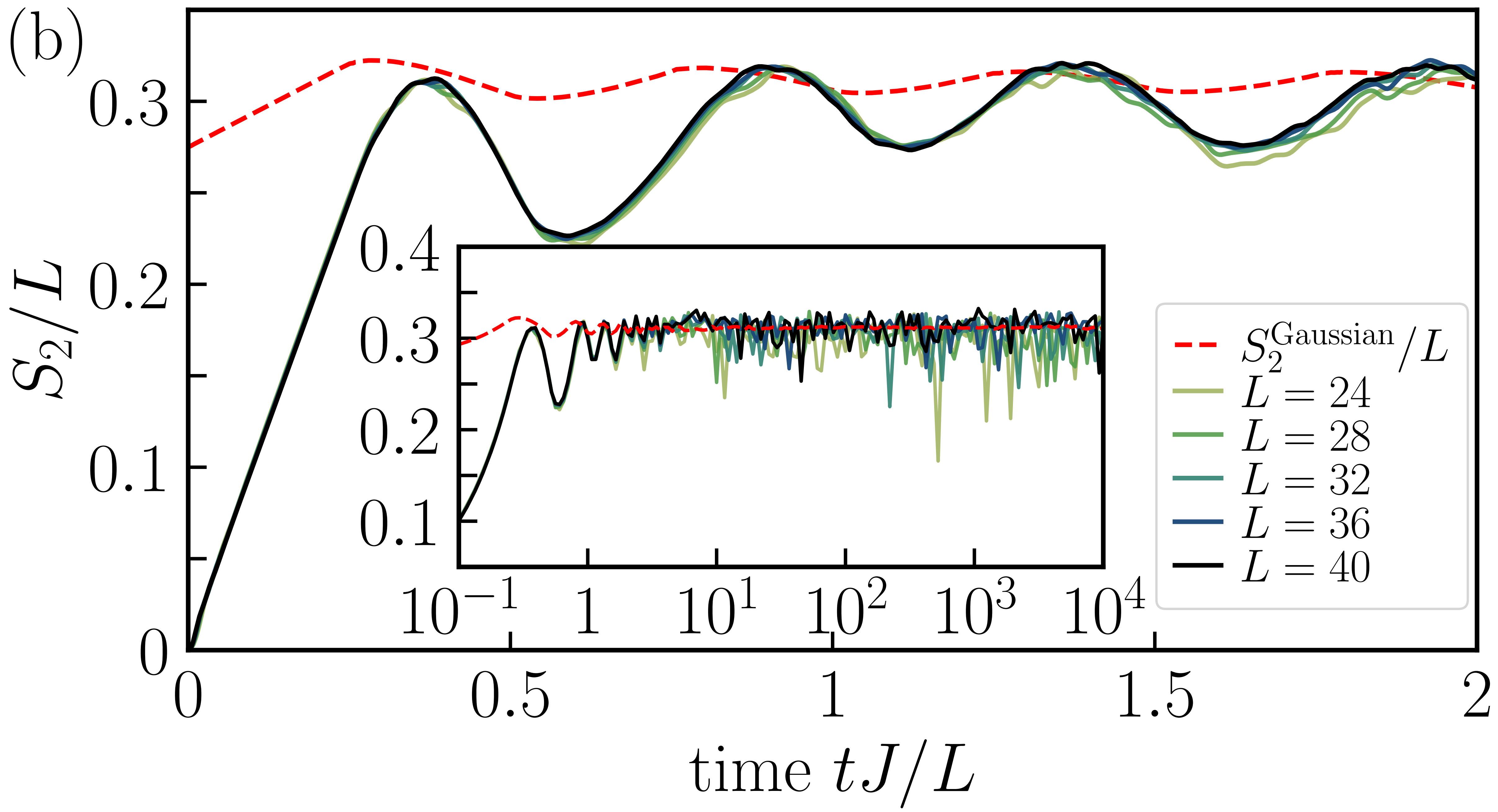
$U = 0$ quench: MI and CDW states relax to a Gaussian state in the thermodynamic limit

Gaussian state: characterized by $\langle \hat{b}_j^\dagger \hat{b}_l \rangle$ $\quad S_2 = \sum_\mu \ln \left[(n_\mu + 1)^2 - n_\mu^2 \right]$ $\quad n_\mu$: eigenvalue of $\langle \hat{b}_j^\dagger \hat{b}_l \rangle$

MI initial state



CDW initial state



S_2 converges to S_2^{Gaussian}.

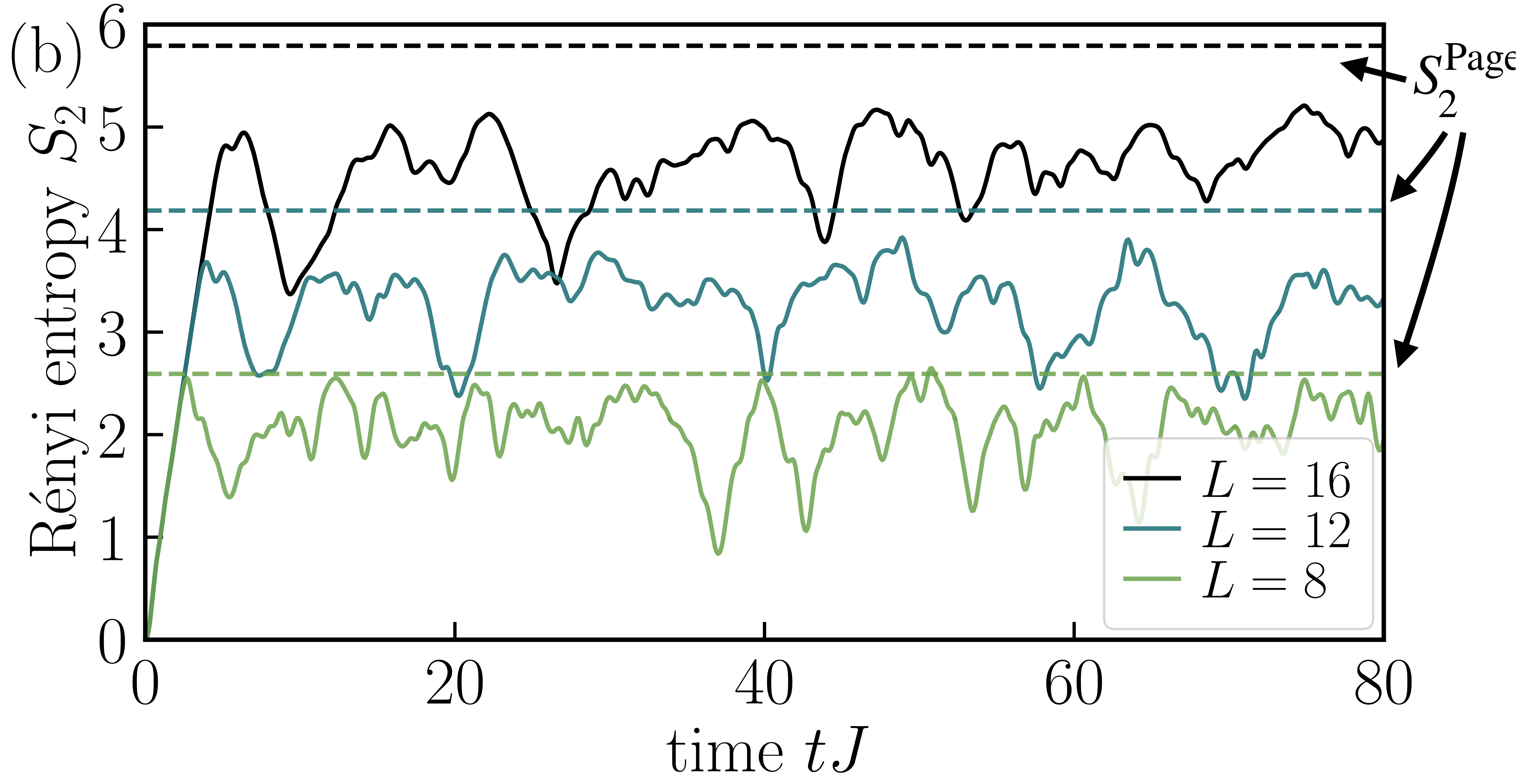
Comparison with Page values (CDW state)

$U > 0$ quench: S_2 converges to the Page value (Thermalization)

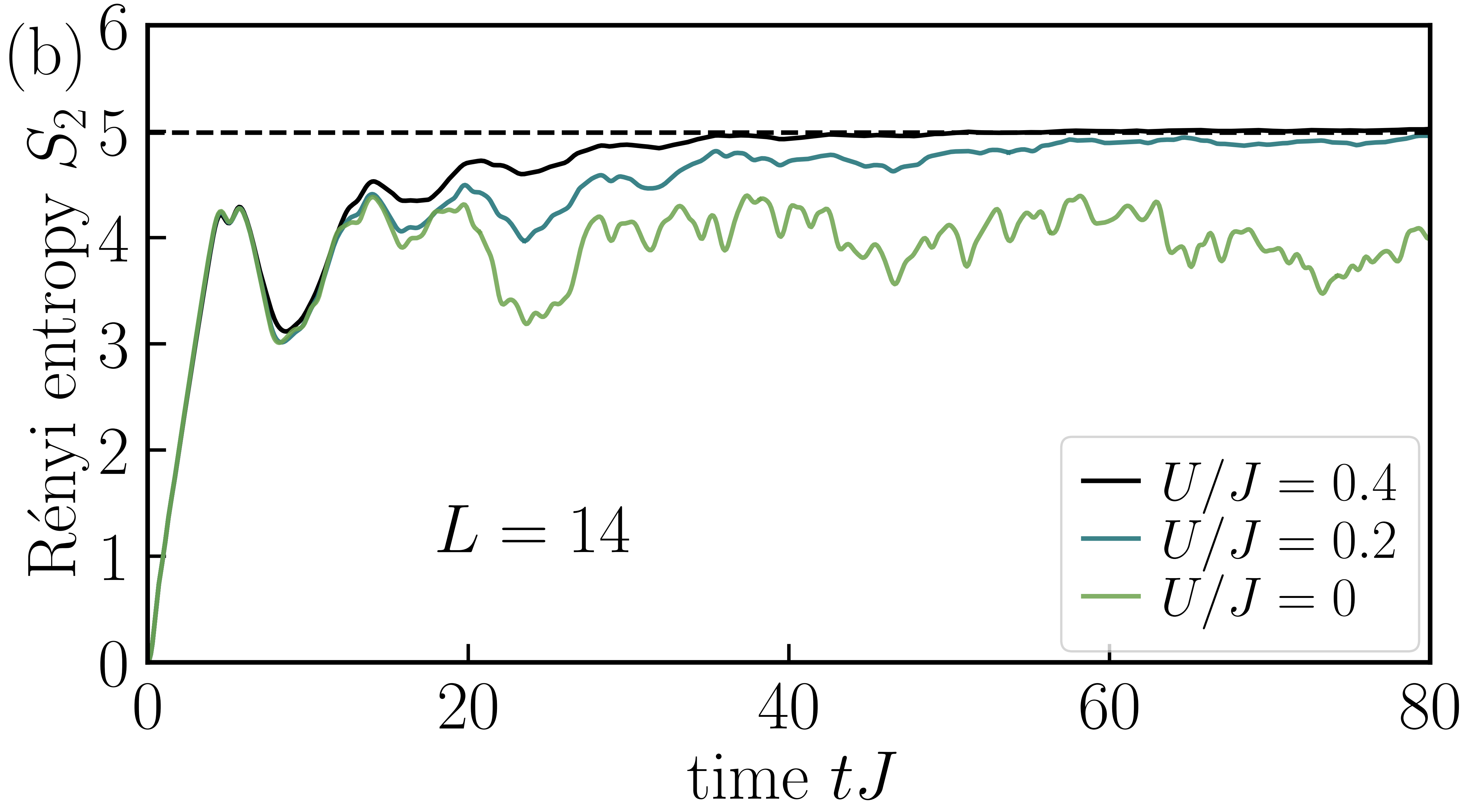
Page value: entropy of random states (obtained by averaging 1024 samples)

D. N. Page, PRL (1973)

M. Kunimi and I. Danshita, PRA (2021)



$U = 0$ quench (integrable): no thermalization

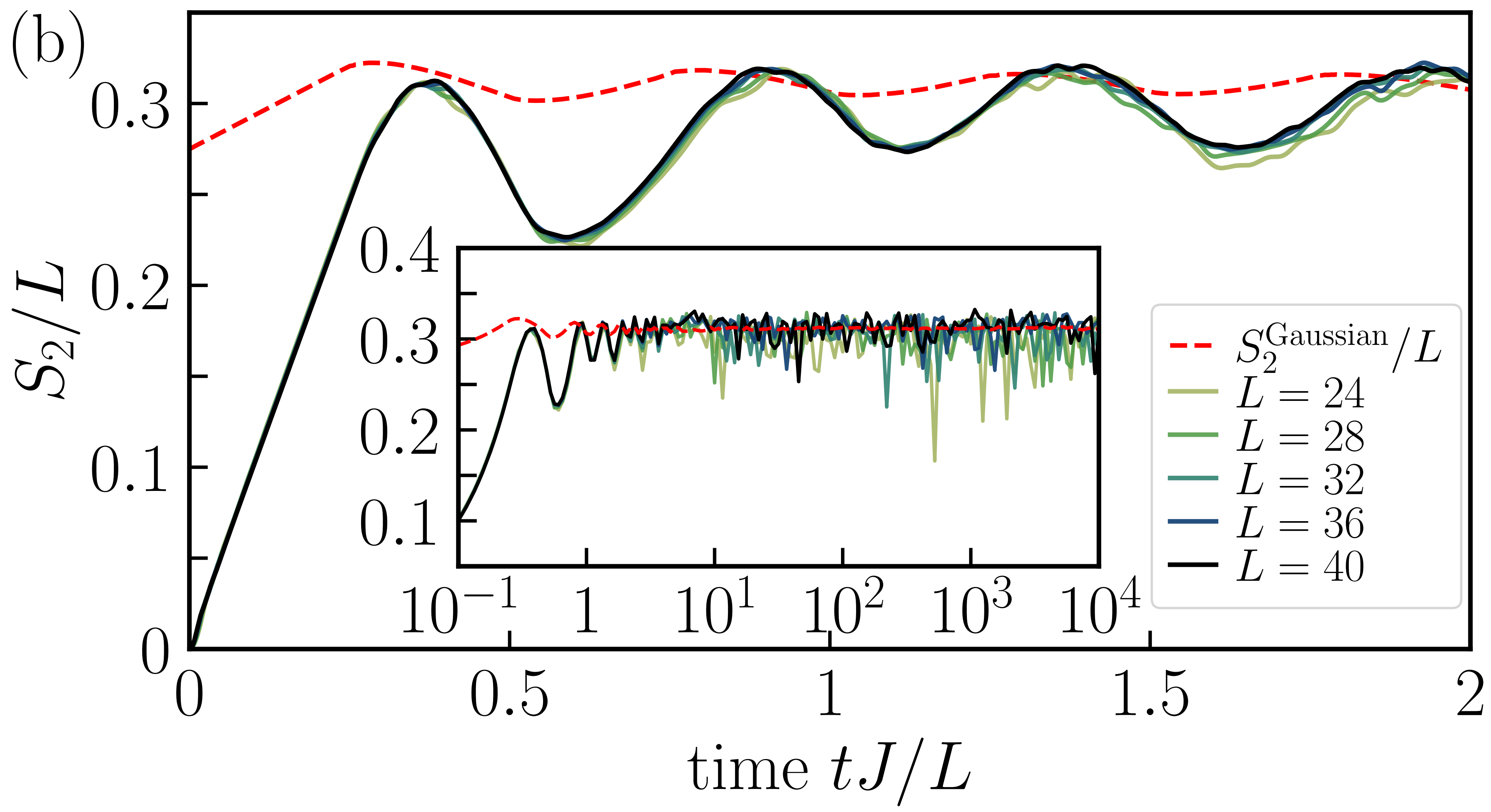


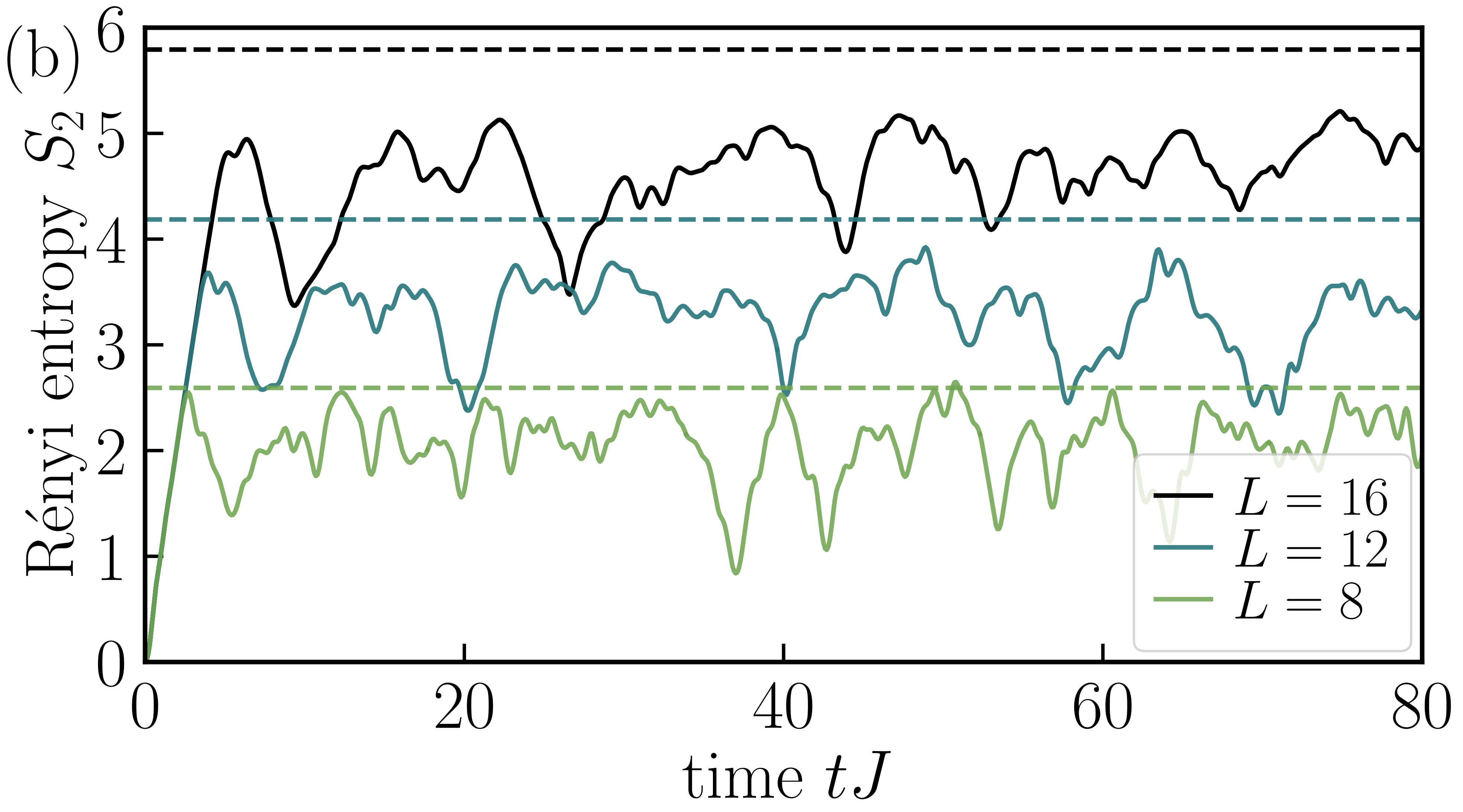
$U > 0$ quench (nonintegrable): thermalization

$$S_2(t \to \infty) < S_2^{\mathrm{Page}}$$

$S_2(t \to \infty)$ obeys the volume law but it is not a thermal one because we are considering an integrable case.

Summary of numerical result part





Our method well reproduces
the well-known results.

- t-linear growth in a short time
- Saturation of S_2 at an $\mathcal{O}(L)$ value (volume-law scaling)
- Converged S_2 is not a thermal one, S_2^{Page}.

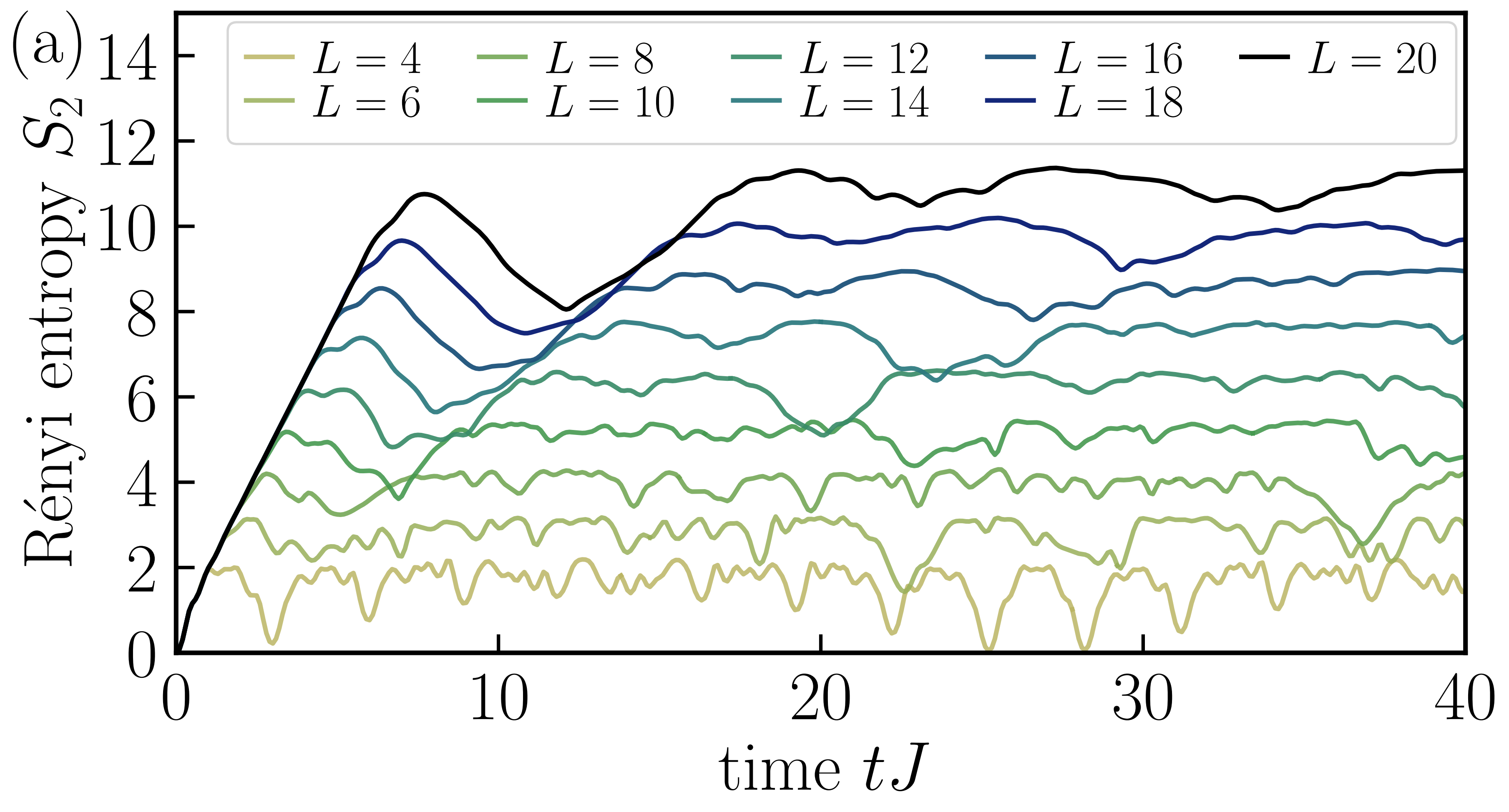
We believe our method
would be useful in studying
more interesting situations.

Summary of part 1

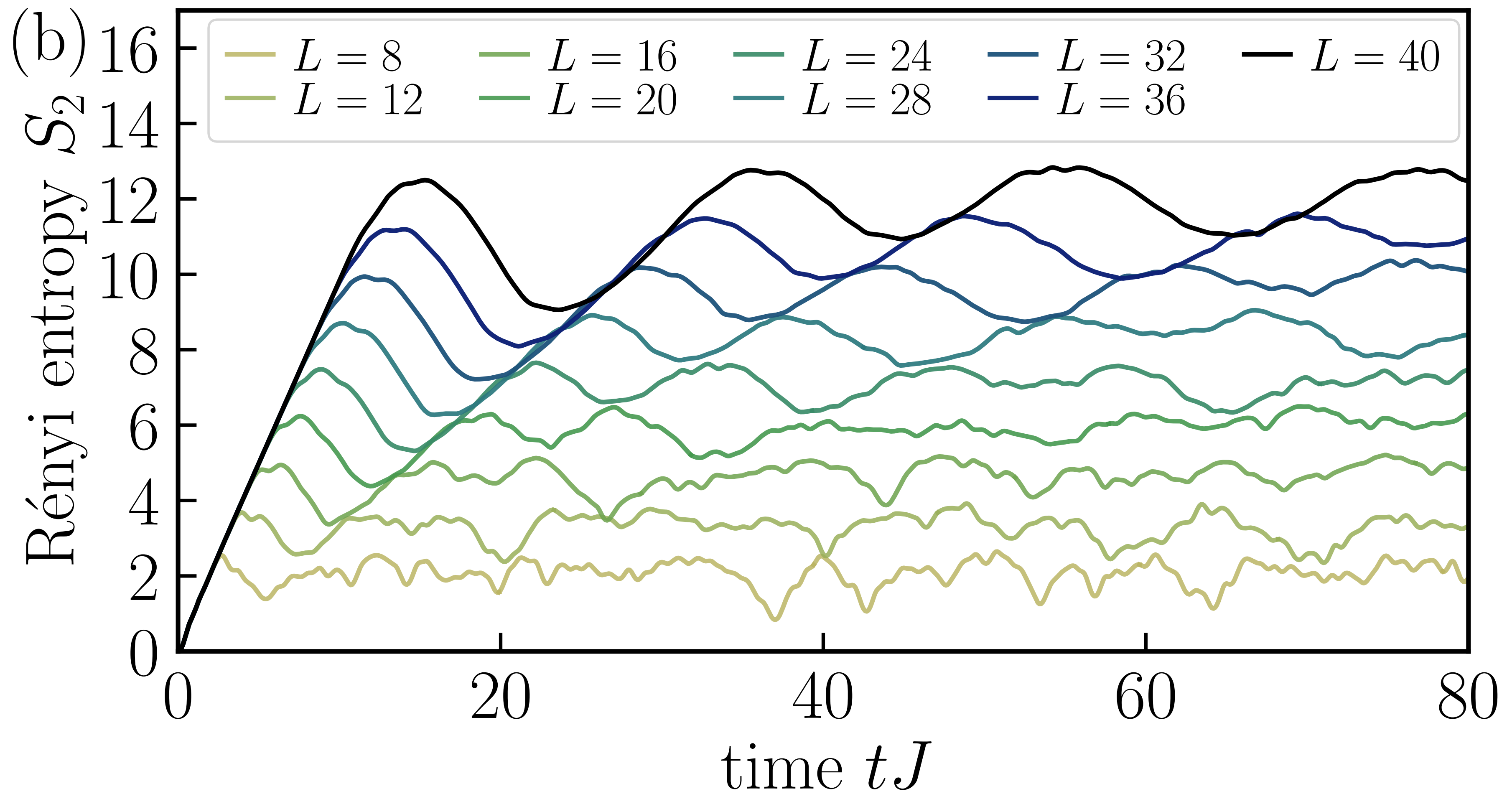
Ref) Phys. Rev. A **107**, 033305 (2023)

- We have investigated Rényi entanglement entropy in 1D bosons on an optical lattice quenched to $U = 0$ from insulating states.
- We have obtained an analytical expression for Rényi entanglement entropy and the condition for the volume-law entanglement scaling.
- We have performed a numerical evaluation of S_2 of large systems.

MI initial state



CDW initial state



Future Problem

Our approach would be applicable to many situations

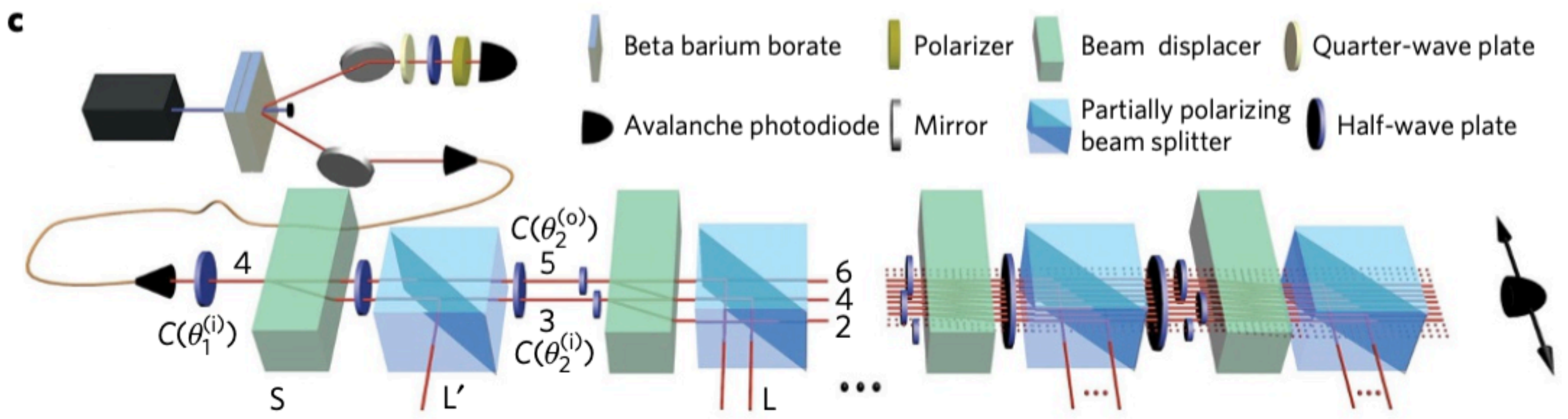
- Non-Hermitian systems (can be applied)

  DK, R. Kaneko, K. Mochizuki, and I. Danshita, in preparation
- Dissipative (Lindblad) dynamics of free bosons
- Larger systems such as in higher dimension

Rényi entanglement entropy of non-Hermitian systems

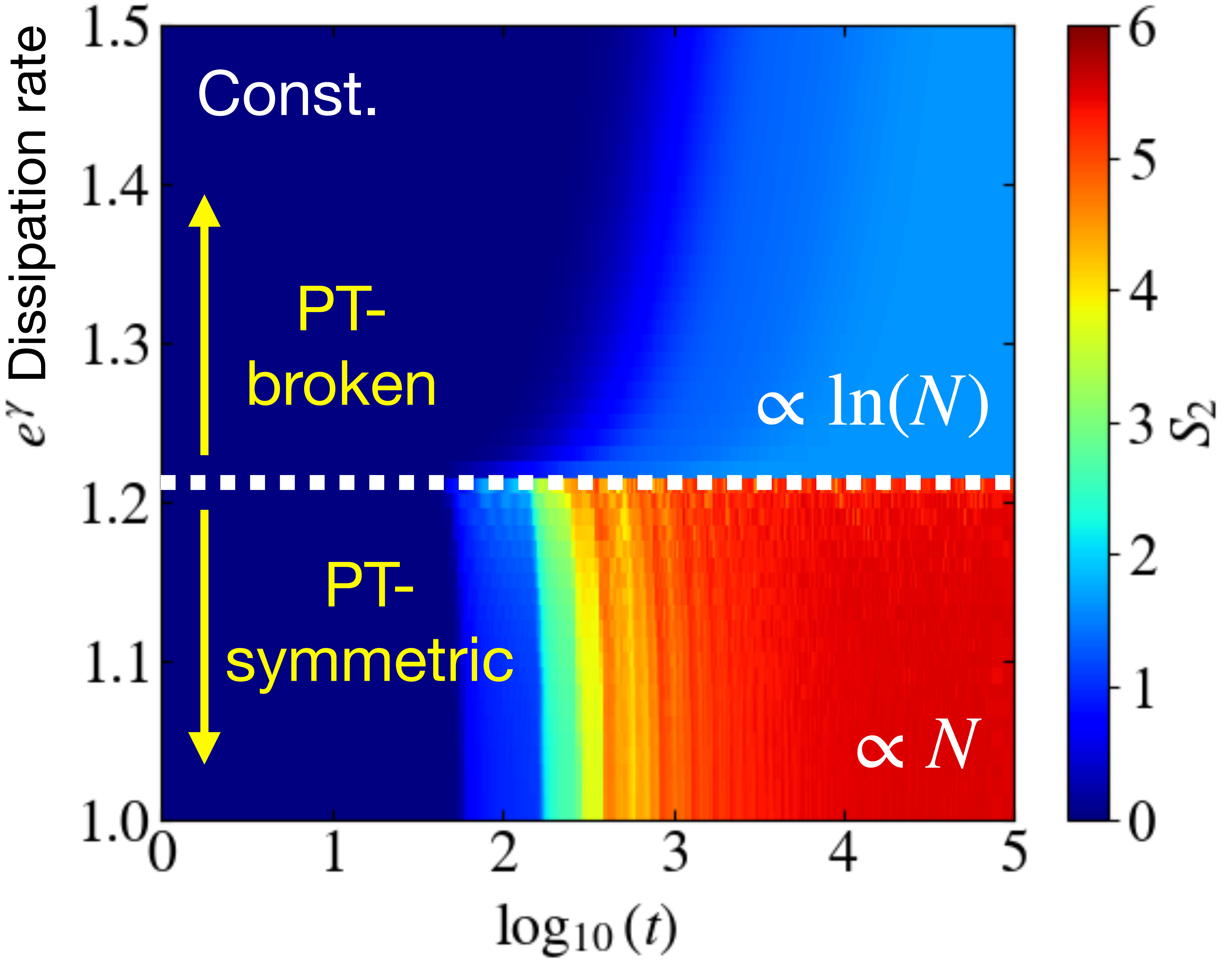
Photons do not have interactions ➡ Many-body non-interacting Bose system
Our method is applicable!

The parity-time-reversal (PT) symmetric non-Hermite photonics system was realized.



L. Xiao, *et al.*, Nat. Phys. (2017)

K. Mochizuki and R. Hamazaki, PRR **5**, 013177 (2023)



We can analytically compute S_2.

DK, R. Kaneko, K. Mochizuki, and I. Danshita, in preparation

Outline

1. Introduction
   -Brief review of entanglement
   -Measurement of entanglement entropy
   -Motivation of this study

2. Entanglement dynamics of non-interacting Bosons
   -Model and quench protocol
   -Analytical result of Rényi entanglement entropy
   -Numerical results

   Ref) Phys. Rev. A **107**, 033305 (2023)

3. **Entanglement dynamics of strongly-interacting Bosons**
   -Model and low-energy effective theory
   -Analytical result of Rényi entanglement entropy

   Ref) arXiv:2209.13340.

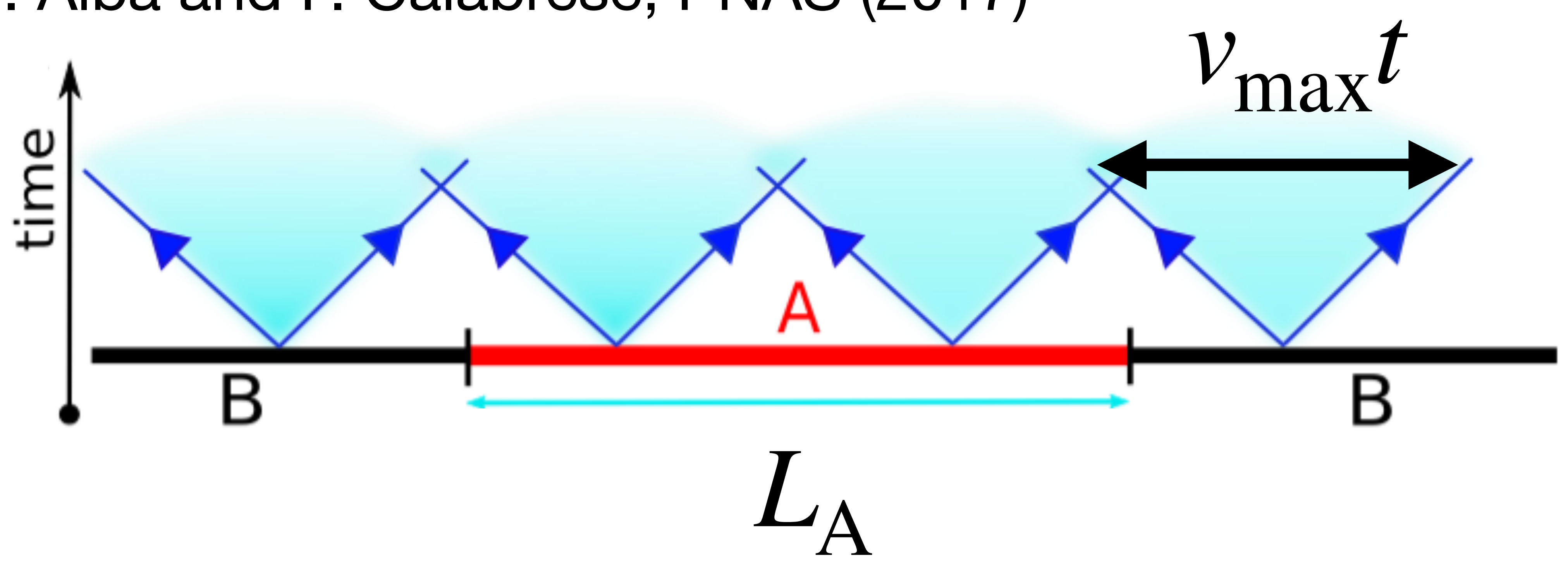
4. Summary

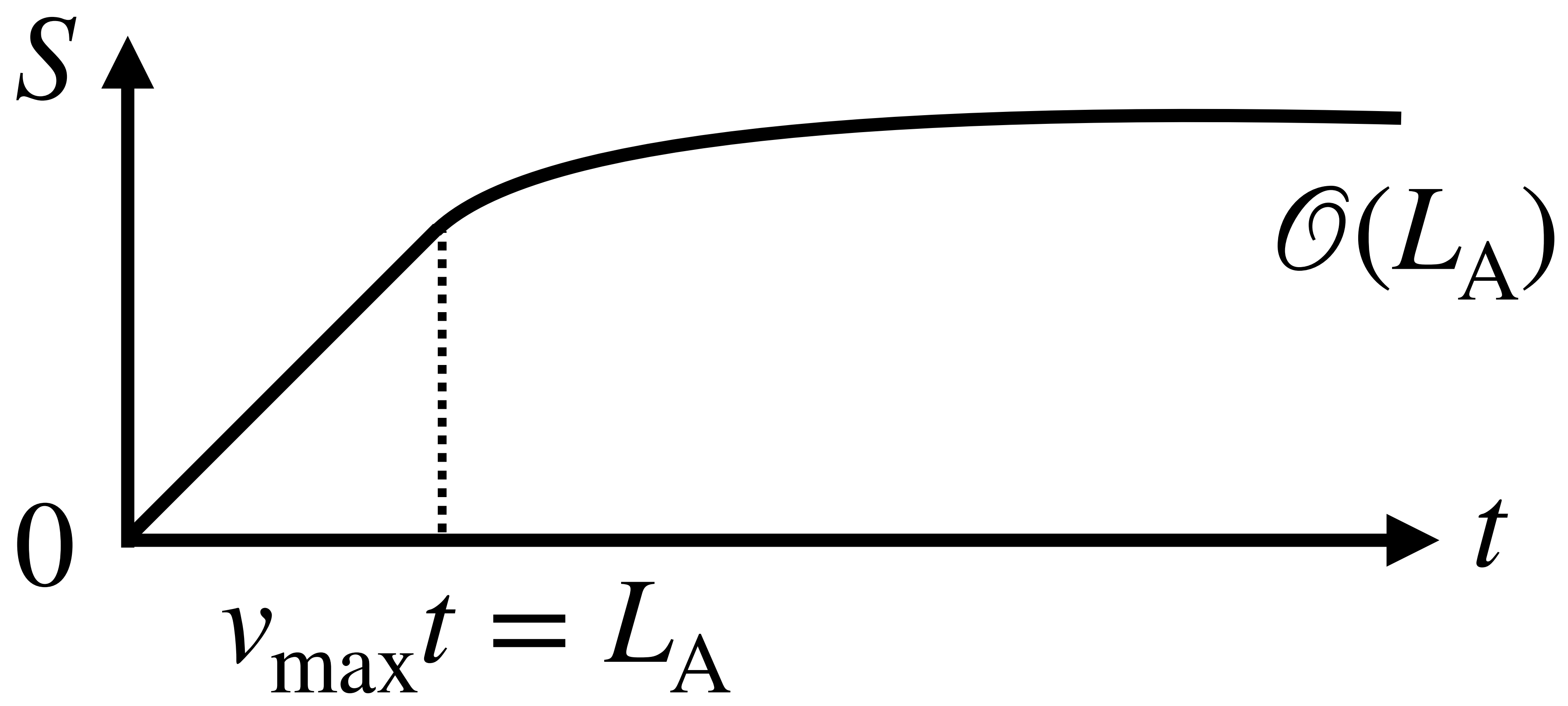
Quasiparticle picture for quench dynamics of entanglement

The quench dynamics of entanglement of integrable models is well explained by quasiparticle picture

P. Calabrese and J. Cardy, J. Stat. Mech. (2005)

V. Alba and P. Calabrese, PNAS (2017)





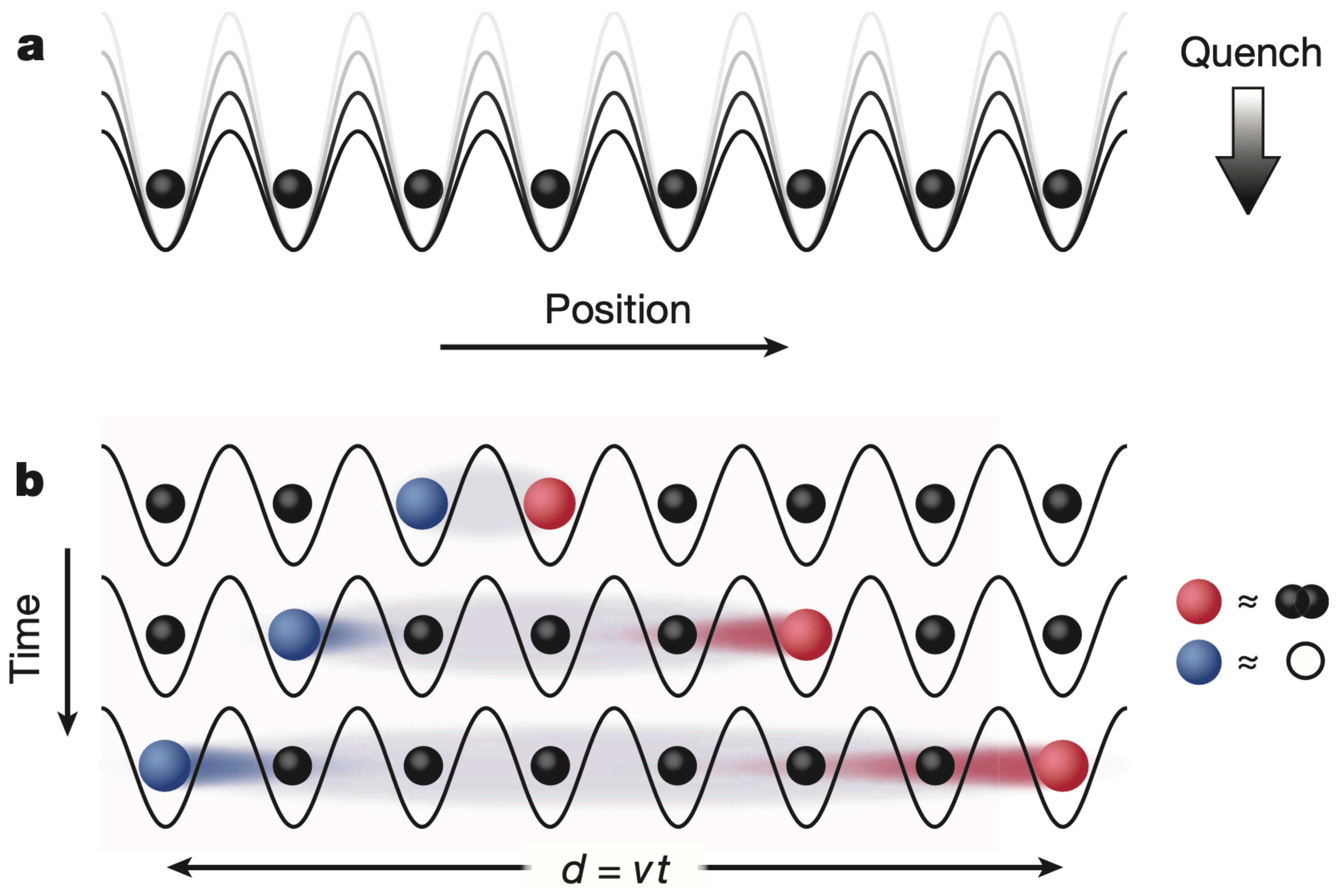
- Quasiparticles are excited in pairs by a quench and they have opposite momentum
- Quasiparticle pairs contribute to entanglement entropy when they are across the boundary between subsystem A and B

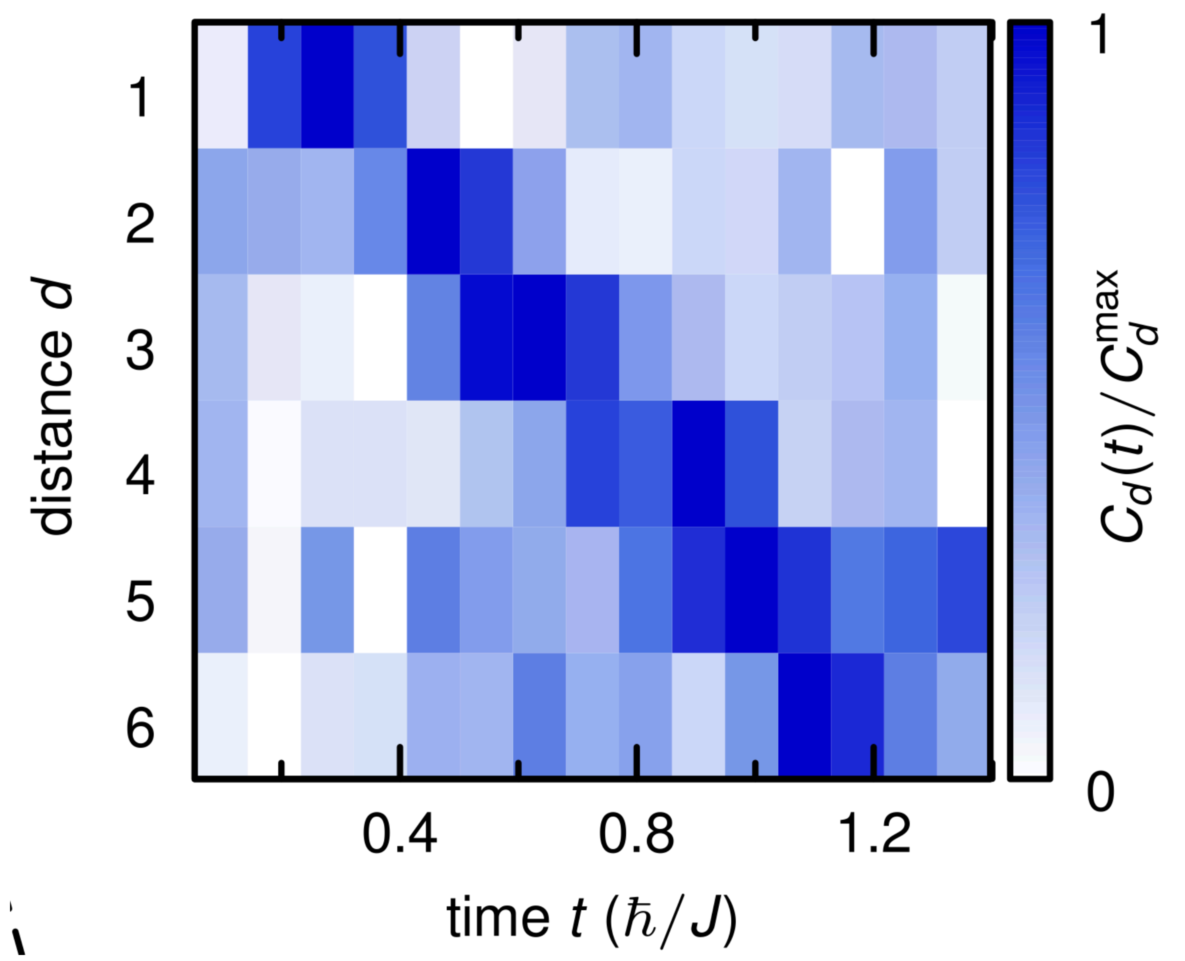
- Linear growth for $v_{\mathrm{max}}t < L_{\mathrm{A}}$
- Saturation at an $\mathcal{O}(L_{\mathrm{A}})$ value

v_{max}: maximal velocity of pairs

Quasiparticle dynamics in strongly-interacting bosons

Quasiparticle dynamics in strongly-interacting bosons

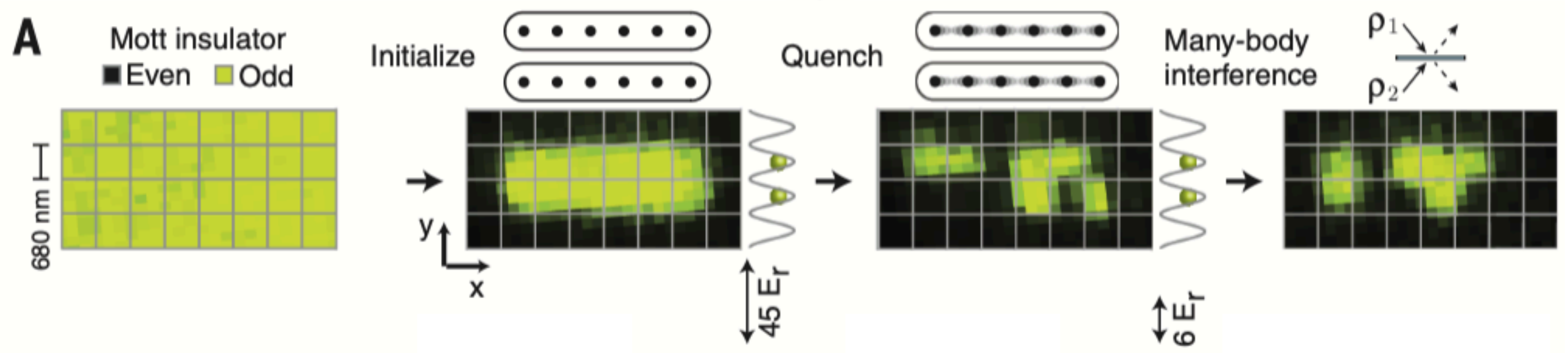


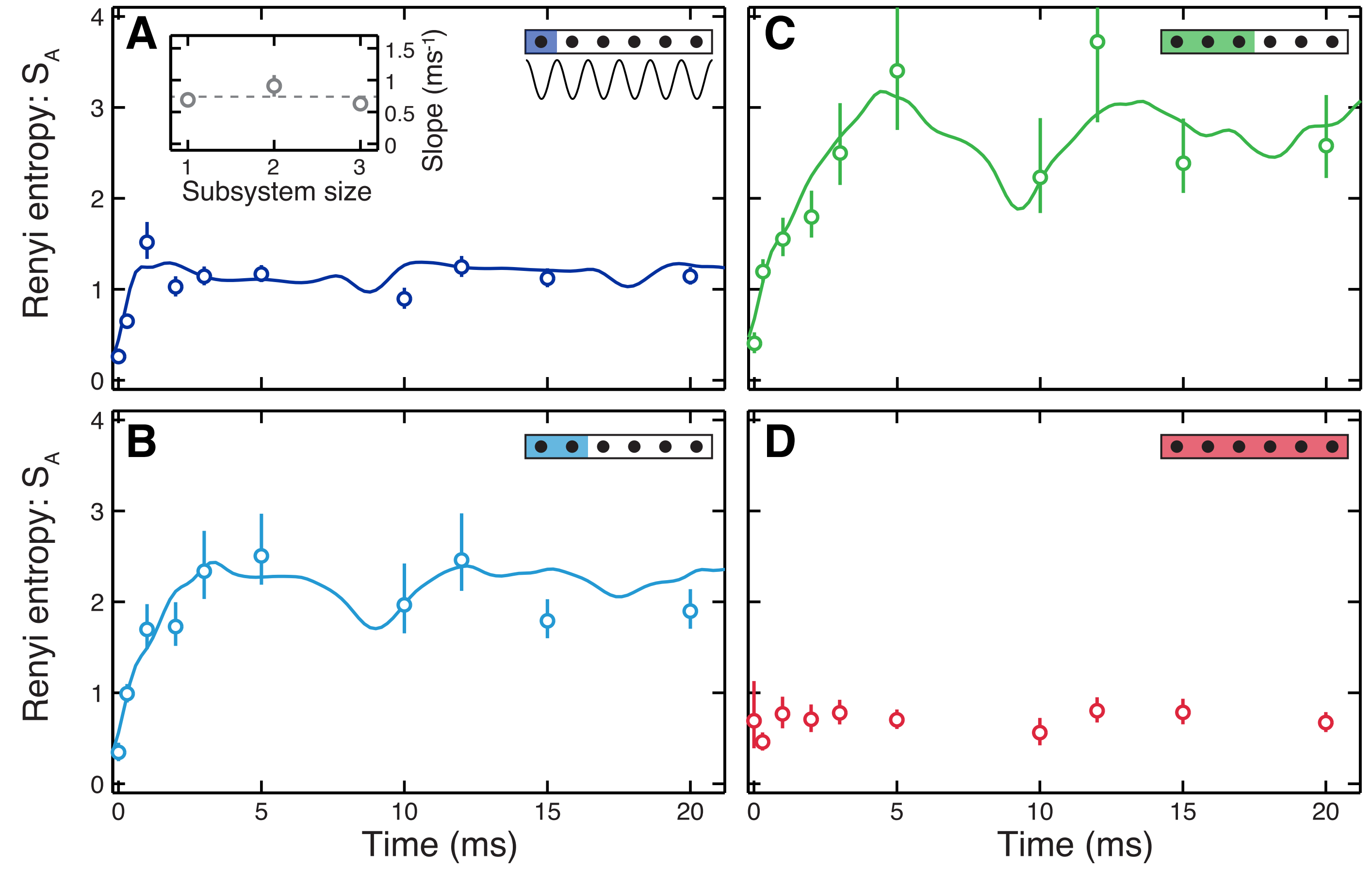


M. Cheneau, *et al.,* Nature (2012)

Quasiparticle (doublon and holon)

Measurement of entanglement





R. Islam *et al.*, Nature (2015);
A. M. Kaufman *et al.*, Science (2016)

<u>We can experimentally check the quasiparticle picture in this system.</u>

Mapping to Effective Hamiltonian from BHM

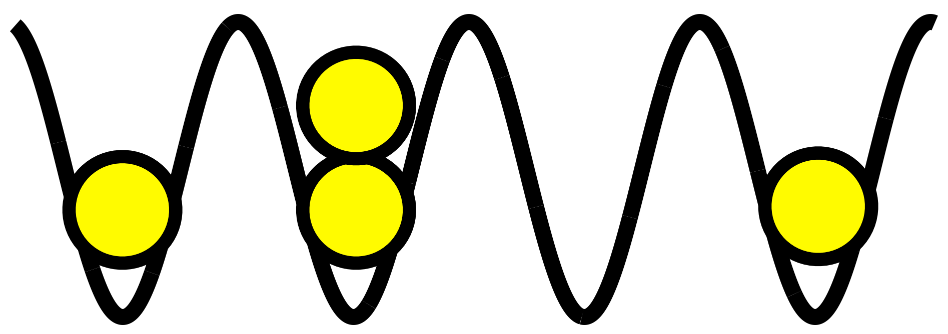
Bose-Hubbard model (BHM)

$$\hat{H}_{\mathrm{BHM}} = -J\sum_{j=1}^{L-1}\left(\hat{b}_j^\dagger \hat{b}_{j+1} + \hat{b}_{j+1}^\dagger \hat{b}_j\right) + \frac{U}{2}\sum_{j=1}^{L}\hat{n}_j(\hat{n}_j - 1)$$

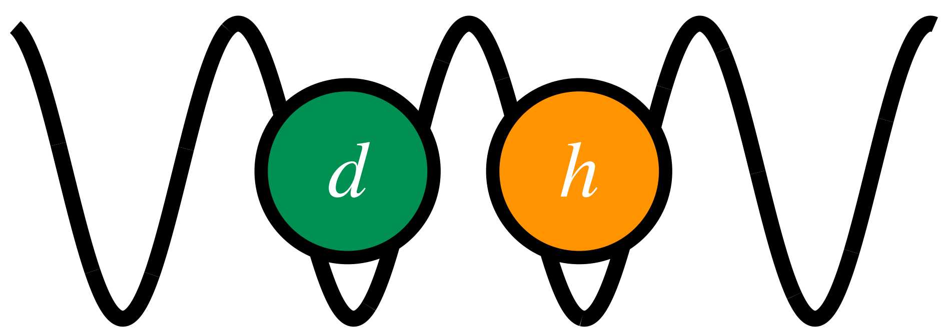
$U \gg J$

- restrict to the truncated Hilbert space spanned by $|0\rangle_j, |1\rangle_j, |2\rangle_j$.
- introduce **fermion** quasiparticle, doublon and holon, by the generalized Jordan-Wigner transformation C. D. Batista and G. Ortiz, PRL (2001)
- Relax the constraint of double occupancy of doublon and holon

$$|2\rangle_j \to \hat{d}_j^\dagger |\mathrm{vac}\rangle$$
$$|0\rangle_j \to \hat{h}_j^\dagger |\mathrm{vac}\rangle$$
$$|\mathrm{MI}\rangle \to |\mathrm{vac}\rangle$$



Boson picture



Doublon-holon picture

Constraint

$$\hat{d}_j^\dagger \hat{d}_j \hat{h}_j^\dagger \hat{h}_j = 0$$
$$\mathcal{O}\left((J/U)^4\right)$$

Effective Hamiltonian

$$\hat{H}_{\mathrm{eff}} = J\sum_j \left[2\hat{d}_{j+1}^\dagger \hat{d}_j + \hat{h}_{j+1}^\dagger \hat{h}_j + \sqrt{2}\left(\hat{d}_j^\dagger \hat{h}_{j+1}^\dagger - \hat{h}_j \hat{d}_{j+1}\right) + \mathrm{h.c.}\right] + \frac{U}{2}\sum_j (\hat{d}_j^\dagger \hat{d}_j + \hat{h}_j^\dagger \hat{h}_j)$$

Wavefunction and entangled pairs

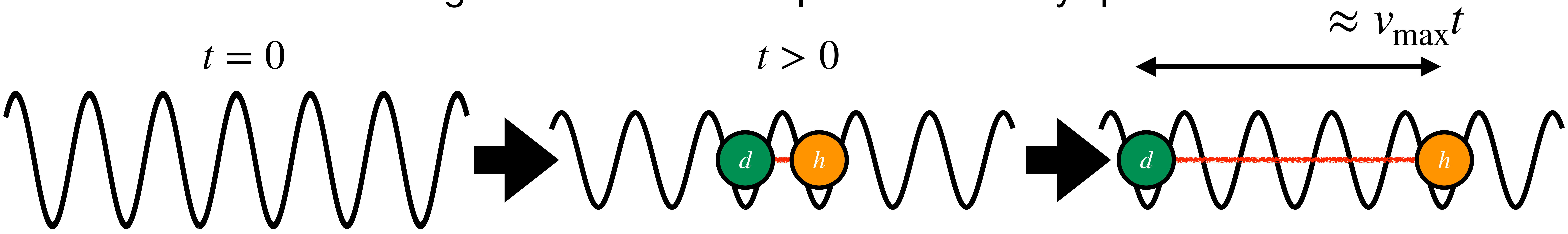
Bogoliubov transformation: $\hat{H}_{\text{eff}} = \sum_k (\varepsilon_{d,k}\hat{\gamma}^\dagger_{d,k}\hat{\gamma}_{d,k} + \varepsilon_{h,k}\hat{\gamma}^\dagger_{h,k}\hat{\gamma}_{h,k})$

Wave function

$$|\Psi(t)\rangle = e^{-i\hat{H}_{\text{BHM}}t}|\text{MI}\rangle \to e^{-i\hat{H}_{\text{eff}}t}|\text{vac}\rangle$$

$$= |\text{vac}\rangle + \frac{J}{U}\sum_k \sin(k)\left[1 - e^{-i(\varepsilon_{d,k}+\varepsilon_{k,h})t)}\right](\hat{d}^\dagger_k\hat{h}^\dagger_{-k} + \hat{h}^\dagger_k\hat{d}^\dagger_{-k})|\text{vac}\rangle + \mathcal{O}((J/U)^2)$$

Entangled doublon-holon pairs excited by quench



Gaussian state and entanglement entropy

- $|\Psi(t)\rangle = e^{-i\hat{H}_{\rm eff}t}|\text{vac}\rangle$ is a Gaussian state.
- The reduced density matrix $\hat{\rho}_{\rm A} = {\rm Tr}_{\rm B}\hat{\rho}$ can be constructed from correlation functions M

$$M_{i,j} = \langle \Psi(t)|\hat{a}_i\hat{a}_j^\dagger|\Psi(t)\rangle,\ \hat{a} = (\hat{d}, \hat{h}, \hat{d}^\dagger, \hat{h}^\dagger)^T$$

I. Peschel, J. Phys. A: Math. Gen. (2003)
I. Frérot and T. Roscilde PRB (2015)

- An effective Hamiltonian is valid when $J/U \ll 1$

Main Result

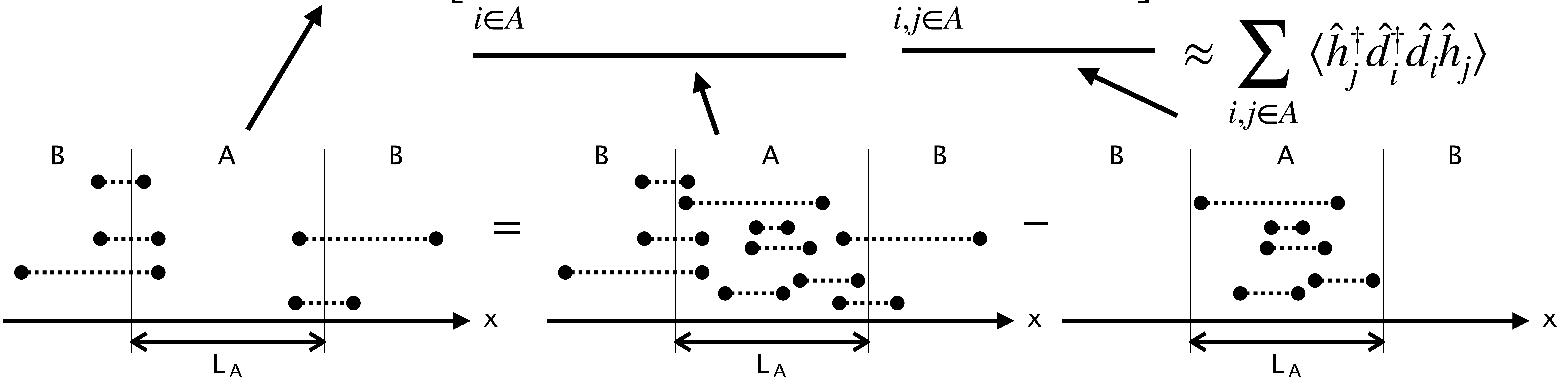
$$S_2 = 2\Big[\sum_{i\in A}(\langle \hat{d}_i^\dagger\hat{d}_i\rangle + \langle \hat{h}_i^\dagger \hat{h}_i\rangle) - \sum_{i,j\in A}|\langle \hat{d}_i\hat{h}_j\rangle|^2\Big] + \mathcal{O}((J/U)^3)$$

$$S_2 = 16\left(\frac{J}{U}\right)^2(L_{\rm A}+1) - 32\left(\frac{J}{U}\right)^2\cos(Ut)\frac{\mathcal{J}_1(Jt)}{3Jt} - 16\left(\frac{J}{U}\right)^2\sum_{n=1}^{L_{\rm A}}(L_{\rm A}-n)n^2\left[\frac{\mathcal{J}_n(jt)}{3Jt}\right]^2 + \mathcal{O}\left(\left(\frac{J}{U}\right)^3\right)$$

Interpretation of Rényi entanglement entropy

$$S_2 = 2\Big[\sum_{i\in A}(\langle \hat{d}_i^\dagger \hat{d}_i\rangle + \langle \hat{h}_i^\dagger \hat{h}_i\rangle) - \sum_{i,j\in A} |\langle \hat{d}_i \hat{h}_j\rangle|^2\Big] + \mathcal{O}((J/U)^3)$$

$$\approx \sum_{i,j\in A} \langle \hat{h}_j^\dagger \hat{d}_i^\dagger \hat{d}_i \hat{h}_j\rangle$$



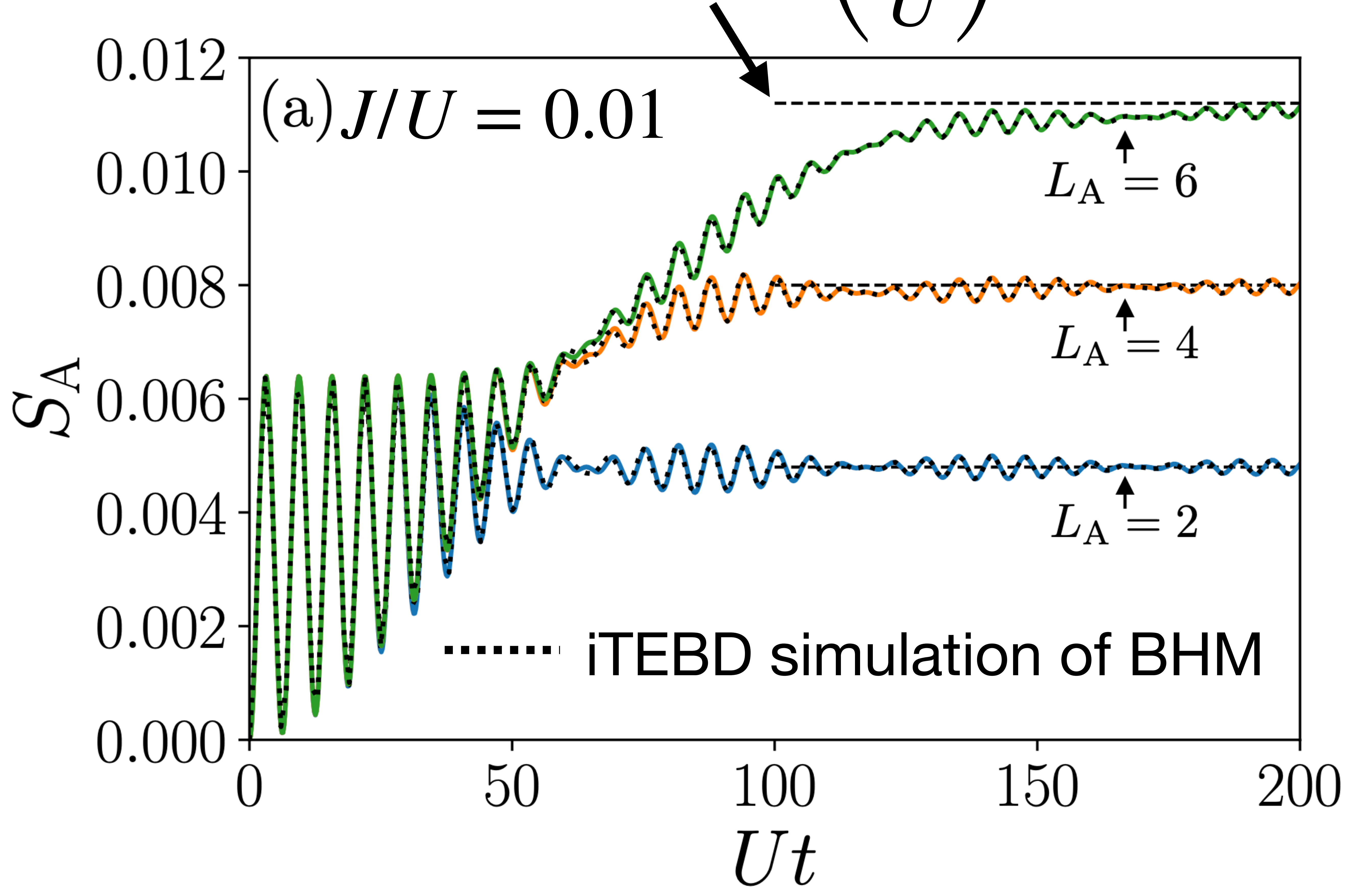
# of pairs across boundary

# of quasiparticles in A
= # of pairs across boundary
+ # of pairs in A

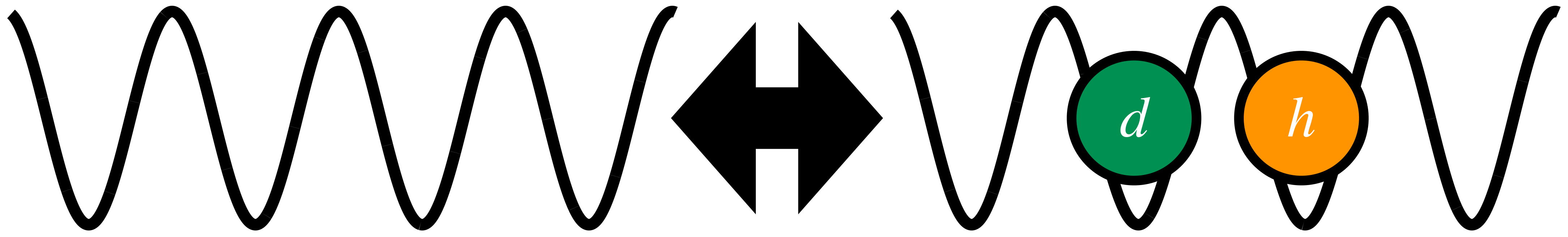
# of pairs in A

Short time dynamics of Rényi entanglement entropy

steady-state value: $16\left(\frac{J}{U}\right)^2 (L_A + 1)$



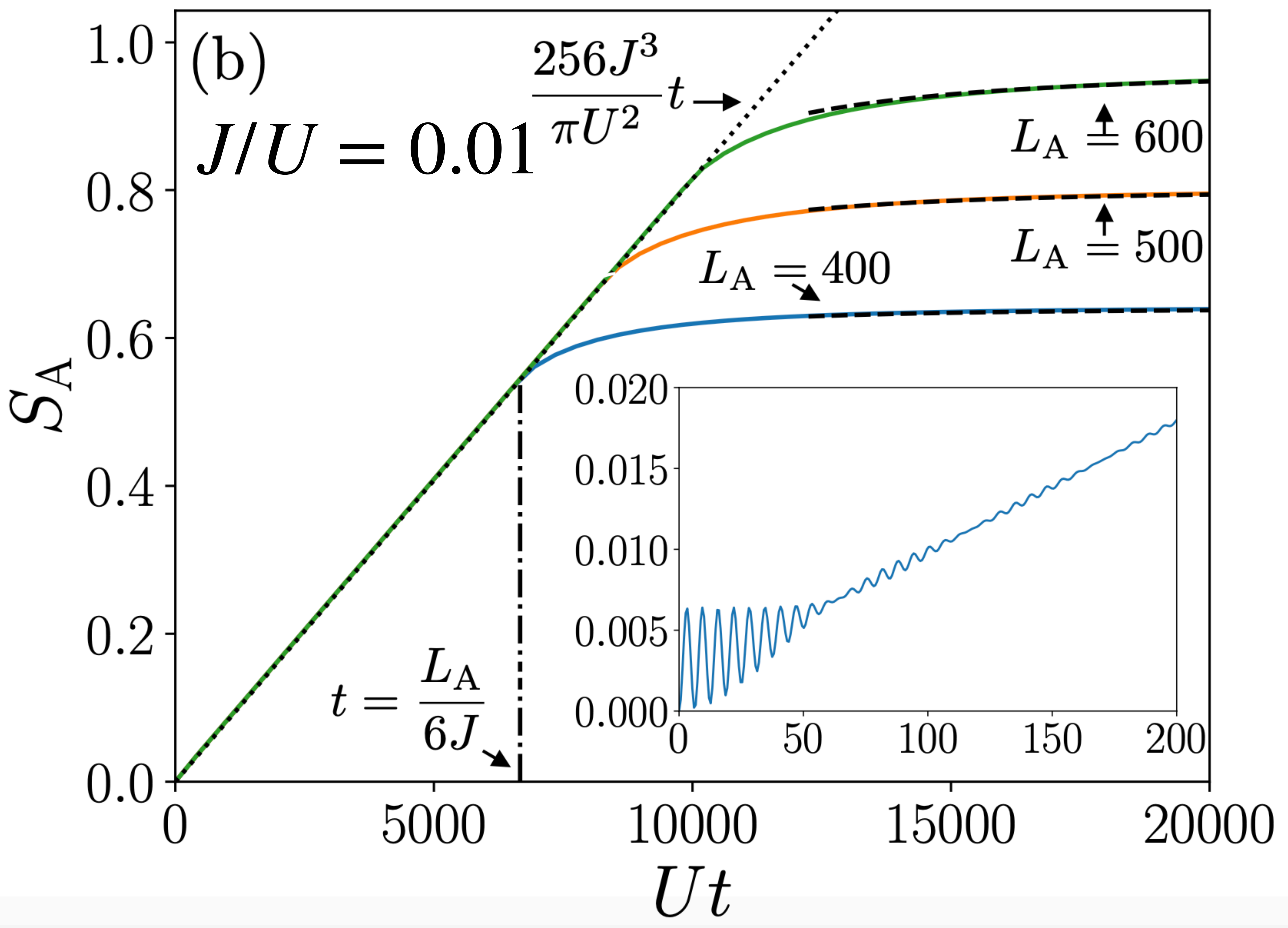
- Good agreement with iTEBD simulation of the original Bose-Hubbard model
- Oscillation with the frequency $2\pi/U$



$\Delta E = U$

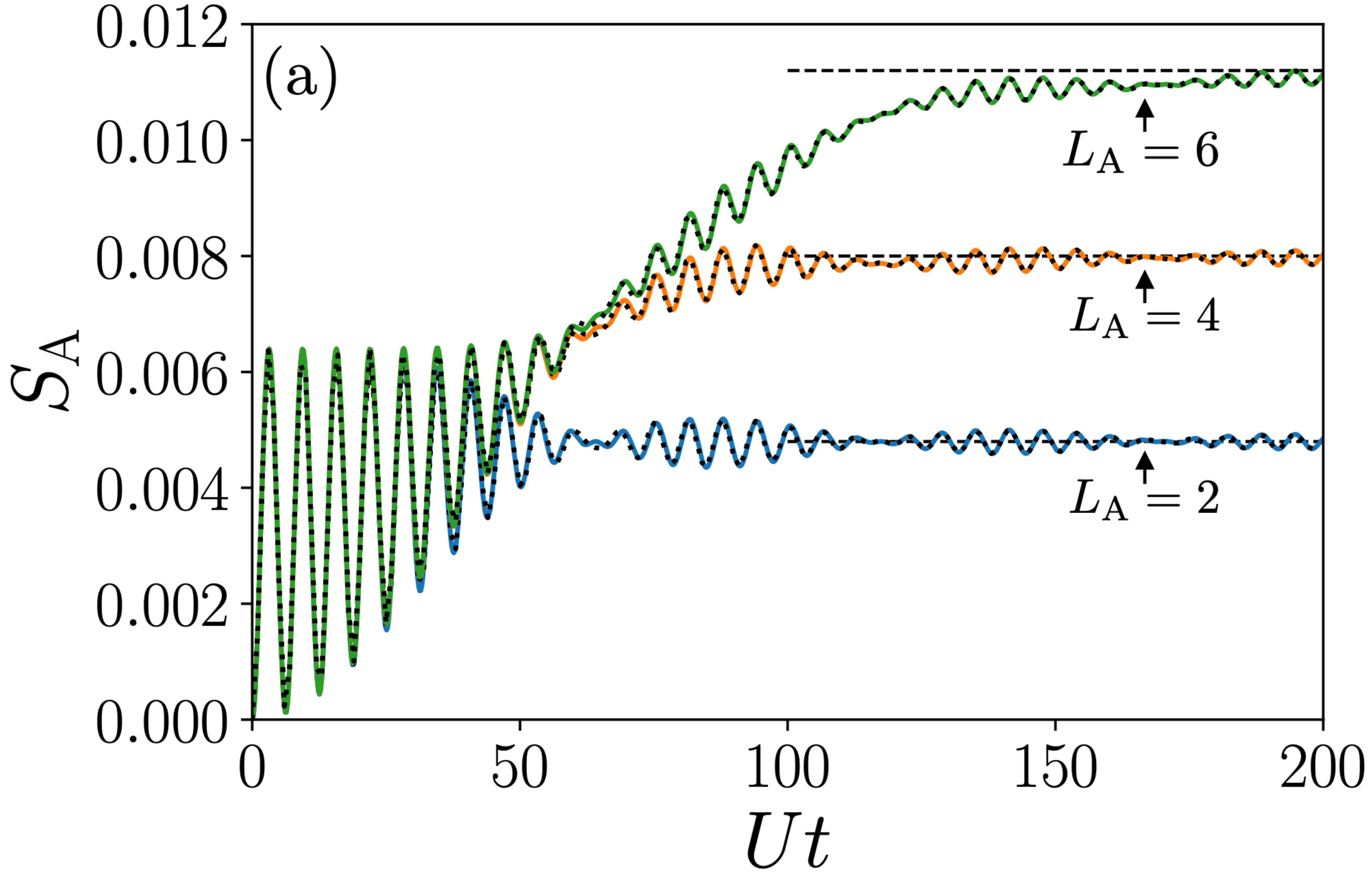
- Saturated at $\mathcal{O}(L_A)$ steady state value

Long time dynamics of Rényi entanglement entropy



- Linear growth for $v_{\max}t = 6Jt < L_A$
- Saturation at an $\mathcal{O}(L_A)$ value

Consistent with the quasiparticle picture



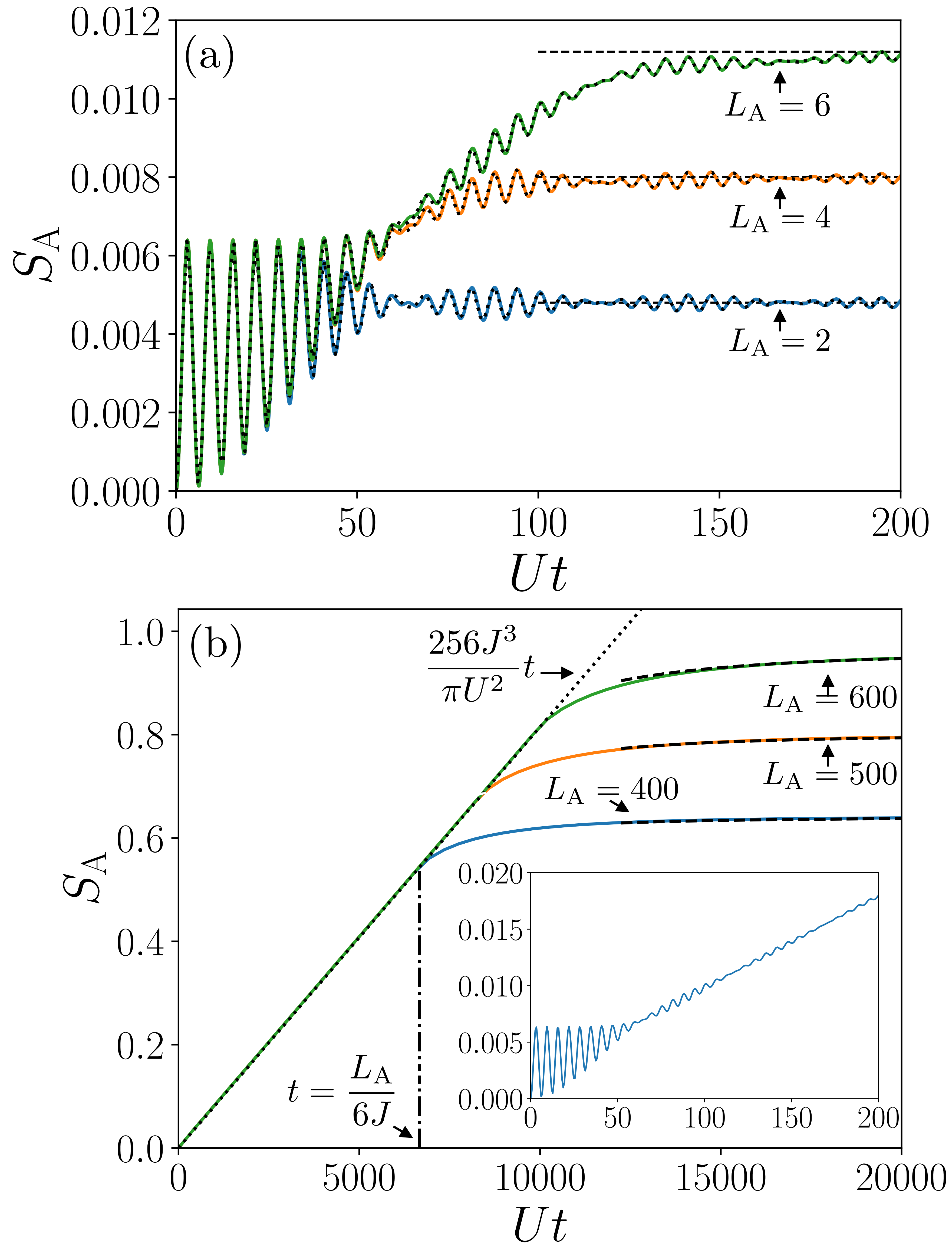
Measuring
- Saturation value
- Timescale of saturation

contribute to the verification of the quasiparticle picture.

Summary of part 2

Ref) arXiv:2209.13340.

- We have investigated Rényi entanglement entropy in 1D bosons in an optical lattice quenched $J/U \gg 1$ from Mott insulating states.
- We have obtained an analytical expression for 2nd order Rényi entanglement entropy.
- Our prediction could be checked by experiment.



Future Problem

- Our approach would be applicable to free fermion systems for small quench

$$S_2 = 2\Big[\sum_{i\in A}\langle \hat{a}_i^\dagger \hat{a}_i\rangle - \sum_{i,j\in A} |\langle \hat{a}_i \hat{a}_j\rangle|^2\Big] + \mathcal{O}(J^3)$$

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   Ref) Phys. Rev. A **107**, 033305 (2023)

3. Entanglement dynamics of strongly-interacting Bosons
   - -Model and low-energy effective theory
   - -Analytical result of Rényi entanglement entropy

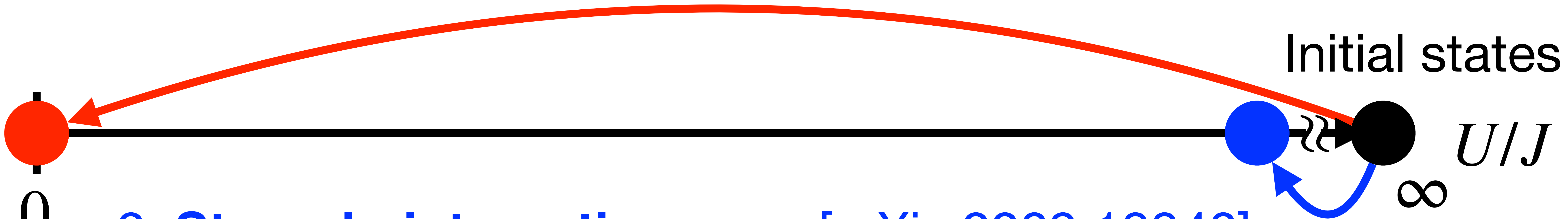
   Ref) arXiv:2209.13340.

4. **Summary**

Summary of this talk

$$S_2 = -\ln \mathrm{perm} A_Z,\ A_Z = \begin{pmatrix} I-Z & Z \\ Z & I-Z \end{pmatrix},\ z_{j,l} = \sum_{m\in \mathrm{A}} y^*_{j,m}(t) y_{l,m}(t)$$

1. **<u>Non-interacting case</u>** [Phys. Rev. A **107**, 033305 (2023)]



2. **<u>Strongly-interacting case</u>** [arXiv:2209.13340]

$$S_2 = 2\left[\sum_{i\in A}(\langle \hat{d}_i^\dagger \hat{d}_i\rangle + \langle \hat{h}_i^\dagger \hat{h}_i\rangle) - \sum_{i,j\in A} |\langle \hat{d}_i \hat{h}_j\rangle|^2\right] + \mathcal{O}((J/U)^3)$$